LECTURES ON THE SHAFAREVICH CONJECTURE ON UNIFORMIZATION

PHILIPPE EYSSIDIEUX

1. Introduction

This is a set of lecture notes for a course given at the Summer School " Uniformisation de familles de variétés complexes" organised by L. Meersseman in Dijon (France) August, 31-September 11, 2009 and funded by the ANR project "Complexe" (ANR-08-JCJC-0130-01), the École doctorale Carnot and the Institut Mathématique de Bourgogne.

These notes are meant to serve as an introduction to non abelian Hodge theory with a focus on its use in the Shafarevich problem.

Definitions and notations. For background information on Kähler manifolds and Hodge Theory, a useful reference is [Voi02]. We will not give any details on the facts and definitions already contained there.

In the sequel, X denotes a compact connected Kähler manifold and ω a Kähler form on X. Its universal covering space will be denoted by $\pi : \widetilde{X^{un}} \to X$.

Uniformization in several complex variables. Basic examples. Uniformization in complex geometry aims at understanding the universal covering space, the fundamental group and hence the various covering spaces of complex manifolds. In these notes, I will focus on the compact Kähler case.

The Riemann uniformization theorem, whose first complete proof was given independantly by Koebe and Poincaré in 1907, states that a simply connected Riemann surface is isomorphic to $\mathbf{P^1}(\mathbb{C})$, $\mathbb{C}$ or $\Delta = \{z \in \mathbb{C} | \ |z| < 1\}$. As a corollary, one can describe the universal covering space of a compact Riemann surface according to the following:

Theorem 1.0.1. *Let C be a compact connected Riemann surface of genus g and $\pi : U \to C$ be its universal covering space.*

- *If* $g = 0$**:** *then C is simply connected, $U = C$ and $C \simeq \mathbf{P^1}(\mathbb{C})$.*
- *If* $g = 1$**:** *then $U \simeq \mathbb{C}$, $C \simeq \Lambda\backslash\mathbb{C}$ where $\Lambda \simeq \mathbb{Z}^2$ is a rank 2 discrete subgroup of $\mathbb{C}$.*
- *If* $g \geq 2$**:** *then $U \simeq \Delta$, $C \simeq \Gamma\backslash\Delta$ where $\Gamma \subset PU(1,1)$ is a torsion-free cocompact discrete subgroup isomorphic to*

$$< a_1, b_1, \ldots, a_g, b_g \quad | \quad a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 >$$

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It was realized early that the situation is much more complicated in several complex variables. For instance, the bidisk and the complex two ball are not isomorphic as complex manifolds but can be realized as the universal covering space of a complex projective surface of general type.

Compact Kähler manifolds with infinite fundamental groups are not too abundant. Nevertheless, a nice zoology of examples can be displayed.

Complex tori: Let $\Lambda \in \mathbb{C}^n$ be a rank $2n$ lattice. The complex torus $\Lambda\backslash\mathbb{C}^n$ is a compact Kähler manifold whose universal covering space is $\mathbb{C}^n$. It is a projective manifold (and hence an abelian variety) iff the weight -1 $\mathbb{Z}$-Hodge structure determined by $\Lambda \in \mathbb{C}^n$ can be polarized.

Hermitian locally symmetric spaces: Let Ω be a bounded symmetric domain (cf. [Sat80]). Familiar examples of irreducible bounded symmetric domains are complex balls (aka complex hyperbolic spaces), Siegel upper half planes, period domains for K3 surfaces.

Let Γ be a cocompact torsion free lattice[1] in $\mathrm{Aut}(\Omega)$, then $\Gamma\backslash\Omega$ is a canonically polarized projective manifold whose universal covering space is Ω.

Kuga varieties: When the bounded symmetric domain Ω is classical, provided $\Gamma \in \mathrm{Aut}(\Omega)$ is well chosen (see [Sat80] for details), then $\Gamma\backslash\Omega$ appears as a fine moduli space for a family of abelian varieties of genus g with non trivial $\mathbb{Q}$-endomorphism ring, polarization type and level structure. This gives rise to an abelian scheme $\pi : A_\Gamma \to \Gamma\backslash\Omega$. The universal covering of the smooth projective manifold A is then biholomorphic to $\Omega \times \mathbb{C}^g$.

Kodaira surfaces: A Kodaira surface is a projective surface S endowed with a smooth fibration $p : S \to C$ whose fibers have genus ≥ 2. Then the universal covering space can be realized as a bounded pseudoconvex domain in $\mathbb{C}^2$ ([Gri71], lemma 6.2, p.39).

Mostow-Siu surfaces and Deraux threefold: In [MoSiu80], exotic (i.e. non complex hyperbolic) projective surfaces of negative curvature are constructed. Improving this construction, M. Deraux was able to produce an example in dimension 3 in [Der05]. In this examples, the universal covering space can be expressed as an infinite ramified covering space over a complex ball.

In all these cases, the universal covering space is Stein and contractible and the compact Kähler manifold is an Eilenberg- Mac Lane $K(\pi, 1)$.

Shafarevich conjecture on holomorphic convexity. Further examples of compact Kähler manifolds with Stein universal covering space can easily be constructed. For instance, any smooth submanifold Y in a projective manifold X with Stein universal covering space has Stein universal covering space, too. On the other hand, even in case the universal covering space of X is contractible, the universal covering space of a sufficiently ample smooth complete intersection of dimension $d \geq 2$ has a non trivial π_d. In particular, its universal covering space is not contractible.

In this example, the manifold has a 1-connected holomorphic embedding into a compact Kähler $K(\pi, 1)$. Toledo's example of a compact Kähler manifold with non residually finite Kähler group comes equiped with an embedding into a smooth

[1] This always exist by a theorem of A. Borel.

quasiprojective $K(\pi, 1)$ [Tol93]. However, the fundamental groups of some projective manifolds constructed in [DPS09] do not have a finite CW-complex as a $K(\pi, 1)$. In particular these manifolds cannot have a 1-connected holomorphic embedding into a compactifiable complex manifold with contractible universal covering space.

On the other hand, the universal covering space of a compact Kähler manifold may contain positive dimensional compact complex analytic subvarieties. The universal covering space of the blow-up at the origin of the complex torus $T = \Lambda\backslash\mathbb{C}^n$ is the blow up of $\mathbb{C}^n$ at all lattice points in Λ and contains an infinite collection of copies of $\mathbf{P}^{n-1}(\mathbb{C})$.

The Shafarevich conjecture takes these examples into account and predicts that the universal covering space of a complex projective manifold (compact and embeddable in $\mathbf{P}^N(\mathbb{C})$) should be holomorphically convex. [2] This problem is still open, in spite of the recent positive results contained in [Eys04] [EKPR09].

Definition 1.0.2. *A complex analytic space S is holomorphically convex if there is a proper holomorphic morphism $\pi : S \to T$ with $\pi_*\mathcal{O}_S = \mathcal{O}_T$ such that T is a Stein space. T is then called the* Cartan-Remmert reduction *of S.*

Remark 1.0.3. *If S is a normal holomorphically convex complex analytic space (in particular a complex manifold), then its Cartan-Remmert reduction T is a normal Stein space.*

If S is holomorphically convex and $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in S escaping to infinity there exists a holomorphic function f such that

$$\lim_{n\to\infty} |f(x_n)| = +\infty.$$

Let us give some evidence towards the Shafarevich conjecture.

- If S is a compact Kähler surface with $\kappa(S) \leq 1$ then its universal covering space is holomorphically convex [GurSha85].
- If X_1 and X_2 have holomorphically convex universal coverings so has $X_1 \times X_2$.
- If X and Y are bimeromorphic compact Kähler manifolds, then X has a holomorphically convex universal covering iff Y has too.
- If $f : X \to Y$ is a holomorphic map such that $f_* : \pi_1(X) \to \pi_1(Y)$ has finite kernel and Y has a holomorphically convex universal covering then X has too.

In the last statement, one cannot drop the restriction on f_* since the universal covering space of a holomorphically convex manifold needs not be holomorphically convex[3].

This defect can be used as an excuse to consider a slightly more general problem. Let $H \subset \pi_1(X)$ be a normal subgroup. Say (X, H) satisfies (HC) iff $H\backslash\widetilde{X^{un}}$ is holomorphically convex. Obviously, the Shafarevich conjecture states that $(X, \{e\})$ should satisfy (HC) if X is a complex projective manifold.

[2]This problem was actually not formulated as a conjecture in the last chapter in [Sha74].

[3]The minimal resolution of a small Stein neighborhood of an elliptic surface singularity is obviously holomorphically convex. The singularity can have a rational nodal curve as an exceptionnal divisor. This rational curve then unfolds in the universal covering space as a connected infinite chain of rational curves on which holomorphic functions are constant. Hence this covering space cannot be holomorphically convex.

Lemma 1.0.4. *If (X,H) satisfies (HC) and $f: Y \to X$ is an holomorphic map from a compact Kähler manifold then $(Y, f_*^{-1}H)$ satisfies (HC).*

If (X,H) satisfies (HC), then there is a proper holomorphic mapping with connected fibers

$$\widetilde{s^H} : H\backslash\widetilde{X^{un}} \to \widetilde{S_H(X)}.$$

It contracts precisely the compact connected analytic subspaces of $H\backslash\widetilde{X^{un}}$. The mapping s^H is equivariant under the Galois group $G = H\backslash\pi_1(X)$ which acts properly and cocompactly on $\widetilde{S_H(X)}$.

The quotient map $s^H : X \to G\backslash\widetilde{S_H(X)}$ is called the H-Shafarevich morphism. In the influential article [Kol93], J. Kollár made it clear that constructing the H-Shafarevich morphism is the first step to settle when trying to prove (HC). The second step is to prove that the normal complex space $\widetilde{S_H(X)}$ is Stein - the problem can be reduced to constructing a strongly plurisubharmonic exhaustion function on $\widetilde{S_H(X)}$. In fact the best general result in the direction of this first step was given in the independant and simultaneous works of J. Kollár and F. Campana:

Theorem 1.0.5. *(*[Cam94]*,* [Kol93] *in the projective case) One can construct a G-equivariant meromorphic map $\widetilde{s^H} : H\backslash\widetilde{X^{un}} \to \widetilde{S_H(X)}$ which is proper and holomorphic outside $(\widetilde{s^H})^{-1}(Z)$ where $Z \subset \widetilde{S_H(X)}$ is a proper complex analytic G-invariant subvariety such that the general fiber of $\widetilde{s^H}$ is a maximal compact connected analytic subvariety of $H\backslash\widetilde{X^{un}}$.*

In this theorem, H needs not be a normal subgroup. Nevertheless, these cycle theoretic methods are unlikely to help in producing non constant holomorphic functions and thus will not help for the second step. Indeed (HC) depends on the group H in view of the Cousin example:

Example 1.0.6. *Let X be a simple abelian variety and $\rho : \pi_1(X) = H_1(X) \to \mathbb{Z}$ be a surjective homomorphism. Then* $\ker(\rho)\backslash\widetilde{X^{un}}$ *has no positive dimensional compact complex subvariety but does not carry any non constant holomorphic function either.*

Hence the Shafarevich problem is an instance of the problem of determining which pairs (X,H) satisfy (HC). The articles [Eys04] [EKPR09] construct for every complex projective manifold X natural invariant subgroups $H \triangleleft \pi_1(X)$ such that (X,H) satisfy (HC). As a corollary, this gives the Shafarevich conjecture provided the fundamental group embeds in $GL_N(\mathbb{C})$ for some $N \in \mathbb{N}$.

These notes are organized as follows. Section 2 and 3 survey the relevant aspects of non abelian Hodge theory in the archimedian and the non archimedian setting. Section 4 survey the construction of the Shafarevich morphism from [Eys04] and section 5 the final section of that article. Section 6 surveys the main ideas of the proof of the linear Shafarevich conjecture [EKPR09]. The last section describes some open problems and directions for future research.

2. Non Abelian Hodge Theory in the archimedian case

Let G be an affine algebraic group over a field K. Let T be a connected topological space endowed with a base point $t \in T$. We assume T is homeomorphic to a simplicial complex- this is the case if T is a complex analytic space. We can construct the sheaf $\underline{G(K)}$ of locally constant functions with values in $G(K)$. Non abelian cohomology is well defined in degree 1 and the cohomology set $H^1(T, \underline{G(K)})$ can be identified with the set

$$Hom(\pi_1(T,t), G(K))/G(K)$$

where $G(K)$ acts by conjugation on the set of representations of $\pi_1(T,t)$ in $G(K)$.

Assume for a short while that $G = \mathbb{G}_a$ is the additive group. This boils down to the well known isomorphism $H^1(T, K_T) \simeq Hom(\pi_1(T,t), K)$. Assume also that $K = \mathbb{R}$ and that T is homeomorphic to a compact Riemannian manifold (M, g). The Hodge theorem then gives an isomorphism:

$$H^k(M, \mathbb{R}) \longrightarrow \mathcal{H}^k(M, \mathbb{R}) \subset \mathcal{Z}^k(M, \mathbb{R})$$

where $\mathcal{H}^k(M, \mathbb{R})$ is the space of harmonic k-forms on M and $\mathcal{Z}^k(M, \mathbb{R})$ is the space of closed k-forms. This Hodge isomorphism is a right inverse for the natural integration map $\mathcal{Z}^k(M, \mathbb{R}) \to H^k(M, \mathbb{R}) = Hom(H_k(M, \mathbb{R}), \mathbb{R})$. Hence, for $k = 1$, we have canonical one-forms representing 1-cohomology classes with values in $\mathbb{G}_a(\mathbb{R})$.

The theory of harmonic mappings is an analog of the Hodge theorem for 1-cohomology classes with values in $G(K)$ if G is reductive and K is either $\mathbb{R}$ or $\mathbb{C}$ or a (locally compact) local field (i.e.: an algebraic extension of $\mathbb{Q}_p$ or $\mathbb{F}_l((T))$ - p, l being prime numbers.).

In the Kähler case, special phenomena occur. They are rich enough to deserve the name of non abelian Hodge theory (in degree 1).

2.1. **Harmonic mappings into Symmetric spaces.**

2.1.1. *The Riemannian symmetric space attached to a reductive real algebraic group.* Let G be a reductive algebraic group over $\mathbb{R}$ such that $G(\mathbb{R})$ is connected. In particular $G(\mathbb{R}) = Z(G).S$ where the identity component of the center $Z(G)$ is isomorphic to $(S^1)^p \times (\mathbb{R}^*_{>0})^q$ and S is a connected semisimple Lie group with $Z(G) \cap S$ finite.

Note that given an algebraic group G' over $\mathbb{C}$, $G'(\mathbb{C})$ may be wieved as the set of $\mathbb{R}$-points of $G = Res_{\mathbb{C}|\mathbb{R}}(G')$ (Weil's restriction of scalars). The main example we have in mind is $G = Res_{\mathbb{C}|\mathbb{R}}(GL_N)$ so that $G(\mathbb{R}) = GL_N(\mathbb{C})$.

Consider K a maximal compact subgroup $G(\mathbb{R})$. The riemannian symmetric space attached to G can be defined set theoretically as the space of left cosets $Riem(G) = G(\mathbb{R})/K$.

Theorem 2.1.1. *The space $R = Riem(G)$ with its quotient topology is a contractible manifold that can be endowed with a complete Riemannian metric ds^2_R of nonpositive sectional curvature. The natural left action of $G(\mathbb{R})$ on R is a proper isometric action. This metric is symmetric in the sense that its curvature tensor is Levi-Civita parallel.*

Actually $R = \mathbb{R}^q \times \prod_{i=1}^k R_i$ where $S = S_1....S_k$ is a decomposition of S in almost simple factors and $R_i = Riem(S_i)$. The metric splits accordingly and gives an euclidean metric on the flat factor $\mathbb{R}^q$ and a symmetric metric well defined up to a scalar on R_i. In particular ds^2_R is not unique: it can be rescaled on each simple factor and be changed to another euclidean metric on the flat factor.

A standard reference for the theory of Riemannian symmetric spaces is [Hel78].

In case $G(\mathbb{R}) = GL_N(\mathbb{C})$, $R = GL_N(\mathbb{C})/U(N) = \mathcal{H}_N^{>0}$ the space of positive definite $N \times N$-hermitian symmetric matrices[4]. One has $\mathcal{H}_N^{>0} = \mathbb{R} \times \mathcal{H}_N$ where $\mathcal{H}_N = SL_N(\mathbb{C})/SU(N)$ is the space of positive definite hermitian matrices of determinant 1.

The $GL_N(\mathbb{C})$ case is somehow universal since there always exist a complex faithful representation $\sigma : G(\mathbb{R}) \to GL_N(\mathbb{C})$, for $N \gg 0$, [5] which determines a totally geodesic embedding [6]

$$\bar{\sigma} : Riem(G) \rightarrowtail GL_N(\mathbb{C})/U(N).$$

2.1.2. *The energy integral.* Now, let (M, g) be a compact connected Riemannian manifold endowed with a base point $m_0 \in M$ and let $\pi : (\widetilde{M^{un}}, \tilde{m}_0) \to (M, m_0)$ be the universal covering space . Consider furthermore $\rho : \pi_1(M, m_0) \to G(\mathbb{R})$ a representation.

Proposition 2.1.2. *Let $k \in \mathbb{N}$. The space $C^k(\widetilde{M^{un}}, Riem(G))_\rho$ of C^k-mappings ϕ from $\widetilde{M^{un}}$ to $Riem(G)$ such that*

$$\forall (\gamma, \tilde{m}) \in \pi_1(M, m_0) \times \widetilde{M^{un}}, \qquad \phi(\gamma.\tilde{m}) = \rho(\gamma).\phi(\tilde{m})$$

is a non empty contractible Banach manifold.

Proof: The space $C^k(\widetilde{M^{un}}, Riem(G))_\rho$ is the space of C^k sections of the fiber bundle $p : \pi_1(M, m_0)\backslash R \times \widetilde{M^{un}} \to M$ where the fundamental group acts by

$$\gamma.(x, \tilde{m}) = (\rho(\gamma).x, \gamma.\tilde{m}).$$

If $m_k \in \mathbb{R}_{\geq 0}$ with $\sum_k m_k = 1$ and $x_1, \ldots, x_k \in R$ we can define

$$\beta = \beta((x_1, m_1), \ldots, (x_k, m_k)) \in R$$

as the point where $x \mapsto \sum_{i=1}^k m_i d^2(x, x_i)$ achieves its minimum. Since the curvature of R is seminegative, this function is smooth, proper and strongly convex on R hence β is uniquely determined and defines a smooth mapping.

Using local sections s_i of p on a covering of M by open sets U_i and a partition of unity ϕ_i one can construct a section s of p by the formula $s(m) = \beta((s_i(m), \phi_i(m))_i)$ for $m \in M$.

