Existence and stability of viscous vortices

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Abstract

Vorticity plays a prominent role in the dynamics of incompressible viscous flows. In two-dimensional freely decaying turbulence, after a short transient period, evolution is essentially driven by interactions of viscous vortices, the archetype of which is the self-similar Lamb-Oseen vortex. In three dimensions, amplification of vorticity due to stretching can counterbalance viscous dissipation and produce stable tubular vortices. This phenomenon is illustrated in a famous model originally proposed by Burgers, where a straight vortex tube is produced by a linear uniaxial strain field. In real flows vortex lines are usually not straight, and can even form closed curves, as in the case of axisymmetric vortex rings which are very common in nature and in laboratory experiments. The aim of this chapter is to review a few rigorous results concerning existence and stability of viscous vortices in simple geometries.

1 Introduction

Since the pioneering work of Helmholtz [24], vorticity has been widely recognized as a quantity of fundamental importance in fluid dynamics, especially for turbulent flows. According to a famous quote by Küchermann [28], "vortices are the sinews and muscles of fluid motions". Intuitively, vorticity describes the local rotation of fluid particles at a given point. In the Eulerian representation, if $\mathbf{u}(\mathbf{x},t)$ denotes the velocity of the fluid at point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$, the vorticity is the vector $\boldsymbol{\omega}(\mathbf{x},t) = \operatorname{curl} \mathbf{u}(\mathbf{x},t) = \nabla \wedge \mathbf{u}(\mathbf{x},t)$. Under the evolution given by the Navier-Stokes equations, the vorticity satisfies

$$\partial_t \boldsymbol{\omega}(\mathbf{x},t) + (\mathbf{u}(\mathbf{x},t),\nabla)\boldsymbol{\omega}(\mathbf{x},t) - (\boldsymbol{\omega}(\mathbf{x},t),\nabla)\mathbf{u}(\mathbf{x},t) \;=\; \nu\Delta\boldsymbol{\omega}(\mathbf{x},t)\,, \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, i.e. the ratio of the viscosity to the fluid density. In the incompressible case considered here, the velocity field satisfies $\operatorname{div} \mathbf{u}(\mathbf{x},t) = 0$ and is thus entirely determined by the vorticity distribution up to an irrotational flow. The Biot-Savart law is a reconstruction formula that expresses $\mathbf{u}$ in terms of $\boldsymbol{\omega}$, depending on the geometry of the fluid domain and the boundary conditions. In the whole space $\mathbb{R}^3$, if the vorticity distribution is sufficiently localized, the Biot-Savart formula reads

$$\mathbf{u}(\mathbf{x},t) \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\mathbf{x}-\mathbf{y})\wedge\boldsymbol{\omega}(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}\mathbf{y}\,. \tag{1.2}$$

In viscous fluids, vorticity is usually created within boundary layers near walls or interfaces, or in the vicinity of a stirring device. Once produced, vorticity can be substantially amplified by the local strain in the fluid, through a genuinely three-dimensional mechanism that is often referred to as "vortex stretching". A Taylor expansion of the velocity field at a given point $\mathbf{x}_0$ reveals that

$$\mathbf{u}(\mathbf{x},t) \;=\; \mathbf{u}(\mathbf{x}_0,t) + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}_0,t)\wedge(\mathbf{x}-\mathbf{x}_0) + (D\mathbf{u}(\mathbf{x}_0,t))(\mathbf{x}-\mathbf{x}_0) + \mathcal{O}(|\mathbf{x}-\mathbf{x}_0|^2)\,,$$

where $D\mathbf{u} = \frac{1}{2}\big((\nabla\mathbf{u}) + (\nabla\mathbf{u})^\top\big)$ is the deformation tensor, whose eigenvalues $\gamma_1, \gamma_2, \gamma_3$ are called the principal strains at $\mathbf{x}_0$. Incompressibility implies that $\gamma_1+\gamma_2+\gamma_3 = 0$, so that two generic situations may occur. If two principal strains (say, γ_1 and γ_2) are negative and the third one is positive, vorticity gets amplified at $\mathbf{x}_0$ in the direction of the principal strain axis corresponding to γ_3. That stretching mechanism can compensate the viscous dissipation and result in the formation of stable vortex filaments, a typical example being the Burgers vortex [2] which will be studied in Section 4. In contrast, if two principal strains are positive at $\mathbf{x}_0$, the stretching effect leads to the formation of vortex sheets, which are also commonly observed in turbulent flows although they undergo the Kelvin-Helmholtz instability at high Reynolds numbers. Vortex sheets play a prominent role in interfacial motion and boundary layer theory, and the interested reader is referred to the chapter entitled "The Inviscid Limit and Boundary Layers for the Navier-Stokes Flows" for further information.

In the present chapter, emphasis is put on vortex tubes or filaments, for which vorticity is essentially concentrated along a curve with no endpoints in the fluid. In general, the curve will evolve with time, because it is advected by the flow. According to Helmholtz's first law, the *total circulation* of such a vortex filament is constant along its length, and is also independent of time as long as the viscous effects can be neglected. This very important quantity, often denoted by Γ, can be defined as the flux of the vorticity vector through any cross section of the vortex tube, or equivalently (in view of Stokes' theorem) as the circulation of the velocity along any closed curve enclosing that tube. The ratio $\alpha = \Gamma/\nu$ of the total circulation to the kinematic viscosity is a dimensionless quantity, sometimes referred to as the circulation Reynolds number, which measures the strength of the vortex and plays a crucial role in stability issues.

Since vortex filaments have no endpoints, they must either extend to the fluid boundary, or to infinity, or form closed curves. In the simple situation, already considered in [24], where all vortex lines are straight and parallel to each other, the velocity and vorticity fields take the particular form

$$\mathbf{u}(\mathbf{x},t) \;=\; \begin{pmatrix} u_1(x_1,x_2,t) \\ u_2(x_1,x_2,t) \\ 0 \end{pmatrix}\,, \qquad \boldsymbol{\omega}(\mathbf{x},t) \;=\; \begin{pmatrix} 0 \\ 0 \\ \omega(x_1,x_2,t) \end{pmatrix}\,, \tag{1.3}$$

where $x = (x_1,x_2) \in \mathbb{R}^2$ and $\omega = \partial_1 u_2 - \partial_2 u_1$. Here the coordinates have been chosen so that the third axis coincides with the direction of the vortex filaments. The evolution equation for the scalar vorticity $\omega(x,t)$ is

$$\partial_t\omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) \;=\; \nu\Delta\omega(x,t)\,, \tag{1.4}$$

where $u = (u_1,u_2)$ satisfies $\partial_1 u_1 + \partial_2 u_2 = 0$. If the vorticity distribution is sufficiently localized, the two-dimensional velocity field $u(x,t)$ is given by the 2D Biot-Savart law

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,\mathrm{d}y\,, \tag{1.5}$$

where $x^\perp = (-x_2,x_1)$ and $|x|^2 = x_1^2 + x_2^2$. Eq. (1.4) is just an advection-diffusion equation for the scalar quantity ω, hence (by the maximum principle) no amplification of vorticity can occur in the two-dimensional case. As a consequence, all localized vortex structures will eventually spread out and decay, since there is nothing to counterbalance the effect of viscosity. A typical example is provided by the *Lamb-Oseen vortex*, an exact self-similar solution to (1.4) of the form

$$\omega(x,t) \,=\, \frac{\Gamma}{\nu t}\, G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \qquad u(x,t) \,=\, \frac{\Gamma}{\sqrt{\nu t}}\, v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \tag{1.6}$$

where the vorticity and velocity profiles are explicitly given by

$$G(\xi) \,=\, \frac{1}{4\pi}\, e^{-|\xi|^2/4}\,, \qquad v^G(\xi) \,=\, \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big)\,, \qquad \xi \in \mathbb{R}^2\,. \tag{1.7}$$

Note that $\int_{\mathbb{R}^2} G(\xi)\,\mathrm{d}\xi = 1$, so that $\int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x = \Gamma$ for all $t > 0$, in agreement with the general definition of the total circulation Γ. The Lamb-Oseen vortex plays a distinguished role in the dynamics of the two-dimensional vorticity equation (1.4), for two main reasons. First, it deserves the name of fundamental solution, in the sense that it is the unique solution of (1.4) with initial data $\omega_0 = \Gamma\delta_0$, where δ_0 denotes the Dirac measure at the origin. Next, it describes to leading order the long-time asymptotics of all solutions of (1.4) with integrable initial data and nonzero circulation [19]. If self-similar variables are used, the Lamb-Oseen vortex becomes a stationary solution of some rescaled equation, and its stability properties can then be studied using spectral theory and other standard techniques. This analysis is presented in Section 2 below, and serves as a model for further existence and stability results in more complex situations.

Another relatively simple and mathematically tractable situation is the axisymmetric case without swirl, where the velocity field is invariant under rotations about a given axis, and under reflections by any plane containing the axis. Here all vortex lines are circles centered on the symmetry axis and normal to it. Using cylindrical coordinates (r,θ,z), so that r represents the distance to the symmetry axis and z the position along the axis, the velocity and vorticity fields are given by

$$\mathbf{u}(\mathbf{x},t) \,=\, u_r(r,z,t)\,\mathbf{e}_r + u_z(r,z,t)\,\mathbf{e}_z\,, \qquad \boldsymbol{\omega}(\mathbf{x},t) \,=\, \omega_\theta(r,z,t)\,\mathbf{e}_\theta\,, \tag{1.8}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ denote unit vectors in the radial, toroidal, and vertical directions, respectively. As in the two-dimensional case, the vorticity vector has only one nonzero component ω_θ, which satisfies the evolution equation

$$\partial_t\omega_\theta + u\cdot\nabla\omega_\theta - \frac{u_r}{r}\omega_\theta \,=\, \nu\Big(\Delta\omega_\theta - \frac{\omega_\theta}{r^2}\Big)\,, \tag{1.9}$$

where $u\cdot\nabla = u_r\partial_r + u_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ denotes the Laplace operator in cylindrical coordinates. The velocity $u = (u_r, u_z)$ can be expressed in terms of the axisymmetric vorticity ω_θ by solving the linear elliptic system

$$\partial_r u_r + \frac{1}{r}u_r + \partial_z u_z \,=\, 0\,, \qquad \partial_z u_r - \partial_r u_z \,=\, \omega_\theta\,, \tag{1.10}$$

in the half-plane $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0\,,\ z \in \mathbb{R}\}$, with boundary conditions $u_r = \partial_r u_z = 0$ at $r = 0$. Explicit formulas for the axisymmetric Biot-Savart law exist, see e.g. [6, 16], but are more involved than in the two-dimensional case. The analogue of the Lamb-Oseen vortex for axisymmetric flows is the solution of (1.9) with a vortex filament as initial data. This means that $\omega_\theta(\cdot,\cdot,0) = \Gamma\delta_{(\bar r,\bar z)}$ where $\delta_{(\bar r,\bar z)}$ denotes the Dirac measure located at some point $(\bar r,\bar z) \in \Omega$. Existence of a global solution to (1.9) with such initial data was recently shown by Feng and Šverák [6], and uniqueness can be established using, in particular, the approach presented in

Section 2, see [17]. The reader is referred to Section 3 below for up-to-date results on existence of axisymmetric vortex rings.

The third and final case considered here is a famous model for vortex filaments in turbulent flows, originally proposed by Burgers [2]. It is assumed that the velocity field has the form $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_s(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)$, where $\mathbf{u}_s(\mathbf{x})$ is a stationary straining flow of the form

$$\mathbf{u}_s(\mathbf{x}) \;=\; \begin{pmatrix}\gamma_1 x_1\\ \gamma_2 x_2\\ \gamma_3 x_3\end{pmatrix} \;=\; \mathcal{M}\mathbf{x}\,, \qquad \mathcal{M} \;=\; \begin{pmatrix}\gamma_1 & 0 & 0\\ 0 & \gamma_2 & 0\\ 0 & 0 & \gamma_3\end{pmatrix}\,, \tag{1.11}$$

where $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\gamma_1, \gamma_2 < 0$, $\gamma_3 > 0$. According to the discussion above, the strain (1.11) describes to leading order the deformation rate of any smooth, incompressible velocity field near the origin, at a given time. Burgers' model is crude in the sense that it assumes that the strain $\mathbf{u}_s(\mathbf{x})$ is independent of time and extends all the way to infinity in space, which is certainly not realistic in turbulent flows. Nevertheless, the model is interesting because it clearly illustrates the vortex stretching effect, which in the present case produces a family of stationary solutions that can be compared with observations in experiments.

If $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_s(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)$, the vorticity equation (1.1) can be written in equivalent form

$$\partial_t \boldsymbol{\omega}(\mathbf{x},t) + (\mathbf{v}(\mathbf{x},t),\nabla)\boldsymbol{\omega}(\mathbf{x},t) - (\boldsymbol{\omega}(\mathbf{x},t),\nabla)\mathbf{v}(\mathbf{x},t) \;=\; \mathcal{L}\boldsymbol{\omega}(\mathbf{x},t)\,, \tag{1.12}$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\boldsymbol{\omega} \;=\; \nu\Delta\boldsymbol{\omega} - (\mathcal{M}\mathbf{x},\nabla)\boldsymbol{\omega} + \mathcal{M}\boldsymbol{\omega}\,. \tag{1.13}$$

As $\operatorname{div}\mathbf{v} = 0$ and $\operatorname{curl}\mathbf{v} = \boldsymbol{\omega}$, the Biot-Savart law (1.2) can be used to reconstruct the time-dependent velocity field $\mathbf{v}$ from the vorticity distribution $\boldsymbol{\omega}$. In addition to the Laplacian, the linear operator $\mathcal{L}$ includes an advection term that depends linearly on the space variable $\mathbf{x}$, and a zero order term involving the strain matrix $\mathcal{M}$ whose main effect is to amplify the third component ω_3 while attenuating ω_1 and ω_2.

The Burgers vortex is a stationary solution of (1.12) which results from the balance between the amplification of vorticity due to stretching and the dissipation due to viscosity. In the axisymmetric case where $\gamma_1 = \gamma_2 = -\gamma/2$ and $\gamma_3 = \gamma > 0$, it has the explicit form

$$\boldsymbol{\omega}(\mathbf{x}) \;=\; \Gamma\,\frac{\gamma}{\nu}\,\mathbf{G}\Bigl(x\sqrt{\gamma/\nu}\Bigr)\,, \qquad \mathbf{v}(\mathbf{x}) \;=\; \Gamma\sqrt{\frac{\gamma}{\nu}}\,\mathbf{v}^G\Bigl(x\sqrt{\gamma/\nu}\Bigr)\,, \tag{1.14}$$

where Γ is the total circulation and

$$\mathbf{G}(\xi) \;=\; \begin{pmatrix}0\\ 0\\ G(\xi)\end{pmatrix}\,, \qquad \mathbf{v}^G(\xi) \;=\; \frac{1}{2\pi|\xi|^2}\Bigl(1 - e^{-|\xi|^2/4}\Bigr)\begin{pmatrix}-\xi_2\\ \xi_1\\ 0\end{pmatrix}\,. \tag{1.15}$$

The striking similarity with the corresponding expressions (1.6), (1.7) for the Lamb-Oseen vortex is of course not an accident. Indeed, if the pair $\boldsymbol{\omega}(\mathbf{x},t), \mathbf{v}(\mathbf{x},t)$ is a solution of (1.12) that is two-dimensional in the sense that $\partial_3\boldsymbol{\omega} = \partial_3\mathbf{v} \equiv 0$ and $\omega_1 = \omega_2 \equiv 0$, then the pair $\omega(x,t), u(x,t)$ defined by *Lundgren's transformation* [31]

$$\omega(x,t) \;=\; \frac{1}{\gamma t}\,\omega_3\Bigl(\frac{x}{\sqrt{\gamma t}}\,,\,\frac{1}{\gamma}\log(\gamma t)\Bigr)\,, \qquad u(x,t) \;=\; \frac{1}{\sqrt{\gamma t}}\,v\Bigl(\frac{x}{\sqrt{\gamma t}}\,,\,\frac{1}{\gamma}\log(\gamma t)\Bigr)\,, \tag{1.16}$$

satisfies the two-dimensional vorticity equation (1.4). In other words, the two-dimensional solutions of equation (1.12), which includes an axisymmetric linear straining field, are in one-to-one correspondence with those of the two-dimensional vorticity equation (1.4), via a self-similar change of variables. This observation plays a crucial role both in Section 2, where stability of the Lamb-Oseen vortex is studied, and in Section 4 where the corresponding results for the axisymmetric Burgers vortex are presented. There is however an important difference between both situations: although the Burgers vortex is a two-dimensional stationary solution of (1.12), there is no reason to restrict the stability analysis to perturbations in the same class. Quite the contrary, the Burgers vortex can be a relevant model for tubular structures in turbulent flows only if one can prove stability with respect to general three-dimensional perturbations, and this is a difficult problem that has no counterpart in the two-dimensional case, see Section 4 for a detailed discussion.

In the asymmetric case where $\gamma_1 \neq \gamma_2$, Burgers vortices still exist, but their profiles satisfy a genuinely nonlinear equation and explicit formulas such as (1.14), (1.15) are no longer available. Thus even existence of such stretched vortices is a challenging mathematical question, which will also be discussed in Section 4. More generally, all existence, uniqueness, and stability results available for the axisymmetric Burgers vortex are expected to remain true in the asymmetric case too, although rigorous proofs are not always available.

