Hysteresis for ferromagnetism: asymptotics of some 2-scale Landau-Lifshitz model

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Abstract. We study a 2-scale version of the Landau-Lifshitz system of ferromagnetism, introduced by Starynkevitch to modelize hysteresis: the response of the magnetization is fast compared to a slowly varying applied magnetic field. Taking the exchange term into account, in space dimension 3, we prove that, under some natural stability assumption on the equilibria of the system, the strong solutions follow the dynamics of these equilibria. We also give explicit examples of relevant equilibria and exterior magnetic fields, when the ferromagnetic medium occupies some ellipsoidal domain.

1 Introduction

Hysteresis is a widely studied, yet not completely understood phenomenon. It has played a role from the very beginning of the works on magnetism. Lord Rayleigh [11] proposed a model for ferromagnetic hysteresis in 1887, while the most achieved micromagnetism theory goes back to Landau and Lifshitz, in 1935 (see [8]).

In [15], Visintin gives many historical references, underlines the links between several forms of hysteresis (in particular, from plasticity, and from ferromagnetism), and how it is related to phase transitions. He performs a mathematical study of the so-called hysteresis operators, including the most famous one, due to Preisach.

Recently in [3], Carbou, Effendiev and Fabrie have proved the existence of strong solutions to a model of ferromagnetic hysteresis due to Effendiev.

In this paper, we rather investigate properties of a two-scale model introduced by Starynkevitch in [13]. This model describes the dynamics obtained when some exterior magnetic field is applied to the ferromagnetic material under consideration, while the response of the magnetization occurs on a much shorter time scale (say, denoted by $\varepsilon > 0$). Mathematically, such models, associated to ordinary differential equations, had been studied in the nonstandard analysis

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framework, leading to "canard cycles" (see [6]). Considering a Landau-Lifshitz model in 0 space dimension (thus, an ODE), Starynkevitch studies the possible equilibria of the system, and the asymptotic behavior of the solutions (as the above mentioned parameter ε goes to zero) when the exterior magnetic field slowly varies.

Our aim is to extend Starynkevitch's approach to the Landau-Lifshitz model in space dimension three, taking exchange term into account. This means, giving the asymptotic description of solutions to the slow-fast corresponding system of partial differential equations. Here, we prove such a result away from the bifurcation points of hysteresis loops. More precisely, assuming that the system (described by its magnetization) possesses at each time t some stable equilibrium $m_{\text{eq}}(t)$, and is submitted to some slowly varying exterior magnetic field, we show that the magnetization follows the dynamics of m_{eq}. We also give explicit examples (for ellipsoidal domains) of relevant equilibria and exterior magnetic fields.

The paper is organized as follows: in Section 2, we present the model and our results; Section 3 is devoted to some preliminary functional analysis lemmas, and to remarks about equilibria of the Landau-Lifshitz equation; Section 4 contains the proof of the main theorem, while Section 5 contains the proofs of other related results; finally, we give the proofs of several lemmas in Section 6.

2 Statement of the results

The initial and boundary value problem associated to the 2 scale Landau-Lifshitz equation considered reads:

$$\left\{\begin{array}{l} \varepsilon\partial_t m^\varepsilon = m^\varepsilon \wedge h^\varepsilon - \alpha m^\varepsilon \wedge (m^\varepsilon \wedge h^\varepsilon), \ \text{ for } t \geqslant 0,\ x \in \Omega, \\ \partial_\nu m^\varepsilon_{|\partial\Omega} = 0, \\ m^\varepsilon_{|t=0} = m_0. \end{array}\right. \tag{2.1}$$

The unknown is the magnetization m^ε, function of the time variable $t \geqslant 0$ and of the space variable $x \in \Omega$, with values in the sphere $S^2 \subset \mathbb{R}^3$. The domain Ω occupied by the ferromagnet is a subset of $\mathbb{R}^3$. Furthermore, $h^\varepsilon = h(t, m^\varepsilon(t))$, where the total magnetic field h is defined by

$$h(t,m) = \overline{\Delta m} + h_{\text{d}}(m) + h_{\text{ext}}(t). \tag{2.2}$$

Here, the first term Δm is the "exchange term", which tends to impose a constant magnetization (in domains called "Weiss domains"); $\overline{\Delta m}$ denotes the extension of Δm by 0 out of Ω. The second term, yielding spatial variations of the magnetization, is the "demagnetizing field" $h_{\text{d}}(m)$, which results from a quasi-stationary approximation of Maxwell's equations; it is defined (at least, for $m \in L^2(\Omega, \mathbb{R}^3)$, as an element of $L^2(\mathbb{R}^3, \mathbb{R}^3)$) by

$$\operatorname{curl} h_{\text{d}}(m) = 0 \quad \text{and} \quad \operatorname{div}(h_{\text{d}}(m) + \overline{m}) = 0 \quad \text{in } \mathbb{R}^3.$$

Classical properties of the mapping $m \mapsto h_{\text{d}}(m)$ are recalled in Section 3.1. The third term, h_{ext}, denotes some given exterior field, which is assumed to depend on time (and possibly on space). The positive constant α is some damping coefficient, which appears in the model when passing from a microscopic to a macroscopic description. The small parameter $\varepsilon > 0$ expresses the fact that, while the exterior field h_{ext} depends on t, and has time variations at scale 1, the magnetization m^ε essentially depends on t/ε, and thus has variations at the much more rapid scale ε.

Throughout this paper, for any $s \in \mathbb{N}$, we denote by $H^s(\Omega)$ the usual Sobolev space of functions with values in some vector space $\mathbb{R}^N$, whereas $H^s(\Omega, S^2)$ is the Sobolev space of functions with values in the sphere S^2 (which is not a vector space),

$$H^s(\Omega, S^2) = \{m \in H^s(\Omega) \mid |m| \equiv 1 \text{ almost everywhere}\}.$$

Finally, for $s \geqslant 2$, $H^s_N(\Omega)$ denotes the subspace of functions in $H^s(\Omega)$ with homogeneous von Neumann boundary condition,

$$H^s_N(\Omega) = \{m \in H^s(\Omega) \mid \partial_\nu m_{|\partial\Omega} = 0\}, \quad H^s_N(\Omega, S^2) = H^s(\Omega, S^2) \cap H^s_N(\Omega).$$

All these spaces (even if not vector spaces) inherit the (metric) topology given by the usual norm on $H^s(\Omega)$.