A retraction to $s \in C^k(\widetilde{M^{un}}, Riem(G))_\rho$ can be given by the explicit formula $H(t, s') = \beta((s, 1-t), (s', t))$ for $t' \in [0, 1]$, $s' \in C^k(\widetilde{M^{un}}, Riem(G))_\rho$. □

In case $G(\mathbb{R}) = GL_N(\mathbb{C})$ the space $C^k(\widetilde{M^{un}}, Riem(G))_\rho$ coincides with the space of C^k-hermitian metrics on the vector bundle

$$\mathcal{V}_\rho = \pi_1(M, m)\backslash(\mathbb{C}^N \times \widetilde{M^{un}}) \to M$$

underlying the local system $\mathbb{V}_\rho$ attached to ρ.

In case $N = 1$ and $k \geq 1$ an element of $C^k(\widetilde{M^{un}}, Riem(G))_\rho$ is a C^k map $\phi : \widetilde{M^{un}} \to \mathbb{R}$ such that:

$$\phi(\gamma.\tilde{m}) = \phi(m) + 2 \log |\rho(\gamma)|$$

[4] In the degenerate case where $N = 1$, then $R = \mathbb{R}$ and $z \in GL_1(\mathbb{C}) = \mathbb{C}^*$ acts by $t \mapsto t + 2\log|z|$.

[5] arising from a $\mathbb{R}$-algebraic group morphism $G \to Res_{\mathbb{C}|\mathbb{R}} GL_N$

[6] An embbeding $i \colon (N, ds^2_N) \rightarrowtail (N', ds^2_{N'})$ of Riemannian manifolds which is isometric ($i^* ds^2_{N'} = ds^2_{N'}$) is totally geodesic iff it is and sends any geodesic ray of N to a geodesic ray of N'. Equivalently the second fundamental form of N in N' vanishes. See [Hel78].

In particular $d\phi \in \mathcal{Z}^1(\widetilde{M^{un}}, \mathbb{R})^{\pi_1(M,m)} = \mathcal{Z}^1(M, \mathbb{R})$ and $\int_\gamma d\phi = 2\log|\rho(\gamma)|$. Hence $d\phi$ is a De Rham representative of the cohomology class $2\log|\rho(\gamma)|$. Conversely for any C^{k-1} De Rham representative α and integration constant $c \in \mathbb{R}$ we can produce an element of $C^k(\widetilde{M^{un}}, \mathbb{R})_{2\log|\rho|}$ setting

$$\phi(\tilde{m}) = \int_{\tilde{m}_0}^{\tilde{m}} \alpha + c.$$

Let $\phi \in C^1(\widetilde{M^{un}}, Riem(G))_\rho$ and $\tilde{m} \in \widetilde{M^{un}}$. Then $d\phi_{\tilde{m}} \in Hom_{\mathbb{R}}(T_{\tilde{m}}\widetilde{M^{un}}, T_{\phi(\tilde{m})})$ and both vector spaces have the norms p^*g_m, resp. $ds^2_{R,\phi(\tilde{m})}$ and one can define a real number $e(\phi, \tilde{m})$ as the Hilbert-Schmid norm of $d\phi_{\tilde{m}}$. Since ϕ is equivariant $\tilde{m} \mapsto e(\phi, \tilde{m})$ is $\pi_1(M, m_0)$-equivariant and descend to $e(\phi) \in C^0(M, \mathbb{R})$. One defines $E(\phi) = \int_M e(\phi) dVol(g)$.

Then $E : C^1(\widetilde{M^{un}}, Riem(G))_\rho \to \mathbb{R}$ is a functional called the energy integral.

2.1.3. *Basic existence theorems of Eells-Sampson and Corlette.* The following proposition, deduced once again from the fact that R has negative curvature, suggests existence of global minimizers of the energy functional.

Proposition 2.1.3. [EeSa64] *E is a convex functionnal.*

In fact, it is not trivial to find a global minimizer. If the Zariski closure of $\rho(\pi_1(M, m_0))$ is a reductive subgroup of G, we will say ρ is *reductive*. It is unfortunate that this terminology competes in the litterature with “ semisimple”.

Theorem 2.1.4. [Cor88] *E has a global minimizer if and only if the Zariski closure of $\rho(\pi_1(M, m_0))$ is a reductive subgroup of G. Furthermore, these global minimizers are smooth.*

In case $G(\mathbb{R}) = \mathbb{C}^*$, for $\phi \in C^k(\widetilde{M^{un}}, \mathbb{R})_{2\log|\rho|}$, $E(\phi) = \int_M |d\phi|^2$ hence the differential of a global minimizer is precisely the harmonic representative of the cohomology class $2\log|\rho|$.

In case $\rho(\pi_1(M, m_0)) \subset G(\mathbb{R})$ is discrete and cocompact, it is Zariski dense and Corlette's theorem applies. Note that this fact was already a consequence of [EeSa64].

2.1.4. *The Euler Lagrange equations for the energy integral.* The definition of the energy integral generalizes to the case of an arbitrary Riemannian manifold (N, h) in place of (R, ds^2_R). In the next two paragraphs, we consider this more general case. Denote by $Isom(N, h)$ the isometry group of (N, h) . Let $\rho : \pi_1(M, m_0) \to Isom(N, h)$ be a representation of the fundamental group of M taking values in this isometry group.

Introduce local coordinates (x^a) on M and (y^i) on N and write using Einstein's summation convention

$$g = g_{ab}dx^a dx^b, \qquad h = h_{ij}dy^i dy^j.$$

In local coordinates $\phi \in C^1(\widetilde{M^{un}}, N)_\rho$ can be written as $\phi = (\phi^i)$ and

$$E(\phi) = \int_M h_{ij}(\phi) g^{ab} \phi^i_a \phi^j_b |g| dx^1 ... dx^m$$

where $\phi^i_a = \frac{\partial}{\partial x^a}\phi^i = \partial_a \phi^i$ and $|g| = det(g_{ab})^{\frac{1}{2}}$.

Consider $\Phi = (\phi_t)_{t\in]-\epsilon,+\epsilon[}$ a one parameter smooth variation of $\phi = \phi_0$. We assume that $\phi_t = \phi$ outside of the coordinate chart on M where the local coordinates are defined. If ϕ is a critical point of E we have at $t = 0$:

$$\int_M \frac{\partial}{\partial t}(h_{ij}(\phi) g^{ab}\phi^i_a\phi^j_b)|g|dx^1...dx^m = 0, \tag{1}$$

Now compute:

$$\begin{aligned}\frac{\partial}{\partial t}(h_{ij}(\phi)g^{ab}\phi^i_a\phi^j_b) &= h_{ij}g^{ab}\frac{\partial^2}{\partial x^a\partial t}\phi^i\frac{\partial}{\partial x^b}\phi^j \\ &\quad + h_{ij}g^{ab}\frac{\partial}{\partial x^a}\phi^i\frac{\partial^2}{\partial x^b\partial t}\phi^j + \partial_k h_{ij}\partial_t\phi^k g^{ab}\phi^i_a\phi^j_b\end{aligned}$$

Integrating by parts in (1) gives:

$$\int_M (-|g|^{-1}\partial_a(|g|h_{ij}g^{ab}\phi^j_b) - |g|^{-1}\partial_b(|g|h_{ji}g^{ab}\phi^j_a)$$
$$+\partial_i h_{kj}g^{ab}\phi^k_a\phi^j_b)\phi^i_t|g|dx = 0.$$

Further derivating, we get:

$$\int_M (+\partial_i h_{kj}g^{ab}\phi^k_a\phi^j_b - \partial_k h_{ij}g^{ab}\phi^k_a\phi^j_b - \partial_k h_{ji}g^{ab}\phi^k_b\phi^j_a$$
$$-2.h_{ij}|g|^{-1}\partial_a(|g|g^{ab}\phi^j_b))\phi^i_t|g|dx = 0.$$

Changing dummy indices k, j in third term, we get:

$$\int_M (+\partial_i h_{kj}g^{ab}\phi^k_a\phi^j_b - \partial_k h_{ij}g^{ab}\phi^k_a\phi^j_b - \partial_j h_{ki}g^{ab}\phi^j_b\phi^k_a$$
$$-2.h_{ij}|g|^{-1}\partial_a(|g|g^{ab}\phi^j_b))\phi^i_t|g|dx = 0.$$

Now the Christoffel symbol Γ^k_{ij} of (N, h) satisfies:

$$\partial_i h_{kj} - \partial_k h_{ij} - \partial_j h_{ki} = -2\Gamma^l_{kj}h_{li}.$$

And the Laplace-Beltrami operator of (M, g) is:

$$\Delta = |g|^{-1}\partial_a(|g|g^{ab}\partial_b)$$

Hence the expression simplifies to:

$$\int_M (-2\Gamma^l_{kj}h_{li}g^{ab}\phi^k_a\phi^j_b - 2.h_{ij}\Delta\phi^j)\phi^i_t|g|dx = 0.$$

Which can be rewritten as:

$$\int_M (-2\Gamma^j_{kl}h_{ji}g^{ab}\phi^k_a\phi^l_b - 2.h_{ij}\Delta\phi^j)\phi^i_t|g|dx = 0,$$

Or as:

$$\int_M (-2\Gamma^j_{kl}g^{ab}\phi^k_a\phi^l_b - 2.\Delta\phi^j)h_{ij}\phi^i_t|g|dx = 0.$$

Since (ϕ^i_t) is arbitrary, ϕ satisfies the semilinear partial differential equation:

$$\Delta\phi^j + \Gamma^j_{kl}(\phi)g^{ab}\phi^k_a\phi^l_b = 0 \tag{2}$$

Corollary 2.1.5. *Let $Riem(G) = \mathbb{R}^q \times \prod_i R_i$ be the decomposition of R into a flat factor $\mathbb{R}^q$ and irreducible symmetric spaces R_i of non compact type (i.e.: such that $Aut(R_i)$ is almost simple). Every twisted harmonic map from M to $Riem(G)$ decomposes as a product of harmonic maps taking their values in each factor.*

Hence the notion of a harmonic mapping with values in $Riem(G)$ does not depend of the choice of a symmetric metric ds^2_R.

Remark 2.1.6. *Schauder estimates reduce the proof of smoothness of ϕ to a C^1-estimate.*

Remark 2.1.7. *It is possible to carry out the calculation in a coordinate free manner.*

Let us indeed sketch this calculation. The differential of Φ is a section of $\mathrm{Hom}(T_M, \Phi^* T_N)$. Endow $\Phi^* T_N$ with the connection $\Phi^*\nabla^N$ induced by the Levi-Civita connection of N and with the metric $\Phi^* h$ induced from h. In a similar way, endow T_M with the Levi-Civita connection of g and with the metric g. The usual construction produces a connection ∇ and a metric on the Hom-bundle $\mathrm{Hom}(T_M, \Phi^* T_N)$. The restriction of $\Phi^*\nabla^N$ to the slice $\{t = 0\}$ gives an operator $D : \phi^* T_N \to \mathrm{Hom}(T_M, \Phi^* T_N)$.

Observe that $\nabla_{\frac{\partial}{\partial t}} d\Phi|_{t=0} = Dv$ where $v = \frac{\partial \Phi}{\partial t}|_{t=0}$. Then

$$\begin{aligned}\frac{\partial}{\partial t} E(\phi_t)|_{t=0} &= \int_M \frac{\partial}{\partial t}(d\Phi, d\Phi)|_{t=0} = 2\int_M (\nabla_{\frac{\partial}{\partial t}} d\Phi, d\Phi)|_{t=0} \\ &= 2\int_M (Dv, d\phi) = 2\int_M (v, D^* d\phi),\end{aligned}$$

where D^* denotes the formal adjoint of D. Hence, the harmonic map equation reads $D^* d\phi = 0$. Also, one easily sees that $D^* d\phi$ is the trace with respect to g of the symmetric $\phi^* T_N$-valued tensor $\nabla d\phi$.

2.2. **Kähler case: Pluriharmonicity.**

2.2.1. *Sampson's Bochner formula.* In this paragraph, we follow closely the calculation in [Sam86].

Assume now that (M, g) is a Kähler manifold. Let $\phi : \widetilde{M^{un}} \to N$ be a twisted harmonic mapping, equivariant with respect to a representation $\rho : \pi_1(M, m_0) \to Isom(N, h)$.

Introduce local complex coordinates (z^a) on M and lift them to $\widetilde{M^{un}}$ (with a slight abuse, we will not introduce another notation). In this coordinates the Kähler form expresses as:

$$\omega = \frac{\sqrt{-1}}{2} g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \quad d\omega = 0, \quad g_{a\bar{b}} = (\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}).$$

We endow the holomorphic vector bundles $T^{1,0}_M, \Omega^1_M = \Omega^{1,0}_M = (T^{1,0}_M)^*, \ldots$ and their tensor products with the hermitian metric induced by g and with the connection induced by the Levi-Civita connection of g which extends to $T^{\mathbb{C}}_M$ leaving $T^{1,0}_M$ invariant (since (M, g) is Kähler). This connection D is the Chern connection of the holomorphic hermitian vector bundle $(T^{1,0}_M, g)$: it preserves g and its $(0,1)$-part is Dolbeault's $\bar{\partial}$-operator.

We endow $\phi^* T_N^{\mathbb{C}}$ with the pull back of the Levi-Civita connection of (N,h). Tensoring with D, we get a connection ∇ on $\Omega^{1,0}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}}$. For $f=(f_a^i) \in C^\infty(\widetilde{M^{un}}, \Omega^{1,0}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}})$, the $(1,0)$-part of ∇f will be denoted by $\bar\nabla f = ((\bar\nabla f)^i_{\bar b a}) \in C^\infty(\widetilde{M^{un}}, \Omega^{0,1}_{\widetilde{M^{un}}} \otimes \Omega^{1,0}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}})$ and we will use the following notation:

$$(\bar\nabla f)^i_{\bar b a} \overset{not.}{=} f^i_{a|\bar b} \overset{in\ coord.}{=} \partial_{\bar b} f_a^i + \Gamma^i_{jk}(N,h) f_a^j \phi^k_{\bar b}.$$

The (complexified) differential of ϕ, denoted by $d\phi$ is a smooth section of $T^{*\mathbb{C}}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}} = \Omega^{1,0}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}} \oplus \Omega^{0,1}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}}$. Decompose accordingly

$$d\phi = \partial\phi + \bar\partial\phi.$$

Then we have:

$$\partial\phi = \phi^i_a dz^a \otimes \partial_{x_i} \overset{not}{=} (\phi^i_a) \in C^\infty(\widetilde{M^{un}}, \Omega^{1,0}_{\widetilde{M^{un}}} \otimes \phi^* T_N^{\mathbb{C}}).$$

Lemma 2.2.1. *With these notations the harmonic map equation (2) can be written as:*

$$g^{b\bar c}\phi^i_{b|\bar c} = 0 \qquad (\Leftrightarrow \omega^{n-1} \wedge \bar\nabla\partial\phi = 0).$$

Remark 2.2.2. *This should be seen as a curved analog of the classical formula for the Laplace operator on a n-dimensional Kähler manifold, namely:*

$$\omega^{n-1} \wedge i\bar\partial\partial f = c_n \Delta(f)\omega^n$$

where $c_n < 0$ is a constant depending only on n.

Let us now introduce two tensors q and ξ by the formula:

$$\begin{array}{rclll} q_{ab} & = & h_{ij}\phi^i_a\phi^j_b & & q_{ab}dz^a dz^b = [\phi^* ds^2_{(N,h)}]^{2,0} \in C^\infty(\widetilde{M^{un}}, S^2\Omega^{1,0}_{\widetilde{M^{un}}}), \\ \xi_a & = & g^{b\bar c} q_{ab|\bar c} & & \xi_a dz^a = [id_{\Omega^1} \otimes \frac{\omega^{n-1}}{(n-1)!} \wedge]\nu(\bar\partial q) \in C^\infty(\widetilde{M^{un}}, \Omega^{1,0}_{\widetilde{M^{un}}}). \end{array}$$

where $\nu : S^2\Omega^{1,0} \otimes \Omega^{0,1} \to \Omega^{1,0} \otimes \Omega^{1,1}$ is the natural map.