Remark 1.1 *Although physical constants are useful for dimensional analysis and important for comparison with experiments, they often hinder the mathematical analysis by making formulas needlessly complicated. In Sections 2 and 4 below, dimensionless variables and functions are systematically used, and this amounts to setting $\nu = \gamma = 1$ in all formulas. In particular, the total circulation of a vortex coincides with the circulation Reynolds number, and will be denoted by α.*

2 Stability of Lamb-Oseen vortices

This section is devoted to the stability analysis of the family of Lamb-Oseen vortices (1.6). These are self-similar solutions of the two-dimensional vorticity equation (1.4), and their properties are most conveniently studied if the equation itself is written in self-similar variables $\xi = x/\sqrt{t}$, $\tau = \log(t)$ [18]. Assuming $\nu = 1$ and setting

$$\omega(x,t) \,=\, \frac{1}{t}\, w\Big(\frac{x}{\sqrt{t}}\,,\, \log(t)\Big)\,, \qquad u(x,t) \,=\, \frac{1}{\sqrt{t}}\, v\Big(\frac{x}{\sqrt{t}}\,,\, \log(t)\Big)\,, \tag{2.1}$$

one obtains for the rescaled vorticity $w(\xi,\tau)$ and the rescaled velocity $v(\xi,\tau)$ the following evolution equation

$$\partial_\tau w(\xi,\tau) + v(\xi,\tau)\cdot\nabla w(\xi,\tau) \,=\, Lw(\xi,\tau)\,, \tag{2.2}$$

where L is the linear operator defined by

$$L \,=\, \Delta + \frac{\xi}{2}\cdot\nabla + 1\,. \tag{2.3}$$

The change of variables (2.1) coincides with Lundgren's transformation (1.16), except that it is used here in the opposite way: starting from the two-dimensional vorticity $\omega(x,t)$ and velocity $u(x,t)$, one obtains the rescaled quantities $w(\xi,\tau), v(\xi,\tau)$ whose physical meaning is not immediately obvious. In addition, the rescaled equation (2.2) looks more complicated than the original vorticity equation (1.4) because the Laplace operator Δ is replaced by the Fokker-Planck operator L. However, from a mathematical point of view, the rescaled equation (2.2) has several advantages which greatly simplify the analysis. In particular, the operator L has (partially) discrete spectrum when considered in appropriate function spaces, and that observation is crucial for the stability analysis of the Lamb-Oseen vortex presented below. Moreover, the associated semigroup $e^{\tau L}$ has nice confinement properties, as a consequence of which it is possible to use compactness methods to investigate the long-time behavior of solutions to the rescaled vorticity equation (2.2), see [19].

Due to scale invariance, the Biot-Savart law (1.5) is not affected by the change of variables (2.1). This means that the rescaled velocity $v(\xi, \tau)$ can be reconstructed from the rescaled vorticity $w(\xi, \tau)$ through the formula

$$v(\cdot, \tau) = K_{2D} * w(\cdot, \tau) , \qquad \text{where} \quad K_{2D}(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} . \tag{2.4}$$

By construction, for any $\alpha \in \mathbb{R}$, the Lamb-Oseen vortex $w = \alpha G$, $v = \alpha v^G$ is a stationary solution of (2.2). The dynamical relevance of this family of equilibria is demonstrated by the following global convergence result.

Theorem 2.1 ([19]) *For any initial data $w_0 \in L^1(\mathbb{R}^2)$, the rescaled vorticity equation* (2.2) *has a unique global solution $w \in C^0([0, \infty), L^1(\mathbb{R}^2))$. This solution satisfies $\|w(\tau)\|_{L^1(\mathbb{R}^2)} \le \|w_0\|_{L^1(\mathbb{R}^2)}$ for all $\tau \ge 0$, and*

$$\lim_{\tau \to \infty} \|w(\tau) - \alpha G\|_{L^1(\mathbb{R}^2)} = 0 , \qquad \textit{where} \quad \alpha = \int_{\mathbb{R}^2} w_0(\xi) \, d\xi . \tag{2.5}$$

Theorem 2.1 shows that Lamb-Oseen vortices describe, to leading order, the long-time behavior of all solutions of the two-dimensional Navier-Stokes equations with integrable initial vorticity and nonzero total circulation α. Similar conclusions were previously obtained for small solutions [22], and for large solutions with small circulation [3]. The first step in the proof consists in showing that the original vorticity equation (1.4) is globally well-posed in $L^1(\mathbb{R}^2)$, that the L^1 norm of the solutions is nonincreasing in time, and that the total circulation $\alpha = \int_{\mathbb{R}^2} \omega \, dx$ is a conserved quantity [1]. Since the change of variables (2.1) leaves the L^1 norm invariant, the same conclusions hold for the rescaled vorticity equation (2.2) too. Then, in view of the confinement properties of the linear semigroup $e^{\tau L}$, one can show that the solutions of (2.2) are not only bounded, but also relatively compact in the space $L^1(\mathbb{R}^2)$. Finally, using appropriate Lyapunov functions [19] or monotonicity properties based on rearrangement techniques [8], one can prove that the omega-limit set in $L^1(\mathbb{R}^2)$ of any solution of (2.2) is included in the family of Lamb-Oseen vortices. As the total circulation is conserved, the omega-limit set is in fact reduced to the singleton $\{\alpha G\}$, which proves (2.5). The interested reader is referred to [19, 23] for details.

The global convergence result (2.5) is very general, but the proof sketched above is not constructive, and does not yield any estimate on the time needed to reach the asymptotic regime described by the Lamb-Oseen vortex. Explicit estimates of the convergence time can however be obtained if the vorticity has a definite sign [19] or is strongly localized [15]. In the rest of this section, emphasis is put on local stability results, for which explicit bounds are also available.

2.1 Local stability results

Theorem 2.1 strongly suggests, but does not really prove, that the Lamb-Oseen vortex αG is a stable equilibrium of the rescaled vorticity equation (2.2) for any $\alpha \in \mathbb{R}$. Stability can be established by considering solutions of the form $w = \alpha G + \tilde{w}$, $v = \alpha v^G + \tilde{v}$. The perturbations satisfy the evolution equation

$$\partial_\tau \tilde{w} + \tilde{v} \cdot \nabla \tilde{w} = (L - \alpha \Lambda) \tilde{w} , \tag{2.6}$$

where L is given by (2.3) and Λ is the nonlocal linear operator defined by

$$\Lambda \tilde{w} \;=\; v^G \cdot \nabla \tilde{w} + \tilde{v} \cdot \nabla G\,, \qquad \text{with} \quad \tilde{v} \;=\; K_{2D} * \tilde{w}\,. \tag{2.7}$$

It is possible to prove that the perturbation equation (2.6) is globally well-posed in the space $L^1(\mathbb{R}^2)$, and that the origin $\tilde{w} = 0$ is a stable equilibrium, but at this level of generality little can be said about the long-time behavior of the solutions. However, more precise stability results can be obtained if one assumes that the vorticity is sufficiently localized in space.

Given $m \in [0,\infty]$, let $\rho_m : [0,\infty) \to [1,\infty)$ be the weight function defined by

$$\rho_m(r) \;=\; \begin{cases} 1 & \text{if} \quad m = 0\,, \\ (1 + \frac{r}{4m})^m & \text{if} \quad 0 < m < \infty\,, \\ e^{r/4} & \text{if} \quad m = \infty\,. \end{cases} \tag{2.8}$$

Perturbations will be taken in the weighted L^2 space

$$L^2(m) \;=\; \Big\{ w \in L^2(\mathbb{R}^2) \;\Big|\; \|w\|^2_{L^2(m)} = \int_{\mathbb{R}^2} \rho_m(|\xi|^2)|w(\xi)|^2 \,\mathrm{d}\xi < \infty \Big\}\,, \tag{2.9}$$

which is a (real) Hilbert space equipped with the scalar product

$$\langle w_1\,,\, w_2\rangle_{L^2(m)} \;=\; \int_{\mathbb{R}^2} \rho_m(|\xi|^2) w_1(\xi) w_2(\xi)\,\mathrm{d}\xi\,. \tag{2.10}$$

Elements of $L^2(m)$ are square integrable functions with algebraic decay at infinity if $0 < m < \infty$, and Gaussian decay if $m = \infty$. Hölder's inequality implies that $L^2(m) \hookrightarrow L^1(\mathbb{R}^2)$ if $m > 1$. In that case, it is useful to introduce the closed subspace

$$L^2_0(m) \;=\; \Big\{ w \in L^2(m) \;\Big|\; \int_{\mathbb{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \Big\}\,, \tag{2.11}$$

which happens to be invariant under the action of both linear operators L and Λ.

To study the stability of the origin $\tilde{w} = 0$ for the perturbation equation (2.6), it is useful to compute the spectrum of the linearized operator $L - \alpha\Lambda$ in the (complexified) Hilbert space $L^2(m)$. In the simple case where $\alpha = 0$, the spectrum is explicitly known:

Proposition 2.2 ([18]) *For any $m \in [0,\infty]$, the spectrum of the linear operator* (2.3) *in the weighted space $L^2(m)$ defined by* (2.8), (2.9) *is*

$$\sigma_m(L) \;=\; \Big\{ \lambda \in \mathbb{C} \;\Big|\; \mathrm{Re}(\lambda) \le \frac{1}{2} - \frac{m}{2} \Big\} \cup \Big\{ -\frac{k}{2} \;\Big|\; k \in \mathbb{N} \Big\}\,. \tag{2.12}$$

Moreover, if $m > k+1$ for some $k \in \mathbb{N}$, then $\lambda_k = -k/2$ is an isolated eigenvalue of L, with (algebraic and geometric) multiplicity $k+1$.

It follows in particular from Proposition 2.2 that L has purely discrete spectrum in $L^2(m)$ when $m = \infty$. This is easily understood if one observes that $\rho_\infty(|\xi|^2) = e^{|\xi|^2/4} = (4\pi)^{-1}G(\xi)^{-1}$, and that

$$G^{-1/2}\, L\, G^{1/2} \;=\; \Delta \;-\; \frac{|\xi|^2}{16} \;+\; \frac{1}{2}\,. \tag{2.13}$$

The formal relation (2.13) implies that the operator L in $L^2(\infty)$ is unitarily equivalent to the harmonic oscillator $\Delta - |\xi|^2/16 + 1/2$ in $L^2(\mathbb{R}^2)$, the spectrum of which is the sequence $(\lambda_k)_{k\in\mathbb{N}}$, where $\lambda_k = -k/2$ has multiplicity $k+1$. If $m < \infty$, the discrete part of the spectrum persists because the corresponding eigenfunctions decay rapidly at infinity. In addition, any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) < (1-m)/2$ is an eigenvalue of L in $L^2(m)$ with infinite multiplicity [18], hence the spectrum $\sigma_m(L)$ also includes the closed half-plane $H_m = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) \le (1-m)/2\}$.

In the more interesting case where $\alpha \neq 0$, the spectrum of $L - \alpha\Lambda$ in $L^2(m)$ cannot be computed explicitly. However, upper bounds on the real part of the spectrum are sufficient for the stability analysis, and such estimates can be obtained by combining the following three observations.

Observation 1 : The operator Λ is a *relatively compact* perturbation of L in $L^2(m)$, for any $m \in [0,\infty]$. This is intuitively obvious, because Λ is a first-order differential operator whose coefficients decay to zero at infinity, whereas L involves in particular the Laplace operator Δ. By Weyl's theorem, the *essential spectrum* [25] of $L - \alpha\Lambda$ in $L^2(m)$ does not depend on α, hence coincides with the closed half-plane $H_m = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) \le (1-m)/2\}$ by Proposition 2.2. It thus remains to locate isolated eigenvalues of $L - \alpha\Lambda$ outside H_m.

Observation 2 : The isolated eigenvalues of $L - \alpha\Lambda$ in $L^2(m)$ do not depend on m. Indeed, if $w \in L^2(m)$ satisfies $(L - \alpha\Lambda)w = \lambda w$ for some $\lambda \in \mathbb{C} \setminus H_m$, one can show that w decays sufficiently fast at infinity so that $w \in L^2(\infty)$ [19]. This means that isolated eigenvalues of $L - \alpha\Lambda$ can be located by considering the particular case $m = \infty$, where the spectrum is fully discrete and consists of a sequence of eigenvalues $(\lambda_k(\alpha))_{k\in\mathbb{N}}$ with $\mathrm{Re}(\lambda_k(\alpha)) \to -\infty$ as $k \to \infty$.

Observation 3 : The operator Λ is *skew-symmetric* in $L^2(\infty)$, namely

$$\langle \Lambda w_1 \,,\, w_2\rangle + \langle w_1 \,,\, \Lambda w_2\rangle \;=\; 0\,, \qquad \text{for all} \quad w_1, w_2 \in D(\Lambda) \subset L^2(\infty)\,, \tag{2.14}$$

where $D(\Lambda) \subset L^2(\infty)$ is the (maximal) domain of the operator Λ, and $\langle \cdot\,,\cdot\rangle$ denotes the scalar product in $L^2(\infty)$, which (up to an irrelevant factor) can be written in the form

$$\langle w_1 \,,\, w_2\rangle \;=\; \int_{\mathbb{R}^2} G(\xi)^{-1} w_1(\xi) w_2(\xi)\,\mathrm{d}\xi\,. \tag{2.15}$$

To prove (2.14) one decomposes $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1 w = v^G \cdot \nabla w$ and $\Lambda_2 w = (K_{2D} * w)\cdot \nabla G$. If $w_1, w_2 \in L^2(\infty)$ belong to the domain of Λ, then

$$\begin{aligned}\langle \Lambda_1 w_1, w_2\rangle + \langle w_1, \Lambda_1 w_2\rangle \;&=\; \int_{\mathbb{R}^2} G^{-1}\Big(w_2\, v^G \cdot \nabla w_1 + w_1\, v^G \cdot \nabla w_2\Big)\,\mathrm{d}\xi \\ &=\; \int_{\mathbb{R}^2} G^{-1}\, v^G \cdot \nabla (w_1 w_2)\,\mathrm{d}\xi \;=\; 0\,,\end{aligned}$$

because the vector field $G^{-1}v^G$ is divergence-free. Moreover using the identity $\nabla G = -\frac{1}{2}\xi G$ and the Biot-Savart law (2.4), one obtains

$$\begin{aligned}\langle \Lambda_2 w_1, w_2\rangle + \langle w_1, \Lambda_2 w_2\rangle \;&=\; -\frac{1}{2}\int_{\mathbb{R}^2}\Big((\xi\cdot v_1)w_2 + (\xi\cdot v_2)w_1\Big)\,\mathrm{d}\xi \\ =\; -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}&\left\{\xi\cdot\frac{(\xi-\eta)^\perp}{|\xi-\eta|^2} + \eta\cdot\frac{(\eta-\xi)^\perp}{|\xi-\eta|^2}\right\} w_1(\eta) w_2(\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi \;=\; 0\,,\end{aligned}$$

because the last integrand vanishes identically. This proves (2.14). One can also show that the operator Λ is not only skew-symmetric, but also skew-adjoint in $L^2(\infty)$, see [34].

The observations above lead to the following *spectral stability result* for the Lamb-Oseen vortex in the space $L^2(m)$.

Proposition 2.3 ([19]) *For any $\alpha \in \mathbb{R}$ and any $m \in [1,\infty]$, the spectrum of the linearized operator $L - \alpha\Lambda$ in the space $L^2(m)$ satisfies*

$$\sigma_m(L-\alpha\Lambda) \subset \Big\{\lambda \in \mathbb{C} \,\Big|\, \mathrm{Re}(\lambda) \le 0\Big\}. \tag{2.16}$$

Moreover, if $m \ge 2$, then

$$\sigma_m(L-\alpha\Lambda) \subset \{0\} \cup \Big\{\lambda \in \mathbb{C} \,\Big|\, \mathrm{Re}(\lambda) \le -\frac{1}{2}\Big\}. \tag{2.17}$$

Finally, if $m \ge 3$, then

$$\sigma_m(L-\alpha\Lambda) \subset \{0\} \cup \Big\{-\frac{1}{2}\Big\} \cup \Big\{\lambda \in \mathbb{C} \,\Big|\, \mathrm{Re}(\lambda) \le -1\Big\}. \tag{2.18}$$

Proof. As before let $H_m = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) \le (1-m)/2\}$. By Observation 1 above, if $m \ge 1$, the essential spectrum of $L - \alpha\Lambda$ is included in the half-space H_1. Assume that $\lambda \in \mathbb{C} \setminus H_1$ is an isolated eigenvalue of $L-\alpha\Lambda$, and let $w \in L^2(m)$ be a nontrivial eigenfunction associated with λ. Then $w \in L^2(\infty)$ by Observation 2, and using Observation 3 one finds

$$\mathrm{Re}(\lambda)\langle w, w\rangle \;=\; \mathrm{Re}\langle (L-\alpha\Lambda)w, w\rangle \;=\; \langle Lw, w\rangle \;\le\; 0\,, \tag{2.19}$$

because L is a nonpositive self-adjoint operator in $L^2(\infty)$ and Λ is skew-symmetric. This contradicts the assumption that $\mathrm{Re}(\lambda) > 0$, hence the whole spectrum of $L - \alpha\Lambda$ in $L^2(m)$ is contained in the half-space H_1, as asserted in (2.16).

As $LG = \Lambda G = 0$, it is clear that 0 is an eigenvalue of $L - \alpha\Lambda$ for any $m \ge 0$ and any $\alpha \in \mathbb{R}$. If $m > 1$, one can write $L^2(m) = \mathbb{R}G \oplus L^2_0(m)$, where $L^2_0(m)$ is the hyperplane defined in (2.11), and this decomposition is left invariant by both operators L and Λ. Now, the same argument as above shows that, if $m \ge 2$, the spectrum of the operator $L - \alpha\Lambda$ acting on $L^2_0(m)$ is contained in the half-plane H_2, because $L \le -1/2$ on $L^2_0(\infty)$. This proves (2.17).