Theorem 2.1. *Let Ω be an open and bounded subset of $\mathbb{R}^3$, with smooth boundary. Let $T > 0$, and $h_{\text{ext}} \in C^1([0,T], C^\infty(\mathbb{R}^3))$, bounded with bounded derivatives. Assume that there exist $m_{\text{eq}} \in C^1([0,T], H^2_N(\Omega, S^2))$ and $m_0 \in H^2_N(\Omega, S^2)$ such that*
(i) for all $t_0 \in [0,T]$, $m_{\text{eq}}(t_0)$ is an equilibrium for

$$\partial_t m = m \wedge h(t_0, m) - \alpha m \wedge \Big(m \wedge h(t_0, m)\Big) \tag{2.3}$$

(see (3.1));
(ii) the solution n_0 to the initial and boundary value problem

$$\begin{cases} \partial_t n_0 = n_0 \wedge h(0, n_0) - \alpha n_0 \wedge \Big(n_0 \wedge h(0, n_0)\Big), \\ \partial_\nu n_{0|\partial\Omega} = 0, \\ n_{0|t=0} = m_0, \end{cases} \tag{2.4}$$

is global ($n_0 \in C([0,\infty), H^2_N(\Omega, S^2))$), with $\nabla \Delta n_0 \in L^2((0,\infty) \times \Omega)$, and $n_0(t)$ converges in $H^2(\Omega)$, as t goes to ∞, towards $m_{\text{eq}}(0)$;
(iii) the linearized operator $\mathcal{L}(\mathrm{m_{eq}})$ given by (4.29) has the following dissipation property:

(2.5)
there exist $C_{\text{lin}} > 0$ and $\eta > 0$ such that,
for all $\delta \in C([0,T], H^\infty(\Omega))$ with $|m_{\text{eq}} + \delta| \equiv 1$ and $\partial_\nu \delta_{|\partial\Omega} = \partial_\nu \Delta \delta_{|\partial\Omega} = 0$,
$\sup_{t \in [0,T]} \|\delta(t)\|_{H^2(\Omega)} \leqslant \eta$ *implies:*

$$\forall t \in [0,T], \quad \Big(\mathcal{L}(\mathrm{t}, \mathrm{m_{eq}}(\mathrm{t}))\, \delta(t) \mid \delta(t)\Big)_{H^2(\Omega)} \leqslant -C_{\text{lin}} \|\delta(t)\|^2_{H^2(\Omega)}.$$

Then, there is $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the solution m^ε to (2.1) exists up to time T ($m^\varepsilon \in C([0,T], H^2_N(\Omega, S^2))$), and converges in $L^2((0,T), H^2(\Omega)) \cap C([t,T], H^2(\Omega))$ towards $m_{\rm eq}$ as ε goes to zero, for all $t \in (0,T)$.

To prove Theorem 2.1, we first show that m^ε converges to $m_{\rm eq}(0)$ within an initial layer of size $t_\varepsilon = c\varepsilon \ln(1/\varepsilon)$. This is achieved *via* classical energy estimates (in H^2), carefully controlling the dependence upon ε –more technically speaking, the quasilinear and parabolic degenerate system of PDE's in (2.1) is first converted into a perturbation of some linear, strongly elliptic system, yielding the usual smoothing properties, and a Galerkin approximation is used. In a second step, we prove that m^ε converges towards $m_{\rm eq}$ on the whole time interval $[t_\varepsilon, T]$. This amounts to proving a long-time existence and return to equilibrium result for small initial data. Toward this end, we use again energy estimates, together with the stability assumption (2.5).

Figure 1 illustrates this corresponding asymptotic behaviour.

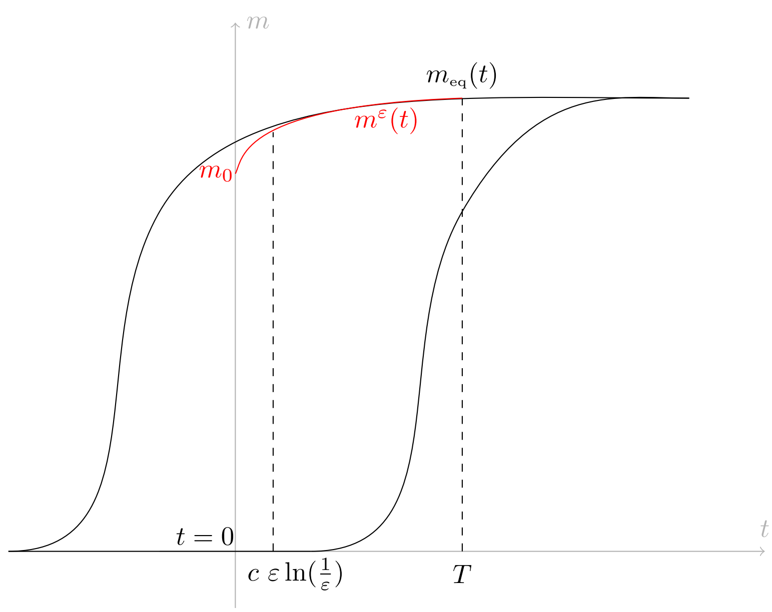

Figure 1: Dynamics of the magnetization away from bifurcation points.

The above assumptions on the equilibrium $m_{\rm eq}$ are discussed in Section 3.2 below. In particular, Assumption (ii) in Theorem 2.1 may be understood as a choice of 'prepared' data m_0 allowing to deal with the initial layer $(0, c\varepsilon \ln(1/\varepsilon))$. The dissipation property (2.5) expresses, for all $t_0 \in [0,T]$, the stability of the linearization around $m_{\rm eq}(t_0)$ of (2.1), with $\varepsilon = 1$ and with $h_{\rm ext}$ replaced with $h_{\rm ext}(t_0)$, independent of time. This is a strong assumption, which ensures global existence of the solutions to the corresponding Landau-Lifshitz equation, for initial data close to $m_{\mathrm{eq}}(t_0)$:

Proposition 2.2. *Let Ω be an open and bounded subset of $\mathbb{R}^3$, with smooth boundary. Consider an exterior magnetic field $h_{\mathrm{ext}} \in C^\infty(\mathbb{R}^3)$ (independent of time) bounded with bounded derivatives. Assume that there exists $m_{\mathrm{eq}} \in H^2_N(\Omega, S^2)$ (independent of time) satisfying the equilibrium condition*

$$m_{\mathrm{eq}} \wedge (\Delta m_{\mathrm{eq}} + h_{\mathrm{d}}(m_{\mathrm{eq}}) + h_{\mathrm{ext}}) = 0 \quad \text{on } \Omega, \tag{2.6}$$

as well as the stability condition

$$\begin{aligned} &\text{there exist } C_{\mathrm{lin}} > 0 \text{ and } \eta > 0 \text{ such that,} \\ &\text{for all } \delta \in H^\infty(\Omega) \text{ with } |m_{\mathrm{eq}} + \delta| \equiv 1 \text{ and } \partial_\nu \delta_{|\partial\Omega} = \partial_\nu \Delta\delta_{|\partial\Omega} = 0, \\ &\|\delta(t)\|_{H^2(\Omega)} \leqslant \eta \text{ implies:} \\ &\Big(\mathcal{L}(0, m_{\mathrm{eq}})\delta \mid \delta\Big)_{H^2(\Omega)} \leqslant -C_{\mathrm{lin}} \|\delta\|^2_{H^2(\Omega)}, \end{aligned} \tag{2.7}$$

for the linearized operator $\mathcal{L}(0, m_{\mathrm{eq}})$ given by (4.29) (with $m_{\mathrm{eq}}(0)$ and $h_{\mathrm{ext}}(0)$ replaced with m_{eq} and h_{ext}, respectively).