The divergence of ξ is the scalar:

$$\delta\xi = g^{a\bar d}\xi_{a|\bar d} \qquad \delta\xi\frac{\omega^n}{n!} = d(*\xi) = \bar\partial(*\xi).$$

Lemma 2.2.3. *Denoting by R^N the curvature tensor of (N,h), we have:*

$$\delta\xi = (h_{ij}\phi^i_{a|\bar c}\phi^j_{b|\bar d} - R^N_{jklm}\phi^j_b\phi^k_a\phi^l_{\bar c}\phi^m_{\bar d})g^{a\bar d}g^{b\bar c}.$$

Proof: In order to clarify the computation, we simplify the expression of ξ_a.

$$\begin{array}{rcl} \xi_a & = & g^{b\bar c}(h_{ij}\phi^i_a\phi^j_b)_{|\bar c} \\ & = & g^{b\bar c}h_{ij}(\phi^i_{a|\bar c}\phi^j_b + \phi^i_a\phi^j_{b|\bar c}) \end{array}$$

since covariant derivatives leave g, h inert.

$$\begin{array}{rcl} \xi_a & = & g^{b\bar c}h_{ij}\phi^i_{a|\bar c}\phi^j_b + h_{ij}\phi^i_a g^{b\bar c}\phi^j_{b|\bar c} \\ \xi_a & = & g^{b\bar c}h_{ij}\phi^i_{a|\bar c}\phi^j_b \end{array}$$

using lemma 2.2.1. Continuing this way,

$$\xi_{a|\bar d} = g^{b\bar c}h_{ij}(\phi^i_{a|\bar c|\bar d}\phi^j_b + \phi^i_{a|\bar c}\phi^j_{b|\bar d}).$$

The curvature of ∇ is the sum of the pull back of the curvature tensor of N and of the curvature of the Levi-Civita connection of (M, g). Coming back to the very definition of curvature as the defect of commutation of mixed covariant partial derivatives, we have:

$$\phi^i_{a|\bar{c}|\bar{d}} - \phi^i_{a|\bar{d}|\bar{c}} = (\nabla_{\bar{c}}\nabla_{\bar{d}} - \nabla_{\bar{d}}\nabla_{\bar{c}})\phi^i_a = -R^i_{klm}\phi^k_a\phi^l_{\bar{c}}\phi^m_{\bar{d}}.$$

since the curvature of the Levi-Civita connection of (M, g) has no $(0, 2)$ component. In particular,

$$\xi_{a|\bar{d}} = g^{b\bar{c}}h_{ij}(\phi^i_{a|\bar{d}|\bar{c}}\phi^j_b + \phi^i_{a|\bar{c}}\phi^j_{b|\bar{d}} - R^i_{klm}\phi^j_b\phi^k_a\phi^l_{\bar{c}}\phi^m_{\bar{d}}).$$

$$\begin{aligned} g^{a\bar{d}}\xi_{a|\bar{d}} &= g^{b\bar{c}}h_{ij}(g^{a\bar{d}}\phi^i_{a|\bar{d}|\bar{c}}\phi^j_b + g^{a\bar{d}}\phi^i_{a|\bar{c}}\phi^j_{b|\bar{d}} - g^{a\bar{d}}R^i_{klm}\phi^j_b\phi^k_a\phi^l_{\bar{c}}\phi^m_{\bar{d}}) \\ &= g^{b\bar{c}}h_{ij}(g^{a\bar{d}}\phi^i_{a|\bar{c}}\phi^j_{b|\bar{d}} - g^{a\bar{d}}R^i_{klm}\phi^j_b\phi^k_a\phi^l_{\bar{c}}\phi^m_{\bar{d}}) \\ &= g^{b\bar{c}}g^{a\bar{d}}h_{ij}\phi^i_{a|\bar{c}}\phi^j_{b|\bar{d}} - g^{b\bar{c}}g^{a\bar{d}}R^N_{jklm}\phi^j_b\phi^k_a\phi^l_{\bar{c}}\phi^m_{\bar{d}}. \end{aligned}$$

□

Working in normal coordinates at $\tilde{m} \in \widetilde{M^{un}}$, $\phi(\tilde{m})$ so that $h_{ij} = \delta_{ij}, g_{a\bar{b}} = \delta_{ab}$, we obtain:

$$\begin{aligned} \delta\xi(\tilde{m}) &= \sum_{i,a,b} \phi^i_{a|\bar{b}}\phi^i_{b|\bar{a}} - R^N_{jklm}\phi^j_b\phi^k_a\phi^l_{\bar{b}}\phi^m_{\bar{a}} \\ &= \sum_{i,a,b} |\phi^i_{a|\bar{b}}|^2 - \sum_{a,b} R^N(\phi_b, \phi_a, \overline{\phi_b}, \overline{\phi_a}) \end{aligned}$$

Proposition 2.2.4. *If (N, h) satisfies $\forall x, y \in T^{\mathbb{C}}_N$ $R(x, y, \bar{x}, \bar{y}) \leq 0$ and (M, g) is compact, then:*

$$\bar{\nabla}\partial\phi = 0 \quad \forall a, b\ R^N(\phi_b, \phi_a, \overline{\phi_b}, \overline{\phi_a}) = 0.$$

Proof: Indeed ξ descends to M and is a 1-form with non negative divergence. Stokes theorem implies $\int_M \delta\xi = 0$ hence $\delta\xi = 0$. □

Remark 2.2.5. *Under the hypotheses of Proposition 2.2.4, the tensor q introduced above is holomorphic and defines a holomorphic symmetric quadratic differential also denoted by $q \in H^0(M, S^2\Omega^1_M)$.*

We say (N, h) has *non positive curvature in the complexified sense* if the curvature condition of propostion 2.2.4 holds.

We say that a map ϕ from a Kähler manifold X to a Riemannian manifold (N, h) is *pluriharmonic* if $\bar{\nabla}\partial\phi = 0$. This notion does only depend on the complex structure of X and we have:

Lemma 2.2.6. *A pluriharmonic map ϕ on a Kähler manifold X is harmonic for every Kähler metric of X. For every holomorphic map $f : X' \to X$, $\phi \circ f$ is pluriharmonic for every Kähler metric on X'.*

The fact that a pluriharmonic map is harmonic for every Kähler metric is a consequence of Lemma 2.2.1.

Let us finish this section by the simplest example of this phenomenon. If $N = \mathbb{R}$, then it has non positive curvature in the complexified sense. Assume furthermore that ρ acts in an orientation preserving way so that ρ is actually a homomorphism

$\pi_1(X) \to \mathbb{R}$ defining a cohomology class $\alpha \in H^1(X, \mathbb{R})$ and $\pi_1(X)$ acts by translations on $\mathbb{R}$. Then, if X is compact Kähler, Proposition 2.2.4 applied to the harmonic map representing α (which actually is a harmonic function $h : \widetilde{X^{un}} \to \mathbb{R}$) shows that h is pluriharmonic, i.e. locally the real part of a holomorphic function. Actually, since $H_1(\widetilde{X^{un}}) = 0$, it follows that h is globally the real part of a holomorphic function f. Lemma 2.2.6 thus generalzes the fact that holomorphic or pluriharmonic functions are harmonic for every Kähler metric and are preserved by holomorphic pull-back. Also the harmonic representative of α is the form $\underline{dh} = Re(\underline{df})$ where $\underline{\beta}$ denotes the form on X obtained by descending to X an equivariant form β on $\widetilde{X^{un}}$. From this, we recover the fact that a real harmonic 1-form on a compact Kähler manifold is the real part of a (closed) holomorphic 1-form.

2.2.2. *Curvature tensor of Riemannian symetric spaces of non compact type.*

Proposition 2.2.7. *The Riemannian symmetric space $R = Riem(G)$ has non positive curvature in the complexified sense.*

Proof: With no loss of generality, we can reduce to the case where $R = G(\mathbb{R})/K$ is irreducible of the non compact type. Then G is semisimple. Denote by $\mathfrak{k} \subset \mathfrak{g}$ the differential at the origin of the embedding $K \subset G(\mathbb{R})$. The left coset $o = eK$ can serve as an origin for $Riem(G)$. The Cartan involution θ is an involution of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{k} = \ker(\theta - id)$.

If $G = SL_n$ and $K = SO(n)$, θ is just $A \mapsto -A^t$. If $G = Res_{\mathbb{C}|\mathbb{R}} SL_n$ and $K = SU(n) \subset SL_N(\mathbb{C})$ θ is $A \mapsto -\bar{A}^t$.

Then $\mathfrak{p} = \ker(\theta + id)$ defines the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the Lie bracket satisfies the basic relations:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The Killing form of $\mathfrak{g}$ is a quadratic form defined by

$$(X, Y) = Tr_{\mathfrak{g}}(ad(X)ad(Y))$$

makes the Cartan decomposition orthogonal, is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$ (the reader may check this in the above examples).

The derivative of the exponential map at origin identifies $\mathfrak{p}$ with $T_o Riem(G)$ and, after a convenient normalisation, the curvature tensor of $Riem(G)$ identifies to $R^{\mathfrak{p}}$:

$$\forall (X, Y, Z, T) \in \mathfrak{p} \quad R^{\mathfrak{p}}(X, Y, Z, T) = ([X, Y], [Z, T]).$$

In particular if $\xi, \eta \in \mathfrak{p}^{\mathbb{C}}$ we have:

$$R^{\mathfrak{p}}(\xi, \eta, \bar{\xi}, \bar{\eta}) = ([\xi, \eta], \overline{[\xi, \eta]}) \leq 0.$$

Note also that $R^{\mathfrak{p}}(\xi, \eta, \bar{\xi}, \bar{\eta}) = ([\xi, \eta], \overline{[\xi, \eta]}) = 0.$ iff $[\xi, \eta] = 0$.

□

2.2.3. *Carlson-Toledo Theorem.*

Corollary 2.2.8. *The twisted harmonic map ϕ in theorem 2.1.4 is pluriharmonic and verifies, if $\phi(\tilde{m}) = o$:*

$$\mathfrak{a} = \partial\phi(T^{1,0}_{\tilde{m}} M) \subset \mathfrak{p}^{\mathbb{C}} \quad \text{is an abelian subspace, i.e. : } [\mathfrak{a}, \mathfrak{a}] = 0.$$

In general this holds true with respect to the Cartan decomposition at $\phi(\tilde{m})$.

Let us remark the following striking consequence - which does not hold for general complex manifolds.

Corollary 2.2.9. *Let X be a compact Kähler manifold. Let $\rho : \pi_1(X) \to G(\mathbb{R})$ be reductive. Let X' be a Kähler manifold and $f : X' \to X$ be holomorphic. Then $f^*\rho : \pi_1(X') \to G(\mathbb{R})$ is reductive too.*

It is easy to find a counterexample to the corresponding statement for complex manifolds. Let indeed $H \subset SL_3$ be the subgroup scheme over $\mathbb{Z}$ consisting of all upper triangular matrices with coefficients equal to 1. Consider the ring $\mathbb{Z}[\sqrt{-1}]$ of Gaussian integers. Then $H(\mathbb{Z}[\sqrt{-1}]) \subset H(\mathbb{C})$ is a cocompact lattice and similarly $SL_3(\mathbb{Z}[\sqrt{-1}]) \subset SL_3(\mathbb{C})$ are cocompact discrete subgroups. Hence $M' = H(\mathbb{Z}[\sqrt{-1}])\backslash H(\mathbb{C})$ is a complex submanifold of the compact complex parallelizable manifold $M = SL_3(\mathbb{Z}[\sqrt{-1}])\backslash SL_3(\mathbb{C})$. Since $SL_3(\mathbb{C})$ is simply connected $\pi_1(M) = SL_3(\mathbb{Z}[\sqrt{-1}])$. Consider the tautological representation $\rho : \pi_1(M) = SL_3(\mathbb{Z}[\sqrt{-1}]) \to SL_3(\mathbb{C})$. Then ρ is reductive but its restriction to $\pi_1(M')$ is not since the Zariski closure of its image is H a unipotent algebraic group.

2.2.4. *Kähler superrigidity.* We finish with a historical remark. The discovery of the Bochner technique for harmonic mappings in Kähler geometry was the main ingredient of the celebrated:

Theorem 2.2.10. [Siu80] *Let $X_\Gamma = \Gamma\backslash\Omega$ be a compact quotient of an irreducible bounded symmetric domain of dimension ≥ 2. A compact Kähler manifold homotopy equivalent to X_Γ is either biholomorphic or conjugate biholomorphic to it.*

A culmination of this beautiful line of thought is [MSY93].

2.3. **Higgs bundles and Simpson's ubiquity theorem.** In a groundbreaking work, [Sim88], C. Simpson proved a converse to Corlette's Theorem 2.1.4.

2.3.1. *Homogenous bundles on Riemannian symmetric spaces.* Consider G as above and $Riem(G) = G(\mathbb{R})/K$. Then the K-principal bundle $p : G(\mathbb{R}) \to Riem(G)$ has a canonical $G(\mathbb{R})$-equivariant connection. Recall that a connection on a principal K-bundle is a smooth right K-equivariant distribution of horizontal subspaces [7]. This connection can be easily described using a Cartan decomposition, choose $\mathfrak{p} \subset \mathfrak{g}$ as the value of the horizontal distribution at the origin $e \in G(\mathbb{R})$ and move it around using the left translation of G on itself. This defined a distribution which is straightforwardly checked to be right K-equivariant and horizontal for p.

Consider $\alpha : K \to GL_{\mathbb{R}}(V)$ a real representation. The canonical connection on the principal K-bundle $p : G(\mathbb{R}) \to Riem(G)$ induces a Koszul connection ∇ on the real vector bundle $q : \mathcal{V} := G \times_K V \to Riem(G)$.

Lemma 2.3.1. *The curvature of $(\mathcal{V}, \nabla)$ at $o = eK$ is given by the following formula, where we identify $q^{-1}(o)$ with V and T_oR with $\mathfrak{p}$:*

$$\forall X, Y \in \mathfrak{p} \qquad \Theta(X, Y) = -\alpha_*([X, Y]),$$

where $\alpha_* : \mathfrak{k} \to \mathfrak{gl}_{\mathbb{R}}(V)$ is the Lie algebra representation arising from α.

This lemma extends naturally to the complex representations.

[7] An horizontal subspace is a subspace supplementary to the relative tangent bundle.

2.3.2. *The notion of Higgs bundle.* Simpson found a very useful reformulation of the properties of the harmonic mapping in terms of Higgs bundles. As far as I know, the first instance of this construction in the litterature is [Don87a], Donaldson's post-scriptum to [Hit87].

Definition 2.3.2. *Let N be a complex manifold. A Higgs bundle on N is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a holomorphic vector bundle on N and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_N$ satisfies*

$$\theta \wedge \theta = 0 \in \mathrm{End}(\mathcal{E}) \otimes \Omega^2_N.$$

If, in local coordinates, $\theta = \theta_a dz^a$, the condition $\theta \wedge \theta = 0$ means $[\theta_a, \theta_b] = 0$ for all indices a, b.

Proposition 2.3.3. *Let X be a compact Kähler manifold, $\rho : \pi_1(X) \to G(\mathbb{R})$ be a reductive representation $\phi : \widetilde{X^{un}} \to Riem(G)$ the twisted pluriharmonic map and $\alpha : K \to V$ a representation.*

Then the pull back of $(\mathcal{V}, \nabla)$ is $\pi_1(X)$-equivariant and descends to a real vector bundle with a connection $(E, d) \to X$. The complexified connection $d^{\mathbb{C}} = \partial + \bar{\partial}$ satisfies $\bar{\partial}^2 = 0$. Hence it defines a holomorphic vector bundle $\mathcal{E}$ on X.

If α is the restriction of a representation $\beta : G(\mathbb{R}) \to V$ then one can construct a holomorphic Higgs field on $\mathcal{E}$ by the formula $\theta = \beta(\partial\phi)$.

Proof: Left to the reader as an interesting exercise. □

2.3.3. *Simpson's Kobayashi-Hitchin correspondance.* Let (X, ω) be a compact Kähler manifold. A Higgs bundle $(\mathcal{E}, \theta)$ is *stable* if for every proper *Higgs* subsheaf $\mathcal{F}$ of the sheaf of its holomorphic sections:

$$rk(\mathcal{E})c_1(\mathcal{F}).\omega^{n-1} < rk(\mathcal{F})c_1(\mathcal{E}).\omega^{n-1}.$$

A Higgs bundle $(\mathcal{E}, \theta)$ is polystable if it is a direct sum of stable Higgs bundles $(\mathcal{E}_i, \theta_i)$ of the same slope $\mu(\mathcal{E}_i) = \frac{c_1(\mathcal{E}_i).\omega^{n-1}}{rk(\mathcal{E}_i)}$.

Theorem 2.3.4. *A Higgs bundle $(\mathcal{E}, \theta)$ on (X, ω) arises from a reductive representation of $\pi_1(X, x)$ in $GL_N(\mathbb{C})$ iff:*

(1) $\int_X c_1(\mathcal{E}).\omega^{n-1} = 0$,
(2) $\int_X c_2(\mathcal{E}).\omega^{n-2} = 0$,
(3) $(\mathcal{E}, \theta)$ is polystable.

The proof [Sim88] uses methods of gauge theory. The dimension 1 case is due to Hitchin [Hit87]. Let us give a brief account of the ideas involved. Consider a Higgs bundle $(\mathcal{E}, \theta)$ on X. Let h be a hermitian metric on $\mathcal{E}$. Consider its Chern connection, the unique connection D_h leaving h invariant such that $D_h^{0,1} = \bar{\partial}$. Consider furthermore the connection:

$$D(h) = D_h^{1,0} + \theta^{*_h} + \bar{\partial} + \theta,$$

where θ^{*_h} is the adjoint of θ with respect to h. The hardest part of the proof is to show that there is a solution h_{YM} to the anti-self-duality equations:

$$\omega^{n-1} i F_{D(h)} = \omega^{n-1}(i\Theta_h + [\theta, \theta^{*_h}]) = C\omega^n$$

where C is a topological constant and $F_{D(h)}$ is the curvature of $D(h)$ whenever $(\mathcal{E}, \theta)$ is stable.

Under (1), $C = 0$. In fact under (2) a Chern class argument gives $F_{D(h_{YM})} = 0$ hence $D(h_{YM})$ is a flat connection whose holonomy gives a representation of the fundamental group.

As a corollary, we see that (poly)stability of a Higgs bundle with $c_1 = c_2 = 0$ does not depend on the Kähler class.

The vector bundle underlying a stable Higgs bundle may not be stable. Indeed $(\mathcal{E}, 0)$ is stable iff $\mathcal{E}$ is stable. In fact the representation given by Theorem 2.3.4 is unitary iff $\theta = 0$. This case of Simpson's theorem is a celebrated result of Uhlenbeck-Yau [UhlYau88], the dimension 1 case being due to [NS65] and the case of projective algebraic manifolds to [Don85, Don87b] - see also [Kob87] for a nice exposition and [LüTe06] for a nice recent exposition in the more general setting of hermitian manifolds.

We finish this paragraph by a brief discussion of the rank one case. A rank one Higgs bundle is a pair (L, α) where L is a holomorphic line bundle and α a holomorphic one form. It is automatically stable. When $c_1(L) = 0$ and X is compact Kähler, Hodge theory implies the existence of unique flat hermitian metric h on the line bundle L. Since GL_1 is abelian, $h = h_{YM}$.

2.3.4. *Moduli spaces.* The *character scheme* $M_B(X, GL_N)$ is defined as the GIT quotient of the (affine) representation scheme $Hom(\pi_1(X), GL_N)$ by the conjugation action of GL_N (we work over $\bar{\mathbb{Q}}$). Its points over an algebraically closed field of characteristic zero $\bar{k}$ are in bijection with conjugacy classes of reductive representations of $\pi_1(X)$ over $\bar{k}$.

Assume that X is projective. There is a coarse moduli space $M_{Dol}(X, GL_N)$ for rank N semistable Higgs bundles with vanishing Chern classes (see [Sim94]). Its complex points are in bijection with isomorphism classes of polystable Higgs bundles.

The correspondance of theorem 2.3.4 underlies a real analytic homeomorphism $kh : M_B(X, GL_N)(\mathbb{C}) \to M_{Dol}(X, GL_N)(\mathbb{C})$.

2.3.5. *Simpson's ubiquity theorem.* For every Higgs bundle $(\mathcal{E}, \theta)$, we may define its characteristic polynomial χ_θ of θ. We have:

$$\chi_\theta(X) = X^N + \sum_{k=0}^{N-1} \chi_k T^k \quad \chi_k \in H^0(X, S^{N-k}\Omega^1_X).$$

Theorem 2.3.5. *Assume that X is projective. Then, the Hitchin map*

$$\left(M_{Dol}(X, GL_N) \to \prod_{k=0}^{N-1} H^0(X, S^{N-k}\Omega^1_X), \qquad [(\mathcal{E}, \theta)] \mapsto (\chi_k)_k \right)$$

is a proper morphism of quasiprojective varieties.