Finally, it is easy to verify that $L(\partial_i G) = -\frac{1}{2}\partial_i G$ for $i = 1, 2$, and differentiating the identity $v^G \cdot \nabla G = 0$ one finds that $\Lambda(\partial_i G) = 0$ for $i = 1, 2$. This means that $-1/2$ is an eigenvalue of $L - \alpha\Lambda$ for any $m \ge 0$ and any $\alpha \in \mathbb{R}$. As above, if $m > 2$, one has the invariant decomposition

$$L^2(m) \;=\; \{\alpha G \,|\, \alpha \in \mathbb{R}\} \oplus \{\beta_1\partial_1 G + \beta_2\partial_2 G \,|\, \beta_1, \beta_2 \in \mathbb{R}\} \oplus L^2_{00}(m)\;,$$

where

$$L^2_{00}(m) \;=\; \Big\{w \in L^2_0(m) \,\Big|\, \int_{\mathbb{R}^2} \xi_i w(\xi)\,\mathrm{d}\xi = 0 \text{ for } i = 1, 2\Big\}. \tag{2.20}$$

As $L \le -1$ on $L^2_{00}(\infty)$, the same argument shows that the spectrum of the operator $L - \alpha\Lambda$ acting on $L^2_{00}(m)$ is contained in the half-plane H_3, if $m \ge 3$. This proves (2.18). ☐

The linear operator $L - \alpha\Lambda$ is the generator of a strongly continuous semigroup in the space $L^2(m)$ for any $\alpha \in \mathbb{R}$ and any $m \in [0,\infty]$ [18]. The following *linear stability result* is a natural consequence of Proposition 2.3 and its proof.

Proposition 2.4 ([19]) *For any $\alpha \in \mathbb{R}$ and any $m > 1$, there exists a positive constant C such that*

$$\|e^{\tau(L-\alpha\Lambda)}\|_{L^2(m)\to L^2(m)} \;\le\; C\,, \qquad \textit{for all } \tau \ge 0\,. \tag{2.21}$$

Moreover, if $m > 2$, then

$$\|e^{\tau(L-\alpha\Lambda)}\|_{L^2_0(m)\to L^2_0(m)} \;\le\; C\,e^{-\tau/2}\,, \qquad \textit{for all } \tau \ge 0\,. \tag{2.22}$$

Finally, if $m > 3$, then

$$\|e^{\tau(L-\alpha\Lambda)}\|_{L^2_{00}(m)\to L^2_{00}(m)} \;\le\; C\,e^{-\tau}\,, \qquad \textit{for all } \tau \ge 0\,. \tag{2.23}$$

When studying the stability of the Lamb-Oseen vortex, there is no loss of generality in considering perturbations with zero total circulation. Indeed, if $w = \alpha G + \tilde{w}$ for some $\tilde{w} \in L^2(m)$ with $m > 1$, then defining $\tilde{\alpha} = \int_{\mathbb{R}^2} \tilde{w}(\xi)\,\mathrm{d}\xi$ one can write $w = (\alpha + \tilde{\alpha})G + (\tilde{w} - \tilde{\alpha}G)$, where by construction $\tilde{w} - \tilde{\alpha}G \in L^2_0(m)$. Thus perturbations with nonzero circulation of the vortex αG can be considered as perturbations with zero circulation of the modified vortex $(\alpha+\tilde{\alpha})G$. As the total circulation is a conserved quantity, the subspace $L^2_0(m)$ is invariant under the evolution defined by the full perturbation equation (2.6). By Proposition 2.4, the linear semigroup $e^{\tau(L-\alpha\Lambda)}$ is exponentially decaying in $L^2_0(m)$ if $m > 2$, and using that information it is routine to deduce the following *asymptotic stability result*, which is the main outcome of this section.

Proposition 2.5 ([19]) *Fix $\alpha \in \mathbb{R}$ and $m \in (2, \infty]$. There exist positive constants ϵ and C such that, for all $\tilde{w}_0 \in L^2_0(m)$ satisfying $\|\tilde{w}_0\|_{L^2(m)} \le \epsilon$, the rescaled vorticity equation* (2.2) *has a unique global solution $w \in C^0([0,\infty), L^2(m))$ with initial data $w_0 = \alpha G + \tilde{w}_0$. Moreover, the following estimate holds*

$$\|w(\tau) - \alpha G\|_{L^2(m)} \;\le\; C\|w_0 - \alpha G\|_{L^2(m)}\, e^{-\tau/2}\;, \qquad \tau \ge 0\;. \tag{2.24}$$

If $m > 2$, the codimension 3 subspace $L^2_{00}(m)$ is also invariant under the evolution defined by the full perturbation equation (2.2). As a consequence, if $\tilde{w}_0 \in L^2_{00}(m)$, the solution of (2.2) given by Proposition 2.5 satisfies $w(\tau) - \alpha G \in L^2_{00}(m)$ for all $\tau \ge 0$. If $m > 3$, one can then use (2.23) to conclude that

$$\|w(\tau) - \alpha G\|_{L^2(m)} \;\le\; C\|w_0 - \alpha G\|_{L^2(m)}\, e^{-\tau}\;, \qquad \tau \ge 0\;.$$

As is shown in [19, 11], if $\alpha \neq 0$, the assumption that w_0 has vanishing first order moments does not really restrict the generality, because this condition can always be met by a suitable translation of the initial data.

Remark 2.6 *If $m = \infty$, one can show that the Lamb-Oseen vortex αG is uniformly stable for all $\alpha \in \mathbb{R}$ in the sense that the constants ϵ and C in Proposition 2.5 do not depend on α [11]. This is in sharp contrast with what happens for shear flows, such as the Poiseuille flow in a cylindrical pipe or the Couette-Taylor flow between two rotating cylinders. In such examples, the laminar stationary flow undergoes an instability, of spectral or pseudospectral nature, when the Reynolds number is sufficiently large. In contrast, a fast rotation has rather a stabilizing effect on vortices, as the analysis below reveals.*

2.2 Large Reynolds number asymptotics

Proposition 2.3 above gives uniform estimates on the spectrum of the linearized operator $L - \alpha\Lambda$, which are sufficient to prove stability of the Lamb-Oseen vortex for all values of the circulation parameter $\alpha \in \mathbb{R}$. However, such estimates do not describe how the spectrum changes as the circulation parameter varies. The most relevant regime for turbulent flows is of course the high Reynolds number limit where $|\alpha| \to \infty$, which deserves a special consideration. As the essential spectrum of $L - \alpha\Lambda$ in the space $L^2(m)$ does not depend on m, it is most convenient to work in the limiting space $X = L^2(\infty)$, equipped with the scalar product (2.15). In that space, as was already mentioned, the spectrum of $L - \alpha\Lambda$ is discrete, and consists of a sequence of eigenvalues $(\lambda_k(\alpha))_{k\in\mathbb{N}}$ with $\mathrm{Re}(\lambda_k(\alpha)) \to -\infty$ as $k \to \infty$. It follows from Proposition 2.2 that $\lambda_k(0) = -k/2$, for any $k \in \mathbb{N}$, and the goal of this section is to investigate the behavior of the real part of $\lambda_k(\alpha)$ as $|\alpha| \to \infty$.

The starting point of the analysis is the determination of the kernel of the skew-symmetric operator Λ. Let $X_0 \subset X$ denote the closed subspace containing all radially symmetric functions. If $w \in X_0$, the associated velocity field $v = K_{2D} * w$ satisfies $\xi \cdot v(\xi) = 0$, and it follows that $\Lambda w = 0$, hence $X_0 \subset \ker(\Lambda)$. On the other hand, it was already observed that $\Lambda(\partial_i G) = 0$ for $i = 1, 2$. The following result asserts that the kernel of Λ does not contain any more elements :

Lemma 2.7 ([34]) $\ker(\Lambda) = X_0 \oplus \{\beta_1 \partial_1 G + \beta_2 \partial_2 G \,|\, \beta_1, \beta_2 \in \mathbb{R}\}$.

In view of Lemma 2.7, the subspace $\ker(\Lambda) \subset X$ is invariant under the action of both operators L and Λ, and the orthogonal complement $\ker(\Lambda)^\perp$ is invariant too because L is self-adjoint and Λ is skew-adjoint. Inside $\ker(\Lambda)$, the spectrum of $L - \alpha\Lambda \equiv L$ does not depend on the circulation parameter α, and consists of all negative integers in addition to the double eigenvalue $-1/2$. In fact, for any $n \in \mathbb{N}$, the eigenfunction corresponding to the eigenvalue $-n$ is the radially symmetric Hermite function $\Delta^n G$. The only difficult task is therefore to study the spectrum of $L_\perp - \alpha\Lambda_\perp$, which is defined as the restriction of $L - \alpha\Lambda$ to the orthogonal complement $\ker(\Lambda)^\perp$. That spectrum does depend in a nontrivial way upon the parameter α. It happens that the real parts of all eigenvalues converge to $-\infty$ as $|\alpha| \to \infty$, which is of course compatible with the uniform bounds given by Proposition 2.3. This phenomenon illustrates the *stabilizing effect* of fast rotation on Lamb-Oseen vortices.

Two natural quantities can be introduced to accurately measure the effect of fast rotation. For any $\alpha \in \mathbb{R}$, one can define the *spectral lower bound*

$$\Sigma(\alpha) \;=\; \inf\Big\{\mathrm{Re}(z) \,\Big|\, z \in \mathrm{spec}(-\mathcal{L}_\perp + \alpha\Lambda_\perp)\Big\}\;,$$

or the *pseudospectral bound*

$$\Psi(\alpha) \;=\; \Big(\sup_{\lambda\in\mathbb{R}} \|(\mathcal{L}_\perp - \alpha\Lambda_\perp - i\lambda)^{-1}\|_{X\to X}\Big)^{-1}\;.$$

In the definition of $\Sigma(\alpha)$, the sign of the linearized operator has been changed to obtain a positive quantity. Although the spectral and pseudospectral bounds are of rather different nature, there is a simple one-sided relation between them :

Lemma 2.8 *For any $\alpha \in \mathbb{R}$ one has $\Sigma(\alpha) \ge \Psi(\alpha) \ge 1$.*

Proof. Fix $\alpha \in \mathbb{R}$. By Lemma 2.7, one has $\ker(\Lambda)^\perp \subset L^2_{00}(m)$, hence $\Sigma(\alpha) \ge 1$ by (2.18). On the other hand, if $(L - \alpha\Lambda + \lambda)w = 0$ for some $\lambda \in \mathbb{C}$ and some $w \in \ker(\Lambda)^\perp$ such that $\langle w, w\rangle = 1$, then $(L - \alpha\Lambda + i\,\mathrm{Im}(\lambda))w = -\mathrm{Re}(\lambda)w$, hence

$$\mathrm{Re}(\lambda) \;\ge\; \|(L_\perp - \alpha\Lambda_\perp + i\,\mathrm{Im}(\lambda))^{-1}\|^{-1} \;\ge\; \Psi(\alpha)\;.$$

This proves that $\Sigma(\alpha) \ge \Psi(\alpha)$. Finally, the proof of Proposition 2.3 shows that the operator $L_\perp - \alpha\Lambda_\perp + 1$ is m-dissipative [26]. This in particular implies that $\|(L_\perp - \alpha\Lambda_\perp - i\lambda)^{-1}\| \le 1$ for all $\lambda \in \mathbb{R}$, hence $\Psi(\alpha) \ge 1$. □

The stabilizing effect in the large Reynolds number limit is qualitatively illustrated by the following result :

Proposition 2.9 ([34]) *One has $\Psi(\alpha) \to \infty$ and $\Sigma(\alpha) \to \infty$ as $|\alpha| \to \infty$.*

The proof given in [34] actually shows that $\Sigma(\alpha) \to \infty$ as $|\alpha| \to \infty$, but can be easily modified to yield the stronger conclusion that $\Psi(\alpha) \to \infty$. For the stability analysis of the Lamb-Oseen vortex αG, the divergence of the spectral bound means that the decay rate in time of perturbations in $\ker(\Lambda)^\perp$ becomes arbitrarily large as $|\alpha| \to \infty$. On the other hand, using the divergence of the pseudospectral bound, one can show that the basin of attraction of the Lamb-Oseen vortex, in the weighted space $L^2(\infty)$, becomes arbitrarily large as $|\alpha| \to \infty$. It should be emphasized, however, that the argument used in [34] is nonconstructive and does not provide any explicit estimate on the quantities $\Psi(\alpha)$ or $\Sigma(\alpha)$ for large $|\alpha|$.

In fact, there are good reasons to conjecture that $\Sigma(\alpha) = \mathcal{O}(|\alpha|^{1/2})$ and $\Psi(\alpha) = \mathcal{O}(|\alpha|^{1/3})$ as $|\alpha| \to \infty$. First of all, extensive numerical calculations performed by Prochazka and Pullin [40, 41] indicate that $\Sigma(\alpha) = \mathcal{O}(|\alpha|^{1/2})$ as $|\alpha| \to \infty$. Next, the conjecture is clearly supported by rigorous analytical results on model problems [9]. In particular, for the simplified linear operator $L - \alpha\Lambda_1$ where the nonlocal part Λ_2 has been omitted, it can be proved that $\Psi(\alpha) = \mathcal{O}(|\alpha|^{1/3})$ as $|\alpha| \to \infty$ [4]. The same result holds for the full linearized operator $L - \alpha\Lambda$ restricted to a smaller subspace than $\ker(\Lambda)^\perp$, where a finite number of Fourier modes with respect to the angular variable in polar coordinates have been removed [5]. The general case is still under investigation [7].

Assuming that the conjecture above is true, it is worth noting that the pseudospectral bound $\Psi(\alpha)$ and the spectral bound $\Sigma(\alpha)$ have different growth rates as $|\alpha| \to \infty$. This reflects the fact that the linearized operator $L - \alpha\Lambda$ becomes highly non-selfadjoint in the fast rotation limit. Indeed, for selfadjoint or normal operators, it is easy to verify that the spectral and pseudospectral bounds always coincide.

2.3 Lamb-Oseen vortices in exterior domains

As was already mentioned, the Lamb-Oseen vortex plays a double role in the dynamics of the Navier-Stokes equations in the whole space $\mathbb{R}^2$: it is the unique solution of the system when the initial vorticity is a Dirac measure, and it describes the long-time asymptotics of all solutions for which the vorticity distribution is integrable and has nonzero total circulation. The proofs given in [19] demonstrate that both properties are closely related, due to scale invariance. Now, if the fluid is contained in a two-dimensional domain $\Omega \subset \mathbb{R}^2$, and satisfies (for instance) no-slip boundary conditions on $\partial\Omega$, scale invariance is broken and there is no simple relation anymore between the Cauchy problem for singular initial data and the long-time asymptotics of general solutions. Both questions are interesting and, at present time, largely open. In this section, the relatively simple case of a two-dimensional exterior domain is considered, where a few results concerning the long-time behavior of solutions with nonzero circulation at infinity have been obtained recently.

Let $\Omega \subset \mathbb{R}^2$ be a smooth exterior domain, namely an unbounded connected open set with a smooth compact boundary $\partial\Omega$. The Navier-Stokes equations in Ω with no-slip boundary conditions can be written in the following form :

$$\begin{cases} \partial_t u + (u\cdot\nabla)u \,=\, \Delta u - \nabla p\ , \quad \operatorname{div} u \,=\, 0\ , & \text{for } x \in \Omega\,, \quad t > 0\,, \\ u(x,t) \,=\, 0\ , & \text{for } x \in \partial\Omega\,, \quad t > 0\,, \\ u(x,0) \,=\, u_0(x)\ , & \text{for } x \in \Omega\,, \end{cases} \tag{2.25}$$

where p denotes the ratio of the pressure to the fluid density. The vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ still satisfies the simple evolution equation (1.4) (with $\nu = 1$), but the assumption that $u = 0$ on $\partial\Omega$ translates into a nonlinear, nonlocal boundary condition for ω, which is very difficult to handle. So, whenever possible, it is preferable to work directly with the velocity formulation (2.25).

If the initial velocity u_0 belongs to the energy space

$$L^2_\sigma(\Omega) \;=\; \Big\{u \in L^2(\Omega)^2 \,\Big|\, \operatorname{div} u = 0 \text{ in } \Omega\,,\;\; u\cdot n = 0 \text{ on } \partial\Omega\Big\}\,,$$

where n denotes the unit normal on $\partial\Omega$, it is well known that system (2.25) has a unique global solution satisfying the energy identity

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \;=\; \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}\,, \qquad t \ge 0\,.$$

That solution converges to zero in $L^2_\sigma(\Omega)$ as $t \to \infty$ [38], which means that the long-time behavior of all finite energy solutions is trivial. However, as in the whole plane $\mathbb{R}^2$, one can consider flows with nonzero circulation at infinity:

$$\alpha \;=\; \lim_{R\to\infty} \oint_{|x|=R} (u_1\,\mathrm{d}x_1 + u_2\,\mathrm{d}x_2) \;\neq\; 0\,,$$

in which case the kinetic energy is necessarily infinite and the long-time behavior may be nontrivial.

To construct such solutions, it is convenient to introduce a smooth cut-off function $\chi : \mathbb{R}^2 \to [0,1]$ such that χ vanishes in a neighborhood of $\mathbb{R}^2 \setminus \Omega$ and $\chi(x) = 1$ whenever $|x|$ is sufficiently large. For technical reasons, one also assumes that χ is radially symmetric and nondecreasing along rays. The *truncated Oseen vortex*

$$u^\chi(x,t) \;=\; \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)\chi(x)\,, \qquad x \in \mathbb{R}^2, \quad t \ge 0\,, \tag{2.26}$$

is a divergence-free velocity field which vanishes identically in a neighborhood of $\mathbb{R}^2 \setminus \Omega$ and coincides with the Lamb-Oseen vortex (with unit circulation) far away from the origin. In particular $u^\chi \notin L^2(\Omega)$. The corresponding vorticity distribution $\omega^\chi = \partial_1 u_2^\chi - \partial_2 u_1^\chi$ reads:

$$\omega^\chi(x,t) \;=\; \frac{1}{4\pi(1+t)}\,e^{-\frac{|x|^2}{4(1+t)}}\,\chi(x) + \frac{1}{2\pi}\,\frac{1}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)x\cdot\nabla\chi(x)\,, \tag{2.27}$$

and satisfies $\int_\Omega \omega^\chi(x,t)\,\mathrm{d}x = 1$ for all $t \ge 0$. Of course the velocity field u^χ is not an exact solution of the Navier-Stokes equations (2.25) (unless $\Omega = \mathbb{R}^2$ and $\chi \equiv 1$), but the following result shows that it is a globally stable asymptotic solution.