Then, there exists $\eta_0 > 0$ such that, for all $m_0 \in H^2_N(\Omega, S^2)$ satisfying

$$\|m_0 - m_{\mathrm{eq}}\|_{H^2(\Omega)} \leqslant \eta_0,$$

the solution n to the initial and boundary value problem

$$\left\{ \begin{aligned} &\partial_t n = n \wedge h(0, n) - \alpha n \wedge \Big(n \wedge h(0, n)\Big), \\ &\partial_\nu n_{|\partial\Omega} = 0, \\ &n_{|t=0} = m_0, \end{aligned} \right. \tag{2.8}$$

is global ($n \in C([0, \infty), H^2_N(\Omega, S^2))$), with $\nabla\Delta(n - m_{\mathrm{eq}}) \in L^2((0, \infty) \times \Omega)$, and $n(t)$ converges in $H^2(\Omega)$, as t goes to ∞, towards m_{eq}.

In the case of $m_{\mathrm{eq}}(0)$ constant over Ω, Proposition 2.2 expresses that in Theorem 2.1, assumptions (i) and (iii) imply assumption (ii), so that we get:

Corollary 2.3. *Let Ω be an open and bounded subset of $\mathbb{R}^3$, with smooth boundary. Let $T > 0$, and $h_{\mathrm{ext}} \in C^1([0, T], C^\infty(\mathbb{R}^3))$, bounded with bounded derivatives. Assume that there exist $m_{\mathrm{eq}} \in C^1([0, T], H^2_N(\Omega, S^2))$ satisfying assumptions (i) and (iii) from Theorem 2.1. Assume furthermore that $m_{\mathrm{eq}}(0)$ is constant over Ω.*

Then, there exist $\eta_0, \varepsilon_0 > 0$ such that, for all $m_0 \in H^2_N(\Omega, S^2)$ such that

$$\|m_0 - m_{\mathrm{eq}}(0)\|_{H^2(\Omega)} \leqslant \eta_0,$$

and for all $\varepsilon \in (0, \varepsilon_0)$, the solution m^ε to (2.1) exists up to time T ($m^\varepsilon \in C([0, T], H^2_N(\Omega, S^2))$), and converges in $L^2((0, T), H^2(\Omega)) \cap C([t, T], H^2(\Omega))$ towards m_{eq} as ε goes to zero, for all $t \in (0, T)$.

In Lemma 3.5 below, we give examples (in ellipsoidal domains) of equilibria m_{eq} satisfying the assumptions of Corollary 2.3. We do not know other situations for which this assumption can be checked. Finding other classes of examples is an interesting problem. The difficulty comes from the non-local demagnetizing operator, which generally produces a non explicit demagnetizing magnetic field, so that computations become more intricate.

3 Preliminaries

3.1 Some functional analysis

In this section, we recall some functional analysis results useful in the sequel. The first of them deals with the continuity properties of the demagnetizating field operator h_{d}. Continuity on Sobolev spaces H^s is immediately deduced from the Fourier representation $\widehat{h_{\mathrm{d}}(u)}(\xi) = -\left(\xi \cdot \hat{u}(\xi)\right)\dfrac{\xi}{|\xi|^2}$. *Via* singular integral representation, this operator also acts on L^p (see e.g. [14], chap. VI, Theorem 3.1). The proof of the result on $W^{1,p}$ and $W^{2,p}$ is given in [4].

Lemma 3.1 (h_d properties)**.** *Let Ω be an open subset of $\mathbb{R}^3$. For all s in $\mathbb{N}$ and u in $H^s(\Omega)$, one has*

$$\|h_{\mathrm{d}}(u)\|_{H^s(\mathbb{R}^3)} \leqslant \|u\|_{H^s(\Omega)}.$$

Furthermore, for all v in $L^2(\Omega)$ we have

$$(h_{\mathrm{d}}(u) \mid v)_{L^2(\Omega)} = -(u \mid h_{\mathrm{d}}(v))_{L^2(\Omega)}.$$

Finally, for all $p \in (1,\infty)$ and $s = 0, 1$ or 2, there exists $C > 0$ such that, for all u in $W^{s,p}(\Omega)$, the restriction of $h_{\mathrm{d}}(u)$ to Ω belongs to $W^{s,p}(\Omega)$, and we have

$$\|h_{\mathrm{d}}(u)\|_{W^{s,p}(\Omega)} \leqslant \|u\|_{W^{s,p}(\Omega)}.$$

In addition to the usual Sobolev embeddings, we recall the following estimate, which results from the coercivity of the operator $A = 1 - \Delta$, with domain $D(A) = \{m \in H^2(\Omega) \mid \partial_\nu m_{|\partial\Omega} = 0\}$ (see for example [5])

Lemma 3.2. *Let Ω be a smooth bounded open set in $\mathbb{R}^3$. There exists a constant $C > 0$ such that for all u in $H^2_N(\Omega)$ one has*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

In the sequel, we will need the following definition.

Definition 3.3. *Let Ω be a smooth bounded open set in $\mathbb{R}^3$. For $k \in \mathbb{N}^\star$, let P_k be the $L^2(\Omega)$-orthogonal projection onto V_k, the vector space spanned by the first k eigenfunctions of $A = 1 - \Delta$, with domain $D(A) = \{m \in H^2(\Omega) \mid \partial_\nu m_{|\partial\Omega} = 0\}$.*

The family of operators $(P_k)_{k\in\mathbb{N}}$ satisfies useful properties:

Lemma 3.4. *The following properties hold.*

(i) $\forall k \in \mathbb{N}^\star$, $\forall u \in D(A)$, $\Delta P_k u = P_k \Delta u$,

(ii) $\forall k \in \mathbb{N}^\star$, $\forall s \in \mathbb{N}$, $\forall u \in H^s(\Omega)$, $P_k u \in H^s(\Omega)$ *(and* $P_k u \in H^s_N(\Omega)$ *when* $s \geqslant 2$*),*

(iii) $\forall s \in \mathbb{N}$, $\lim_{k\to\infty} \|(1-P_k)u\|_{H^s(\Omega)} = 0$ *for all* $u \in H^s(\Omega)$ *when* $s = 0, 1$, *and for all* $u \in H^s_N(\Omega)$ *when* $s \geqslant 2$.

3.2 About equilibria

Global solutions and equilibria. In [1, Th. 4.3], in the case of ellipsoidal domains $\Omega \subset \mathbb{R}^3$ and under a smallness assumption (on $\|h_{\text{ext}}\|_{L^\infty}$ and $\|\Delta m_0\|_{L^2}$), Alouges and Beauchard construct global smooth solutions to (2.1). Furthermore, these solutions satisfy

$$\forall T > 0, \quad \|\Delta m(T)\|^2_{L^2(\Omega)} + C\int_0^T \|\nabla \Delta m\|^2_{L^2(\Omega)} \leqslant \|\Delta m_0\|_{L^2(\Omega)},$$

so that $\nabla\Delta m$ belongs to $L^2((0,\infty)\times\Omega)$. This is a part of our assumptions on the equilibrium m_{eq}, when requiring the existence of the global solution n_0. Saying that $m_{\text{eq}}(t_0)$ is an equilibrium for (2.3) means

$$\left\{\begin{array}{l} m_{\text{eq}}(t_0) \wedge h(t_0, m_{\text{eq}}(t_0)) = 0, \\ \partial_\nu m_{\text{eq}}(t_0)_{|\partial\Omega} = 0, \end{array}\right. \tag{3.1}$$

and requiring H^2 convergence of $n_0(t)$ towards $m_{\text{eq}}(0)$ as t goes to ∞ implies that $m_{\text{eq}}(0)$ is an equilibrium for (2.3) with $t_0 = 0$.