Corollary 2.3.6. *Assume that X is projective. Let $(\mathcal{E}, \theta)$ be a polystable Higgs bundle with vanishing Chern classes. Then, for $t \in \mathbb{C}^*$ $(\mathcal{E}, t.\theta)$ is a polystable Higgs bundle with vanishing Chern classes. Furthermore $\lim_{t\to 0} [(\mathcal{E}, t\theta)] = [(\mathcal{E}', \theta')]$ exists and is a fixed point of the action of the multiplicative group GL_1 on $M_{Dol}(X, GL_N)$ given by $t.[(\mathcal{E}, \theta)] = [(\mathcal{E}, t\theta)]$.*

Proof: Indeed, $[(\mathcal{E}, t\theta)]$ is an orbit of an algebraic GL_1-action on $M_{Dol}(X, GL_N)$ and its projection by the Hitchin map has a linit as $t \to 0$. □

In fact, these results combine in a statement known as Ubiquity of Variations of Hodge Structure. This statement will be one of the two main ingredients of our approach.

Definition 2.3.7. *A $\mathbb{C}$-VHS (polarized complex variation of Hodge structures) on X of weight $w \in \mathbb{Z}$ is a 5-tuple $(X, \mathbb{V}, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, S)$ where:*

1. *$\mathbb{V}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces,*
2. *S a non degenerate flat sesquilinear pairing on $\mathbb{V}$,*
3. *$\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ a biregular decreasing filtration of the flat vector bundle $(\mathcal{V}, d)$ underlying $\mathbb{V}$ by holomorphic subbundles such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$,*
4. *$\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ a biregular decreasing filtration of the flat vector bundle underlying $\mathbb{V}$ by antiholomorphic subbundles such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1} \otimes \Omega^1_{\bar{X}}$,*
5. *for every point $x \in X$ the fiber at x $(\mathbb{V}_x, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-Hodge Structure polarized by S_x.*

In other words $\mathcal{V} = \sum_{p+q=w} H^{p,q}$ with an S-orthogonal decomposition $H^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{G}}^q$ with $(-1)^p S|_{H^{pq}} > 0$.

Observe that the axioms are slightly redundant since $\overline{\mathcal{G}}^q$ is the S-orthogonal of $\mathcal{F}^{w-q+1}$. In particular condition (4) follows from condition (3) aka Griffiths' transversality.

If, furthermore, $A \subset \mathbb{C}$ is a ring, we have $\mathbb{V} = \mathbb{V}_A \otimes \mathbb{C}$ where $\mathbb{V}_A$ is a local system of finite rank projective A-modules, we have a A-VHS or polarized variation of A-Hodge structure. If $A \subset \mathbb{R}$, we need to require that $\overline{\mathcal{G}}^\bullet$ is the complex conjugate of $\mathcal{F}^\bullet$. In an influential series of articles culminating with [Gri73], Griffiths discovered a structure of $\mathbb{Z}$-VSH on the monodromy of a polarized smooth family of projective varieties.

Since $H^{p,q} \simeq \mathcal{F}^p/\mathcal{F}^{p+1}$, we can endow the smooth vector bundle $\mathcal{V}$ with a holomorphic structure which does not in general coincide with the holomorphic structure underlying the flat connection. Let us call $\mathcal{E}$ the resulting holomorphic vector bundle.

Then second fundamental form of the flat connexion d for $\mathcal{F}^p$ takes values in $\mathcal{F}^{p-1}/\mathcal{F}^p$ and vanishes on $\mathcal{F}^{p+1}$, thanks to condition (3). This defines $\nabla'_p : H^{p,q} \to H^{p-1,q+1} \otimes \Omega^1_X$. It is easy to see that $(\mathcal{E}, \oplus_p \nabla'_p)$ is a polystable Higgs bundle with torsion Chern classes.

The pluriharmonic map attached to a $\mathbb{C}$-VHS has very nice properties. By definition the monodromy of a $\mathbb{C}$-VSH is a representation $\sigma : \pi_1(X, x) \to U(S_x)$ and every framing of $\mathcal{V}_x$ defines an embedding $i : U(S_x) \to GL_N(\mathbb{C})$.

Let $P = \sum_{p\equiv 0[2]} \dim_{\mathbb{C}} H^{p,q}_x$ and $Q = \sum_{p\equiv 1[2]} \dim_{\mathbb{C}} H^{p,q}_x$. Then $U(S_x) \simeq U(P, Q)$. The maximal compact subgroup of $U(P, Q)$ is $U(P) \times U(Q)$ and we have a totally geodesic embedding of Riemannian symmetric spaces

$$Riem(U(P, Q)) = U(P, Q)/U(P) \times U(Q) \xrightarrow{\eta} Riem(GL_N(\mathbb{C})) = GL_N(\mathbb{C})/U(N).$$

Now define $\mathcal{D} = U(P, Q)/\prod U(H^{p,q}_x)$. The space $\mathcal{D}$ parametrizes polarized Hodge filtrations on $(\mathbb{V}_x, S_x)$. It is actually an open $U(P, Q)$-orbit of a complex rational homogenous space $\widehat{\mathcal{D}}$. $\widehat{\mathcal{D}}$ is the subvariety of the flag variety parametrizing flags (F^p) with $\dim_{\mathbb{C}} F^p/F^{p+1} = \dim_{\mathbb{C}} H^{p,q}$ defined by the condition $S(F^p, F^{w-p+1}) = 0$.

A $\mathbb{C}$-VHS defines a *holomorphic* map

$$\left(q : \widetilde{X^{un}} \to \mathcal{D} \qquad \tilde{y} \mapsto (\text{Parallel transport at } \tilde{x})((\mathcal{F}^{p}_{\tilde{y}})_p))\right).$$

Now introduce the quotient map $p : \mathcal{D} \to Riem(U(P,Q))$ [8].

Proposition 2.3.8. *The composed map $\eta \circ p \circ q$ is the $i \circ \sigma$-equivariant pluriharmonic map.*

In [GriSch69], the construction is put in the larger context of groups of Hodge type and the geometry of the period mapping is extensively studied from a global perspective. The important point is that Griffiths' transversality means that the period mapping is tangent to a holomorphic distribution $T_h(\mathcal{D})$[9], transverse to the fibers of p.

Theorem 2.3.9. [Sim88, Sim92] *Assume that X is projective. Fixed points of the action of GL_1 on $M_{Dol}(X, GL_N)$ correspond to the isomorphism classes of the holonomy representation of a complex polarized variation of Hodge structures of weight zero. Hence, every reductive representation of a Kähler group can be deformed to a $\mathbb{C}$-VHS.*

In fact, before Simpson's ubiquity theorem, we had almost no way of constructing $\mathbb{C}$-VHS except using monodromy representations.

For a rank one Higgs bundle (L, α) with flat unitary metric h

$$\lim_{t \to 0} (L, t.\alpha, h) = (L, 0, h)$$

and the limit harmonic bundle is just a flat $U(1)$ bundle with a vanishing Higgs field. These are the $\mathbb{C}$-VHS of rank one and their period mappings are constant.

Let us give an example of the power of Simpson's ubiquity theorem. Let C be a compact Riemann surface of negative Euler characteristic. A spin structure on C is given by a line bundle Θ together with an isomorphism $c : \Theta^2 \to K_C$. Tensoring with Θ^{-1}, we produce $c' : \Theta \to K_C \otimes \Theta^{-1}$. We also get a rank 2 Higgs bundle, setting $E = \Theta \oplus \Theta^{-1}$ and $\theta = \begin{pmatrix} 0 & c' \\ 0 & 0 \end{pmatrix}$. This Higgs bundle is stable since its rank one proper Higgs subsheaves are contained in Θ^{-1} which has negative degree. Furthermore it is $\mathbb{C}^*$-stable. Hence, we get a $\mathbb{C}$-VHS of weight 1 with $H^{1,0} = \Theta$, $H^{0,1} = \Theta^{-1}$ and the graded part of the Gauss-Manin connection is c'. Its period mapping lands in $U(1,1)/U(1) \times U(1) = \Delta$ and defines an equivariant holomorphic mapping from the universal covering space of C to the unit disk. Since c' computes the derivative of the period mapping and never vanishes it follows that the period mapping is unramified. Using this, we conclude that C has a hyperbolic metric, hence that is universal covering space is Δ. It is easily seen that the period mapping actually defines an uniformization.

The last part of Theorem 2.3.9 is likely to hold in the general Kähler case, although Simpson's construction of the moduli spaces is only valid in the projective case. Actually, the factorisation theorem in [Zuo96] can be used to reduce the

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[8] The knowledgeable reader will have noticed that $U(P,Q)/U(P) \times U(Q)$ is a hermitian symmetric spaces, also known as the bounded symmetric domain $D^{I}_{p,q}$, hence carries a canonical complex structure. He should beware that the fibers of p are holomorphic submanifolds of $\mathcal{D}$ but the map p itself is not holomorphic.

[9] For a recent survey of the EDS perspective on period mappings, see [CGG09].

statement to the projective case. I believe that Zuo's theorem is correct but it seems to me that the proof has a weak point. I hope that this will be remedied by a work in preparation in collaboration with B. Claudon.

3. Non-abelian Hodge Theory in the non archimedean case

3.1. **Valued fields.** Recall that a discrete valuation on a field K is a map $v : K^* \to \mathbb{Z}$ such that:

- $\forall x, y \quad v(xy) = v(x) + v(y)$
- $\forall x, y \quad v(x+y) \geq \min(v(x), v(y))$.

Denote by $O_v = \{x \in K|\ v(x) \geq 0\}$, $m_v = \{x \in K|\ v(x) \geq 1\}$, $k(v) = O_v/m_v$. Then, $k(v)$ is called the residual field.

Example 3.1.1. *Let C/k be a smooth algebraic curve over a field k, and $P \in C(\bar{k})$ be a closed point. Then for $f \in k(C)^*$, we can define $v(f) = ord_P(f)$. This defines a discrete valuation v on $k(C)$.*

For p a prime number and $x \in \mathbb{Q}^*$, we define $v_p(x)$ to be the multiplicity of p in a reduced prime decomposition of x. This defines a discrete valuation v_p on $\mathbb{Q}$.

A local field K_v is a field equipped with a surjective discrete valuation $v : K_v^* \to \mathbb{Z}$ which is a complete metric space with respect to the distance $d(x, y) = e^{-v(x-y)}$. Local compacity of a complete valued field is equivalent to the residual field being finite.

Example 3.1.2. *A locally compact local field is either isomorphic to a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers or to $\mathbb{F}_q((t))$ for some power q of a prime number.*

3.2. **The affine building attached to a reductive algebraic group over a valued field.** Let G/K_v be a connected semisimple algebraic group defined over the local field K_v. Then $G(K_v)$ acts isometrically on its Bruhat-Tits building $\Delta^{BT}(G(K_v))$. Following the tradition of [GroSch92], we will use a slightly non-standard notation here and will denote by $\Delta^{BT}(G(K_v))$ the geometric realization endowed with its Tits distance of the usual Bruhat-Tits building, a polysimplicial complex which will be denoted by $s\Delta^{BT}(G(K_v))$ [BruTit72].

This action is the good analog of the action of a real semisimple algebraic group on its Riemannian symmetric space. In fact the naïve analog $G(K_v)/G(O_v)$ is discrete and has a rather poor geometric structure. It is realized as a subset of the set of simplices of $s\Delta^{BT}(G(K_v))$.

3.2.1. *The Bruhat-Tits building for $G = SL_N$.* The description of $\Delta^{BT}(G(K_v))$ is rather involved. For simplicity, we will just give a brief description of $\Delta^{BT}(SL_N(K_v))$.

First of all we describe a simplicial complex $s\Delta^{BT}(SL_N(K_v))$ whose geometric realization will be homeomorphic to $\Delta^{BT}(SL_N(K_v))$.

A *lattice* $L \subset K_v^N$ is a free O_v-submodule of rank N. Two lattices L, L' are *homothetic* iff there exists $a \in K_v^*$ such that $aL = L'$. We denote by $v\Delta^{BT}(SL_N(K_v))$ the set of equivalence classes of homothetic lattices. Note that the equivalence class of the lattice L is $\{\pi^k L\}_{k \in \mathbb{Z}}$ where π is any fixed element of K_v^* such that $v(\pi) = 1$. Given two lattices L, L' the minimum of $\alpha \in \mathbb{Z}$ such that $\pi^\alpha L' \subset L$ will be denoted by $\kappa(L', L)$.

Say that a couple of distinct lattice classes $([L], [L'])$ is *adjacent* if there are representatives L, L' satisfying:

$$\pi L \subset L' \subset L.$$

Note that we have $\kappa(L', L) = 0$ in the latter case. It should be noted that if $([L], [L'])$ is adjacent $([L'], [L])$ is adjacent too and we say that the lattices $[L], [L']$ are *adjacent*.

Lemma 3.2.1. *Assume $[L'], [L''], [L]$ are pairwise adjacent. Choose representatives such that:*

$$\pi L \subset L' \subset L, \qquad \pi L \subset L'' \subset L.$$

Then $L' \subset L''$ or $L'' \subset L'$.

Proof: Assume $L' \not\subset L''$. Then $\kappa(L', L'') = 1$. If L' and L'' are adjacent, then $\pi L'' \subset \pi L' \subset L''$. Hence $L'' \subset L'$. □

Corollary 3.2.2. *Let $[L]$ be a lattice class and $[L_1], \ldots, [L_p]$ be pairwise adjacent lattice classes. Then there is $\sigma \in S_p$ and representatives L_k of $[L_{\sigma(k)}]$ such that:*

$$\pi L \subsetneqq L_1 \subsetneqq L_2 \subsetneqq \ldots L_p \subsetneqq L.$$

In particular (maximal) sets of pairwise adjacent classes of lattice classes adjacent to a given $[L]$ are in one-to-one correspondance with (complete) flags in the $k(v)$-vector space $L/\pi L$.

Definition 3.2.3. *We define $s\Delta^{BT}(SL_N(K_v))$ to be the $N-1$-dimensional simplicial complex whose vertex set is the set of lattice classes and p-simplices are $p+1$-tuples of pairwise adjacent lattice classes. This simplicial complex carries an action of $GL_N(K_v)$ given by $(g, [L]) \mapsto [g.L]$.*

The $N-1$-dimensional simplices are called the chambers of $s\Delta^{BT}(SL_N(K_v))$.

Fix a particular lattice L, for instance $L = O_v^N$. Given another lattice L' there is some $g \in GL_N(K_v)$ such that $L' = g.L$. The congruence class $c(g) = v(det(g))[N] \in \mathbb{Z}/NZ$ is independant of g and defines $c(L') \in \mathbb{Z}/NZ$ the *label* of L'. The labelling $c : v\Delta^{BT}(SL_N(K_v)) \to \mathbb{Z}/N\mathbb{Z}$ is not preserved by $GL_N(K_v)$ but is preserved by $SL_N(K_v)$.

Lemma 3.2.4. *Each chamber has precisely one vertex of given label.*

Proof: Left to the reader. □

Let $B = (e_1, \ldots, e_N)$ be a basis of K_v^N. Then we can consider a full subcomplex sA_B of $s\Delta^{BT}(SL_N(K_v))$, called the *apartment* corresponding to B, whose vertices are:

$$[L_{r_1,\ldots,r_N}] = [\pi^{r_1} O_v e_1 + \cdots + \pi^{r_N} O_v e_N] \quad r_i \in \mathbb{Z}.$$

Obviously the vertex set of sA_B identifies to $\Lambda_N = \mathbb{Z}^N/\mathbb{Z}.E$ where $E = (1, \ldots, 1)$. Consider the natural embedding j of Λ_N as a lattice in the real vector space $V = \mathbb{R}^N/\mathbb{R}E$. This extends linearly on each simplex to a continuous map $\bar{j} : |sA_B| \to V$ [10].

Lemma 3.2.5. *$\bar{j}$ is a homeomorphism.*

[10] If $\mathbb{X}$ is a simplicial complex, we denote by $|\mathbb{X}|$ its geometric realization.

Proof: Although it is not that obvious, it will be left as an exercise. □

Obviously $\bar{j}$ is equivariant under the action of S_N by permutation of the coordinates and by the translation action of Λ_N. In fact it is equivariant by the action of the semidirect product $W_N = \Lambda_N \ltimes S_N$.

The action of S_N on V being irreducible, there is a unique (up to multiplication by a scalar constant) euclidian norm on V such that W_N acts by isometries.

The length of the 1-simplices between adjacent lattices $L, L' \in sA_B^0$ depends only on $c(L) - c(L')$, hence we can normalise the euclidian norm requiring that the length of a 1-simplex $[L, L']$ is 1 if $c(L) - c(L') = \pm 1$.

This defines on $A = |sA_B| \subset |s\Delta^{BT}(SL_N(K_v))|$ a distance function d_A. This distance functions glues into a function $d : |s\Delta^{BT}(SL_N(K_v))|^2 \to \mathbb{R}_{\geq 0}$ thanks to the fact that the following building axioms are fulfilled:

Proposition 3.2.6. *Any two simplices of $s\Delta^{BT}(SL_N(K_v))$ are contained in an apartment. Given two apartments A, A' and σ, σ' two simplices in their intersection then there is a simplicial isometry $A \to A'$ keeping σ, σ' fixed.*

Proof: See the standard expositions [Bro89], [Ron89] or [Gar97]. □

Theorem 3.2.7. *The function d is a distance function for which apartments are isometrically embedded. The resulting metric space is complete. It is locally compact iff K_v is locally compact.*

Given two points P, Q in $|s\Delta^{BT}(SL_N(K_v))|$ there is a unique geodesic $\gamma : [0,1] \to |s\Delta^{BT}(SL_N(K_v))|$, i.e.: a continuous path such that $\gamma(0) = P, \quad \gamma(1) = Q$ and $d(P, \gamma(t)) = t.d(P, Q)$. This geodesic is just an arc length parametrized line segment in any apartment containing P and Q.

Given three points P, Q, R in $|s\Delta^{BT}(SL_N(K_v))|$ define $P_t = \gamma(t)$ where γ is the geodesic from Q to R. Then,

$$\forall t \in [0,1], \quad d^2(P, P_t) \leq t d^2(P, R) + (1-t) d^2(P, Q) - t(1-t) d^2(Q, R). \tag{3}$$

Equality holds iff P, Q, R lie in a common apartment.