Theorem 2.10 ([14, 35]) *Fix $q \in (1,2]$, and let $\mu = 1/q - 1/2$. There exists a constant $\epsilon > 0$ such that, for all initial data of the form $u_0 = \alpha u^\chi(\cdot,0) + v_0$ with $|\alpha| \le \epsilon$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the Navier-Stokes equations* (2.25) *have a unique global solution which satisfies*

$$\|u(\cdot,t) - \alpha u^\chi(\cdot,t)\|_{L^2(\Omega)} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla u^\chi(\cdot,t)\|_{L^2(\Omega)} \;=\; \mathcal{O}(t^{-\mu})\,, \tag{2.28}$$

as $t \to +\infty$. Moreover, if $q = 2$, then $\|u(\cdot,t) - \alpha u^\chi(\cdot,t)\|_{L^2(\Omega)} \to 0$ as $t \to +\infty$.

Several comments are in order. Existence and uniqueness of global solutions to the Navier-Stokes equations (2.25) for a class of infinite-energy initial data including those considered in Theorem 2.10 were established by Kozono and Yamazaki in [27]. The novelty here is the description of the long-time asymptotics for a specific family of solutions, corresponding to spatially localized perturbations of the truncated Oseen vortex. Theorem 2.10 is a global stability result, in the sense that arbitrary large perturbations v_0 of the vortex can be considered. There is, however, a limitation on the size of the circulation parameter α, which is probably of technical nature. To remove that restriction it seems rather natural to use the vorticity formulation and the nice properties of the linearized operator established in Section 2.1, but this is difficult in the present case because the boundary condition for the vorticity is very awkward. The parameter q in Theorem 2.10 measures the spatial decay of the initial perturbations v_0 to the Oseen vortex, and is directly related to the decay rate in time (called μ) of the corresponding solutions. If $q < 2$, then $\mu > 0$ and it is shown in [14] that the constant ϵ depends only on q, and not on the domain Ω. The limiting case where $q = 2$ was treated in [29, 35].

The main original ingredient in the proof of Theorem 2.10 is a logarithmic energy estimate that is worth discussing briefly. For solutions of the Navier-Stokes equations (2.25) of the form $u(x,t) = \alpha u^\chi(x,t) + v(x,t)$, the perturbation v satisfies

$$\partial_t v + \alpha(u^\chi, \nabla)v + \alpha(v,\nabla)u^\chi + (v,\nabla)v \;=\; \Delta v + \alpha R^\chi - \nabla q\,, \qquad \operatorname{div} v \;=\; 0\,, \tag{2.29}$$

where the source term $R^\chi = \Delta u^\chi - \partial_t u^\chi$ measures by how much the truncated vortex u^χ fails to be an exact solution of (2.25). Taking into account the uniform bounds

$$\|\nabla u^\chi(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \;\le\; \frac{b}{1+t}\,, \qquad \|R^\chi(\cdot,t)\|_{L^2(\mathbb{R}^2)} \;\le\; \frac{\kappa}{1+t}\,,$$

which hold for some positive constants b, κ depending only on the cut-off χ, a standard energy estimate yields the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)} \;\le\; \frac{b|\alpha|}{1+t}\,\|v(t)\|^2_{L^2(\Omega)} + \frac{\kappa|\alpha|}{1+t}\,\|v(t)\|_{L^2(\Omega)}\,, \tag{2.30}$$

which predicts a polynomial growth of the L^2 norm $\|v(t)\|_{L^2(\Omega)}$. This naive estimate can be substantially improved if one observes that the truncated Oseen vortex u^χ decays like $|x|^{-1}$ as $|x| \to \infty$, and thus nearly belongs to the energy space. The optimal result is :

Proposition 2.11 *There exists a constant $K > 0$ such that, for any $\alpha \in \mathbb{R}$ and any $v_0 \in L^2_\sigma(\Omega)$, the solution of* (2.29) *with initial data v_0 satisfies, for all $t \ge 1$,*

$$\|v(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla v(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \;\le\; K\Big(\|v_0\|^2_{L^2(\Omega)} + \alpha^2\log(1+t) + \alpha^2\log(2+|\alpha|)\Big)\,. \tag{2.31}$$

In the proof of Theorem 2.10, the logarithmic bound (2.31) is combined with standard energy estimates for a fractional primitive of the velocity field to prove that $v(\cdot,t)$ converges to zero in $L^2(\Omega)$ as $t \to \infty$, see [14, 12]. The optimal decay rate in (2.28) is then obtained by a direct study of small solutions to the perturbation equation (2.29).

Although Theorem 2.10 is established using the velocity formulation of the Navier-Stokes system, it is instructive to see what it implies for the vorticity distribution ω. Assume for instance that the initial vorticity $\omega_0 = \partial_1(u_0)_2 - \partial_2(u_0)_1$ is sufficiently localized so that

$$\int_\Omega (1+|x|^2)^m|\omega_0(x)|^2\,\mathrm{d}x \;<\; \infty\,,$$

for some $m > 1$. By Hölder's inequality this implies that $\omega_0 \in L^1(\Omega)$. Then denoting $v_0 = u_0 - \alpha u^\chi(\cdot,0)$ where $\alpha = \int_\Omega \omega_0(x)\,\mathrm{d}x$, it follows that $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for any $q \in (1,2)$ such that $q > 2/m$ [18]. In particular, if $|\alpha| \le \epsilon$, the conclusion of Theorem 2.10 holds. Moreover, the vorticity satisfies

$$\int_\Omega |\omega(x,t) - \alpha\omega^\chi(x,t)|\,\mathrm{d}x \;=\; \mathcal{O}(t^{-\mu}\log t)\,, \qquad \text{as} \quad t \to \infty\,, \tag{2.32}$$

where $\omega^\chi(x,t)$ is defined in (2.27), see [12]. In both convergence results (2.28), (2.32), one can replace the truncated Oseen vortex by the original Lamb-Oseen vortex for which $\chi \equiv 1$, because the additional error converges to zero like $\mathcal{O}(t^{-1})$ as $t \to \infty$.

3 Axisymmetric vortex rings and filaments

When restricted to axisymmetric flows without swirl, the three-dimensional Navier-Stokes equations bear some similarity with the two-dimensional situation considered in the previous section. The only nonzero component of the vorticity vector satisfies Eq. (1.9), which can be written in the equivalent form

$$\partial_t\omega_\theta + \partial_r(u_r\omega_\theta) + \partial_z(u_z\omega_\theta) \;=\; \nu\Big((\partial_r^2 + \partial_z^2)\omega_\theta + \partial_r\frac{\omega_\theta}{r}\Big)\,. \tag{3.1}$$

The analogy is most striking if one introduces the related quantity $\eta = \omega_\theta/r$, which satisfies the advection-diffusion equation

$$\partial_t\eta + u\cdot\nabla\eta \;=\; \Delta\eta + \frac{2}{r}\,\partial_r\eta\,, \tag{3.2}$$

where $u\cdot\nabla = u_r\partial_r + u_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$. Equation (3.2) is considered in the half-plane $\Omega = \{(r,z)\in\mathbb{R}^2\,|\,r>0\,,\; z\in\mathbb{R}\}$, with homogeneous Neumann boundary conditions on $\partial\Omega$.

It is clear from (3.2) that $\eta(r,z,t)$ obeys the parabolic maximum principle, and this provides a priori estimates on the solutions which imply that the Cauchy problem for the axisymmetric Navier-Stokes equations is globally well-posed, without any restriction on the size of the initial data. The first results in this direction were obtained by Ladyzhenskaya [30] and by Ukhovskii and Yudovich [45], for finite energy solutions. Recently, it was shown in [16] that the vorticity equation (3.1) is globally well-posed in the scale invariant space $L^1(\Omega, \mathrm{d}r\,\mathrm{d}z)$, equipped with the norm

$$\|\omega_\theta\|_{L^1(\Omega)} \;=\; \int_\Omega |\omega_\theta(r,z)|\,\mathrm{d}r\,\mathrm{d}z \;=\; \int_\Omega |\eta(r,z)|\,r\,\mathrm{d}r\,\mathrm{d}z\,.$$

The proof follows remarkably the same lines as in the two-dimensional case, and in particular uses very similar function spaces. Solutions constructed in this way have infinite energy in general, but if the initial vorticity ω_θ^0 decays somewhat faster at infinity than what is necessary to be integrable, the velocity field becomes square integrable for all positive times.

It is also possible to solve the Cauchy problem for Eq. (3.1) in the more general situation where the initial vorticity is a finite measure. Global existence and uniqueness are established in [16] assuming that the total variation norm of the atomic part of the initial vorticity is small compared to the viscosity parameter. The general case is open to the present date, but interesting results have been obtained for circular *vortex filaments*, which correspond to the situation where the initial vorticity is a Dirac measure. The resulting solutions can be considered as the analogue of the family of Lamb-Oseen vortices in $\mathbb{R}^2$. These solutions cannot be written in explicit form, but small-time asymptotic expansions can be computed which involve the two-dimensional profiles G and v^G defined in (1.7). If the circulation parameter Γ is small compared to the viscosity ν, the results of [16] imply the existence of a unique global solution to (3.1) with initial vorticity $\omega_\theta^0 = \Gamma\delta_{(\bar r,\bar z)}$, for any $(\bar r,\bar z)\in\Omega$. For larger circulations, the following existence result was recently established by Feng and Šverák:

Proposition 3.1 ([6]) *Fix* $\Gamma > 0$, $(\bar r,\bar z)\in\Omega$, *and* $\nu > 0$. *Then the axisymmetric vorticity equation* (3.1) *has a nonnegative global solution such that* $\omega_\theta(t) \rightharpoonup \Gamma\delta_{(\bar r,\bar z)}$ *as* $t\to 0$. *Moreover,* *this solution satisfies, for all $t > 0$,*

$$\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \;\le\; \Gamma\,, \qquad \int_\Omega r^2\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \;=\; \Gamma\,\bar r^2\,. \tag{3.3}$$

Needless to say, the assumption $\Gamma > 0$ does not restrict the generality, because the corresponding result for $\Gamma < 0$ can be obtained by symmetry. Proposition 3.1 is proved by a very general approximation argument, which provides global existence without any restriction on the size of the circulation parameter, but does not imply uniqueness and does not give any precise information on the qualitative behavior of the solution for short times. Using more sophisticated techniques, a more accurate result can be established :

Theorem 3.2 ([17]) *Fix $\Gamma \in \mathbb{R}$, $(\bar r, \bar z) \in \Omega$, and $\nu > 0$. Then the axisymmetric vorticity equation* (3.1) *has a unique global mild solution $\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$ such that*

$$\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} \;<\; \infty\;, \qquad \textit{and} \quad \omega_\theta(t) \rightharpoonup \Gamma\,\delta_{(\bar r,\bar z)} \quad \textit{as } t \to 0\;. \tag{3.4}$$

In addition, there exists a constant $C > 0$ such that the following estimate holds :

$$\int_\Omega \Big|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\,e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4\nu t}}\Big|\,\mathrm{d}r\,\mathrm{d}z \;\le\; C\,|\Gamma|\,\frac{\sqrt{\nu t}}{\bar r}\,\log\frac{\bar r}{\sqrt{\nu t}}\;, \tag{3.5}$$

as long as $\sqrt{\nu t} \le \bar r/2$.

Since the existence of a global solution to (3.1) satisfying (3.4) is already asserted by Proposition 3.1, the main contributions of Theorem 3.2 are the uniqueness of that solution and its asymptotic behavior as $t \to 0$, as described in (3.5). The first step in the proof is a localization estimate, which can be established using a Gaussian upper bound on the fundamental solution of the "linear" equation (3.1) for ω_θ, where the velocity field $u = (u_r, u_z)$ is considered as given. It is found that, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|\omega_\theta(r,z,t)| \;\le\; \frac{C_\epsilon|\Gamma|}{\nu t}\,\exp\Big(-\frac{(r-\bar r)^2 + (z-\bar z)^2}{(4+\epsilon)\nu t}\Big)\,, \qquad (r,z) \in \Omega\,, \quad t > 0\,. \tag{3.6}$$

Moreover $\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z$ converges to Γ as $t \to 0$. The second step consists in introducing self-similar variables, in the spirit of (2.1). The rescaled vorticity f and velocity v are defined by

$$\omega_\theta(r,z,t) \;=\; \frac{\Gamma}{\nu t}\,f\Big(\frac{r-\bar r}{\sqrt{\nu t}}\,,\frac{z-\bar z}{\sqrt{\nu t}}\,,t\Big)\,, \qquad u(r,z,t) \;=\; \frac{\Gamma}{\sqrt{\nu t}}\,v\Big(\frac{r-\bar r}{\sqrt{\nu t}}\,,\frac{z-\bar z}{\sqrt{\nu t}}\,,t\Big)\,,$$

and the following dimensionless quantities are introduced :

$$R \;=\; \frac{r-\bar r}{\sqrt{\nu t}}\;, \qquad Z \;=\; \frac{z-\bar z}{\sqrt{\nu t}}\;, \qquad \epsilon \;=\; \frac{\sqrt{\nu t}}{\bar r}\;, \qquad \alpha \;=\; \frac{\Gamma}{\nu}\;.$$

The evolution equation for the new function $f(R,Z,t)$ reads

$$tf_t + \alpha\Big(\partial_R(v_R f) + \partial_Z(v_Z f)\Big) \;=\; Lf + \epsilon\partial_R\Big(\frac{f}{1+\epsilon R}\Big)\,, \tag{3.7}$$

where as in (2.3)

$$L \;=\; \partial_R^2 + \partial_Z^2 + \frac{R}{2}\,\partial_R + \frac{Z}{2}\,\partial_Z + 1\,.$$

Note that equation (3.7) lives in the time-dependent domain where $1 + \epsilon R > 0$, but using the homogeneous Dirichlet boundary condition one can extend the rescaled vorticity by zero outside that domain and consider it as defined on the whole plane $\mathbb{R}^2$. In the small time limit $\epsilon \to 0$, the system formally reduces to the equation for perturbations around Oseen's vortex, which was studied in detail in Section 2.1, and the proof of Theorem 3.2 consists in showing that this intuition is indeed correct. The Gaussian bound (3.6) provides a uniform control on the solution of (3.7) in the weighted space X_t defined by the norm

$$\|f(t)\|_{X_t}^2 \;=\; \int_{1+\epsilon R>0} f(R,Z,t)^2\, e^{(R^2+Z^2)/4}\,\mathrm{d}R\,\mathrm{d}Z\,, \qquad t>0\,,$$

which coincides when $t = 0$ with the norm of the space $L^2(\infty)$ introduced in (2.9). A compactness argument, as in the proof of Theorem 2.1, can then be invoked to show that $f(R, Z, t)$ necessarily converges to the Oseen vortex profile as $t \to 0$:

$$\lim_{t\to 0} \|f(t) - \Gamma G\|_{X_t} \;=\; 0\,, \qquad \text{where} \quad G(R,Z) \;=\; \frac{1}{4\pi}\, e^{-(R^2+Z^2)/4}\,. \tag{3.8}$$

The final step is an energy estimate which shows that, for some positive constants C and δ,

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{X_t}^2 \;\le\; -\delta\|f(t)\|_{X_t}^2 + C\epsilon^2|\log\epsilon|^2\,, \tag{3.9}$$

when $t > 0$ is sufficiently small. The differential inequality (3.9) relies on the spectral properties of the linear operator L in the space $L^2(\infty)$, which were established in Section 2.1. As $X_t \hookrightarrow L^1(\mathbb{R}^2)$, it immediately implies estimate (3.5) in Theorem 3.2. Moreover, a similar argument applied to the difference $f_1 - f_2$ of two solutions of (3.1) satisfying (3.4) leads to the conclusion that $f_1 \equiv f_2$, which yields uniqueness.

Theorem 3.2 shows that the two-dimensional Lamb-Oseen vortex naturally appears in the axisymmetric case too, where it describes the short time behavior of solutions arising from vortex filaments as initial data, see (3.5). However, the long-time asymptotics are very different in both situations, as can be seen from the following result:

Proposition 3.3 ([16]) *Assume that the initial vorticity $\omega_\theta^0 \in L^1(\Omega)$ is nonnegative and has finite impulse:*

$$\mathcal{I} \;=\; \int_\Omega r^2\omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z \;<\; \infty\;. \tag{3.10}$$

Then the unique global solution of (3.1) *satisfies*

$$\lim_{t\to\infty}\,\sup_{(r,z)\in\Omega}\left|t^2\omega_\theta(r\sqrt{t},z\sqrt{t},t) - \frac{\mathcal{I}}{16\sqrt{\pi}}\,r\,e^{-\frac{r^2+z^2}{4}}\right| \;=\; 0\,. \tag{3.11}$$

In particular $\|\omega_\theta(t)\|_{L^\infty(\Omega)} = \mathcal{O}(t^{-2})$ as $t \to \infty$.

Proposition 3.3 applies in particular to the vortex rings constructed in Proposition 3.1 and Theorem 3.2. It shows that the long-time asymptotics are described, to leading order, by a self-similar solution of the *linearized* equation obtained by setting $u = 0$ in (3.1). This is in sharp contrast with what happens in the two-dimensional case.