Energy minimization. It is worth noting that energy decay occurs along the evolution of $n_0(t)$, with n_0 given by (2.4), so that one may hope at least H^1 convergence of $n_0(t)$ towards some local minimum of the energy, as t goes to ∞. To the Landau-Lifshitz system (2.1) is associated the energy

$$\mathcal{E}(t,m) = \frac{1}{2}\int_\Omega |\nabla m|^2 - \frac{1}{2}\int_\Omega m\cdot h_{\text{d}}(m) - \int_\Omega m \cdot h_{\text{ext}}(t),$$

and when m^ε is solution to (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t, m^\varepsilon(t)) = -\frac{\alpha}{\varepsilon}\|m^\varepsilon(t)\wedge h(t, m^\varepsilon(t))\|^2_{L^2(\Omega)} - \int_\Omega m^\varepsilon(t)\cdot\partial_t h_{\text{ext}}(t).$$

Since the exterior magnetic field does not depend on time during the evolution of n_0, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t, n_0(t)) = -\alpha\|n_0(t)\wedge h(t, n_0(t))\|^2_{L^2(\Omega)}.$$

In the case of ellipsoidal domains, special configurations are available. See [10], and references therein: there exists a real 3×3 definite positive diagonalizable matrix D giving the demagnetizing field resulting from any magnetization constant over Ω (this fact characterizes ellipsoids amongst domains in $\mathbb{R}^3$; this was known as the Eshelby conjecture, and recently solved in [7] and [9]):

$$\forall v \in \mathbb{R}^3, \quad h_{\mathrm{d}}(v)_{|\Omega} \equiv -Dv.$$

Hence, if $u \in S^2$ is an eigenvector of D associated to the eigenvalue $d > 0$, and if the exterior magnetic field is $h_{\mathrm{ext}} = \lambda u$ for some $\lambda > 0$ (or $h_{\mathrm{ext}}(x) = \lambda\chi(x)u$ for some $\chi \in C_c^\infty(\mathbb{R}^3, [0, 1])$ to get a spatially localized field), then the system possesses two explicit equilibria m^+_{eq} and m^-_{eq}:

$$m^{\pm}_{\mathrm{eq}} = \pm u. \tag{3.2}$$

One easily computes the energy associated to perturbations of these equilibria: for all $\delta \in H^2_N(\Omega, \mathbb{R}^3)$ such that $|m^{\pm}_{\mathrm{eq}} + \delta| = 1$ a.e.,

$$\mathcal{E}(m^{\pm}_{\mathrm{eq}} + \delta) - \mathcal{E}(m^{\pm}_{\mathrm{eq}}) = \frac{1}{2}\int_\Omega |\nabla\delta|^2 - \frac{1}{2}\int_\Omega \delta \cdot h_{\mathrm{d}}(\delta) + \frac{1}{2}(\pm\lambda - d)\int_\Omega |\delta|^2.$$

The first two terms are non-negative, so that for λ large enough ($\lambda > d$), m^+_{eq} is a global minimum of $\mathcal{E}$; but for λ small, it may fail to be even a local minimum. Concerning m^-_{eq}, for all $\lambda > 0$, if d is the largest eigenvalue of D, and δ is constant in space, then the difference of energies above is less than $-\dfrac{\lambda}{2}\displaystyle\int_\Omega |\delta|^2$, thus negative, whereas for δ with large variations, the gradient term dominates, and the energy difference becomes positive. Hence, m^-_{eq} is always a saddle point for $\mathcal{E}$.

The dissipation property (2.5). We have the following lemma, the proof of which is postponed to Section 6.1:

Lemma 3.5. *For $\lambda > 0$ large enough, the equilibrium m^+_{eq} from (3.2) satisfies the dissipation property (2.5) (for some constant C_{lin} depending on λ).*

For m^-_{eq}, it is shown in Section 6.1 that for λ large, we have on the contrary:

Lemma 3.6. *For $\lambda > 0$ large enough, there exist $C = C(\alpha, \lambda) > 0$ and $\eta = \eta(\alpha, \lambda) > 0$ such that, for all $\delta \in C([0, T], H^\infty(\Omega))$ with $|m_{\mathrm{eq}} + \delta| \equiv 1$ and $\partial_\nu \delta_{|\partial\Omega} = \partial_\nu \Delta\delta_{|\partial\Omega} = 0$, when $\|\delta\|_{H^2(\Omega)} \leqslant \eta$, we have:*

$$\forall t \in [0, T], \quad \Big(\mathcal{L}(t, m^-_{\mathrm{eq}}(t))\delta(t) \mid \delta(t)\Big)_{H^2(\Omega)} \geqslant C\|\delta\|^2_{H^2(\Omega)}.$$

4 Proof of Theorem 2.1

First, consider the solution n_0 to the Cauchy problem (2.4), and define n^ε by

$$\forall t \geqslant 0, \quad n^\varepsilon(t) = n_0(t/\varepsilon).$$

Then, $n^\varepsilon \in C_b([0,\infty), H^2(\Omega))$ (with $\nabla\Delta n^\varepsilon \in L^2((0,\infty)\times\Omega)$), and we know that

$$t_\varepsilon/\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \infty \quad \Longrightarrow \quad n^\varepsilon(t_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} m_{\mathrm{eq}}(0) \text{ in } H^2(\Omega). \tag{4.1}$$

Next, as in [4], we observe that for smooth functions m with constant modulus (w.r.t. x), one has $m\cdot\Delta m = -2|\nabla m|^2$, so that smooth solutions to (2.1) equivalently satisfy

$$\begin{cases} \varepsilon\partial_t m^\varepsilon - \alpha\Delta m^\varepsilon = \mathcal{F}(t, m^\varepsilon), \\ \partial_\nu m^\varepsilon_{|\partial\Omega} = 0, \\ m^\varepsilon_{|t=0} = m_0, \end{cases} \tag{4.2}$$

where

$$\mathcal{F}(t,m) = m\wedge h(t,m) + \alpha|\nabla m|^2 m - \alpha m\wedge\Big(m\wedge(h_{\mathrm{d}}(m) + h_{\mathrm{ext}}(t))\Big). \tag{4.3}$$

Furthermore, smooth ($L^\infty_t H^2_x$) solutions to (4.2) issued from m_0 with constant modulus, equal to one, are shown to keep the same modulus for all time, (due to uniqueness of the solution $a = |m|^2$ to $\varepsilon\partial_t a = \alpha\Delta a + 2\alpha|\nabla u|^2(a-1)$, $\partial_\nu a_{|\partial\Omega} = 0$, $a_{|t=0} = 1$). We thus solve (4.2) in the Banach space $C([0,T], H^2_N(\Omega))$, and deduce from this conservation that the solution actually belongs to the space $C([0,T], H^2_N(\Omega, S^2))$.