Proof: This is proved in [BruTit72]. See also the standard references. □

Definition 3.2.8. *The Bruhat-Tits building $\Delta^{BT}(SL_N(K_v))$ is $|s\Delta^{BT}(SL_N(K_v))|$ endowed with the distance function d. The group $GL_N(K_v)$ acts on $\Delta^{BT}(SL_N(K_v))$ by simplicial isometries.*

Proposition 3.2.9. *A subgroup of $SL_N(K_v)$ fixes a point in $\Delta^{BT}(SL_N(K_v))$ iff it is bounded (i.e.: conjugate to a subgroup of $GL_N(O_v)$).*

Remark 3.2.10. *$\Delta^{BT}(SL_N(K_v))$ can also be identified with the Goldman-Iwahori space of non archimedian norms of volume one on K_v^N in perfect analogy with the realisation of $SL_N(\mathbb{C})/SU(N)$ as the space of hermtian metrics of volume 1.*

The simplest case of this construction is $N = 2$, see [Ser77] for a beautiful exposition. In that case, $\mathbf{T} = \Delta^{BT}(SL_2(K_v))$ is a tree. Let q be the cardinal of the residue field k_v. Then, $\mathbf{T}$ is the unique tree such that all vertices are $q+1$-valent and all edges have length one. Its apartments are geodesic lines.

3.2.2. *Remarks on the general case.*

Definition 3.2.11. *A complete geodesic metric space $(\mathcal{X}, d)$ is non positively curved (NPC) if and only if inequality 3 in theorem 3.2.7 is satisfied.*

For a general simply connected semisimple group G/K_v the Bruhat Tits building $\Delta^{BT}(G(K_v)$ is a NPC metric space with an action of $G(K_v)$ which is the realization of a simplicial complex, is a union of affine euclidean apartments and satisfies Proposition 3.2.6 and Theorem 3.2.7.

Although GL_N is not semisimple, it is convenient to give the following definition:

Definition 3.2.12. *The (extended) Bruhat-Tits building $\Delta^{BT}(GL_N(K_v))$ is the product metric space $\Delta^{BT}(SL_N(K_v)) \times \mathbb{R}$, with distance function*

$$d = (d^2_{\Delta^{BT}(SL_N(K_v))} + d^2_{\mathbb{R}})^{1/2}$$

and an action of $GL_N(K_v)$ given by the product action of the action described above on $\Delta^{BT}(SL_N(K_v))$ with the translation action by the group morphism $g \mapsto v(det(g))$.

Every apartment $A \subset \Delta^{BT}(SL_N(K_v))$ gives rise to an isometrically embedded apartment $A' = A \times \mathbb{R} \subset \Delta^{BT}(GL_N(K_v))$.

3.3. **Harmonic mappings into affine buildings.**

3.3.1. *Korevaar-Schoen's energy and harmonic mappings into NPC spaces.* Given Ω a compact Riemannian manifold with boundary and $(\mathcal{X}, d)$ a NPC metric space, [KoSc93] carried out in detail the program outlined in [GroSch92] of constructing a Sobolev space $W^1_2(\Omega, \mathcal{X})$ containing the space of Lipschitz maps from Ω to $\mathcal{X}$ endowed with a lower semicontinuous Dirichlet integral E which is the integral of a locally defined absolutely continuous energy density measure.

In case $\mathcal{X}$ is a ball of finite radius in $\Delta^{BT}(SL_N(K_v))$, K_v locally compact, consider $j : \mathcal{X} \to \mathbb{R}^M$ an embedding such that each simplex is isometrically embedded. We have:

$$E(u) = \int_\Omega |\nabla(j \circ u)|^2.$$

This definition is the working definition adopted in [GroSch92] and suffices for most purposes.

Given an apropriate[11] boundary value $w : \partial\Omega \to \mathcal{X}$, there is a unique energy minimizer for E on the subclass of $u \in W^1_2(\Omega, \mathcal{X})$ such that $u|_{\partial\Omega} = w$. This minimizer is locally Lipschitz in the interior of Ω.

Definition 3.3.1. *Let (M^*, g) be a Riemannian manifold and $f : M^* \to \mathcal{X}$ is a locally Lipschitz mapping. We say f is harmonic iff for every $\Omega \subset M^*$ a sufficiently small ball $f|_\Omega$ is the energy minimizer relative to $f|_{\partial\Omega}$.*

Proposition 3.3.2. *Assume $\Phi : \mathcal{X} \to \mathbb{R}$ is a continuous function which is convex when restricted to geodesics. If $f : M^* \to \mathcal{X}$ is harmonic then $\Phi \circ f$ is subharmonic.*

Again, let (M, g) be a compact connected Riemannian manifold endowed with a base point $m_0 \in M$ and let $\pi : (\widetilde{M^{un}}, \tilde{m}_0) \to (M, m_0)$ be the universal covering space . Consider furthermore $\rho : \pi_1(M, m_0) \to Isom(\mathcal{X})$ a representation.

The space $Lip(\widetilde{M^{un}}, \mathcal{X})_\rho$ of Lipschitz continuous mappings ϕ from $\widetilde{M^{un}}$ to $\mathcal{X}$ such that

$$\forall (\gamma, \tilde{m}) \in \pi_1(M, m_0) \times \widetilde{M^{un}}, \qquad \phi(\gamma.\tilde{m}) = \rho(\gamma).\phi(\tilde{m})$$

carries a convex energy functional too and its global minimizers, if any, are equivariant harmonic mappings. The fact that this space is non empty and contractible

[11]Lipschitz continuous for instance.

and that the energy is convex is a consequence of the NPC property of $\mathcal{X}$, just like in section 2.

The minimizing sequences for this problem are studied in general in [KoSc97]. For our purposes, it will be enough to state the following consequence of [GroSch92]:

Theorem 3.3.3. *Assume $\rho : \pi_1(M, m_0) \to GL_N(K_v)$ is a reductive representation. Then there exists a Lipschitz continuous equivariant harmonic mapping*

$$h_\rho : \widetilde{M^{un}} \to \Delta(GL_N(K_v)).$$

This sufficient condition for existence of twisted harmonic mappings is not a necessary condition.

3.3.2. *Gromov-Schoen's regularity theorem for harmonic mappings into locally compact Bruhat-Tits buildings.*

Definition 3.3.4. *Let $h : M^* \to \Delta(GL_N(K_v))$ be a harmonic mapping.*

The regular locus of h is the set of all points $x \in M^*$ such that there exists $r > 0$ and an apartment $A \subset \Delta(GL_N(K_v))$ such that $h(B(x,r)) \subset A$ where $B(x,r)$ is the open Riemannian ball of radius r centered at x.

In this case $h|_{B(x,r)} = i_A \circ h_x^A$ where i_A is the isometric embedding of A and h_x^A is a harmonic function defined on $B(x,r)$ with values in A - hence is smooth.

Say h is *regular* if $M^* - (M^*)^{reg}$ is of Hausdorff codimension ≥ 2.

Theorem 3.3.5. *The harmonic mapping constructed in Theorem 3.3.3 is regular provided K_v is locally compact.*

Proof: See [GroSch92]. The general non locally compact case seems to be true, see p.564 in the introduction to [KoSc93]. To the best of our knowledge, the only published reference treats the case of $\mathbb{R}$-trees and applies here only in case $N = 2$ [Sun03]. □

As pointed out by the referee, a local example of a regular harmonic mapping with values in a tree is given by $\mathbb{R}^2 \to G_4,\ (x,y) \mapsto (x|y|, y|x|)$ where G_4 is the graph embedded in $\mathbb{R}^2$ as the union of the diagonal and the antidiagonal. See also [GroSch92, pp. 178-180] for another enlighting example.

3.3.3. *Pluriharmonic mappings in the sense of Gromov-Schoen.*

Definition 3.3.6. *Assume (M^*, g) is Kähler and let $h : M^* \to \Delta(GL_N(K_v))$ be a regular harmonic mapping.*

We define h to be pluriharmonic if the following holds:

For $x \in (M^*)^{reg}$, choose $r > 0$ and an apartment $A_x \subset \Delta(GL_N(K_v))$ such that $h(B(x,r)) \subset A_x$. Write $h|_{B(x,r)} = i_A \circ h_x^A$ where i_A is the isometric embedding of A_x and h_x^A is a harmonic function defined on $B(x,r)$ with values in A. Then h_x^A is pluriharmonic.

Theorem 3.3.7. *If (M, g) is Kähler, the harmonic mapping constructed in Theorem 3.3.3 is pluriharmonic provided K_v is locally compact.*

Proof: See [GroSch92]. The general non locally compact case is again likely to be true but we cannot give a reference. □

Proposition 3.3.8. *A pluriharmonic map ϕ is harmonic for every Kähler metric. Its singular locus is contained in a proper closed complex analytic subset. For every holomorphic map $f : X' \to M^*$, $\phi \circ f$ is pluriharmonic for every Kähler metric on X'.*

Proof: A detailled proof of these easy consequences of [GroSch92] is given in [Eys04]. □

3.4. **Spectral coverings and associated constructions.** In this paragraph, we fix $\rho_v : \pi_1(X, x) \to GL_N(K_v)$ to be a reductive representation.

Denote by $h_v : \widetilde{X^{un}} \to \Delta^{BT}(GL_N(K_v))$ the harmonic mapping given by theorem 3.3.3. Assume that the conclusion of theorem 3.3.7 holds (e.g. $N = 2$ or K_v locally compact). There is a proper closed analytic subset $S \subset X$ such that $(\widetilde{X^{un}})^{reg} \supset \widetilde{X^{un}} - \pi^{-1}(S)$ thanks to Proposition 3.3.8.

3.4.1. *Construction of the spectral covering.* As announced in the last paragraph of [GroSch92], one can construct a finite Galois ramified covering $Sp_{\rho_v}(X) \to X$ such that the lift of the foliation defined by $\ker(\partial h_v)$ is given by a finite system of holomorphic one forms. We now describe this object called the spectral covering of X attached to ρ_v.

Fix once for all an apartment A of $\Delta^{BT}(GL_N(K_v))$. This apartment split canonically as $A = A' \times \mathbb{R}$ where A' is an apartment of $\Delta^{BT}(SL_N(K_v))$. We denote by $A_{\mathbb{C}}$ the complexification and by A^* the real dual vector space of the underlying vector space of the affine space A.

Fix $x \in X - S$ and choose a lift U_x in $\widetilde{X^{un}}$ of a small neighborhood V_x of x such that the properties in definition 3.3.6 are satisfied. This gives an apartment A_x and a pluriharmonic mapping $U_x \to A_x$. The apartment A_x splits canonically as $A = A'_x \times \mathbb{R}$ where A'_x is an apartment of $\Delta^{BT}(SL_N(K_v))$.

We denote by $i_x : A \to A_x$ an isometry respecting the canonical splittings and the simplicial structure of A'_x, i_x being chosen arbitrarily. This choice enables to define a pluriharmonic mapping $h_x = i_x^{-1} \circ h_x^A$.

Any other choice of lift and apartment gives another pluriharmonic mapping $h'_x : V_x \to A$ and we have:

$$h'_x = w_a.h_x, \quad w_a \in W_{aff}$$

where W_{aff} is the Weyl group of $\Delta^{BT}(GL_N(K_v))$. Here $W_{aff} = W'_{aff} \times \mathbb{Z}$ preserves the canonical splitting, W'_{aff} is the Coxeter group of type $\widetilde{A_{N-1}}$ acting on A' preserving the triangulation described above. We have an exact sequence

$$1 \to \Lambda = \mathbb{Z}^{N-1} \to W'_{aff} \overset{Lin}{\to} W = S_N \to 1$$

and $\Lambda \times \mathbb{Z}$ acts by translation on A. The map from W'_{aff} to W is denote by Lin because it maps an affine isometry to its linear part.

In particular, $\partial h'_x = w.\partial h_x$, $w \in W$. Hence, the union of the graphs of the $\{w.\partial h_x\}_{w \in W}$ glue into a W invariant locally closed submanifold $Z_x^{1,reg} \subset \mathrm{Tot}(A_{\mathbb{C}} \otimes \Omega^1_X|_{V_x})$. Here we denote by $\mathrm{Tot}(A_{\mathbb{C}} \otimes \Omega^1_X)$ the total space of the holomorphic vector bundle $A_{\mathbb{C}} \otimes \Omega^1_X$. In fact, the $\{Z_x^{1,reg}\}_{x \in X}$ glue into a W invariant locally closed submanifold $Z^{1,reg} \subset \mathrm{Tot}(A_{\mathbb{C}} \otimes \Omega^1_X)$, where W acts on $\mathrm{Tot}(A_{\mathbb{C}} \otimes \Omega^1_X)$ by $w \otimes \mathrm{id}_{\Omega^1_X}$. The canonical projection $q : \mathrm{Tot}(A_{\mathbb{C}} \otimes \Omega^1_X) \to X$ restricted to Z yields an étale covering $q : Z^{1,reg} \to X - S$.

Proposition 3.4.1. *The closure Z^1 of $Z^{1,reg}$ is a W-invariant complex analytic subspace of* $\mathrm{Tot}(A\otimes\Omega^1_X)$ *and the natural map $q: Z^1 \to X$ identifies X with Z^1/W.*

Proof: Choose $\alpha \in A^*$. Consider $\{<\alpha, w.\partial h_x>\}_{w\in W}$. This is a collection of $|W|$ one forms on V_x. Denote by Σ_k, for $1 \le k \le |W|$, the symmetric function:

$$\Sigma_k(X_1,\dots,X_{|W|}) = \sum_{1\le i_1<\dots<i_k\le |W|} X_{i_1}\dots X_{i_k}.$$

Then $\sigma_k^x = \Sigma_k(\{<\alpha, w.\partial h_x>\}_{w\in W})$ is a well defined holomorphic k-th order symmetric differential defined on V_x. In fact, the $\{\sigma_k^x\}_{x\in X}$ glue into an element $\sigma^k_{reg} \in H^0(X-S, S^k\Omega^1_X)$.

Lemma 3.4.2. *σ^k_{reg} extends to a global holomorphic symmetric differential $\sigma^k \in H^0(X, S^k\Omega^1_X)$.*

Proof: Assume L is a global Lipschitz constant for h_v. Then, uniformly in $x \in X-S$, we have $|\sigma^k_{reg}(x)| \le C_k.L^k$ where C_k is an universal constant. As is well known, a bounded holomorphic section of a vector bundle defined on a Zariski open subset of a complex manifold extends holomorphically to its closure. □

As usual denote by $T^*_X = \mathrm{Tot}(\Omega^1_X)$ the holomorphic tangent bundle of X and by $q: T^*_X \to X$ the canonical projection. Denote by $\lambda \in H^0(T^*_X, q^*\Omega^1_X)$ the Liouville form (or tautological form). In local canonical coordinates (z^i,ζ_i), we have $\lambda = \sum_i \zeta_i dz^i$. Hence λ is holomorphic and so is:

$$\chi_\alpha = \lambda^{|W|} + \sum_{k=1}^{|W|}(-1)^k.\lambda^k\sigma_k \in H^0(T^*_X, q^*S^{|W|}\Omega^1_X).$$

We have the obvious:

Lemma 3.4.3. *Denote by $\underline{\alpha}$ the map* $\mathrm{Tot}(A_{\mathbb{C}}\otimes\Omega^1_X) \to T^*_X$ *defined by* $\alpha\otimes\mathrm{id}$. *Then,*

$$Z^{1,reg} = \cap_{\alpha\in A^*}\underline{\alpha}^{-1}(\chi_\alpha^{-1}(\{0\})) \cap q^{-1}(X-S).$$

In particular, Z^1 is the union of the irreducible components of the analytic subvariety $\cap_{\alpha\in A^*}\underline{\alpha}^{-1}(\chi_\alpha^{-1}(\{0\}))$ meeting $q^{-1}(X-S)$.

To prove that $Z^1/W = X$, consider $Z^1_\nu \to Z^1$ the normalization. It is a W-equivariant holomorphic map. Denote by $a: Z^1_\nu/W \to Z^1/W$ its quotient by W. W permutes the irreducible components of Z^1_ν since they agree with those of Z^1. Consider the natural map $b: Z^1/W \to X$. Then $b\circ a$, the natural map $Z^1_\nu/W \to X$, is a dominant holomorphic map of normal complex spaces which is an isomorphism over a dense Zariski open subset. Hence it is an isomorphism. It follows easily that b is an isomorphism. □

Since the harmonic mapping h_v is uniquely defined up to a geodesic translation, it follows that the construction of the W-equivariant covering $Z^1 \to X$ is independant of the choice of h_v.

Observe that the group morphism $h: W \to Aut(Z^1_\nu/X)$ need not be injective hence this cover need not be a W-Galois covering in general [12].

One can slightly refine the construction using an idea due to Klingler[Kli03].

Assume that the harmonic map is non degenerate which means that the image of the h_x's is not contained in a wall of the apartment A.

[12]It is a $W/\ker(h)$-Galois covering.

Lemma 3.4.4. *The data $g_{xy} = i_x^{-1} \circ i_y$ is a 1-cocycle with values in W_{aff} relative to the covering $\mathfrak{V} = \{V_x\}_{x \in X-S}$ of $X-S$. Its cohomology class in $H^1(X-S, W_{aff})$ is independant of the choices made above.*

Proof: Left to the reader. □

The cocycle $Lin(g_{xy})$ gives a cohomology class $\kappa \in H^1(X-S, W)$ hence a W-torsor $T \to X-S$, i.e.: a finite etale covering $T \to X-S$ with $W = Aut(T/X-S)$.

Lemma 3.4.5. *One can construct a W-equivariant etale covering $T \to Z^{1,reg}$ which is unique up to the action of W.*

Proof: Left to the reader. □

Definition 3.4.6. *The Galois ramified covering $sp : Z^2 = Z^1_\nu \to X$ is called the spectral covering attached to ρ_v and will be denoted by $sp : Sp_{\rho_v}(X) \to X$.*

Example 3.4.7. *Assume $N = 2$ and that the representation ρ_v has trivial determinant. In this case $\Delta^{BT}(SL_2(K_v))$ is a tree and $W = \{\pm 1\}$. The spectral covering is then completely determined by a quadratic differential $\eta = (dz)^2 \in H^0(X, S^2\Omega^1_X)$.*

The notation is meant to emphasize a slighly subtle issue. Except if $\dim(X) = 1$, *a non vanishing quadratic differential need not be locally a square but η is always of rank ≤ 1 and $sp^*\eta$ is the square of a holomorphic one form on the double covering $Sp_{\rho_v}(X) \to X$.*

If η is already the square of a holomorphic one form on X then $Sp_{\rho_v}(X)$ is the union of two copies of X.

Except in degenerate cases, the quadratic differential comes from a Riemann surface S via a map $X \to S$ see [Sim93b].

Remark 3.4.8. *The theory extends almost verbatim to more general groups than GL_N.*

3.4.2. *Holomorphic* 1*-forms on normal varieties.* A compact Kähler normal complex space is a compact normal complex space having a Kähler class in the sense of definition 5.1.2. Such a space admits a Kähler resolution of singularities.