4 Existence and stability of Burgers vortices

As was mentioned in the introduction, the Burgers vortex is a simple but important model in fluid mechanics, describing the balance between the dissipation due to the viscosity and the vorticity stretching through the action of a background straining flow. By rescaling variables in a suitable manner (see, e.g. [20]), one can assume without loss of generality that the rates of the linear strain in (1.11) have the following form

$$\gamma_1 \,=\, -\frac{1+\lambda}{2}\,, \qquad \gamma_2 \,=\, -\frac{1-\lambda}{2}\,, \qquad \gamma_3 = 1\,. \tag{4.1}$$

Here $\lambda \in [0,1)$ is a free parameter that represents the asymmetry of the strain, and the case $\lambda = 0$ corresponds to an axisymmetric strain. The Burgers vortex with circulation α and asymmetry λ is a two-dimensional stationary vorticity field of the form $\boldsymbol{\omega}_{\lambda,\alpha} = (0, 0, \omega_{\lambda,\alpha})^\top$. In view of Eq. (1.12)–(1.13), this means that the third component $\omega_{\lambda,\alpha}$ depends only on the horizontal variable $x = (x_1, x_2) \in \mathbb{R}^2$ and satisfies the following elliptic problem in $\mathbb{R}^2$:

$$L_\lambda \omega - (v, \nabla)\omega \,=\, 0\,, \qquad v \,=\, K_{2D} * \omega\,, \qquad \int_{\mathbb{R}^2} \omega \, \mathrm{d}x \,=\, \alpha\,, \tag{4.2}$$

where $K_{2D}(x) = x^\perp/(2\pi|x|^2)$ and L_λ is the two-dimensional differential operator defined by

$$L_\lambda \,=\, \Delta + \frac{1+\lambda}{2} x_1 \partial_1 + \frac{1-\lambda}{2} x_2 \partial_2 + 1 \,=\, L + \lambda M\,, \qquad M \,=\, \frac{x_1}{2}\partial_1 - \frac{x_2}{2}\partial_2\,. \tag{4.3}$$

When $\lambda = 0$, Eq. (4.2) has the explicit solution $\omega = \alpha G$, which is the classical axisymmetric Burgers vortex [2] with circulation α. Note that αG is in fact the unique solution of (4.2) in the space $L^1(\mathbb{R}^2)$, as can be deduced from Theorem 2.1. Due to its simple explicit expression, the axisymmetric Burgers vortex is often used for comparison with experiments. However, the vortex tubes observed in real flows or numerical simulations usually exhibit an elliptical core region, rather than a circular one, because the local strain due to the background flow is not axisymmetric in general. It is therefore important to propose a model which takes into account the asymmetry of the strain in an appropriate way, and allows one to understand its influence on the shape of the vortex tubes. This motivates the study of the Burgers vortex in the general case where the asymmetry parameter λ is nonzero [39, 41, 42]. In that situation, solutions of (4.2) cannot be written in explicit form, and have to be constructed by a rigorous mathematical argument. The aim of this section is to give an overview of the mathematical results available by now about the existence of asymmetric Burgers vortices (Section 4.1) and their stability with respect to two or three-dimensional perturbations (Sections 4.2 and 4.3).

4.1 Existence and uniqueness of asymmetric Burgers vortices

Since an explicit representation is no longer available for asymmetric Burgers vortices, existence of such solutions is the first question to address. One of the key observations is that, as the asymmetry parameter λ in (4.1) is increased, the localizing effect due to the linear strain becomes weaker in the x_2 direction. This phenomenon is illustrated by the shape of the function

$$\mathcal{G}_\lambda(x) \,=\, \frac{\sqrt{1-\lambda^2}}{4\pi}\, e^{-\frac{1+\lambda}{4}x_1^2 - \frac{1-\lambda}{4}x_2^2}\,, \qquad x \in \mathbb{R}^2\,, \tag{4.4}$$

which solves the equation $L_\lambda \mathcal{G}_\lambda \,=\, 0$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \mathcal{G}_\lambda \, \mathrm{d}x \,=\, 1$. The form of $\mathcal{G}_\lambda$ indicates that asymmetric Burgers vortices, if they exist, still have a Gaussian decay at infinity, but with a rate that becomes slower as λ increases. Therefore, the function space $L^2(\infty)$ defined in (2.9) has to be modified in an appropriate way to allow for a general asymmetry parameter $\lambda \in [0,1)$. In view of (4.4) it is rather natural to introduce the function

$$G_\lambda(x) \;=\; \frac{1-\lambda}{4\pi}\, e^{-\frac{1-\lambda}{4}|x|^2}\,, \qquad x \in \mathbb{R}^2\,, \tag{4.5}$$

and the weighted L^2 spaces

$$L^2(\infty;\lambda) \;=\; \Bigl\{ f \in L^2(\mathbb{R}^2) \;\Big|\; \|f\|^2_{L^2(\infty;\lambda)} := \int_{\mathbb{R}^2} |f(x)|^2 \frac{\mathrm{d}x}{G_\lambda(x)} < \infty \Bigr\}\,, \tag{4.6}$$

$$L^2_0(\infty;\lambda) \;=\; \Bigl\{ f \in L^2(\infty;\lambda) \;\Big|\; \int_{\mathbb{R}^2} f \,\mathrm{d}x = 0 \Bigr\}\,, \tag{4.7}$$

together with the associated weighted Sobolev space $W^{1,2}(\infty;\lambda)$ equipped with the norm

$$\|f\|_{W^{1,2}(\infty;\lambda)} \;=\; \|\rho f\|_{L^2(\infty;\lambda)} + \|\nabla f\|_{L^2(\infty;\lambda)}\,, \qquad \rho(x) = (1+|x|^2)^{\frac12}\,. \tag{4.8}$$

When $\lambda = 0$ these spaces are simply denoted by $L^2(\infty)$, $L^2_0(\infty)$, and $W^{1,2}(\infty)$, respectively.

In the analysis of the Burgers vortex, the linear operators L_λ in (4.3) and Λ_f defined by

$$\Lambda_f\,\omega \;=\; \bigl(K_{2D} * f, \nabla\bigr)\omega + \bigl(K_{2D} * \omega, \nabla\bigr) f \tag{4.9}$$

play essential roles. The operator Λ_f naturally appears as the linearization of the quadratic term $(v,\nabla)\omega$ in (4.2) around a given vorticity profile f. Note that Λ_G is nothing but the operator Λ defined in (2.7), but in the present section the general notation Λ_G will be preferred in order to emphasize the dependence upon G. The operators L_λ (when $\lambda = 0$) and Λ_f (when f is radially symmetric) are invariant under rotations about the origin in $\mathbb{R}^2$. It is thus natural to use polar coordinates (r,θ) in the plane and to expand all functions in Fourier series with respect to the angular variable θ. In this way, one can introduce the projections $\mathbb{P}_n$ $(n \in \mathbb{Z})$ defined by

$$(\mathbb{P}_n g)(r,\theta) \;=\; g_n(r) e^{in\theta}\,, \qquad g_n(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, e^{-in\theta}\,\mathrm{d}\theta\,, \tag{4.10}$$

and the projected spaces $\mathbb{P}_n X = \{\mathbb{P}_n g \,|\, g \in X\}$ for any function space X such as $L^2(\infty;\lambda)$. By construction the projections $\mathbb{P}_n$ commute with both operators L and Λ_G.

As long as the asymmetry parameter λ lies in $[0,1)$, the linear strain (4.1) localizes the vorticity in the horizontal directions, because $\gamma_1 < 0$ and $\gamma_2 < 0$. Starting from this observation and relying on numerical calculations, Robinson and Saffman [42] conjectured the existence of asymmetric Burgers vortices, i.e., solutions to (4.2), for all values of the parameters $\lambda \in [0,1)$ and $\alpha \in \mathbb{R}$, at least in the regime where $\frac{\lambda}{1+|\alpha|}$ is small enough. This fundamental question has been settled by now as follows.

Theorem 4.1 (Existence) *For all $\lambda \in [0,1)$ and all $\alpha \in \mathbb{R}$, there exists at least one asymmetric Burgers vortex $\omega_{\lambda,\alpha} \in L^2(\infty;\lambda)$ satisfying* (4.2)*.*

This existence result is established in [21] when $0 \le \lambda \ll \frac12$ and $\alpha \in \mathbb{R}$, in [20] when $\lambda \in [0,1)$ and $|\alpha| \le \kappa(\lambda) \ll 1$, and in [32] when $\lambda \in [0,\frac12)$ and $|\alpha| \ge R(\lambda) \gg 1$. Here $\kappa(\lambda)$ and $R(\lambda)$ are positive numbers satisfying $\kappa(\lambda) \to 0$ as $\lambda \to 1$ and $R(\lambda) \to \infty$ as $\lambda \to 1/2$. The proofs in [20, 21, 32] are based on a detailed analysis of some linearized operators, and existence of solutions to (4.2) is established using the Banach fixed point theorem, which also provides uniqueness in a suitable class of functions, see Theorem 4.3 below. In contrast, the general existence result for all $\lambda \in [0, 1)$ and all $\alpha \in \mathbb{R}$ established in [33] relies on the Leray-Schauder fixed point theorem, which does not give any information about uniqueness.

Since Burgers vortices are used to model tubular structures in turbulent flows, it is highly interesting to study their asymptotic shape as $|\alpha| \to \infty$ in the presence of asymmetric linear strains. Numerical calculations by Prochazka and Pullin [40, 41] and by Robinson and Saffman [42] indicate that a large circulation α has a symmetrizing effect on the vortex, so that the leading order of the flow in a bounded fluid region is the axisymmetric Burgers vortex αG, even when $\lambda \neq 0$. Using formal asymptotic expansions, Moffatt, Kida, and Ohkitani [39] explained this phenomenon analytically and obtained for the Burgers vortex $\omega_{\lambda,\alpha}$ an expression of the form

$$\omega_{\lambda,\alpha} \;=\; \alpha G + W_\lambda + R_{\lambda,\alpha}\,, \qquad \text{as} \quad |\alpha| \to \infty\,, \tag{4.11}$$

where W_λ is independent of α and $R_{\lambda,\alpha} = \mathcal{O}(|\alpha|^{-1})$ as $|\alpha| \to \infty$. The second order term W_λ in (4.11) is especially relevant, since it describes how the asymmetry of the background flow modifies the shape of the vortex.

Inserting the formal expansion (4.11) into (4.2) and using the cancellation $(K_{2D}*G, \nabla)G = 0$, one obtains the relation

$$L_\lambda\big(\alpha G + W_\lambda + R_{\lambda,\alpha}\big) \;=\; \alpha \Lambda_G W_\lambda + \mathcal{O}(1)\,, \qquad \text{as} \quad |\alpha| \to \infty\,. \tag{4.12}$$

Since $L_\lambda G = (L + \lambda M)G = \lambda M G$, the terms proportional to α in both sides of (4.12) are equal if and only if $W_\lambda = \lambda w_\infty$, where w_∞ satisfies

$$\Lambda_G\, w_\infty \;=\; MG\,. \tag{4.13}$$

Eq. (4.13) is derived and studied numerically in [39], and discussed in [21] within a rigorous functional framework. More precisely, it is shown in [21, Proposition 3.1] that there exists a unique w_∞ in $\mathbb{P}_2 W^{1,2}(\infty) + \mathbb{P}_{-2} W^{1,2}(\infty)$ solving (4.13), while the expansion (4.11) has been mathematically verified as follows.

Theorem 4.2 (Large circulation asymptotics) *For any $\lambda \in [0, 1)$, there exists $R_0(\lambda) > 0$ such that, for all $\alpha \in \mathbb{R}$ with $|\alpha| \geq R_0(\lambda)$, there exists an asymmetric Burgers vortex $\omega_{\lambda,\alpha} \in \mathbb{P}^e L^2(\infty; \lambda)$ solving (4.2) and satisfying*

$$\|\omega_{\lambda,\alpha} - \alpha G - \lambda w_\infty\|_{W^{1,2}(\infty;\lambda)} \;\leq\; \frac{\lambda C(\lambda)}{1 + |\alpha|}\,. \tag{4.14}$$

Here $\mathbb{P}^e$ is the even projection defined by $\mathbb{P}^e = \oplus_{n\in\mathbb{Z}} \mathbb{P}_{2n}$, and the constants $R_0(\lambda), C(\lambda)$ satisfy

$$\lim_{\lambda\to 1} R_0(\lambda) \;=\; \lim_{\lambda\to 1} C(\lambda) \;=\; \infty\,.$$

The conclusion of Theorem 4.2 was first established in [21] assuming $0 \leq \lambda \ll \frac{1}{2}$, in which case estimate (4.14) holds in the stronger norm of $W^{1,2}(\infty)$ and not just in $W^{1,2}(\infty; \lambda)$. The result was then extended to the larger range $\lambda \in [0, \frac{1}{2})$ in [32], using again the space $W^{1,2}(\infty)$, and the general case where $\lambda \in [0, 1)$ was finally settled in [33]. The basic strategy in [21, 32, 33] is to construct a solution of (4.2) as a perturbation of the leading order approximation $\alpha G + \lambda w_\infty$, in such a way that estimate (4.14) holds. To explain this idea more precisely, it is convenient to introduce the perturbation $w^{(1)} = \omega_{\lambda,\alpha} - \alpha G - \lambda w_\infty$, which has to solve the system

$$\begin{aligned} \big(L_\lambda - \alpha\Lambda_G - \lambda\Lambda_{w_\infty}\big) w^{(1)} \;&=\; (K_{2D} * w^{(1)}, \nabla) w^{(1)} + \lambda f_\lambda\,, \qquad \int_{\mathbb{R}^2} w^{(1)}\,\mathrm{d}x \;=\; 0\,, \\ f_\lambda \;&=\; -L_\lambda w_\infty + \lambda (K_{2D} * w_\infty, \nabla) w_\infty\,. \end{aligned} \tag{4.15}$$

To show that $w^{(1)}$ is of order $\mathcal{O}(|\alpha|^{-1})$ as $|\alpha| \to \infty$, the key observation is that the source term f_λ in (4.15) also belongs to the range of Λ_G. Indeed, a similar argument as in [21, Proposition 3.1] implies the existence of a unique $h_\lambda \in (I - P_0)\mathbb{P}^e W^{1,2}(\infty)$ satisfying $\Lambda_G h_\lambda = f_\lambda$. Then f_λ is decomposed as

$$f_\lambda \,=\, \Lambda_G h_\lambda \,=\, -\frac{1}{\alpha}\big(L_\lambda - \alpha\Lambda_G - \lambda\Lambda_{w_\infty}\big)h_\lambda + \frac{1}{\alpha}\big(L_\lambda - \lambda\Lambda_{w_\infty}\big)h_\lambda\,,$$

and Eq. (4.15) can thus be reduced to the following system for $w^{(2)} = w^{(1)} + \frac{\lambda}{\alpha}h_\lambda$:

$$\begin{aligned}\big(L_\lambda - \alpha\Lambda_G - \lambda\Lambda_{w_\infty - \frac{1}{\alpha}h_\lambda}\big)w^{(2)} \,&=\, (K_{2D} * w^{(2)}, \nabla)w^{(2)} + \frac{\lambda}{\alpha}F_{\lambda,\alpha}\,, \qquad \int_{\mathbb{R}^2} w^{(2)}\,\mathrm{d}x \,=\, 0\,,\\ F_{\lambda,\alpha} \,&=\, \big(L_\lambda - \lambda\Lambda_{w_\infty}\big)h_\lambda + \frac{\lambda}{\alpha}(K_{2D} * h_\lambda, \nabla)h_\lambda\,.\end{aligned} \tag{4.16}$$

It is clear that the source term $\frac{\lambda}{\alpha}F_{\lambda,\alpha}$ is of order $\mathcal{O}(\frac{\lambda}{|\alpha|})$ in $L^2_0(\infty;0)$ as $|\alpha| \to \infty$. Since $\frac{\lambda}{\alpha}\Lambda_{h_\lambda}$ is a lower order perturbation that becomes small in the regime where $|\alpha| \gg 1$, the crucial step to establish (4.14) is to prove the invertibility of the operator

$$L_\lambda - \alpha\Lambda_G - \lambda\Lambda_{w_\infty}\,, \qquad \text{in the space } L^2_0(\infty;\lambda)\,, \tag{4.17}$$

together with a uniform estimate for its inverse when $|\alpha| \gg 1$.

This is an easy task if $\lambda \in [0, \frac{1}{2})$ is small, because one can then work in the space $L^2_0(\infty)$ instead of $L^2_0(\infty;\lambda)$, and consider the operator in (4.17) as a small perturbation of the more familiar operator $L - \alpha\Lambda_G$, which has been thoroughly studied in Section 2. As was mentioned in (2.14), the operator Λ_G is skew-symmetric in $L^2(\infty)$, namely

$$\langle \Lambda_G f, g\rangle_{L^2(\infty)} + \langle f, \Lambda_G g\rangle_{L^2(\infty)} \,=\, 0\,, \qquad \text{for all } f, g \in W^{1,2}(\infty)\,.$$

Moreover, it follows from (2.13) that $-L$ is a self-adjoint operator in $L^2(\infty)$ with compact resolvent, which satisfies the lower bound $-L \geq 0$ in $L^2(\infty)$ and $-L \geq \frac{1}{2}$ in $L^2_0(\infty)$. These two observations yield the uniform lower bound

$$\langle\big(-L + \alpha\Lambda_G\big)f, f\rangle_{L^2(\infty)} \geq \frac{1}{2}\|f\|^2_{L^2(\infty)}\,, \qquad f \in L^2_0(\infty) \cap D(L)\,, \tag{4.18}$$

which in turn implies that $\|(-L + \alpha\Lambda_G)^{-1}\|_{L^2_0(\infty)\to L^2_0(\infty)} \leq 2$ for all $\alpha \in \mathbb{R}$. A straightforward perturbation argument then gives a uniform estimate on $(-L_\lambda + \alpha\Lambda_G + \lambda\Lambda_{w_\infty})^{-1}$ in $L^2_0(\infty)$ if λ is sufficiently small, see [21, Proposition 2.1].