It is worth noting that (2.1) is an initial and boundary value problem for some quasilinear and parabolic degenerate operator, which is seen in (4.2) as a perturbation of a linear and strongly parabolic one.

Standard energy estimates ensure local-in-time existence and uniqueness of solutions continuous in time, with values in $H^2(\Omega)$) (with an existence time depending on ε): see for example [1] or [4]. By the usual continuation argument, we simply need to bound the H^2 norm of m^ε to ensure existence up to time T. Actually, we shall prove convergence (as ε goes to zero) at the same time, *via* energy estimates.

We first show that, after some time t_ε of the form $t_\varepsilon = C\varepsilon\ln(1/\varepsilon)$, m^ε and n^ε are close: $\sup_{[0,t_\varepsilon]}\|m^\varepsilon - n^\varepsilon\|_{H^2(\Omega)}$ goes to zero with ε; thus, for ε small enough, $m^\varepsilon(t_\varepsilon)$ is as close (in $H^2(\Omega)$) to $m_{\mathrm{eq}}(0)$ as desired. We then use the stability property of $m_{\mathrm{eq}}(t)$ to show that $m^\varepsilon(t)$ stays close to it, for $t\in[t_\varepsilon, T]$.

4.1 First step: the initial layer $[0, t_\varepsilon]$

4.1.1 Galerkin scheme

For $k\in\mathbb{N}^\star$, let P_k be the $L^2(\Omega)$- orthogonal projection onto V_k, the vector space spanned by the first k eigenfunctions of $A = 1-\Delta$, with domain $D(A) = \{m\in H^2(\Omega) \mid \partial_\nu m_{|\partial\Omega} = 0\}$, as in Definition 3.3. Define a Galerkin approximation of (4.2) by:

$$\begin{cases} \varepsilon\partial_t m^\varepsilon_k - \alpha\Delta m^\varepsilon_k = P_k\mathcal{F}(t, m^\varepsilon_k), \\ m^\varepsilon_{k|t=0} = P_k m_0. \end{cases} \tag{4.4}$$

The projection $n_k^\varepsilon = P_k n^\varepsilon$ also satisfies

$$\begin{aligned}\varepsilon\partial_t n_k^\varepsilon - \alpha\Delta n_k^\varepsilon =& P_k\mathcal{F}(0,n_k^\varepsilon) + \alpha[P_k,\Delta]n^\varepsilon + P_k\Big(\mathcal{F}(0,n^\varepsilon) - \mathcal{F}(0,n_k^\varepsilon)\Big)\\ =& P_k\mathcal{F}(0,n_k^\varepsilon) + P_k[P_k,\mathcal{F}(0,\cdot)](n^\varepsilon),\end{aligned}$$

since for $u \in D(A)$, $P_k\Delta u = \Delta P_k u$, according to Lemma 3.4.

Now, perform energy estimates (in L^2) for $\varphi_k^\varepsilon = m_k^\varepsilon - n_k^\varepsilon$, solution to

$$\left\{\begin{array}{l}\varepsilon\partial_t\varphi_k^\varepsilon - \alpha\Delta\varphi_k^\varepsilon = P_k\Big(\mathcal{F}(t,m_k^\varepsilon) - \mathcal{F}(0,n_k^\varepsilon)\Big) - P_k[P_k,\mathcal{F}(0,\cdot)](n^\varepsilon),\\ \varphi_k^\varepsilon{}_{|_{t=0}} = 0.\end{array}\right. \tag{4.5}$$

4.1.2 L^2 estimates

Take the scalar product (in $L^2(\Omega)$) of φ_k^ε with the first equation in (4.5) to get

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\|\varphi_k^\varepsilon\|^2_{L^2(\Omega)}\Big) + \alpha\|\nabla\varphi_k^\varepsilon\|^2_{L^2(\Omega)} = I_1 + I_2 + I_3 + I_4,$$

with

$$\begin{aligned}I_1 &= \Big(\varphi_k^\varepsilon \mid m_k^\varepsilon \wedge h(t,m_k^\varepsilon) - n_k^\varepsilon \wedge h(0,n_k^\varepsilon)\Big)_{L^2(\Omega)},\\ I_2 &= \alpha\Big(\varphi_k^\varepsilon \mid |\nabla m_k^\varepsilon|^2 m_k^\varepsilon - |\nabla n_k^\varepsilon|^2 n_k^\varepsilon\Big)_{L^2(\Omega)},\\ I_3 &= -\alpha\Big(\varphi_k^\varepsilon \mid m_k^\varepsilon \wedge \Big(m_k^\varepsilon \wedge (h_\mathrm{d}(m_k^\varepsilon) + h_\mathrm{ext}(t))\Big)\\ &\qquad\qquad - n_k^\varepsilon \wedge \Big(n_k^\varepsilon \wedge (h_\mathrm{d}(n_k^\varepsilon) + h_\mathrm{ext}(0))\Big)\Big)_{L^2(\Omega)},\\ I_4 &= \Big(\varphi_k^\varepsilon \mid [P_k,\mathcal{F}(0,\cdot)](n^\varepsilon)\Big)_{L^2(\Omega)}.\end{aligned}$$

Estimating I_1**.** Decompose $m_k^\varepsilon = n_k^\varepsilon + \varphi_k^\varepsilon$. For all $\varphi, h \in \mathbb{R}^3$, $\varphi\cdot(\varphi\wedge h) = 0$, so that

$$\begin{aligned}I_1 =& \Big(\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge \Big(\Delta(n_k^\varepsilon + \varphi_k^\varepsilon) + h_\mathrm{d}(n_k^\varepsilon + \varphi_k^\varepsilon) + h_\mathrm{ext}(t)\Big)\\ &\qquad - n_k^\varepsilon \wedge \Big(\Delta n_k^\varepsilon + h_\mathrm{d}(n_k^\varepsilon) + h_\mathrm{ext}(0)\Big)\Big)_{L^2(\Omega)}\\ =& \Big(\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge (\Delta\varphi_k^\varepsilon + h_\mathrm{d}(\varphi_k^\varepsilon))\Big)_{L^2(\Omega)} + \Big(\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge (h_\mathrm{ext}(t) - h_\mathrm{ext}(0))\Big)_{L^2(\Omega)}.\end{aligned}$$

Using the continuity of h_d on $L^2(\Omega)$, we get, for some constant C depending on $\|\partial_t h_\mathrm{ext}\|_{L^\infty_{t,x}}$ and $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$:

$$I_1 \leqslant C\|\varphi_k^\varepsilon\|_{L^2(\Omega)}\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)} + t\Big). \tag{4.6}$$

Estimating I_2***.*** Write

$$\begin{aligned}|\nabla m_k^\varepsilon|^2 m_k^\varepsilon - |\nabla n_k^\varepsilon|^2 n_k^\varepsilon &= (|\nabla m_k^\varepsilon|^2 - |\nabla n_k^\varepsilon|^2) m_k^\varepsilon + |\nabla n_k^\varepsilon|^2 \varphi_k^\varepsilon \\ &= (\nabla(2n_k^\varepsilon + \varphi_k^\varepsilon)\cdot\nabla\varphi_k^\varepsilon)(n_k^\varepsilon + \varphi_k^\varepsilon) + |\nabla n_k^\varepsilon|^2 \varphi_k^\varepsilon.\end{aligned}$$