Definition 3.4.9. *Let Z be a compact Kähler normal complex space. A holomorphic one form on Z is the data of a holomorphic one form $\alpha_{\hat{Z}} \in \Omega^1_{\hat{Z}}$ for every resolution of singularities $\hat{Z} \to Z$ subject to the constraints:*

(1) $\pi^*\alpha_{\hat{Z}} = \alpha_{\hat{Z}'}$ if $\pi : \hat{Z}' \to \hat{Z}$ is a holomorphic map of resolutions.
(2) If F is a fiber of $\hat{Z} \to Z$ then $\alpha_{\hat{Z}}|_{F^{reg}} = 0$.

The space of such one-forms will be denoted by $\Omega^1(Z)$.

Lemma 3.4.10. *Let Z be a normal compact Kähler space. The category of holomorphic map $Z \to A$ where A is a compact complex torus has an initial object $a_Z : Z \to Alb(Z)$. Furthermore $a_Z^* : H^0(Alb(Z), \Omega^1) \to \Omega^1(Z)$ is an isomorphism.*

Proof: Let $\hat{Z} \to Z$ be a resolution of singularities. Consider the largest subtorus B of $Alb(\hat{Z})$ on which vanish the forms which vanish upon restriction to the fibers of $\hat{Z} \to Z$ as considered in the second item of definition 3.4.9 and denote by $Alb(Z)$ the quotient torus $Alb(Z) = Alb(\hat{Z})/B$. Consider $a : \hat{Z} \to Alb(Z)$ the composition of the quotient map with an Albanese morphism $\hat{Z} \to Alb(\hat{Z})$. Since Z is normal every fiber F of $\hat{Z} \to Z$ is connected hence $a(F)$ is a point. In particular a descends to a continuous map $a_Z : Z \to Alb(Z)$ which is holomorphic on a dense Zariski open subset of Z. Hence a_Z is holomorphic. Its further properties are easily established. □

Remark 3.4.11. *Note that Z need not be connected. The Albanese torus of Z, $Alb(Z)$, is the product of the Albanese tori of its components and the Albanese map a_Z maps every irreducible component to the corresponding factor.*

3.4.3. *Canonical system of holomorphic* 1*-forms on the spectral covering.* For every piecewise C^1 path $\gamma : [0,1] \to X$ we denote by $\ell_{\rho_v,X}(\gamma)$ the length of the rectifiable path $h_v \circ \bar{\gamma} : [0,1] \to \Delta^{BT}(GL_N(K_v))$, where $\bar{\gamma}$ is a continuous path in $\widetilde{X^{un}}$ lifting γ. Similarily for $\gamma : [0,1] \to Sp_{\rho_v}(X)$ piecewise C^1 we define $\ell_{rho_v}(\gamma) = \ell_{\rho_v,X}(sp(\gamma))$.

Proposition 3.4.12. *The map $\Lambda : A^* \to \Omega^1(Sp_{\rho_v}(X))$ defined by $\alpha \in A^* \mapsto \underline{\alpha}^*\lambda$ where λ is the Liouville form and $\underline{\alpha}$ is defined in lemma 3.4.3 is $\mathbb{R}$-linear, W-equivariant and satisfies:*

$$\forall \gamma : [0,1] \stackrel{pw.C^1}{\longrightarrow} Sp_{\rho_v}(X) \quad \ell_{\rho_v}(\gamma) = \int_\gamma ds.$$

Here $ds = \sqrt{\sum_{i=1}^N Re(\Lambda(e_i))^2}$ where (e_i) is an orthonormal basis of W^*.

Proof: If γ lies in $Sp_{\rho_v}(X) - sp^{-1}(S)$ this is true by construction. The general case follows by approximation. □

Definition 3.4.13. *The map $\Lambda : A^* \to \Omega^1(Sp_{\rho_v}(X))$ in proposition 3.4.12 is called the canonical system of one-forms attached to ρ_v.*

3.4.4. *Katzarkov-Zuo reduction.* Denote by $\mathcal{A} \subset \Omega^1(Sp_{\rho_v}(X))$ the $\mathbb{C}$-vector space spanned by $\mathrm{Im}(A^* \to \Omega^1(Sp_{\rho_v}(X))$. Denote by B the largest subtorus of $Alb(Sp_{\rho_v}(X))$ with $TB \subset \mathcal{A}$. Obviously B is preserved by the action of W.

Consider the following commutative square:

$$\begin{array}{ccc} Sp_{\rho_v}(X) & \longrightarrow & Alb(Sp_{\rho_v}(X))/B \\ \downarrow & & \downarrow \\ X & \longrightarrow & (Alb(Sp_{\rho_v}(X)/B)/W. \end{array}$$

Definition 3.4.14. *The Katzarkov-Zuo reduction of X attached to ρ_v is the Stein factorization $s_{\rho_v} : X \to S_{\rho_v}(X)$ of the above map $X \to (Alb(Sp_{\rho_v}(X))/B)/W$.*

Proposition 3.4.15. *The Katzarkov-Zuo reduction is characterized uniquely by the following properties:*

(1) $S_{\rho_v}(X)$ is a Kähler normal complex space, projective if X is.
(2) s_{ρ_v} is a fibration, i.e.: a surjective holomorphic map with connected fibers.
(3) Let $Z \subset X$ a closed connected complex analytic subspace. Then $s_{\rho_v}(Z)$ is a point if and only if $\rho_v(\mathrm{Im}(\pi_1(Z) \to \pi_1(X)))$ is a bounded subgroup of $GL_N(K_v)$.

Proof: Easy consequence of propositions 3.2.9, 3.4.12. □

4. Reductive Shafarevich morphisms

4.1. **The rigid case.**

4.1.1. *Variants of Stein factorization.* We recall two classical lemmas to be used over and over (cf [Car]):

Lemma 4.1.1. *Let X be a compact normal complex space. Let $(f_a : X \to S_a)_{a \in A}$ a family of fibrations.*

There is a normal complex space S_∞, a fibration $f_\infty : X \to S_\infty$, and morphisms $e_a = S_\infty \to S_a$ such that:

- $f_n = e_n \circ f_\infty$.
- *For all* $s \in S$ $f_\infty^{-1}(s) = \cap_{n \in \mathbb{N}} f_n^{-1}(e_n(s))$.

We call f_∞ the simultaneous Stein factorization of $(f_a)_{a \in A}$.

Lemma 4.1.2. *Let X, S be complex spaces and $f : X \to S$ be a morphism. Suppose a connected component F of a fibre of f is compact. Then, F has an open neighborood V such that $g(V)$ is a locally closed analytic subvariety S and $V \to g(V)$ is proper.*

Suppose furthermore that X is normal and that every connected component F of a fibre of f is compact. The set of connected components of fibres of f can be endowed with the structure of a normal complex space so that the natural map $e : X \to S$ is a proper holomorphic fibration.

4.1.2. *The Shafarevich morphism of a rigid representation.* Let $G/\mathbb{Q}$ be a reductive algebraic group and $\rho : \pi_1(X) \to G(\mathbb{C})$ be a rigid reductive representation, i.e. : such that $[\rho] \in M_B(X, G)(\mathbb{C})$ is an isolated point. Fix $\alpha : G \to GL_N$ a faithful linear representation defined over $\mathbb{Q}$.

Then the following holds:

(1) There is a number field $\mathbb{Q} \subset L \subset \bar{\mathbb{Q}}$ such that ρ is conjugate to $\rho' : \pi_1(X) \to G(L)$ [Rag72, Prop 6.6 p.90]. Consequently there is a finite set S of non-archimedian places[13] v of L such that $\rho'\pi_1(X) \subset GL_N(O_v)$ iff $v \notin S$.

(2) For every archimedian place [14] w of L $w \circ \rho'$ is the monodromy representation of a $\mathbb{C}$-VHS by Simpson's ubiquity theorem.

Let O_S be the localization of O_L at S. We have $\alpha \circ \rho'(\pi_1(X)) \subset GL_N(O_S)$. It is a well known theorem of Borel and Serre that $GL_N(O_S)$ is a discrete subgroup of $\prod_{w \in Ar(L)} GL_N(L_w) \times \prod_{v \in S} GL_N(L_v)$. We denote by $P : G(L) \to \prod_{w \in Ar(L)} GL_N(L_w)$ the product $P = \prod_{w \in Ar(L)} w \circ \alpha$.

Lemma 4.1.3. *Let F be a connected fiber of the Stein factorization of*

$$\prod_{v \in S} s_{\rho_v} : X \longrightarrow \prod_{v \in S} S_{\rho_v}(X).$$

Then $P : \rho'(\pi_1(F)) \to \prod_{w \in Ar(L)} GL_N(L_w)$ has finite kernel and its image is a discrete subgroup.

Proof: Indeed for $v \in S$, $\mathrm{Im}(\alpha \circ \rho' : \pi_1(F) \to GL_N(L_v))$ is contained in a compact subgroup. □

[13]A non-archimedian place of L is a prime ideal of the ring O_L of algebraic integers in L. We denote by L_v the corresponding locally compact local field and by v the corresponding valuation.

[14]An archimedian place of L is a real embedding of L or a complex embedding up to conjugacy. We set $L_w = \mathbb{R}$ or $\mathbb{C}$ accordingly. The set of archimedian places will be denoted by $Ar(L)$.

Denote by $\pi_\rho : \widetilde{X_\rho} = \widetilde{X^{un}}/\ker(\rho) \to X$ the intermediate covering correponding to ρ. For $w \in Ar(L)$ denote by $p_w : \widetilde{X_\rho} \to D_w$ the period mapping attached to the $\mathbb{C}$-VHS whose holonomy is $w \circ \rho'$.

Corollary 4.1.4. *The map* $\prod_{w\in Ar(L)} p_w : \pi_\rho^{-1}(F) \to \prod_{w\in Ar(L)} D_w$ *is proper.*

Proof: Immediate consequence of Lemma 4.1.3. □

Define a holomorphic mapping

$$q = \prod_{w\in Ar(L)} p_w \times \prod_{v\in S} s_{\rho_v} : \widetilde{X_\rho} \to \prod_{w\in Ar(L)} D_w \times \prod_{v\in S} S_{\rho_v}(X).$$

Corollary 4.1.5. *Every connected component of a fiber of q is compact.*

Proof: Indeed it is contained in $\pi_\rho^{-1}(F)$ with F as in lemma 4.1.3. □

Proposition 4.1.6. *There is a proper fibration $\tilde{s}_\rho : \widetilde{X_\rho} \to \widetilde{S_\rho}$ which contracts every compact connected analytic subspace of $\widetilde{X_\rho}$ to a point.*

Proof: This map is what one gets applying lemma 4.1.2 to q. □

Observe that $\rho(\pi_1(X)) = Gal(\widetilde{X_\rho}/X)$ acts properly on $\widetilde{S_\rho}$ and that we can construct the quotient space $Sh_\rho(X) = \widetilde{S_\rho}/\rho(\pi_1(X))$.

Proposition 4.1.7. *The fibration $sh_\rho : X \to Sh_\rho(X)$ contracts $Z \subset X$ to a point if and only if $\rho(\pi_1(Z))$ is finite.*

Remark 4.1.8. *$Sh_\rho(X)$ is a normal Kähler space, projective if X is.*

4.2. **The non-rigid case.**

Theorem 4.2.1. *Fix $N \in \mathbb{N}$. Consider $M \subset M_B(X, GL_N)(\mathbb{C})$ a set of conjugacy classes of reductive linear representations.*

Let $\widetilde{X_M} = \widetilde{X^{un}}/H_M$ be the covering space corresponding to the intersection H_M of kernels of all elements in M.

There is a proper holomorphic fibration $\widetilde{X_M} \to \widetilde{S_M}$ where $\widetilde{S_M}$ is a normal complex space with no positive dimensional compact complex analytic subspaces.

The quotient of $\widetilde{S_M}$ by the proper action induced from that of $Gal(\widetilde{X_M}/X) = \pi_1(X)/H_M$ is a normal Kähler space $sh_M(X)$, which is projective if X is.

The natural holomorphic map $X \to sh_M(X)$ is called the Shafarevich morphism attached to M.

The general strategy of the proof will be described in the rest of this section. It is analogous to the rigid case using representations with values in algebraic groups over function field to deal with non rigid representations.

4.2.1. *Factoring out the non-rigidity.* We circumvent the unavailability of theorem 3.3.7 in the non locally compact case by reduction modulo p and get another version of the Katzarkov-Zuo reduction.

Lemma 4.2.2. *Let L be a number field. Consider the field $L((t))$ endowed with its natural valuation. Let $\rho : \pi_1(X) \to GL_N(L((t)))$ be a reductive representation.*

Then there is a holomorphic fibration $X \to S_\rho(X)$ such that:

(1) $S_\rho(X)$ is a Kähler normal complex space, projective if X is.

(2) *Let $Z \subset X$ a closed connected complex analytic subspace. Then $s_\rho(Z)$ is a point if and only iff $\rho(\mathrm{Im}(\pi_1(Z) \to \pi_1(X)))$ is a bounded subgroup of $GL_N(L((t)))$.*

Proof: Since $\pi_1(X)$ is finitely generated there exists a finite set S of non-archimedian places of L such that $\rho(\pi_1(X)) \subset GL_N(O_S((t)))$. For $v \notin S$, one has $O_S \subset O_v$ and we can construct by reduction modulo v a representation $\rho_v : \pi_1(X) \to GL_N(k_v((t)))$ where k_v is the residue field of L_v. Enlarging S if necessary we can assume that ρ_v is reductive for $v \notin S$. Since $k_v((t))$ is a locally compact local field one can construct the Katzarkov Zuo reduction $s_{\rho_v} : X \to S_{\rho_v}(X)$. The simultaneous Stein factorization of the family $\{s_{\rho_v}\}_{v \notin S}$ then satisfies the required properties.

□

Lemma 4.2.3. *Let $T/\bar{\mathbb{Q}}$ be an irreducible algebraic variety over $\bar{\mathbb{Q}}$ and consider a rational map $r : T \to M_B(X, GL_N)/\bar{\mathbb{Q}}$ and let $\rho_T : \pi_1(X) \to GL_N(\bar{\mathbb{Q}}(T))$ be a reductive representation in the function field of T whose conjugacy class is a generic point of the image of r . Then there is a holomorphic fibration $s_T : X \to S_T(X)$ with the following property*

(1) *$S_T(X)$ is a Kähler normal complex space, projective if X is.*
(2) *Let $Z \subset X$ a closed connected complex analytic subspace. Then $s_T(Z)$ is a point if and only iff the rational map $T \to M_B(Z, GL_N)$ is locally constant.*

Proof:

Assume first $T/\bar{\mathbb{Q}}$ be a complete connected algebraic curve. There is a finite set S of $\bar{\mathbb{Q}}$-points P of T such that if $P \notin S$, $\rho_T(\pi_1(X)) \in GL_N(O_P)$ and if $P \in S$, the localization of ρ_T at P denoted by $\rho_{T,P} : \pi_1(X) \to GL_N(Frac(\widehat{O_P}))$ is a reductive representation. For all P, $Frac(\widehat{O_P})$ is isomorphic to $\bar{\mathbb{Q}}((t))$ and we can once again perform the simultaneous Stein factorisation of the $s_{\rho_{T,P}}$ to get a fibration $s_T : X \to S_T(X)$. Let Z be a fiber of S_T. Since for all $P \in S$ $\rho_{T,P}(\pi_1(Z))$ is bounded, it follows that for all $\gamma \in \pi_1(Z)$ $Tr(\rho_{T,P}(\gamma)) \in \widehat{O_P}$. This implies that $Tr(\rho_T(\gamma)) \in \bar{\mathbb{Q}}((T))$ is a regular function at every point $P \in T(\bar{\mathbb{Q}})$. It follows that $Tr(\rho_T(\gamma))$ is regular function on T, hence a constant since T is complete. By a theorem of Procesi, $\{Tr(\rho(\gamma))\}_{\gamma \in \pi_1(Z)}$ generate the coordinate ring of the affine scheme $M_B(Z, GL_N)/\bar{\mathbb{Q}}$. Hence, indeed $T \mapsto M_B(Z, GL_N)$ is constant. Conversely if this map is constant, performing a finite cover of T if necessary, we may assume that $\rho_T|_{\pi_1(Z)}$ is conjugate to a constant representation and certainly $\rho_{T,P}$ will be bounded at every point $P \in T(\bar{\mathbb{Q}})$. This forces $s_T(Z)$ to be a point.

The general case follows by simultaneous Stein factorization of the s_C, C being a curve over $\bar{\mathbb{Q}}$ mapping to T.

□

4.2.2. *Case where M is a $\bar{\mathbb{Q}}$-point.* We now prove theorem 4.2.1 when $M = \{[\rho]\}$ with $\rho : \pi_1(X) \to G(L)$ where L is a number field.

For every complex embedding $w \in Ar(L)$, we can deform $w \circ \rho$ to a $\mathbb{C}$-VHS that will be denoted by ρ_w^{VHS} and we may construct $p_w : \widetilde{X^{un}} \to D_w$ its period mapping. Then the holomorphic map $\prod_w p_w : \widetilde{X^{un}} \to \prod_w D_w$ descends to the intermediate covering space with Galois group Γ_ρ^+ defined by:

$$X_\rho^+ = \widetilde{X^{un}} / \ker(\rho) \cap \cap_{w \in Ar(L)} \ker(\rho_w^{VHS}).$$

Consider the collection $\mathcal{Z}$ of compact Kähler manifold over X, $f : Z \to X$ such that $\rho(f_* \pi_1(Z)) = \{e\}$ and let H be the normal subgroup of $\pi_1(X)$ generated by the $\{f_*\pi_1(Z)\}_{Z\in\mathcal{Z}}$. Let $Q^1 \subset R(\pi_1(X), GL_N)$ be the preimage of the origin by the natural morphism $R(\pi_1(X), GL_N) \to R(H, GL_N)$ and $Q = Q^1//GL_N \in M_B(X, GL_N)$. Then, for every component T of Q, we have the fibration $s_T : X \to S_T(X)$ constructed in lemma 4.2.3.

There is a finite set S of non-archimedian places v of L such that $\rho_v(\pi_1(X))$ is unbounded.

We consider the holomorphic map defined on X_ρ^+ by

$$q = \prod_{w\in Ar(L)} p_w \times \prod_T s_T \circ \pi_\rho^+ \times \prod_{v\in S} s_{\rho_v} \circ \pi_\rho^+.$$

The argument in section 4.1.2 can be done with q and we get a fibration $\tilde{X}_\rho^+ \to \widetilde{S_\rho^+}$ where $\widetilde{S_\rho^+}$ has no positive dimensional compact complex subspace (one should observe that $\rho_w^{VHS} \in Q^1$, see [Eys04].).