For larger values of λ, the term $\lambda\Lambda_{w_\infty}$ is not a small perturbation anymore, and the invertibility of the operator $L_\lambda - \alpha\Lambda_G - \lambda\Lambda_{w_\infty}$ in $L^2_0(\infty;\lambda)$ is not known for arbitrary $\alpha \in \mathbb{R}$. However, if $|\alpha|$ is sufficiently large (depending on λ), one can exploit the stabilizing effect that was already discussed in Section 2.2 for the simpler operator $L - \alpha\Lambda_G$, see Proposition 2.9. For the modified operator $L_\lambda - \alpha\Lambda_G$ with $\lambda \in [0, 1)$, one has the following estimates

$$\begin{aligned}&\lim_{|\alpha|\to\infty} \|(I - \mathbb{P}_0)(-L_\lambda + \alpha\Lambda_G)^{-1}\mathbb{P}^e\|_{L^2_0(\infty;\lambda)\to L^2_0(\infty;\lambda)} = 0\,,\\ &\lim_{|\alpha|\to\infty} \|\mathbb{P}_0(-L_\lambda + \alpha\Lambda_G)^{-1}(I - \mathbb{P}_0)\mathbb{P}^e\|_{L^2_0(\infty;\lambda)\to L^2_0(\infty;\lambda)} = 0\,,\end{aligned} \tag{4.19}$$

which are proved in [32] for $\lambda \in [0, \frac{1}{2})$ and in [33] for arbitrary $\lambda \in [0, 1)$. Roughly speaking, this means that the non-radially symmetric elements of $\mathbb{P}^e L^2(\infty;\lambda)$ are strongly attenuated under the action of $(-L_\lambda + \alpha\Lambda_G)^{-1}$ when $|\alpha|$ is large. In addition, since the function w_∞ belongs to $\mathbb{P}_2 L^2(\infty) + \mathbb{P}_{-2}L^2(\infty)$, one has the identity

$$\mathbb{P}_0 \Lambda_{w_\infty} \mathbb{P}_0 f \,=\, 0 \qquad \text{hence} \quad \mathbb{P}_0\Lambda_{w_\infty} f \,=\, \mathbb{P}_0\Lambda_{w_\infty}(I - \mathbb{P}_0)f\,, \tag{4.20}$$

for all $f \in \mathbb{P}^e W^{1,2}(\infty;\lambda)$. Combining (4.19) and (4.20), it is possible to show that $\lambda\Lambda_{w_\infty}$ is a relatively small perturbation of $L_\lambda - \alpha\Lambda_G$ in $\mathbb{P}^e L^2_0(\infty;\lambda)$ if $|\alpha|$ is sufficiently large (depending on λ), and this implies the invertibility of $L_\lambda - \alpha\Lambda_G + \lambda\Lambda_{w_\infty}$ in $\mathbb{P}^e L^2_0(\infty;0)$ and provides a uniform bound for the inverse when $|\alpha| \gg 1$.

When $\lambda \in [0, \frac{1}{2})$, the resolvent estimates (4.19) also hold in the smaller space $L^2_0(\infty)$, instead of $L^2_0(\infty,\lambda)$, and are substantially easier to prove because one can then use the convenient property that Λ_G is skew-symmetric, see [21, 32]. However, when $\lambda \in [\frac{1}{2}, 1)$, the Burgers vortex does not belong to $L^2(\infty)$ anymore, as can be seen from the shape of the function $\mathcal{G}_\lambda$ in (4.4). One is then forced to work in a wider space such as $L^2(\infty;\lambda)$, where the operator Λ_G is no longer skew-symmetric. The key idea in [33] to overcome this difficulty is to construct explicitly a bounded and invertible operator T so that $\Lambda_G T$, the right action of T on Λ_G, becomes skew-symmetric. With this skew-symmetrizer T, the equation $(L_\lambda - \alpha\Lambda_G)w = f$ is written in the equivalent form

$$(L_\lambda - \alpha\Lambda_G T)T^{-1}w \,=\, -L_\lambda(T-I)T^{-1}w + f\,.$$

Using the skew-symmetry of $\Lambda_G T$, one can show that the operator $L_\lambda - \alpha\Lambda_G T$ satisfies resolvent bounds similar to (4.19), and the additional term $L_\lambda(T - I)$ can be considered as a relatively small perturbation. This implies the invertibility of $L_\lambda - \alpha\Lambda_G T + L_\lambda(T-I)$, hence of $L_\lambda - \alpha\Lambda_G$. The argument also yields uniform estimates on the inverse $(L_\lambda - \alpha\Lambda_G)^{-1}$, which in turn make it possible to treat $\lambda\Lambda_{w_\infty}$ as a relatively small perturbation when $|\alpha|$ is sufficiently large (depending on λ), thus concluding the proof of Theorem 4.2.

The uniqueness of asymmetric Burgers vortices in a suitable class of functions is also available for some range of parameters (λ, α).

Theorem 4.3 (Uniqueness) (i) Case $0 \le \lambda \ll \frac{1}{2}$ and $\alpha \in \mathbb{R}$: *There exist $\lambda_0 \in (0, \frac{1}{2})$ and $\tau_0 > 0$ such that, if $\lambda \in [0, \lambda_0]$ and $\alpha \in \mathbb{R}$, there exists at most one asymmetric Burgers vortex $\omega_{\lambda,\alpha}$ in the set $\{f \in L^2(\infty) \mid \|f - \alpha G\|_{W^{1,2}(\infty)} \le \tau_0\}$.*
(ii) Case $0 \le \lambda < 1$ and $|\alpha| \ll 1$: *For all $\lambda \in [0,1)$ there exists $\kappa_0 = \kappa_0(\lambda) > 0$ such that, for any $\alpha \in \mathbb{R}$ with $|\alpha| \le \kappa_0$, there exists at most one asymmetric Burgers vortex $\omega_{\lambda,\alpha}$ in the set $\{f \in L^2(\infty) \mid \|f - \alpha\mathcal{G}_\lambda\|_{L^2(\infty)} \le \kappa_0\}$.*
(iii) Case $0 \le \lambda < 1$ and $|\alpha| \gg 1$: *For all $\lambda \in [0,1)$ and all $\tau > 0$ there exists $R'(\lambda,\tau) \ge R_0(\lambda)$ such that, for any $\alpha \in \mathbb{R}$ with $|\alpha| \ge R'(\lambda,\tau)$, there exists at most one asymmetric Burgers vortex $\omega_{\lambda,\alpha}$ in the set $\{f \in \mathbb{P}^e L^2(\infty,\lambda) \mid \|f - \alpha G - \lambda w_\infty\|_{W^{1,2}(\infty;\lambda)} \le \tau\}$. Here $R_0(\lambda)$ is as in Theorem 4.2, and $R'(\lambda,\tau)$ satisfies $\lim_{\tau\to\infty} R'(\lambda,\tau) \,=\, \infty$.*

The statement (i) of Theorem 4.3 is proved in [21] using the uniform estimate (4.18) for the inverse of $L - \alpha\Lambda_G$ in $L^2_0(\infty)$, while (iii) is established in [32, 33] using the stabilization effect at large circulations described in (4.19). The uniqueness in the case (ii) is obtained in [20] in the more general framework of the polynomially weighted spaces $L^2(m)$. The key point in the proof of (ii) is an estimate for the inverse L_λ^{-1} in $L^2_0(m)$ when m is large enough, which enables to apply the Banach fixed point theorem when $|\alpha|$ is sufficiently small. Remark that existence of asymmetric Burgers vortices is also established in [20, 21, 32, 33] for all three cases (i), (ii), and (iii) above, whereas uniqueness for the parameter regions not covered by Theorem 4.3 is an interesting but difficult question, which is essentially open.

4.2 Two-dimensional stability of asymmetric Burgers vortices

In the parameter regions where existence and uniqueness have been established, the next important issue is stability. Since the Burgers vortex itself is a two-dimensional vorticity field, it is possible to study its stability within the class of purely two-dimensional flows, and this is the point of view adopted in this subsection. The axisymmetric case where $\lambda = 0$ was already discussed in detail in Section 2, hence the main focus here will be on the asymmetric case $\lambda \neq 0$.

The evolution equations for the perturbations are obtained from system (1.12), where $\nu = 1$ and $\gamma_1, \gamma_2, \gamma_3$ are as in (4.1), by expanding the vorticity vector $\boldsymbol{\omega}(\mathbf{x}, t)$ around the stationary Burgers vortex $\boldsymbol{\omega}_{\lambda,\alpha}(\mathbf{x}) = (0, 0, \omega_{\lambda,\alpha}(x))^\top$. When the vorticity field $\mathbf{w}(\mathbf{x}, t) = (0, 0, w(x, t))^\top$ of the perturbation is two-dimensional, the problem is reduced to the following equations for the scalar function w:

$$\begin{cases} \partial_t w - \big(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}\big) w + (v, \nabla) w \;=\; 0\,, \qquad v \;=\; K_{2D} * w\,, \qquad t > 0\,, \\ w|_{t=0} \;=\; w_0\,, \quad \displaystyle\int_{\mathbb{R}^2} w_0 \,\mathrm{d}x \;=\; 0\,. \end{cases} \tag{4.21}$$

Here the operators L_λ and Λ_f are defined by (4.3) and (4.9), respectively. As can be expected, the properties of the linearized operator $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ play a crucial role in the stability analysis. It is not difficult to show that L_λ generates a C_0-semigroup in the polynomially weighted space $L^2(m)$ for $m < \infty$, and an analytic semigroup in the Gaussian weighted space $L^2(\infty; \lambda)$. In fact, the semigroup e^{tL_λ} has the following explicit representation

$$\big(e^{tL_\lambda} f\big)(x) \;=\; \frac{e^t}{4\pi\sqrt{a_\lambda(t) a_{-\lambda}(t)}} \int_{\mathbb{R}^2} \exp\Big(- \frac{|x_1 - y_1|^2}{4a_\lambda(t)} - \frac{|x_2 - y_2|^2}{4a_{-\lambda}(t)}\Big) f(y_1 e^{\frac{1+\lambda}{2}t}, y_2 e^{\frac{1-\lambda}{2}t}) \,\mathrm{d}y\,,$$

where $a_\theta(t) = (1 - e^{-(1+\theta)t})/(1 + \theta)$. Since $\Lambda_{\omega_{\lambda,\alpha}}$ is a relatively compact perturbation of L_λ, the full linearized operator $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ is also the generator of a C_0 (or analytic) semigroup, and the main concern is the long-time behavior of that semigroup. The following results have been established in (essentially) the same parameter regions as in Theorem 4.3.

Proposition 4.4 (Linear stability) (i) Case $0 \le \lambda \ll \frac{1}{2}$ and $\alpha \in \mathbb{R}$: *There exists $\lambda_1 \in (0, \frac{1}{2})$ such that, for all $\lambda \in [0, \lambda_1]$ and all $\alpha \in \mathbb{R}$,*

$$\|e^{t(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}})} f\|_{L^2(\infty)} \le C \|f\|_{L^2(\infty)}\, e^{-\frac{1-\lambda}{2}t}\,, \qquad t \ge 0\,, \tag{4.22}$$

for all $f \in L^2_0(\infty)$. Here C is a universal constant independent of $\lambda \in [0, \lambda_1]$ and $\alpha \in \mathbb{R}$.
(ii) Case $0 \le \lambda < 1$ and $|\alpha| \ll 1$: *For all $\lambda \in [0, 1)$ there exists $\kappa_1(\lambda) > 0$ such that, if $|\alpha| \le \kappa_1(\lambda)$, then*

$$\|e^{t(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}})} f\|_{L^2(m)} \le C \|f\|_{L^2(m)} e^{-\frac{1-\lambda}{2}t}\,, \qquad t \ge 0\,, \tag{4.23}$$

for all $f \in L^2_0(m)$, $m > 3$. Here C depends only on $\lambda \in [0, 1)$ and m.
(iii) Case $0 \le \lambda < \frac{1}{2}$ and $|\alpha| \gg 1$: *For all $\lambda \in [0, \frac{1}{2})$ there exists $R_1(\lambda) \ge R_0(\lambda)$ such that, if $|\alpha| \ge R_1(\lambda)$, then*

$$\|e^{t(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}})} f\|_{L^2(\infty)} \;\le\; C \|f\|_{L^2(\infty)} e^{-\frac{1-\lambda}{2}t}\,, \qquad t \ge 0\,, \tag{4.24}$$

for all $f \in L^2_0(\infty)$. Here C depends only on λ and α, while $R_0(\lambda)$ is as in Theorem 4.2.

The statement (i) of Proposition 4.4 is proved in [21], where $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ is regarded as a small perturbation of the simpler operator $L - \alpha\Lambda_G$ for which, as recalled in (4.18), the stability estimate is obtained uniformly in α using the skew-symmetry of Λ_G in $L^2(\infty)$. The case (ii) follows from the analysis developed in [20]. In fact, as is mentioned in the next subsection, the three-dimensional stability is the main concern of [20], but the class of perturbations considered there includes purely two-dimensional flows. In case (ii) the asymmetric Burgers vortex $\omega_{\lambda,\alpha}$ is of order $\mathcal{O}(|\alpha|)$ in $L^2(\infty;\lambda)$, and the operator $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ is handled as a small perturbation of L_λ, for which complete information on the spectrum and the associated semigroup is available. Case (iii) is treated in [32], using in an essential way the stabilizing effect at large circulations described in (4.19). The restriction $\lambda \in [0, \frac{1}{2})$ in (iii) is due to the fact that, when $\lambda \geq \frac{1}{2}$, the operator $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ has to be analyzed in the space $L^2(\infty, \lambda)$, where Λ_G is no longer skew-symmetric. So far this difficulty could not be overcome for the stability problem, although existence and uniqueness of Burgers vortices were established by constructing a suitable skew-symmetrizer, as explained in Section 4.1.

It should be emphasized here that, in all cases (i), (ii), and (iii), the temporal decay estimate for the semigroup $e^{t(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}})}$ involves the exponent $-\frac{1-\lambda}{2}$, which is known to be optimal. Indeed, by differentiating the identity $L_\lambda \omega_{\lambda,\alpha} - (K_{2D} * \omega_{\lambda,\alpha}, \nabla)\omega_{\lambda,\alpha} = 0$ with respect to x_2, one observes that $\partial_2 \omega_{\lambda,\alpha}$ is an eigenfunction of $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ for the eigenvalue $-\frac{1-\lambda}{2}$. Numerical results due to Prochazka and Pullin [41] indicate that $-\frac{1-\lambda}{2}$ is actually the largest eigenvalue of $L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ in $L^2_0(\infty, \lambda)$ for any $\lambda \in [0,1)$, but a mathematical proof of this conjecture is still missing, except in the three cases stated in Proposition 4.4.

The semigroup $e^{t(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}})}$ has standard parabolic smoothing properties. Nonlinear stability with respect to small initial perturbations can thus be obtained by analyzing the integral equation associated with (4.21):

$$w(t) \,=\, e^{t(L_\lambda - \Lambda_{\omega_{\alpha,\lambda}})} w_0 - \int_0^t e^{(t-s)(L_\lambda - \Lambda_{\omega_{\alpha,\lambda}})} \big(K_{2D} * w(s), \nabla\big) w(s)\,\mathrm{d}s\,, \tag{4.25}$$

and applying the conclusions of Proposition 4.4. This gives the following result:

Theorem 4.5 (Local 2D stability) (i) Case $0 \leq \lambda \ll \frac{1}{2}$ and $\alpha \in \mathbb{R}$: *There exists $\epsilon > 0$ such that, for all $\lambda \in [0, \lambda_1]$ and all $\alpha \in \mathbb{R}$, the following statement holds. For all initial data $w_0 \in L^2_0(\infty)$ such that $\|w_0\|_{L^2(\infty)} \leq \epsilon$, Eq. (4.21) admits a unique solution $w \in C^0([0,\infty); L^2_0(\infty))$, which satisfies*

$$\|w(t)\|_{L^2(\infty)} \,\leq\, C\|w_0\|_{L^2(\infty)}\, e^{-\frac{1-\lambda}{2}t}\,, \qquad t \geq 0\,. \tag{4.26}$$

Here the constant C is independent of $\lambda \in [0, \lambda_1]$ and $\alpha \in \mathbb{R}$, while λ_1 is as in Proposition 4.4.
(ii) Case $0 \leq \lambda < 1$ and $|\alpha| \ll 1$: *For all $\lambda \in [0,1)$ there exists $\epsilon = \epsilon(\lambda) > 0$ such that, for any $\alpha \in \mathbb{R}$ with $|\alpha| \leq \kappa_1(\lambda)$, the following statement holds. For all initial data $w_0 \in L^2_0(m)$, $m > 3$, such that $\|w_0\|_{L^2(m)} \leq \epsilon$, Eq. (4.21) admits a unique solution $w \in C^0([0,\infty); L^2_0(m))$, which satisfies*

$$\|w(t)\|_{L^2(m)} \,\leq\, C\|w_0\|_{L^2(m)}\, e^{-\frac{1-\lambda}{2}t}\,, \qquad t \geq 0\,. \tag{4.27}$$

Here C depends only on λ and m, while $\kappa_1(\lambda)$ is as in Proposition 4.4.
(iii) Case $0 \leq \lambda < \frac{1}{2}$ and $|\alpha| \gg 1$: *For all $\lambda \in [0, \frac{1}{2})$ and any $\alpha \in \mathbb{R}$ with $|\alpha| \geq R_1(\lambda)$, there exists $\epsilon = \epsilon(\lambda, \alpha) > 0$ such that the following statement holds. For all initial data $w_0 \in L^2_0(\infty)$ with $\|w_0\|_{L^2(\infty)} \leq \epsilon$, Eq. (4.21) admits a unique solution $w \in C^0([0,\infty); L^2_0(\infty))$, which satisfies*

$$\|w(t)\|_{L^2(\infty)} \,\leq\, C\|w_0\|_{L^2(\infty)}\, e^{-\frac{1-\lambda}{2}t}\,, \qquad t \geq 0\,. \tag{4.28}$$

Here C depends only on λ and α, while $R_1(\lambda)$ is as in Proposition 4.4.