Then, use Sobolev's embeddings, such as

$$\begin{aligned}\Big(\varphi_k^\varepsilon \mid (\nabla n_k^\varepsilon \cdot \nabla \varphi_k^\varepsilon) n_k^\varepsilon\Big)_{L^2(\Omega)} &\leqslant \|\varphi_k^\varepsilon\|_{L^\infty(\Omega)} \|\nabla n_k^\varepsilon\|_{L^2(\Omega)} \|\nabla \varphi_k^\varepsilon\|_{L^4(\Omega)} \|n_k^\varepsilon\|_{L^4(\Omega)} \\ &\lesssim \|\varphi_k^\varepsilon\|_{H^2(\Omega)} \|n_k^\varepsilon\|_{H^1(\Omega)} \|\nabla \varphi_k^\varepsilon\|_{H^1(\Omega)} \|n_k^\varepsilon\|_{H^1(\Omega)},\end{aligned}$$

and

$$\Big(\varphi_k^\varepsilon \mid |\nabla n_k^\varepsilon|^2 \varphi_k^\varepsilon\Big)_{L^2(\Omega)} \leqslant \|\varphi_k^\varepsilon\|_{L^\infty(\Omega)}^2 \|\nabla n_k^\varepsilon\|_{L^2(\Omega)}^2 \lesssim \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2 \|n_k^\varepsilon\|_{H^1(\Omega)}^2,$$

to get the estimate

$$I_2 \leqslant C\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)} + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^3\Big), \tag{4.7}$$

for some constant C depending on $\|n_0\|_{L^\infty((0,\infty),H^1(\Omega))}$.

Estimating I_3***.*** As for I_1, cancellations allow to write

$$\begin{aligned}I_3 = &-\alpha\Big(\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge \Big(n_k^\varepsilon \wedge h_{\mathrm{d}}(\varphi_k^\varepsilon) + \varphi_k^\varepsilon \wedge h_{\mathrm{d}}(n_k^\varepsilon)\Big) + n_k^\varepsilon \wedge \Big(\varphi_k^\varepsilon \wedge h_{\mathrm{ext}}(t)\Big)\Big)_{L^2(\Omega)} \\ &-\alpha\Big(\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge \Big(n_k^\varepsilon \wedge (h_{\mathrm{ext}}(t) - h_{\mathrm{ext}}(0))\Big)\Big)_{L^2(\Omega)}.\end{aligned}$$

Boundedness of h_{d} on L^p for finite p provides the bounds

$$\|n_k^\varepsilon \wedge (n_k^\varepsilon \wedge h_{\mathrm{d}}(\varphi_k^\varepsilon))\|_{L^2(\Omega)} \leqslant \|n_k^\varepsilon\|_{L^6(\Omega)}^2 \|\varphi_k^\varepsilon\|_{L^6(\Omega)},$$

and

$$\|n_k^\varepsilon \wedge (\varphi_k^\varepsilon \wedge h_{\mathrm{d}}(n_k^\varepsilon))\|_{L^2(\Omega)} \leqslant \|n_k^\varepsilon\|_{L^6(\Omega)}^2 \|\varphi_k^\varepsilon\|_{L^6(\Omega)}.$$

The above L^6 norms are controlled by H^1 norms. Thus, for some constant C depending on $\|h_{\mathrm{ext}}\|_{L^\infty_{t,x}}$, $\|\partial_t h_{\mathrm{ext}}\|_{L^\infty_{t,x}}$ and $\|n_0\|_{L^\infty((0,\infty),H^1(\Omega))}$:

$$I_3 \leqslant C\|\varphi_k^\varepsilon\|_{L^2(\Omega)}\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)} + t\Big). \tag{4.8}$$

Estimating I_4***.*** Setting $r_k^\varepsilon = \|[P_k, \mathcal{F}(0,\cdot)](n_k^\varepsilon)\|_{L^2(\Omega)}^2$, we have:

$$I_4 \leqslant \|\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + r_k^\varepsilon, \text{ and } r_k^\varepsilon \underset{k\to\infty}{\longrightarrow} 0 \text{ in } L^1(0,T) \text{ for all } T>0, \text{ with } \varepsilon \text{ fixed.} \tag{4.9}$$

This is a consequence of the following lemma, the proof of which is postponed to Section 6.2.

Lemma 4.1. *For all $T > 0$ and $n \in C([0,T], H_N^2(\Omega)) \cap L^2((0,T), H^3(\Omega))$,*

$$[P_k, \mathcal{F}(0,\cdot)](n) \underset{k\to\infty}{\longrightarrow} 0 \ \textit{in}\ L^2((0,T), H^1(\Omega)).$$

Gathering L^2 estimates. Adding (4.6), (4.7), (4.8) and (4.9), we get

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\varphi_k^\varepsilon\|^2_{L^2(\Omega)}\right)+\alpha\|\nabla\varphi_k^\varepsilon\|^2_{L^2(\Omega)} \leqslant C\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big(t+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}+\|\varphi_k^\varepsilon\|^3_{H^2(\Omega)}+r_k^\varepsilon\Big), \tag{4.10}$$

with r_k^ε from (4.9), and for some constant depending on the quantities $\|h_{\mathrm{ext}}\|_{L^\infty_{t,x}}$, $\|\partial_t h_{\mathrm{ext}}\|_{L^\infty_{t,x}}$ and $\|n_0\|_{L^\infty((0,\infty),H^1(\Omega))}$.

4.1.3 H^2 estimates

Next, take the scalar product in $L^2(\Omega)$ of $\Delta^2\varphi_k^\varepsilon$ with the first equation in (4.5) to get

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}\right)+\alpha\|\nabla\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}=II_1+II_2+II_3+II_4,$$

with

$$\begin{aligned}
II_1&=\Big(\Delta^2\varphi_k^\varepsilon\mid m_k^\varepsilon\wedge h(t,m_k^\varepsilon)-n_k^\varepsilon\wedge h(0,n_k^\varepsilon)\Big)_{L^2(\Omega)},\\
II_2&=\alpha\Big(\Delta^2\varphi_k^\varepsilon\mid|\nabla m_k^\varepsilon|^2m_k^\varepsilon-|\nabla n_k^\varepsilon|^2n_k^\varepsilon\Big)_{L^2(\Omega)},\\
II_3&=-\alpha\Big(\Delta^2\varphi_k^\varepsilon\mid m_k^\varepsilon\wedge\Big(m_k^\varepsilon\wedge(h_{\mathrm{d}}(m_k^\varepsilon)+h_{\mathrm{ext}}(t))\Big)\\
&\qquad\qquad-n_k^\varepsilon\wedge\Big(n_k^\varepsilon\wedge(h_{\mathrm{d}}(n_k^\varepsilon)+h_{\mathrm{ext}}(0))\Big)\Big)_{L^2(\Omega)},\\
II_4&=\Big(\Delta^2\varphi_k^\varepsilon\mid[P_k,\mathcal{F}(0,\cdot)](n^\varepsilon)\Big)_{L^2(\Omega)}.
\end{aligned}$$