Now we quotient by $\Gamma^+(\rho)$ and get a map $s_\rho^+ : X \to S_\rho^+/\Gamma_\rho^+$. One then uses once again the same trick with the holomorphic map q defined on $\widetilde{X_\rho} = \widetilde{X^{un}}/\ker(\rho)$ by $q = s_\rho^+ \circ \pi_\rho$. This produces a proper fibration $\tilde{s}_\rho : \widetilde{X_\rho} \to \widetilde{S_\rho}$ where $\widetilde{S_\rho}$ has no positive dimensional compact complex subspace.

4.2.3. *General case.* The general case of theorem 4.2.1 is given in [Eys04] and relies on [LasRam96]. The idea is to replace M with M' the $\bar{\mathbb{Q}}$-points of its $\bar{\mathbb{Q}}$-Zariski closure and use the $\{\tilde{s}_\rho\}_{\rho\in M'}$ as above.

5. Reductive Shafarevich conjecture

5.1. **Psh functions attached to pluriharmonic mappings.** In order to prove that the normal Stein space $\widetilde{S_M}$ constructed in theorem 4.2.1 is Stein under some restrictions on M, we need to construct a strongly plurisubharmonic continuous exhaustion function on this space.

On the other hand the theory of pluriharmonic mappings provides continuous weakly plurisubharmonic functions thanks to the following principle: the pull-back of a convex function by a (pluri)harmonic mapping is (pluri)subharmonic. It should be noted that NPC spaces carry several convex functions. For instance, the distance to a fixed point is convex.

Hence, one needs to systematically study the degeneracies of the complex hessian of the plurisubharmonic functions constructed by the method of harmonic mappings. It turns out that nice lower bounds for the complex hessian are satified by plurisubharmonic functions arising from archimedian and non archimedian representations of Kähler groups. Let us now give more details on these constructions.

5.1.1. *The non archimedian case.* Let $\rho_v : \pi_1(X) \to GL_N(K_v)$ be a reductive representation in a local field and assume that Theorem 3.3.7 applies. Denote by $h_v : \widetilde{X^{un}} \to \Delta^{BT}(GL_N(K_v))$ the corresponding pluriharmonic mapping. Consider $\Lambda : A^* \to \Omega^1(Sp_{\rho_v}(X))$ the canonical system of one forms attached to ρ_v. Let (e_i) be an orthonormal basis of A^*. The expression:

$$\omega_{0,\rho_v} = \frac{\sqrt{-1}}{2} \sum_{i=1}^N \Lambda(e_i) \wedge \overline{\Lambda(e_i)}$$

defines a closed real semipositive $(1,1)$ form on the smooth locus of $Sp_{\rho_v}(X)$ and is W-invariant. It therefore descends to a closed real semipositive $(1,1)$ form ω_{1,ρ_v} on $X - S$.

The Lipschitz property of h_v implies that ω_{1,ρ_v} is uniformly bounded hence extends to a closed positive current on X using for instance the Skoda-El Mir theorem. In fact, it is natural to expect that it descends to a closed positive current on the Katzarkov-Zuo reduction of X. Since this Katzarkov-Zuo reduction needs not be smooth, we first have to make sense of the notion of a closed positive current on it! This is not so well known but very easy:

Definition 5.1.1. *Let Z be an irreducible normal complex space.*

A upper semi continuous function $\phi : Z \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if it is not identically $-\infty$ and every point $z \in Z$ has a neighborhood U embeddable as a closed subvariety of the unit ball B of some $\mathbb{C}^M$ in such a way that $\phi|_U$ extends to a psh function on B.

A closed positive current with continuous potentials ω on Z is specified by a data $\{U_i, \phi_i\}_i$ of an open covering $\{U_i\}_i$ of Z, a continuous psh function ϕ_i defined on U_i such that $\phi_i - \phi_j$ is pluriharmonic on $U_i \cap U_j$.

A closed positive current with continuous potentials Z is a Kähler form iff its local potentials can be chosen smooth and strongly plurisubharmonic.

A psh function ϕ on Z is said to satisfy $dd^c\phi \geq \omega$ iff $\phi - \phi_i$ is psh on U_i for every i.

In other words, a closed positive current with continuous potentials is a section of the sheaf $C^0 \cap PSH_Z/Re(O_Z)$.

Definition 5.1.2. *Assume Z to be compact. The class of a closed positive current with continuous potentials is its image in $H^1(Z, Re(O_Z))$.*

A class in $H^1(Z, Re(O_Z))$ is said to be Kähler if it is the image of a Kähler form.

To make contact with the usual terminology observe that if Z is a compact Kähler manifold $H^1(Z, Re(O_Z)) = H^{1,1}(Z, \mathbb{R})$.

Lemma 5.1.3. *There is a closed positive current with continuous potentials on $S_{\rho_v}(X)$ which we denote by ω_{ρ_v} such that $s^*_{\rho_v}\omega_{\rho_v}|_{X-S} = \omega_{1,\rho_v}$.*

Proof: Left as an easy exercise, see [Eys04]. □

The current ω_{ρ_v} will serve as a lower bound for the complex hessian of plurisubharmonic functions constructed by the method of harmonic mappings. Let us be more precise. Let $x_0 \in \Delta^{BT}(GL_N(K_v))$ be an arbitrary point.

Proposition 5.1.4. *The function $\phi_{\rho_v} : \widetilde{X^{un}} \to \mathbb{R}_{\geq 0}$ defined by*

$$\phi_{\rho_v}(x) = d^2(h_v(x), x_0)$$

satisfies the following properties:

1. *ϕ_{ρ_v} descends to a function $\phi^*_{\rho_v}$ on $\widetilde{X_{\rho_v}} = \widetilde{X^{un}}/\ker(\rho_v)$.*
2. *Let $\pi : \widetilde{X^{un}} \to \widetilde{X} \to \widetilde{X}_{\rho_v}$ be a covering space of X dominating $\widetilde{X}_{\rho_v}$. By abuse of notation we denote by ϕ_{ρ_v} the function $\phi^*_{\rho_v} \circ \pi$.*
3. *ϕ_{ρ_v} is locally Lipschitz.*
4. *ϕ_{ρ_v} is psh.*

(5) *Let T be a normal complex space and $\widetilde{X} \xrightarrow{r} T$ a proper holomorphic fibration such that $s_{\rho_v} \circ \pi : \widetilde{X} \to S_{\rho_v}(X)$ factorizes via a morphism $T \xrightarrow{\phi_T} S_\rho(X)$. The function $\phi_{\rho_v} \circ \pi$ is of the form $\phi_{\rho_v} = \phi^T_{\rho_v} \circ r$, ϕ^T_ρ being a continuous weakly psh function on T.*

(6) *Furthermore, $\phi^T_{\rho_v}$ satisfies $dd^c \phi^T_{\rho_v} \geq \phi_T^* \omega_{\rho_v}$.*

Proof: For complete details on this plausible statement see [Eys04]. □

5.1.2. *VHS case.* Let $\rho_w : \pi_1(X) \to GL_N(\mathbb{C})$ be a reductive representation which appears as the monodromy of a $\mathbb{C}$-VHS. We may assume that $\rho_w(\pi_1(X)) \subset U(P_w, Q_w)$ with $P_w + Q_w = N$. Then the pluriharmonic mapping $h_w : \widetilde{X^{un}} \to R_w = Riem(U(P_w, Q_w))$ factors as $P \circ p_w$ where $P : D_w = U(P_w, Q_w)/U \to R_w$ is the natural projection.

Lemma 5.1.5. *The plurisubharmonic function $\phi_{\rho_w} : \widetilde{X^{un}} \to \mathbb{R}_{\geq 0}$ defined by*

$$\phi_{\rho_w}(x) = d^2(h_w(x), x_0)$$

descends to a smooth plurisubharmonic function on the space $\widetilde{S_{\rho_w}}$ which appears in the proper holomorphic fibration $\widetilde{X_{\rho_w}} = \widetilde{X^{un}}/\ker(\rho_w) \to \widetilde{S_{\rho_w}}$ constructed in theorem 4.2.1.

There is a smooth holomorphic hermitian line bundle (L, h) on the Shafarevich variety $Sh_{\rho_w}(X) = \widetilde{S_{\rho_w}}/\Gamma_w$ with a semi positive curvature current ω_{ρ_w} such that, denoting by $\pi_{\rho_w} : \widetilde{S_{\rho_w}} \to Sh_{\rho_w}(X)$ the natural quotient map, there exists $C > 0$ such that $dd^c \phi_{\rho_w} \geq C.\pi^*_{\rho_w} \omega_{\rho_w}$.

Proof: See [Eys04]. However the relevant computation was already made in [GriSch69], in fact (L, h) is a pull-back of a holomorphic hermitian line bundle on D_w with positive curvature along the horizontal directions. . □

5.1.3. *A sufficient condition for $\widetilde{S_M}$ to be Stein.* Consider $M \subset M_B(X, GL_N)(\mathbb{Q})$ a set of conjugacy classes of reductive linear representations. Assume that M is constructible for the $\bar{\mathbb{Q}}$-Zariski topolology. Let $\widetilde{X_M}$ the covering space corresponding to the intersection H_M of kernels of all elements in M. Theorem 4.2.1 constructs a proper holomorphic fibration $\widetilde{X_M} \to \widetilde{S_M}$.

Definition 5.1.6. *A representation of type I attached to M is a reductive $\rho_v : \pi_1(X) \to GL_N(K_v)$ where K_v is a local field obtained by one of the following constructions:*

- *Localization at a non archimedian place of a number field L of a representative of an element of M.*
- *Reduction modulo p of the localization at a point of an affine connected curve $T/\bar{\mathbb{Q}}$ of a representation $\rho_T : \pi_1(X) \to GL_N(\bar{\mathbb{Q}}(T))$ such that the rational map $T \to M_B(X, GL_N)$ maps T into M.*

A representation of type II attached to M is a reductive $\rho_w : \pi_1(X) \to GL_N(\mathbb{C})$ which occurs as the holonomy of a $\mathbb{C}$-VHS obtained by deforming $w \circ \rho$ where w is a real or complex embedding of a number field L and $\rho : \pi_1(X) \to GL_N(L)$ is a reductive representation whose class lies in M.

Proposition 5.1.7. *If the convex hull of the $[\omega_\rho] \in H^1(Sh_M(X), Re(O_{Sh_M(X)}))$ for ρ attached to M contains a Kähler class then $\widetilde{S_M}$ is Stein.*

Proof: Easy. See [Eys04]. □

5.2. **Absolute constructibility.** Assume now that compact Kähler manifold X is projective algebraic and defined over a subfield $\ell \subset \mathbb{C}$.

Let G be an algebraic reductive group defined over $\overline{\mathbb{Q}}$. The *representation scheme* of $\pi_1(X,x)$ is an affine $\overline{\mathbb{Q}}$-algebraic scheme described by its functor of points:

$$R(\pi_1(X,x),G)(\mathrm{Spec}(A)) := \mathrm{Hom}(\pi_1(X,x),G(A))$$

for any $\overline{\mathbb{Q}}$ algebra A. The *character scheme* of $\pi_1(X,x)$ with values in G is the affine scheme

$$M_B(X,G) = R(\pi_1(X,x),G)//G.$$

Character schemes of fundamental groups of complex projective manifolds are rather special. Simpson constructed two additional quasi-projective schemes over ℓ, $M_{DR}(X,G)$ and $M_{Dol}(X,G)$. The $\mathbb{C}$-points of $M_{DR}(X,G)$ are in bijection with the equivalence classes of flat G-connections with reductive monodromy, and the $\mathbb{C}$-points of $M_{Dol}(X,G)$ are in bijection with the isomorphism classes of polystable G-Higgs G-bundles with vanishing first and second Chern class. Whereas the notion of a polystable Higgs bundle depends on the choice of a polarization on X the moduli space $M_{Dol}(X,G)$ does not, i.e. - all moduli spaces one constructs for the different polarizations are naturally isomorphic. $M_{Dol}(X,G)$ is acted upon algebraically by the multiplicative group. There is furthermore a complex analytic biholomorphic map

$$RH : M_B(X,G)(\mathbb{C}) \to M_{DR}(X,G)(\mathbb{C})$$

and a real analytic homeomorphism

$$KH : M_B(X,G)(\mathbb{C}) \to M_{Dol}(X,G)(\mathbb{C}).$$

RH and KH are also indpendant of the choice of a Kähler metric. When $l = \overline{\mathbb{Q}}$, one defines an *absolute constructible subset* of $M_B(X,G)(\mathbb{C})$ to be a subset M such that:

- M is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_B(X,G)$,
- $RH(M)$ is the set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{DR}(X,G)$,
- $KH(M)$ is a $\mathbb{C}^*$-invariant set of complex points of a $\overline{\mathbb{Q}}$-constructible subset of $M_{Dol}(X,G)$.

There is a rich theory describing the structure of absolutely constructible subsets in $M_B(X,G)$. Here we briefly summarize only those properties of absolutely constructible sets that we will need later. Full proofs and details can be found in [Sim93a].

- The full moduli space $M_B(X,G)$ of representations of $\pi_1(X,x)$ in G defined in [Sim94] is absolutely constructible and quasi compact (acqc).
- The closure (in the classical topology) of an acqc subset is also acqc.
- Whenever ρ is an isolated point in $M_B(X,G)$, $\{\rho\}$ is acqc.
- Absolute constructibility is invariant under standard geometric constructions. For instance, for any morphism $f : Y \to X$ of smooth connected projective varieties, the property of a subset being absolutely constructible is preserved when taking images and preimages via $f^* : M_B(X,G) \to M_B(Y,G)$. Similarly, for any homomorphism $\mu : G \to G'$ of reductive groups, taking images and preimages under $\mu_* : M_B(X,G) \to M_B(X,G')$ preserves absolute constructibility.

- Given a dominant morphism $f : Y \to X$ and $i \in \mathbb{N}$ the set $M^i_f(X, GL_n)$ of local systems V on Y such that $R^i f_* V$ is a local system is ac. Also, taking images and inverse images under $R^i f_* : M^i_f(X, GL_n) \to M_B(Y, GL_{n'})$ preserves acqc sets.
- The complex points of a closed acqc set M are stable under the natural $\mathbb{C}^*$ action. As seen before, the fixed point set $M^{\mathrm{VHS}} := M^{\mathbb{C}^*}$ consists of representations underlying polarizable complex Variations of Hodge structures.

The main theorem of [Eys04] is:

Theorem 5.2.1. *Let X be a connected complex projective manifold and $M \subset M_B(X, GL_N(\bar{\mathbb{Q}}))$ an absolute constructible subset. Then $\widetilde{X}_M$ is holomorphically convex.*

Theorem 5.2.1 follows via Proposition 5.1.7 from:

Proposition 5.2.2. *The convex hull of the $[\omega_\rho] \in H^1(Sh_M(X), Re(O_{Sh_M(X)}))$ for ρ attached to M contains a Kähler class.*

5.3. **Rigid integral case.** A representation of a group in $G(\bar{\mathbb{Q}})$ where $G/\bar{\mathbb{Q}}$ is a linear algebraic group is *integral* if for every embedding of G in GL_N its image is conjugate to a subgroup of $GL_N(\bar{\mathbb{Z}})$

Assume $M = \{[\rho]\}$ where $\rho : \pi_1(X) \to G(L)$ is rigid and integral. Hence the non trivial associated representations are all of type II. Then the group $\Gamma_\rho = \rho(\pi_1(X))$ acts discretely on $\prod_{w \in Ar(L)} D_w$ and $sh_\rho : X \to Sh_\rho(X)$ is the Stein factorization of the holomorphic map $p_\rho : X \to \prod_{w \in Ar(L)} D_w/\Gamma_\rho$. This determines a finite holomorphic map $\sigma_\rho : Sh_\rho(X) \to \prod_{w \in Ar(L)} D_w/\Gamma_\rho$ such that $p_\rho = \sigma_\rho \circ sh_\rho$. For every $Z \subset Sh_\rho(X)$ a closed complex subvariety the map $\sigma_\rho|_Z$ is finite onto its image hence generically immersive and horizontal.

Since the line bundle L is a tensor product of line bundles on the D_w that have positive horizontal curvature, it follows that $(\sum_w \omega_{\rho_w})^{\dim(Z)}.Z > 0$.

In particular $[\sum_w \omega_{\rho_w}]$ is a Kähler form by [DemPau04] -which holds true with singularities.

5.4. **Conclusion of the proof of theorem 5.2.1.** As the preceding section suggests, representations of type II attached to M are easily dealt with. The main problem is to show that absolute constructibility entails enough representations of type I.

Consider $sh^I_M : X \to Sh^I_M(X)$ the simultaneous Stein factorization of the representations of type I attached to M. One has a factorization $sh^I_M = \psi \circ sh_M$ and for every representation of type I

$$H^1(Sh_M(X), Re(O_{Sh_M(X)}) \ni [\omega_{\rho_v}] = \psi^*[\omega'_{\rho_v}] \quad [\omega'_{\rho_v}] \in H^1(Sh^I_M(X), Re(O_{Sh^I_M(X)})).$$

Theorem 5.2.1 follows then in the now familiar way of:

Proposition 5.4.1. *Let $M \subset M_B(X, GL_N)(\bar{\mathbb{Q}})$ be absolute constructible. Then, the convex hull of the $[\omega'_{\rho_v}] \in H^1(Sh^I_M(X), Re(O_{Sh^I_M(X)})$ contains a Kähler form.*

Remark 5.4.2. *This is true if $N = 1$. In that case, representations of type II are unitary hence give trivial information for our purpose. In that case, we have $Sh^I_M = Sh_M$.*

The proof is non trivial and relies on an idea of [Sim93b] pushed further in [JosZuo96],[JosZuo00].

6. Linear Shafarevich conjecture

6.1. **Variations of Mixed Hodge Structure.** To treat linear groups, one has to take into account non reductive linear representations of the fundamental group. These do not posses pluriharmonic mappings but a privileged subclass of local systems does have holomorphic period mappings: the Variations of Mixed Hodge structures introduced in [StZ85] [Usu83].