As in Proposition 4.4, the statement (i) of Theorem 4.5 is proved in [21], while (ii) follows from the results of [20]. The case (iii) is obtained in [32]. It should be emphasized here that the basin of attraction in the case (i) is uniform in the circulation number α, as a consequence of the linear estimate (4.22) in Proposition 4.4.

4.3 Three-dimensional stability of Burgers vortices

The stability analysis becomes more complicated when the perturbations are three-dimensional, because the vorticity field is no longer a scalar quantity, and vortex stretching terms already appear in the linearized operator. The problem is highly nontrivial even in the axisymmetric case $\lambda = 0$, where Rossi and Le Dizès [43] have shown that the linearized operator does not have any eigenfunction with nontrivial dependence upon the vertical variable. Numerical evidence of linear stability with exponential decay of the perturbations was obtained by Schmid and Rossi [44], but their analysis also reveals the occurrence of short-time amplification for generic solutions. A mathematical understanding of the underlying mechanisms, leading to a rigorous explanation of these observations, is an important and challenging question, for which significant progress has been made in recent years.

Starting from the vorticity equation (1.12), with $\nu = 1$ and $\gamma_1, \gamma_2, \gamma_3$ as in (4.1), it is easy to write the evolution equation for perturbations $\mathbf{w} = \boldsymbol{\omega} - \boldsymbol{\omega}_{\lambda,\alpha}$, where $\boldsymbol{\omega}_{\lambda,\alpha} = (0,0,\omega_{\lambda,\alpha})^\top$ is the Burgers vortex with circulation α. The result is

$$\begin{cases} \partial_t \mathbf{w} - \big(\mathcal{L}_\lambda - \boldsymbol{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}\big)\mathbf{w} + (\mathbf{v},\nabla)\mathbf{w} - (\mathbf{w},\nabla)\mathbf{v} \,=\, 0\,, \quad \mathbf{v} \,=\, K_{3D} * \mathbf{w}\,, \quad t > 0\,, \\ \mathbf{w}|_{t=0} \,=\, \mathbf{w}_0\,, \qquad \nabla\cdot\mathbf{w}_0 = 0\,, \qquad \displaystyle\int_{\mathbb{R}^2} w_{0,3}(x,x_3)\,\mathrm{d}x \,=\, 0\,, \quad x_3 \in \mathbb{R}\,, \end{cases} \tag{4.29}$$

where K_{3D} is the kernel of the Biot-Savart law (1.2). Here the operator $\mathcal{L}_\lambda$ is given by (1.13) with $\nu = 1$, namely

$$\mathcal{L}_\lambda \,=\, \begin{pmatrix} L_\lambda + \partial_3^2 - x_3\partial_3 - \frac{3+\lambda}{2} \\ L_\lambda + \partial_3^2 - x_3\partial_3 - \frac{3-\lambda}{2} \\ L_\lambda + \partial_3^2 - x_3\partial_3 \end{pmatrix}, \tag{4.30}$$

and L_λ is the two-dimensional differential operator (4.3). On the other hand, the operator $\boldsymbol{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}$ is defined by

$$\begin{aligned} \boldsymbol{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} \,=\, & (K_{3D} * \boldsymbol{\omega}_{\lambda,\alpha}, \nabla)\mathbf{w} + (K_{3D} * \mathbf{w}, \nabla)\boldsymbol{\omega}_{\lambda,\alpha} \\ & - (\mathbf{w},\nabla)K_{3D} * \boldsymbol{\omega}_{\lambda,\alpha} - (\boldsymbol{\omega}_{\lambda,\alpha},\nabla)K_{3D} * \mathbf{w}\,. \end{aligned} \tag{4.31}$$

The divergence-free condition as well as the zero mass condition $\int_{\mathbb{R}^2} w_3(x,x_3)\,\mathrm{d}x = 0$ are preserved under the evolution defined by (4.29). Note that, at least formally, a divergence-free vector field $\mathbf{w} = (w_1,w_2,w_3)^\top$ always satisfies the identity $\frac{\mathrm{d}}{\mathrm{d}x_3}\int_{\mathbb{R}^2} w_3(x,x_3)\,\mathrm{d}x = 0$. This means that the condition $\int_{\mathbb{R}^2} w_{0,3}(x,x_3)\,\mathrm{d}x = 0$ in (4.29) is a natural requirement on the initial data, which does not restrict the generality; see [13, Section 1] for a detailed discussion.

Since the Burgers vortex itself is essentially a two-dimensional flow, it is natural to choose a functional setting that allows for purely two-dimensional perturbations, and more generally for perturbations which do not decay to zero as $|x_3| \to \infty$. For this purpose, the following function spaces are introduced in [20, 13]:

$$X(m) \,=\, BC(\mathbb{R}; L^2(m))\,, \qquad X_0(m) \,=\, BC(\mathbb{R}; L^2_0(m))\,, \tag{4.32}$$

as well as $\mathbb{X}(m) = X(m) \times X(m) \times X_0(m)$. Here $BC(\mathbb{R}; L^2(m))$ denotes the space of all bounded and continuous functions from $\mathbb{R}$ into $L^2(m)$, which is a Banach space equipped with the norm $\|\phi\|_{X(m)} = \sup_{x_3 \in \mathbb{R}} \|\phi(\cdot, x_3)\|_{L^2(m)}$, and $BC(\mathbb{R}; L^2_0(m))$ is the closed subspace defined in a similar way. Since the leading order term $\mathcal{L}_\lambda$ in the evolution equation (4.29) contains the dilation operator $-x_3\partial_3$, one cannot expect that the solutions will be continuous in time in the uniform topology of $\mathbb{X}(m)$. To restore continuity in time, it is convenient to work in $X_{loc}(m)$, which is the very same space $X(m)$ equipped with the weaker topology induced by the countable family of seminorms $\|\phi\|_{X_n(m)} = \sup_{|x_3| \le n} \|\phi(\cdot, x_3)\|_{L^2(m)}$, for $n \in \mathbb{N}$. For vector valued functions, the space $\mathbb{X}_{loc}(m)$ is defined in a similar way by endowing $\mathbb{X}(m)$ with the localized topology.

Using these notations, the local stability results available so far can be summarized as follows.

Theorem 4.6 (Local 3D stability) (i) *For all $\lambda \in [0, 1)$ and all $\mu \in (0, \frac{1-\lambda}{2})$ there exist $\epsilon = \epsilon(\lambda) > 0$ and $\kappa_2(\lambda, \mu) \in (0, \kappa_1(\lambda)]$ such that, for any $\alpha \in \mathbb{R}$ with $|\alpha| \le \kappa_2(\lambda, \mu)$, the following statement holds. For all initial data $\mathbf{w}_0 \in \mathbb{X}(m)$, $m > 3$, with $\nabla \cdot \mathbf{w}_0 = 0$ and $\|\mathbf{w}_0\|_{\mathbb{X}(m)} \le \epsilon$, Eq. (4.29) admits a unique solution $\mathbf{w} \in L^\infty(\mathbb{R}_+; \mathbb{X}(m)) \cap C^0([0, \infty); \mathbb{X}_{loc}(m))$, which satisfies*

$$\|\mathbf{w}(t)\|_{\mathbb{X}(m)} \le C \|\mathbf{w}_0\|_{\mathbb{X}(m)}\, e^{-\mu t}\,, \qquad t \ge 0\,. \tag{4.33}$$

Here C depends only on α, and $\kappa_1(\lambda)$ is as in Proposition 4.4.

(ii) *Let $\lambda = 0$. For all $m \in (2, \infty]$ and all $\alpha \in \mathbb{R}$ there exists $\epsilon = \epsilon(m, \alpha) > 0$ such that the following statement holds. For all initial data $\mathbf{w}_0 \in \mathbb{X}(m)$ with $\nabla \cdot \mathbf{w}_0 = 0$ and $\|\mathbf{w}_0\|_{\mathbb{X}(m)} \le \epsilon$, Eq. (4.29) admits a unique solution $\mathbf{w} \in L^\infty(\mathbb{R}_+; \mathbb{X}(m)) \cap C^0([0, \infty); \mathbb{X}_{loc}(m))$, which satisfies*

$$\|\mathbf{w}(t)\|_{\mathbb{X}(m)} \le C \|\mathbf{w}_0\|_{\mathbb{X}(m)}\, e^{-\frac{t}{2}}\,, \qquad t \ge 0\,. \tag{4.34}$$

Here C depends only on m and α.

The statement (i) of Theorem 4.6 is proved in [20], using estimates on the semigroup $e^{t\mathcal{L}_\lambda}$ generated by the operator $\mathcal{L}_\lambda$. In view of (4.30), one has the representation

$$e^{t\mathcal{L}_\lambda}\mathbf{w} \,=\, \Bigl(e^{-\frac{3+\lambda}{2}t} e^{tS_\lambda} w_1\,,\, e^{-\frac{3-\lambda}{2}t} e^{tS_\lambda} w_2\,,\, e^{tS_\lambda} w_3\Bigr)^\top\,, \tag{4.35}$$

where S_λ is the differential operator defined by $S_\lambda = L_\lambda + \partial_3^2 - x_3\partial_3$. Since the operators L_λ and $\partial_3^2 - x_3\partial_3$ act on different variables, it is possible to obtain the following explicit formula

$$\bigl(e^{tS_\lambda} f\bigr)(\mathbf{x}) \,=\, \frac{1}{\sqrt{4\pi a_1(t)}} \int_{\mathbb{R}} \exp\bigl(-\frac{|x_3 e^{-t} - y_3|^2}{4a_1(t)}\bigr) \Bigl(e^{tL_\lambda} f\,(\cdot, y_3)\Bigr)(x)\, \mathrm{d}y_3\,, \tag{4.36}$$

where $a_1(t) = (1 - e^{-2t})/2$ and e^{tL_λ} is the two-dimensional semigroup encountered in Section 4.2. Useful estimates for the semigroup e^{tS_λ} in $X(m)$ are established in [20], together with elementary spectral properties of the generator S_λ. Note that it is possible to take $\mu = \frac{1-\lambda}{2}$ in estimate (4.33), as can be shown using some arguments borrowed from [13].

The statement (ii) of Theorem 4.6 is established in [13]. Remarkably, as in the two-dimensional case, the local stability holds for all values of the circulation number α, and moreover the rate of convergence $e^{-\frac{t}{2}}$ is uniform in α. Although the result of (ii) is stated only in the purely axisymmetric case $\lambda = 0$, by a standard perturbation argument it is also possible to prove local stability of the asymmetric Burgers vortex $\boldsymbol{\omega}_{\lambda,\alpha}$ if the asymmetry parameter λ is sufficiently small, depending on $|\alpha|$. The proof of (ii) in [13] is based on the analysis of the linearized operator $\mathcal{L} - \alpha\mathbf{\Lambda_G}$ and its associated semigroup, where $\mathcal{L}$ and $\alpha\mathbf{\Lambda_G}$ are shorthand notations for the operators $\mathcal{L}_\lambda$ and $\mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}$, respectively, when $\lambda = 0$. Since $\mathbf{\Lambda_G}$ is a lower order perturbation it is not difficult to construct the semigroup $e^{t(\mathcal{L}-\alpha\mathbf{\Lambda_G})}$ in $\mathbb{X}(m)$, but the main problem is to control the long-time behavior. The following result is the key achievement of [13].

Proposition 4.7 (Axisymmetric linear stability) *For all $m \in (2,\infty]$ and all $\alpha \in \mathbb{R}$ one has*

$$\|e^{t(\mathcal{L}-\alpha\mathbf{\Lambda_G})}\mathbf{f}\|_{\mathbb{X}(m)} \le Ce^{-\frac{t}{2}}\|\mathbf{f}\|_{\mathbb{X}(m)}\,, \qquad t \ge 0\,, \tag{4.37}$$

for all $\mathbf{f} \in \mathbb{X}(m)$, where C depends only on m and α. Moreover, if $\nabla\cdot\mathbf{f} = 0$ then $\nabla\cdot e^{t(\mathcal{L}-\alpha\mathbf{\Lambda})}\mathbf{f} = 0$.

The proof of Proposition 4.7 in [13] is based on two important observations:

(I) As an effect of vortex stretching, the vertical derivatives of the velocity and vorticity fields decay exponentially as $t \to \infty$, so that the long-time asymptotics are governed by a two-dimensional vectorial system.

(II) When restricted to two-dimensional solutions, the linearized operator $\mathcal{L} - \alpha\mathbf{\Lambda_G}$ has symmetry properties which imply uniform stability for all values of the circulation parameter.

In the rest of this section both mechanisms are explained in some detail for the more general semigroup $e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})}$, where $0 \le \lambda < 1$. Proposition 4.7 is stated and proved in [13] in the axisymmetric case $\lambda = 0$, but the arguments are robust and can be used to establish linear stability in the asymmetric case too.

Property (I) above is due to a very specific dependence of the operator $\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}$ upon the vertical variable x_3. Indeed, using the definition in (4.30), it is straightforward to verify that $[\partial_3, \mathcal{L}_\lambda] = -\partial_3$, where $[A,B] = AB - BA$ denotes the commutator of A and B. Moreover, since the Burgers vortex $\boldsymbol{\omega}_{\lambda,\alpha}$ is a two-dimensional stationary solution, one has $[\partial_3, \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}] = 0$. At the level of the semigroup, these identities imply that

$$\partial_3^k\, e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})} \,=\, e^{-kt}\, e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})}\partial_3^k\,, \qquad t \ge 0\,, \tag{4.38}$$

for all integer $k \in \mathbb{N}$. Since the semigroup $e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})}$ grows at most exponentially in time, at a rate that depends only on λ and α, Eq. (4.38) shows that the k^{th} order vertical derivative of any solution to the linearized equation $\partial_t\mathbf{w} = (\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})\mathbf{w}$ decays exponentially as $t \to \infty$, if $k \in \mathbb{N}$ is large enough. By a simple interpolation argument, it follows that any expression involving at least one vertical derivative of the solution becomes negligible in the long-time regime, which means that one can restrict the analysis to the two-dimensional vectorial system obtained by disregarding the vertical dependence of all quantities under consideration. More precisely, in view of (4.31), the operator $\mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}$ can be decomposed as

$$\mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} \,=\, \mathbf{\Lambda}^{(1)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} + \mathbf{\Lambda}^{(2)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} - \mathbf{\Lambda}^{(3)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} - \mathbf{\Lambda}^{(4)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w}\,,$$

where, with the notations $\nabla_h = (\partial_1, \partial_2)^\top$ and $\mathbf{w}_h = (w_1, w_2)^\top$,

$$\begin{aligned} \mathbf{\Lambda}^{(1)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} &= (K_{2D} * \omega_{\lambda,\alpha}, \nabla_h)\mathbf{w}\,, & \mathbf{\Lambda}^{(2)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} &= (K_{3D} * \mathbf{w}, \nabla)\boldsymbol{\omega}_{\lambda,\alpha}\,, \\ \mathbf{\Lambda}^{(3)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} &= (\mathbf{w}_h, \nabla_h)K_{3D} * \boldsymbol{\omega}_{\lambda,\alpha}\,, & \mathbf{\Lambda}^{(4)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} &= \omega_{\lambda,\alpha}\partial_3 K_{3D} * \mathbf{w}\,. \end{aligned} \tag{4.39}$$

The discussion above motivates the following decomposition of $\mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}$ into 2D and 3D parts:

$$\begin{aligned} \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w} \;&=\; \begin{pmatrix}(u_{\lambda,\alpha},\nabla_h)\mathbf{w}_h - (\mathbf{w}_h,\nabla_h)u_{\lambda,\alpha} \\ \Lambda_{\omega_{\lambda,\alpha}} w_3\end{pmatrix} \,+\, \mathbf{R}_{\lambda,\alpha}\mathbf{w}\,, \\ \mathbf{R}_{\lambda,\alpha}\mathbf{w} \;&=\; \Big((K_{3D} * \mathbf{w})_h - K_{2D} * w_3, \nabla_h\Big)\boldsymbol{\omega}_{\lambda,\alpha} \,-\, \mathbf{\Lambda}^{(4)}_{\boldsymbol{\omega}_{\lambda,\alpha}}\mathbf{w}\,. \end{aligned} \tag{4.40}$$

To derive (4.40) one uses the fact that $\boldsymbol{\omega}_{\lambda,\alpha} = (0,0,\omega_{\lambda,\alpha})^\top$ is a two-dimensional vorticity field, so that $K_{3D} * \boldsymbol{\omega}_{\lambda,\alpha} = (u_{\lambda,\alpha},0)^\top$, with $u_{\lambda,\alpha} = K_{2D} * \omega_{\lambda,\alpha}$. The operator $\Lambda_{\omega_{\lambda,\alpha}}$, defined in (4.9), is artificially produced by adding and subtracting the term $(K_{2D} * w_3, \nabla_h)\omega_{\lambda,\alpha}$ in the right-hand side. As is shown in [13, Proposition 4.5], all terms in the operator $\mathbf{R}_{\lambda,\alpha}$ involve at least one derivative with respect to x_3, and hence play a negligible role in long-time asymptotics. Therefore, the problem is now reduced to the analysis of the simpler operator $\mathcal{L}_{\lambda,\alpha}$ defined by

$$\mathcal{L}_{\lambda,\alpha}\mathbf{w} \,=\, \mathcal{L}_\lambda\mathbf{w} \,-\, \begin{pmatrix}(u_{\lambda,\alpha},\nabla_h)\mathbf{w}_h - (\mathbf{w}_h,\nabla_h)u_{\lambda,\alpha} \\ \Lambda_{\omega_{\lambda,\alpha}} w_3\end{pmatrix} \,=\, \mathcal{A}_{\lambda,\alpha}\mathbf{w} \,+\, (\partial_3^2 - x_3\partial_3)\mathbf{w}\,,$$

where

$$\mathcal{A}_{\lambda,\alpha}\mathbf{w} \,=\, \begin{pmatrix}\mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h \\ \mathcal{A}_{\lambda,\alpha,3}w_3\end{pmatrix} \,=\, \begin{pmatrix}\big(L_\lambda - \frac{3+\lambda}{2}\big)w_1 - (u_{\lambda,\alpha},\nabla_h)w_1 + (\mathbf{w}_h,\nabla_h)u_{\lambda,\alpha,1} \\ \big(L_\lambda - \frac{3-\lambda}{2}\big)w_2 - (u_{\lambda,\alpha},\nabla_h)w_2 + (\mathbf{w}_h,\nabla_h)u_{\lambda,\alpha,2} \\ \big(L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}\big)w_3\end{pmatrix}. \tag{4.41}$$

The crucial observation here is that the (vectorial) operator $\mathcal{A}_{\lambda,\alpha}$ acts only on the horizontal variable, so that the semigroup $e^{t\mathcal{L}_{\lambda,\alpha}}$ generated by $\mathcal{L}_{\lambda,\alpha} = \mathcal{A}_{\lambda,\alpha} + \partial_3^2 - x_3\partial_3$ can be expressed in terms of the 2D semigroup $e^{t\mathcal{A}_{\lambda,\alpha}}$ in the same way as in (4.36). As a consequence, the long-time behavior of the semigroup $e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})}$ in $\mathbb{X}(m)$ can be deduced from the spectral analysis of the two-dimensional operator $\mathcal{A}_{\lambda,\alpha}$ acting on $\mathbb{L}^2(m) := L^2(m)^2 \times L^2_0(m)$. This leads to the following criterion [13]:

Stability criterion: *Let $\lambda \in [0,1)$, $\alpha \in \mathbb{R}$, and $m > 2$. If the stability estimate*

$$\|e^{t\mathcal{A}_{\lambda,\alpha}}\|_{\mathbb{L}^2(m)\to\mathbb{L}^2(m)} \le Ce^{-\mu t}\,, \qquad t \ge 0\,, \tag{4.42}$$

holds for some $\mu \in (0,\frac{1-\lambda}{2}]$, then $\|e^{t(\mathcal{L}_\lambda - \mathbf{\Lambda}_{\boldsymbol{\omega}_{\lambda,\alpha}})}\|_{\mathbb{X}(m)\to\mathbb{X}(m)} \le C'e^{-\mu t}$ holds for all $t \ge 0$.