Estimating II_1. Split

$$II_1=II_{1,1}+II_{1,2}+II_{1,3},$$

with

$$\begin{aligned}
II_{1,1}&=\Big(\Delta^2\varphi_k^\varepsilon\mid m_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon+h_{\mathrm{d}}(\varphi_k^\varepsilon))\Big)_{L^2(\Omega)},\\
II_{1,2}&=\Big(\Delta^2\varphi_k^\varepsilon\mid\varphi_k^\varepsilon\wedge h(t,n_k^\varepsilon)\Big)_{L^2(\Omega)},\\
II_{1,3}&=\Big(\Delta^2\varphi_k^\varepsilon\mid n_k^\varepsilon\wedge(h_{\mathrm{ext}}(t)-h_{\mathrm{ext}}(0))\Big)_{L^2(\Omega)}.
\end{aligned}$$

The first term is written

$$II_{1,1}=\Big(\Delta^2\varphi_k^\varepsilon\mid n_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon+h_{\mathrm{d}}(\varphi_k^\varepsilon))\Big)_{L^2(\Omega)}+\Big(\Delta^2\varphi_k^\varepsilon\mid\varphi_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon+h_{\mathrm{d}}(\varphi_k^\varepsilon))\Big)_{L^2(\Omega)}.$$

Integrating by parts,

$$\begin{aligned}
&\left(\Delta^2\varphi_k^\varepsilon \mid n_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon))\right)_{L^2(\Omega)} = \\
&-\left(\nabla\Delta\varphi_k^\varepsilon \mid \nabla n_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon)) + n_k^\varepsilon\wedge\nabla h_{\rm d}(\varphi_k^\varepsilon)\right)_{L^2(\Omega)} \\
&\leqslant \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta\left(\|n_k^\varepsilon\|_{H^2(\Omega)}^2 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2\right)\|\varphi_k^\varepsilon\|_{H^2(\Omega)},
\end{aligned}$$

for all $\eta > 0$, for some (large) constant C_η, using Sobolev's inequalities. From this, we deduce that for all $\eta > 0$, there exists $C_\eta > 0$, depending only on $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$, such that

$$\begin{aligned}
&\left(\Delta^2\varphi_k^\varepsilon \mid n_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon))\right)_{L^2(\Omega)} \leqslant \\
&\qquad \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta\left(1 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2\right)\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2.
\end{aligned} \tag{4.11}$$

Integrating by parts again,

$$\begin{aligned}
&\left(\Delta^2\varphi_k^\varepsilon \mid \varphi_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon))\right)_{L^2(\Omega)} = \\
&-\left(\nabla\Delta\varphi_k^\varepsilon \mid \nabla\varphi_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon)) + \varphi_k^\varepsilon\wedge\nabla h_{\rm d}(\varphi_k^\varepsilon)\right)_{L^2(\Omega)} \\
&\lesssim \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\left(\|\Delta\nabla\varphi_k^\varepsilon\|_{L^2(\Omega)} + \|\nabla\varphi_k^\varepsilon\|_{L^2(\Omega)}\right)\|\varphi_k^\varepsilon\|_{H^2(\Omega)} \\
&\quad + \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2,
\end{aligned}$$

using $\|\nabla h_{\rm d}(\varphi_k^\varepsilon)\|_{L^4(\Omega)} \lesssim \|h_{\rm d}(\varphi_k^\varepsilon)\|_{H^2(\Omega)} \lesssim \|\varphi_k^\varepsilon\|_{H^2(\Omega)}$. Hence, there exists an absolute constant $C > 0$, and for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$\begin{aligned}
&\left(\Delta^2\varphi_k^\varepsilon \mid \varphi_k^\varepsilon\wedge(\Delta\varphi_k^\varepsilon + h_{\rm d}(\varphi_k^\varepsilon))\right)_{L^2(\Omega)} \leqslant \\
&\qquad \left(\eta + C\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\right)\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^4.
\end{aligned} \tag{4.12}$$

Summing up (4.11) and (4.12), one gets $C > 0$ and, for all $\eta > 0$, a constant $C_\eta > 0$ (depending on $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$) such that

$$\begin{aligned}
II_{1,1} \leqslant{}& (\eta + C\|\varphi_k^\varepsilon\|_{H^2(\Omega)})\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 \\
&+ C_\eta\left(1 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2\right)\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2.
\end{aligned} \tag{4.13}$$

The second term is

$$\begin{aligned}
II_{1,2} ={}& -\left(\nabla\Delta\varphi_k^\varepsilon \mid \nabla\varphi_k^\varepsilon\wedge(\Delta n_k^\varepsilon + h_{\rm d}(n_k^\varepsilon) + h_{\rm ext}(t))\right)_{L^2(\Omega)} \\
&-\left(\nabla\Delta\varphi_k^\varepsilon \mid \varphi_k^\varepsilon\wedge(\nabla\Delta n_k^\varepsilon + \nabla h_{\rm d}(\varphi_k^\varepsilon) + \nabla h_{\rm ext}(t))\right)_{L^2(\Omega)}.
\end{aligned}$$

Using Sobolev's inequalities again, we have, for all $\eta > 0$, a constant $C_\eta > 0$ (depending on $\|h_{\rm ext}\|_{L^\infty_{t,x}}$, $\|\nabla h_{\rm ext}\|_{L^\infty_{t,x}}$ and $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$) such that

$$II_{1,2} \leqslant \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta\left(1 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2\right)\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2. \tag{4.14}$$

The third term is

$$II_{1,3} = \Big(\Delta^2\varphi_k^\varepsilon \mid n_k^\varepsilon \wedge (h_{\mathrm{ext}}(t) - h_{\mathrm{ext}}(0))\Big)_{L^2(\Omega)}.$$

Integrating two times by parts, it is easily estimated, thanks to a constant C depending on $\|\partial_t h_{\mathrm{ext}}\|_{L_t^\infty W_x^{2,\infty}}$, as

$$II_{1,3} \leqslant Ct\|n_k^\varepsilon\|_{H^2(\Omega)}\|\varphi_k^\varepsilon\|_{H^2(\Omega)}. \tag{4.15}$$

This gives finally, summing up (4.13), (4.14) and (4.15): there is $C > 0$, and for all $\eta > 0$, there is $C_\eta > 0$ (depending on η, $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$, $\|h_{\mathrm{ext}}\|_{L_t^\infty W_x^{1,\infty}}$ and $\|\partial_t h_{\mathrm{ext}}\|_{L_t^\infty W_x^{2,\infty}}$), such that

$$\begin{aligned} II_1 \leqslant{}& \Big(\eta + C\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big)\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 \\ &+ C_\eta\Big(\Big(1 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2\Big)\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2 + t\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big). \end{aligned} \tag{4.16}$$