Definition 6.1.1. *A $\mathbb{C}$-VMHS on X is a 6-tuple $(X, \mathbb{V}, \mathbb{W}_\bullet, \mathcal{F}^\bullet, \overline{\mathcal{G}}^\bullet, (S_k)_{k\in\mathbb{Z}})$ where:*

(1) $\mathbb{V}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces,
(2) $\mathbb{W}_\bullet = (\mathbb{W}_k)_{k\in\mathbb{Z}}$ is a decreasing filtration of $\mathbb{V}$ by local subsystems,
(3) $\mathcal{F}^\bullet = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_X$ by locally free coherent analytic sheaves such that $d'\mathcal{F}^p \subset \mathcal{F}^{p-1}\otimes\Omega^1_X$,
(4) $\overline{\mathcal{G}}^\bullet = (\overline{\mathcal{G}}^q)_{q\in\mathbb{Z}}$ a biregular decreasing filtration of $\mathbb{V}\otimes_{\mathbb{C}}\mathcal{O}_{\bar{X}}$ by locally free coherent antianalytic sheaves such that $d''\overline{\mathcal{G}}^p \subset \overline{\mathcal{G}}^{p-1}\otimes\Omega^1_{\bar{X}}$,
(5) $\forall x\in X$ the stalk $(\mathbb{V}_x, \mathbb{W}_{\bullet,x}, \mathcal{F}^\bullet_x, \overline{\mathcal{G}}^\bullet_x)$ is a $\mathbb{C}$-MHS,
(6) S_k is flat sesquilinear non degenerate pairing on $Gr^{\mathbb{W}}_k\mathbb{V}$,
(7) $(X, Gr^{\mathbb{W}}_k\mathbb{V}, Gr_k^{\mathbb{W}\otimes_{\mathbb{C}}\mathcal{O}_X}\mathcal{F}^\bullet, Gr_k^{\mathbb{W}\otimes_{\mathbb{C}}\mathcal{O}_{\bar{X}}}\overline{\mathcal{G}}^\bullet, S_k)$ is a $\mathbb{C}$-VHS.

6.2. **Goldmann-Millson's theorem and Hodge Theory.** In this paragraph, we review the construction of [EysSim09] which develops some Hodge theoretic aspects of Goldman-Millson's theory of deformations for representations of Kähler groups [GolMil88].

Fix $N\in\mathbb{N}$ and assume that $G = GL_N$ and $M = M_B(X, GL_N)$. Let $\rho : \pi_1(X,x)\to GL_N(\mathbb{C})$ be the monodromy represention of a $\mathbb{C}$-VHS. Let $\hat{\mathcal{O}}_\rho$ be the complete local ring of $[\rho]\in R(\pi_1(X,x), GL_N)(\mathbb{C})$. Let

$$\mathsf{obs}_2 = [-;-] : S^2H^1(X, \mathrm{End}(\mathbb{V}_\rho)) \to H^2(X, \mathrm{End}(\mathbb{V}_\rho))$$

be the Goldman-Millson obstruction to deforming ρ. Define $I_2, (I_n)_{n\geq 2}, (\Pi_n)_{n\geq 0}$, as follows:

$$\begin{aligned}
\Pi_0 &= \mathbb{C}\\
\Pi_1 &= H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
I_2 &= \mathrm{Im}({}^t\mathsf{obs}_2) \subset S^2H^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
I_n &= I_2S^{n-2}H^1(X, \mathrm{End}(\mathbb{V}_\rho))^* \subset S^nH^1(X, \mathrm{End}(\mathbb{V}_\rho))^*\\
\Pi_n &= S^nH^1(X, \mathrm{End}(\mathbb{V}_\rho))^*/I_n
\end{aligned}$$

Then the complete graded local $\mathbb{C}$-algebra

$$(\widehat{\mathcal{O}}_T, \mathfrak{m}) := (\widehat{\oplus_{n\geq 0}\Pi_n, \oplus_{n\geq 1}}\Pi_n)$$

is the function algebra of a formal scheme T which is the germ at 0 of the quadratic cone

$$\underline{\mathsf{obs}_2}^{-1}(0) \subset H^1(X, \mathrm{End}(\mathbb{V}_\rho)).$$

Here $\underline{\mathsf{obs}_2}$ is the quadratic $H^2(X, \mathrm{End}(\mathbb{V}_\rho))$-valued function defined on $H^2(X, \mathrm{End}(\mathbb{V}_\rho))$ by obs_2. In [GolMil88], the formal local scheme T is realized as a hull of the deformation functor for ρ. They construct in effect a representation $\rho_T^{GM} : \pi_1(X,x)\to GL_N(\widehat{\mathcal{O}}_T)$ such that:

(1) $\rho_T^{GM} \cong \rho \bmod \mathfrak{m}$,
(2) $\rho_T^{GM} \bmod \mathfrak{m}^2$ is the universal first order deformation of ρ,

(3) For every local Artin ring $(A, \mathfrak{m})$ with $A/\mathfrak{m} = \mathbb{C}$ and every $\rho_A : \pi_1(X, x) \to GL_N(A)$ such that $\rho_A \cong \rho \mod \mathfrak{m}$ there is a ring morphism $\psi : \widehat{\mathcal{O}}_T \to A$ such that $\rho_A = \psi(\rho_T^{GM})$.

This hull is well defined up to automorphism but not up to a unique isomorphism.

We can now summarize the results developped by [EysSim09] in the form we shall need:

Definition 6.2.1. *Let $\eta_1, \dots, \eta_b \in E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$ form a basis of the subspace $\mathcal{H}^1(X, \mathrm{End}(\mathbb{V}_\rho))$ of harmonic twisted one forms, each η_i being of pure Hodge type (P_i, Q_i) for the Deligne-Zucker $\mathbb{C}$-Mixed Hodge Complex $E^\bullet(X, \mathrm{End}(\mathbb{V}_\rho))$. Then $\{\eta_i\}$ is a basis of $H^1(X, End(\mathbb{V}_\rho))$ whose dual basis we denote by $(\{\eta_1\}^*, \dots, \{\eta_b\}^*)$.*

The $\mathrm{End}(\mathbb{V}_\rho) \otimes \Pi_1$-valued one-form α_1 is defined by the formula:

$$\alpha_1 = \sum_{i=1}^{b} \eta_i \otimes \{\eta_i\}^*.$$

Proposition 6.2.2. *For $k \geq 2$, we can construct a unique D''-exact form $\alpha_k \in E^1(X, \mathrm{End}(\mathbb{V}_\rho)) \otimes \Pi_k$ such that the following relation holds:*

$$D'\alpha_k + \alpha_{k-1}\alpha_1 + \alpha_{k-2}\alpha_2 + \ldots + \alpha_1\alpha_{k-1} = 0.$$

Proposition 6.2.3. *Let $A = \sum \alpha_k$ acting on the vector bundle underlying the filtered local system $(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T, W_k(\mathbb{V}_\rho \otimes \hat{\mathcal{O}}_T) = \mathbb{V}_\rho \otimes \mathfrak{m}^{k - wght(\mathbb{V}_\rho)})$, whose connection will be denoted by D.*

Then, $D + A^v$ respects this weight filtration, satisfies Griffiths' transversality for the Hodge filtration $\mathcal{F}^\bullet$ defined by

$$\mathcal{F}^p = \bigoplus_{k=-\infty}^{0} \mathcal{F}^p(\mathbb{V}_\rho \otimes \Pi_{-k})$$

and we can construct an anti-Hodge filtration so that the resulting structure is a graded polarizable $\mathbb{C}$-VMHS. Its monodromy representation denoted by $\rho_T^{GM''}$ is a hull of the deformation functor of ρ.

A detailed proof of this proposition is given in [EysSim09].

Definition 6.2.4. *The $\mathbb{C}$-VMHS obtained from that of Proposition 6.2.3 by reduction* $\mod \mathfrak{m}^{n+1}$ *is of finite rank and will be called the n-th deformation of $\mathbb{V}_\rho$ and will be denoted by $\mathbb{D}_n(\mathbb{V}_\rho)$.*

These universal VMHS being constructed, we need to explain how to use them.

Remark 6.2.5. *The restriction $G = GL_N$ in the above considerations was introduced only for convenience. It is not essential. In* [EysSim09]*, similar statements are proven for arbitrary reductive groups G.*

6.3. **Strictness.**

6.3.1. *Subgroups of $\pi_1(X, x)$ attached to an absolute closed subset.* Let G be a reductive algebraic group defined over $\overline{\mathbb{Q}}$. Suppose as before $M \subset M_B(X, G)$ is an absolute closed subset.

Definition 6.3.1. *Let M^{VHS} be the subset of $M(\mathbb{C})$ consisting of the conjugacy classes of $\mathbb{C}$-VHS that is $M^{VHS} := KH^{-1}(M_{Dol}(X, G)^{\mathbb{C}^*}(\mathbb{C})) \cap M(\mathbb{C})$.*

With no risk of confusion we will identify this conjugacy classes with actual representations.

Definition 6.3.2. *The tannakian categories $\mathcal{T}_M^{VHS}$ and $\mathcal{T}_M$ are defined as follows:*

- $\mathcal{T}_M^{VHS}$**:** *is the full Tannakian subcategory of the category of local systems on X generated by the elements of M^{VHS}.*
- $\mathcal{T}_M$**:** *is the full Tannakian subcategory of the category of local systems on X generated by the elements of M.*

Every object in $\mathcal{T}_M^{VHS}$ is isomorphic to an object which is a subquotient of $\alpha_1(\rho_1) \otimes \ldots \otimes \alpha_s(\rho_s)$, where $\rho_1, \ldots, \rho_s$ are elements of M. The objects of $\mathcal{T}_M^{VHS}$ underly polarizable $\mathbb{C}$-VHS.

Definition 6.3.3. *Given X, G, and $M \subset M_B(X, G)$ as above, and $k \in \mathbb{N}$ we define the following natural quotients of $\pi_1(X, x)$:*

- Γ_M^∞**:** *is the quotient of $\pi_1(X, x)$ by the intersection H_M^∞ of the kernels of the monodromy representation of $\mathbb{D}_n(\mathbb{V}_\sigma)$, $\sigma \in M$, $n \in \mathbb{N}$, and of the objects of $\mathcal{T}_M^{VHS}$.*
- Γ_M^k**:** *is the quotient of $\pi_1(X, x)$ by the intersection H_M^k of the kernels of the monodromy representation of $\mathbb{D}_k(\mathbb{V}_\sigma)$, $\sigma \in \mathcal{T}_M^{VHS}$, and of the objects of M.*

Note that we have the inclusions:

$$\Gamma_M^\infty = \bigcap_{k \in \mathbb{N}} \Gamma_M^k \subset \Gamma_M^{k+1} \subset \Gamma_M^k \subset \Gamma_M^0 = \Gamma_M.$$

It should be noted that since H_M^k is normal the various base point changing isomorphisms $\pi_X(X, x') \to \pi_1(X, x)$ respect H_M^k . Hence, dropping the base point dependance in the notation H_M^k is harmless.

6.3.2. *Strictness.* Let $z \in Z$ be a base point in the connected projective variety Z.

Proposition 6.3.4. *For every $f : (Z, z) \to (X, x)$ such that $\pi_1(Z, z) \to \Gamma_M$ is trivial, the following are equivalent:*

(1) For every $\mathbb{V}$ in $\mathcal{T}_M^{VHS}$, $H^1(X, \mathbb{V}) \to H^1(Z, \mathbb{V})$ is trivial,
(2) $\pi_1(Z, z) \to \Gamma_M^1$ is trivial,
(3) For every $\sigma \in \mathcal{T}_M^{VHS}$ and $k \in \mathbb{N}$, for every $\widehat{Z}_i \to Z$ a resolution of singularities of an irreducible component, the VMHS $\mathbb{D}_k(\mathbb{V}_\sigma)_{\widehat{Z}_i}$ is trivial.
(4) $\pi_1(Z, z) \to \Gamma_M^\infty$ is trivial.

Proof: The proof of this generalization of the main point of [Kat97] uses the explicit construction [EysSim09], see [EKPR09]. The only thing to prove is (1) $\Rightarrow$ (4). Since *in degree 1* restrictions of harmonic forms stay harmonic, (1) implies that the restriction to Z of the form α_1^v is zero.

The forms α_k for $k \geq 2$ are obtained by recurrence as follows. Applying the $D'D''$ lemma, we see that the curvature of $D_k = D + \sum_{i=1}^{k-1} \alpha_i$ is $D'D''$-exact. Then $\alpha_k = \pm D'' \beta_k$ where β_k is the unique (up to a parallel section) solution on X of $D'D''\beta_k = (D_k)^2$.

Now a solution of this equation β_k on X restricts on each resolution of singularities $\hat{Z}_i$ of a component of Z to a solution of the same equation on $\hat{Z}_i$. From this we infer that $\alpha_k|_Z = 0$ for $k \geq 2$.

□

6.4. **Linear Shafarevich Conjecture.** The main Theorem of [EKPR09] is:

Theorem 6.4.1. *Let X be complex projective manifold. Let G be a reductive algebraic group defined over $\mathbb{Q}$. Let $M = M_B(X, G)$.*

For every $0 \leq k \leq +\infty$, the covering space $\widetilde{X^{un}}/H^k_M$ is holomorphically convex.

The case $k = 0$ is covered by the main result of [Eys04]. Proposition 6.3.4 reduces the theorem to the case $k = 1$. In fact, the maximal compact connected subspaces of $\widetilde{X^{un}}/H^k_M$, $k \geq 1$, are lifts of those of $\widetilde{X^{un}}/H^1_M$. Hence, the Shafarevich maps for the H^k_M are the same for all $k \geq 1$.

The theorem implies easily the Shafarevich conjecture for complex projective manifolds whose fundamental group posseses a faithfull linear representation.

The idea of the proof is as follows.

Using [Eys04], we construct the Cartan-Remmert reduction $s^0_M : \widetilde{X_M} = \widetilde{X^{un}}/H^0_M \to \widetilde{S^0_M}$. Its composition with the natural projection gives a holomorphic map $s^1_M : \widetilde{X^{un}}/H^1_M \to \widetilde{S^0_M}$.

Fix Σ a finite subset of $\mathcal{T}^{VHS}_M$. For $\sigma \in \Sigma$, the period mapping of $D_1(\mathbb{V}_\sigma)$ is a holomorphic map $p^1_\sigma : \widetilde{X^{un}}/H^1_M \to MD^1_\sigma$ where MD^1_σ is a mixed period domain parametrizing MHS of weights $-1, 0$. This complex manifold is an affine bundle over a Griffiths period domain.

Using this we have a holomorphic map

$$q(\Sigma) = \prod_{\sigma \in \Sigma} p^1_\sigma \times s^1_M : \prod_{\sigma \in \Sigma} \widetilde{X^{un}}/H^1_M \to \prod_{\sigma \in \Sigma} MD^1_\sigma \times \widetilde{S^0_M}.$$

If every connected component of a fiber of $q(\Sigma)$ is compact we may argue as in section 4. The restriction of $q(\Sigma)$ to the lift of (a finite etale cover) of a fiber Z of s^0_M is essentially given by a finite set of abelian integrals of Z. We assume, for simplification, Z is smooth and irreducible (the general case can be reduced to this special case). Their differentials span a vector subspace $P^{1,0}$ of $H^0(Z, \Omega^1_Z)$ and define a $\mathbb{R}$- sub Hodge structure $P = P_\Sigma(Z/X) = P^{1,0} \oplus \overline{P^{1,0}} \subset H^1(Z, \mathbb{C})$.

One sees easily that one can choose Σ_0 in such a way that $P_\Sigma(Z/X) \subset P_{\Sigma_0}(Z/X)$ for all Σ and Z as above.

If $P^{1,0} \subset H^0(Z, \Omega^1_Z)$ is a vector subspace, the holomorphic map they define on their integration covering $\widetilde{Z_P} \to (P^{1,0})^*$ is proper provided P is defined over $\mathbb{Q}$.

This reduces the problem to proving that $P_{\Sigma_0}(Z/X)$ is rational. Under this condition, $P_{\mathbb{Z}} = P \cup H^1(Z, \mathbb{Z})$ is a weight 1 polarized $\mathbb{Z}$-Hodge structure whose Albanese torus A is a quotient of the Albanese torus of Z and the above integration covering is a base change of the holomorphic map $Z \to A$. This is achieved by a Mixed Hodge Theoretic argument in [EKPR09] using [EysSim09] crucially. A simpler argument might actually exist but remains to be discovered.

7. Conclusion and Open Problems

The main theorems of [Eys04] [EKPR09] rely on a projectivity assumption which comes from the use of [Sim93a] by [Eys04]. On the other hand, it is natural to expect these theorems hold in the compact Kähler case.

7.1. **Bogomolov-Katzarkov's examples.** Are the general type surfaces constructed in [BoKa98] counter-examples to the Shafarevich conjecture? This seems to be a hard problem in geometric group theory.

7.2. **Kähler Groups.** The problem, originally posed by Serre, of determining which groups can occur as the fundamental group of a complex projective manifold is wide open. It is not known whether the class coincides with Kähler groups (fundamental groups of compact Kähler manifolds). Non abelian Hodge Theory is very powerful to find restrictions obeyed by Kähler groups. For a survey, see [ABCKT96].

The class of fundamental groups of complex quasiprojective manifolds is studied by [JosZuo96][JosZuo00], see also the important work of T. Mochizuki [Moc07].

7.3. **Rigid representations of Kähler groups.** A semisimple representation in $G(\bar{\mathbb{Q}})$ where G is a semisimple group is called integral if the values of its character are algebraic integers . No non integral rigid representation of a Kähler group is known.

A conjecture of Simpson claims this is impossible. This conjecture is correct if $G = SL_2$. This conjecture seems to be rather difficult in general. Non archimedian non abelian Hodge theory is obviously a tool.

7.4. **Toledo conjecture.** A real semisimple algebraic group is of hermitian symmetric type if its Riemannian symmetric space is a bounded symmetric domain. It is conjectured that a uniform lattice in a real semisimple algebraic group is Kähler iff the group is of hermitian symmetric type.

A related problem is the Toledo conjecture claiming that $H^2(\pi_1(X), \mathbb{R}) \neq 0$ whenever X is a compact Kähler manifold such that $\pi_1(X)$ has an infinite complex linear representation.

Klingler (unpublished), developping an idea of Reznikov, has given an argument reducing this to the case where all linear representations underly a rigid $\mathbb{C}$-VHS. This case is still open despite several attempts.

7.5. **Invariance of Γ-dimension and variation of the Shafarevich morphism.** It is expected, thanks to work of Campana [Cam94] and Claudon [Cla09], that the Γ-dimension of (X, H) namely $\dim \widetilde{S_H(X)}$ as in Theorem 1.0.5 is a deformation invariant.

The linear case of this problem is related to the following admittedly vague question: can one formulate and prove a partial converse to the Kähler case of the Gromov-Schoen theorem, namely reconstruct from an enriched version of the spectral covering the original action of the fundamental group on an affine building?

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Philippe Eyssidieux.
Institut Fourier. Université Joseph Fourier. Grenoble (France).
eyssi@fourier.ujf-grenoble.fr
http://www-fourier.ujf-grenoble.fr/ eyssi/