The key observation (II) concerns the structure of the 2D operator $\mathcal{A}_{\lambda,\alpha}$. From the definition (4.41) it is apparent that the horizontal component $\mathbf{w}_h = (w_1,w_2)^\top$ and the vertical component w_3 are completely decoupled under the action of $\mathcal{A}_{\lambda,\alpha}$. Furthermore, the third component $\mathcal{A}_{\lambda,\alpha,3} = L_\lambda - \Lambda_{\omega_{\lambda,\alpha}}$ acting on w_3 is exactly the linearized operator at the Burgers vortex considered in Section 4.2, when only two-dimensional perturbations are allowed. Proposition 4.4 (iii) thus provides the desired stability estimate for the semigroup generated by $\mathcal{A}_{\lambda,\alpha,3}$, uniformly for all $\alpha \in \mathbb{R}$ if the asymmetry parameter λ is small enough.

One of the main contributions of [13] is the analysis of the horizontal component $\mathcal{A}_{\lambda,\alpha,h}$, which also has a nice structure that allows to obtain a stability estimate for all $\alpha \in \mathbb{R}$, at least if $\lambda = 0$. The argument is as follows. Since the operator $\mathbf{w}_h \mapsto (\mathbf{w}_h,\nabla_h)u_{\lambda,\alpha} - (u_{\lambda,\alpha},\nabla_h)\mathbf{w}_h$ is a relatively compact perturbation of the second order operator L_λ, a standard perturbation argument reproduced in [13, Proposition 3.4] shows that the long-time behavior of the semigroup $e^{t\mathcal{A}_{\lambda,\alpha,h}}$ in $L^2(m)^2$ (with $m > 1$ sufficiently large) is determined by the eigenvalues of the generator $\mathcal{A}_{\lambda,\alpha,h}$ in a Gaussian weighted space such as $L^2(\infty;\lambda)^2$. As usual, when $\lambda \in [0,\frac{1}{2})$, one can use $L^2(\infty)^2$ instead of $L^2(\infty;\lambda)^2$. To locate the eigenvalues of $\mathcal{A}_{\lambda,\alpha,h}$, the following identities play a crucial role:

$$\begin{aligned} x_h \cdot \mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h \;&=\; (L_\lambda - 2)x_h \cdot \mathbf{w}_h - 2\nabla_h \cdot \mathbf{w}_h - \lambda(x_1 w_1 - x_2 w_2) \\ &\quad - (u_{\lambda,\alpha}, \nabla_h)x_h \cdot \mathbf{w}_h + (\mathbf{w}_h, \nabla_h)x_h \cdot u_{\lambda,\alpha}\,, \\ \nabla_h \cdot \mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h \;&=\; (L_\lambda - 1)\nabla_h \cdot \mathbf{w}_h - (u_{\lambda,\alpha}, \nabla_h)\nabla_h \cdot \mathbf{w}_h\,. \end{aligned} \tag{4.43}$$

Here the notation $x_h = x = (x_1, x_2)^\top$ is used. When $\lambda = 0$ the two-dimensional velocity field $u_{\lambda,\alpha} = \alpha v^G$ satisfies $x_h \cdot u_{\lambda,\alpha} = 0$, hence the first identity in (4.43) becomes substantially simpler. If $\mathbf{w}_h \in L^2(\infty)^2 \cap D(L)$ is a nontrivial eigenfunction of $\mathcal{A}_{\lambda,\alpha,h}$ with eigenvalue $\mu \in \mathbb{C}$, one has the obvious relations

$$\mu\mathbf{w}_h \;=\; \mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h\,, \qquad \mu x_h \cdot \mathbf{w}_h \;=\; x_h \cdot \mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h\,, \qquad \mu\nabla_h \cdot \mathbf{w}_h \;=\; \nabla_h \cdot \mathcal{A}_{\lambda,\alpha,h}\mathbf{w}_h\,,$$

which can be combined with (4.43) to obtain valuable information on μ. Indeed, assume for simplicity that $\lambda = 0$. If $\nabla_h \cdot \mathbf{w}_h \not\equiv 0$, the identity

$$\mu\nabla_h \cdot \mathbf{w}_h \;=\; (L - 1)\nabla_h \cdot \mathbf{w}_h - \alpha(v^G, \nabla_h)\nabla_h \cdot \mathbf{w}_h$$

implies that $\mathrm{Re}(\mu) \le -3/2$, in view of the spectral properties of L established in Proposition 2.2 and the fact that the operator $\omega \mapsto (v^G, \nabla)\omega$ is skew-symmetric in $L^2(\infty)$, see (2.14). If $\nabla_h \cdot \mathbf{w}_h \equiv 0$ and $x_h \cdot \mathbf{w}_h \not\equiv 0$, one has the relation

$$\mu x_h \cdot \mathbf{w}_h \;=\; (L - 2)x_h \cdot \mathbf{w}_h - \alpha(v^G, \nabla_h)x_h \cdot \mathbf{w}_h\,,$$

which implies that $\mathrm{Re}(\mu) \le -2$ by the same argument. Finally, if $x_h \cdot \mathbf{w}_h \equiv 0$, the eigenvalue equation reduces to

$$\mu\mathbf{w}_h \;=\; (L - \tfrac{3}{2})\mathbf{w}_h - \alpha(v^G, \nabla_h)\mathbf{w}_h + \alpha\mathbf{w}_h^\perp h\,, \qquad h(x_h) \;=\; \frac{1}{2\pi|x_h|^2}\bigl(1 - e^{-\frac{|x_h|^2}{4}}\bigr)\,,$$

and a simple energy estimate leads to the conclusion that $\mathrm{Re}(\mu) \le -3/2$ in all cases. As a consequence, when $\lambda = 0$, one has the desired stability estimate

$$\|e^{t\mathcal{A}_{\lambda,\alpha,h}}\mathbf{w}_h\|_{L^2(m)^2} \;\le\; C_{m,\alpha}\, e^{-\frac{3}{2}t}\|\mathbf{w}_h\|_{L^2(m)^2}\,, \qquad t \ge 0\,, \tag{4.44}$$

for all $\alpha \in \mathbb{R}$, if $m > 1$ is sufficiently large. Note that the constant in (4.44) depends on α and becomes large as $|\alpha| \to \infty$, which may be related to the short time amplification phenomenon observed numerically by Schmid and Rossi [44].

By a perturbation argument, it is easy to show that the stability estimate also holds if the asymmetry parameter λ is nonzero and small, but it is unclear whether the smallness assumption on λ is uniform with respect to the circulation parameter α. This interesting question is answered affirmatively by Theorem 4.5 if the perturbations are restricted to purely two-dimensional flows. In the general case what is missing so far is a precise information on the eigenvalues of the two-dimensional operator $\mathcal{A}_{\lambda,\alpha,h}$, especially in the regime where $|\alpha| \gg 1$.

5 Conclusion

As can be seen from the results reviewed in the previous sections, the mathematical theory of viscous vortices has reached a certain level of maturity, but many interesting questions remain open to the present date. In the simple case of a single, axisymmetric, straight vortex tube, there are explicit formulas for the vorticity and velocity profiles, and the stability with respect to two-dimensional perturbations is fully understood for all values of the total circulation (Section 2.1), including the large Reynolds number limit where additional stabilization occurs (Section 2.2). In presence of a non-axisymmetric strain, existence of stretched vortices is known for all values of the circulation and asymmetry parameters (Section 4.1), but uniqueness and stability results are not completely satisfactory, except perhaps in the large circulation limit where asymptotic symmetrization and stabilization are observed (Section 4.2). When arbitrary three-dimensional perturbations are allowed, local stability of the axisymmetric Burgers vortex is well understood for all values of the total circulation (Section 4.3), but less is known in the asymmetric case, and the question is essentially open for the self-similar Lamb-Oseen vortex, due to the lack of stretching in the vertical direction.

In real fluids, however, one usually observes the interaction of several vortex tubes, none of which is perfectly straight, and vorticity is also created near the boundaries. All these discrepancies from the ideal situation considered above give rise to difficult mathematical questions, which are essentially open. The rigorous theory of curved vortex filaments in viscous flows is still in its infancy, except perhaps in the axisymmetric case without swirl where existence and uniqueness of vortex rings can be established (Section 3). Interaction of vortices has been studied so far only in the weakly coupled regime where the distance between the vortex centers is much larger than the size of the vortex cores [10, 36, 37]. Stronger interactions, such as vortex merging (in two dimensions) or reconnection of vortex tubes (in three dimensions), play a crucial role in the dynamics of turbulent flows, but a rigorous description of these phenomena seems completely out of reach. Finally, there are no mathematical results yet concerning the interaction of viscous vortices with rigid boundaries, although the existence of self-similar vortices in two-dimensional exterior domains can be established at least for small values of the circulation parameter (Section 2.3).

Cross references

- Self-Similar Solutions to the Nonstationary Navier-Stokes Equations
- Large Time Behavior of The Navier-Stokes Flow
- Models and Special Solutions of the Navier-Stokes Equations
- Inviscid Limit and Boundary Layer of the Navier-Stokes Flow

References

[1] M. Ben-Artzi, Global solutions of two-dimensional Navier-Stokes and Euler equations, Arch. Rational Mech. Anal. **128** (1994), 329–358.

[2] J. M. Burgers, A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech. **1** (1948), 171–199.

[3] A. Carpio, Asymptotic behavior for the vorticity equations in dimensions two and three, Comm. Partial Differential Equations **19** (1994), 827–872.

[4] Wen Deng, Resolvent estimates for a two-dimensional non-self-adjoint operator, Commun. Pure Appl. Anal. **12** (2013), 547–596.

[5] Wen Deng, Pseudospectrum for Oseen vortices operators, Int. Math. Res. Not. IMRN **2013** (2013), 1935–1999.

[6] H. Feng and V. Šverák, On the Cauchy problem for axi-symmetric vortex rings, Arch. Rational Mech. Anal. **215** (2015), 89–123.

[7] I. Gallagher and Th. Gallay, Resolvent estimates for rapidly rotating Oseen vortices, in preparation.

[8] I. Gallagher, Th. Gallay, and P.-L. Lions, On the uniqueness of the solution of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity, Math. Nachr. **278** (2005), 1665–1672.

[9] I. Gallagher, Th. Gallay, and F. Nier, Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator, Int. Math. Res. Notices **2009** (2009), 2147–2199.

[10] Th. Gallay, Interaction of vortices in weakly viscous planar flows, Arch. Ration. Mech. Anal. **200** (2011), 445–490.

[11] Th. Gallay, Stability and interaction of vortices in two-dimensional viscous flows, Discr. Cont. Dyn. Systems Ser. S **5** (2012), 1091-1131.

[12] Th. Gallay, Long-time asymptotics for the Navier-Stokes equation in a two-dimensional exterior domain, comptes-rendus des Journées EDP 2012, Biarritz.

[13] Th. Gallay and Y. Maekawa, Three-dimensional stability of Burgers vortices. Comm. Math. Phys. **302** (2011), 477–511.

[14] Th. Gallay and Y. Maekawa, Long-time asymptotics for two-dimensional exterior flows with small circulation at infinity, Anal. PDE **6** (2013), 973–991.

[15] Th. Gallay and L. M. Rodrigues, Sur le temps de vie de la turbulence bidimensionnelle, Ann. Fac. Sci. Toulouse Math. **17** (2008), 719–733.

[16] Th. Gallay and V. Šverák, Remarks on the Cauchy problem for the axisymmetric Navier-Stokes equations, Confluentes Mathematici **7** (2015), 67–92.

[17] Th. Gallay and V. Šverák, Uniqueness of axisymmetric viscous flows originating from circular vortex filaments, https://arxiv.org/abs/1609.02030.

[18] Th. Gallay and C. E. Wayne, Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$, Arch. Ration. Mech. Anal. **163** (2002), 209–258.

[19] Th. Gallay and C. E. Wayne, Global Stability of vortex solutions of the two dimensional Navier-Stokes equation. Comm. Math. Phys. **255** (2005), 97–129.

[20] Th. Gallay and C.E. Wayne, Three-dimensional stability of Burgers vortices : the Low Reynolds number case", Physica D **213** (2006), 164–180.

[21] Th. Gallay and C.E. Wayne, Existence and stability of asymmetric Burgers vortices, J. Math. Fluid Mech. **9** (2007), 243–261.

[22] Y. Giga and T. Kambe, Large time behavior of the vorticity of two dimensional viscous flow and its application to vortex formation, Comm. Math. Phys. **117**, (1988) 549–568.

[23] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear partial differential equations. Asymptotic behavior of solutions and self-similar solutions*, Progress in Nonlinear Differential Equations and their Applications **79**, Birkhäuser, Boston, 2010.

[24] H. von Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, J. reine angew. Math, **5** (1858), 25–55.

[25] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lectures Notes in Mathematics **840**, Springer, 1981.

[26] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren der mathematischen Wissenschaften **132**, Springer, 1966.

[27] H. Kozono and M. Yamazaki, Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$, Houston J. Math. **21** (1995), 755–799.

[28] D. Küchermann, Report on the I.U.T.A.M Symposium on concentrated vortex motion in fluids, J. Fluid Mech. **21** (1965), 1–20.

[29] D. Iftimie, G. Karch, and C. Lacave, Self-similar asymptotics of solutions to the Navier-Stokes system in two-dimensional exterior domain, J. Lond. Math. Soc. **90** (2014), 785–806.

[30] O. Ladyzhenskaya, Unique solvability in the large of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **7** (1968), 155–177 (in Russian).

[31] T. Lundgren, Strained spiral vortex model for turbulent fine structure, Phys. Fluids **25** (1982), 2193.

[32] Y. Maekawa, On the existence of Burgers vortices for high Reynolds numbers, J. Math. Anal. Appl. **349** (2009), 181–200.

[33] Y. Maekawa, Existence of asymmetric Burgers vortices and their asymptotic behavior at large circulations, Math. Models Methods Appl. Sci. **19** (2009), 669–705.

[34] Y. Maekawa, Spectral properties of the linearization at the Burgers vortex in the high rotation limit, J. Math. Fluid Mech. **13** (2011), 515–532.

[35] Y. Maekawa, On asymptotic stability of global solutions in the weak L^2 space for the two-dimensional Navier-Stokes equations, Analysis **35** (2015), 245–257.

[36] C. Marchioro, On the inviscid limit for a fluid with a concentrated vorticity, Commun. Math. Phys. **196** (1998), 53–65.

[37] C. Marchioro, Vanishing viscosity limit for an incompressible fluid with concentrated vorticity, J. Mathematical Phys. **48** (2007), 065302 (1-16).

[38] K. Masuda, Weak solutions of Navier-Stokes equations, Tohoku Math. J. **36** (1984), 623–646.

[39] H. K. Moffatt, S. Kida and K. Ohkitani, Stretched vortices–the sinews of turbulence; large-Reynolds-number asymptotics, J. Fluid Mech. **259** (1994), 241–264.

[40] A. Prochazka and D. I. Pullin, On the two-dimensional stability of the axisymmetric Burgers vortex, Phys. Fluids. **7** (1995), 1788–1790.

[41] A. Prochazka and D. I. Pullin, Structure and stability of non-symmetric Burgers vortices, J. Fluid Mech. **363** (1998), 199–228.

[42] A. C. Robinson and P. G. Saffman, Stability and structure of stretched vortices, Stud. Appl. Math. **70** (1984), 163–181.

[43] M. Rossi and S. Le Dizès, Three-dimensional temporal spectrum of stretched vortices, Phys. Rev. Lett. **78** (1997), 2567–2569.

[44] P. J. Schmid and M. Rossi, Three-dimensional stability of a Burgers vortex, J. Fluid Mech. **500** (2004), 103–112.

[45] M. Ukhovskii and V. Yudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, J. Appl. Math. Mech. **32** (1968), 52–61.