Estimating II_2***.*** Split

$$II_2 = II_{2,1} + II_{2,2},$$

with

$$\begin{aligned} II_{2,1} &= \Big(\Delta^2\varphi_k^\varepsilon \mid (|\nabla m_k^\varepsilon|^2 - |\nabla n_k^\varepsilon|^2)(n_k^\varepsilon + \varphi_k^\varepsilon)\Big)_{L^2(\Omega)} \\ &= \Big(\Delta^2\varphi_k^\varepsilon \mid \nabla\varphi_k^\varepsilon \cdot (\nabla\varphi_k^\varepsilon + 2\nabla n_k^\varepsilon)m_k^\varepsilon\Big)_{L^2(\Omega)}, \\ II_{2,2} &= \Big(\Delta^2\varphi_k^\varepsilon \mid |\nabla n_k^\varepsilon|^2(m_k^\varepsilon - n_k^\varepsilon)\Big)_{L^2(\Omega)} = \Big(\Delta^2\varphi_k^\varepsilon \mid |\nabla n_k^\varepsilon|^2\varphi_k^\varepsilon\Big)_{L^2(\Omega)}. \end{aligned}$$

Then, using in particular the Sobolev inequality from Lemma 3.2

$$\|\nabla n_k^\varepsilon\|_{L^\infty(\Omega)} \lesssim \|\nabla n_k^\varepsilon\|_{L^2(\Omega)} + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)},$$

we get:

$$\begin{aligned} II_{2,1} ={}& -\Big(\nabla\Delta\varphi_k^\varepsilon \mid \Delta\varphi_k^\varepsilon \cdot (\nabla\varphi_k^\varepsilon + 2\nabla n_k^\varepsilon)(n_k^\varepsilon + \varphi_k^\varepsilon)\Big)_{L^2(\Omega)} \\ &- \Big(\nabla\Delta\varphi_k^\varepsilon \mid \nabla\varphi_k^\varepsilon \cdot (\Delta\varphi_k^\varepsilon + 2\Delta n_k^\varepsilon)(n_k^\varepsilon + \varphi_k^\varepsilon)\Big)_{L^2(\Omega)} \\ &- \Big(\nabla\Delta\varphi_k^\varepsilon \mid \nabla\varphi_k^\varepsilon \cdot (\nabla\varphi_k^\varepsilon + 2\nabla n_k^\varepsilon)(\nabla n_k^\varepsilon + \nabla\varphi_k^\varepsilon)\Big)_{L^2(\Omega)} \\ \leqslant{}& \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\|\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\times \\ &\times\Big(\|\nabla\varphi_k^\varepsilon\|_{L^\infty(\Omega)} + 2\|\nabla n_k^\varepsilon\|_{L^\infty(\Omega)}\Big)\Big(\|n_k^\varepsilon\|_{L^\infty(\Omega)} + \|\varphi_k^\varepsilon\|_{L^\infty(\Omega)}\Big) \\ &+ \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi_k^\varepsilon\|_{L^4(\Omega)}\times \\ &\times\Big(\|\Delta\varphi_k^\varepsilon\|_{L^4(\Omega)} + 2\|\Delta n_k^\varepsilon\|_{L^4(\Omega)}\Big)\Big(\|n_k^\varepsilon\|_{L^\infty(\Omega)} + \|\varphi_k^\varepsilon\|_{L^\infty(\Omega)}\Big) \\ &+ \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi_k^\varepsilon\|_{L^6(\Omega)}\times \\ &\times\Big(\|\nabla\varphi_k^\varepsilon\|_{L^6(\Omega)} + 2\|\nabla n_k^\varepsilon\|_{L^6(\Omega)}\Big)\Big(\|\nabla n_k^\varepsilon\|_{L^6(\Omega)} + \|\nabla\varphi_k^\varepsilon\|_{L^6(\Omega)}\Big) \end{aligned} \tag{4.17}$$

$$\lesssim \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big(\|n_k^\varepsilon\|_{H^2(\Omega)}+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big)\times$$
$$\times\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)}+\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}+\|n_k^\varepsilon\|_{H^2(\Omega)}+\|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}\Big)$$
$$\lesssim \Big(\eta+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big(\|n_k^\varepsilon\|_{H^2(\Omega)}+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big)\Big)\|\nabla\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}$$
$$+C_\eta\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)}+\|n_k^\varepsilon\|_{H^3(\Omega)}+\|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}\Big)^2\times$$
$$\times\Big(\|n_k^\varepsilon\|_{H^2(\Omega)}+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big)^2,$$

for all $\eta>0$, for some $C_\eta>0$.

Also, for all $\eta>0$, there is $C_\eta>0$ such that

$$\begin{aligned}II_{2,2}&=-\Big(\nabla\Delta\varphi_k^\varepsilon\mid 2\nabla n_k^\varepsilon\Delta n_k^\varepsilon\varphi_k^\varepsilon\Big)_{L^2(\Omega)}-\Big(\nabla\Delta\varphi_k^\varepsilon\mid |\nabla n_k^\varepsilon|^2\nabla\varphi_k^\varepsilon\Big)_{L^2(\Omega)}\\
&\leqslant \|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}\Big(2\|\nabla n_k^\varepsilon\|_{L^\infty(\Omega)}\|\Delta n_k^\varepsilon\|_{L^2(\Omega)}\|\varphi_k^\varepsilon\|_{L^\infty(\Omega)}\\
&\qquad\qquad+\|\nabla n_k^\varepsilon\|^2_{L^6(\Omega)}\|\nabla\varphi_k^\varepsilon\|_{L^6(\Omega)}\Big)\\
&\lesssim \eta\|\nabla\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}\\
&\qquad+C_\eta\Big(\|n_k^\varepsilon\|^2_{H^2(\Omega)}+\|\nabla\Delta n_k^\varepsilon\|^2_{L^2(\Omega)}+1\Big)\|n_k^\varepsilon\|^2_{H^2(\Omega)}\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}.\end{aligned}\tag{4.18}$$

Summing up (4.17) and (4.18), we get: there is $C>0$, and for all $\eta>0$, there is $C_\eta>0$ (with C and C_η depending on $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$) such that

$$\begin{aligned}II_2&\leqslant C\Big(\eta+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big(1+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big)\Big)\|\nabla\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}\\
&\quad+C_\eta\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}\Big(1+\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}\Big)\Big(1+\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}+\|\nabla\Delta n_k^\varepsilon\|^2_{L^2(\Omega)}\Big).\end{aligned}\tag{4.19}$$

Estimating II_3***.*** Now,

$$II_3=II_{3,1}+II_{3,2},$$

with

$$II_{3,1}=-\alpha\Big(\Delta^2\varphi_k^\varepsilon\mid m_k^\varepsilon\wedge\Big(m_k^\varepsilon\wedge h_{\mathrm{d}}(m_k^\varepsilon)\Big)-n_k^\varepsilon\wedge\Big(n_k^\varepsilon\wedge h_{\mathrm{d}}(n_k^\varepsilon)\Big)\Big)_{L^2(\Omega)},$$
$$II_{3,2}=-\alpha\Big(\Delta^2\varphi_k^\varepsilon\mid m_k^\varepsilon\wedge\Big(m_k^\varepsilon\wedge h_{\mathrm{ext}}(t)\Big)-n_k^\varepsilon\wedge\Big(n_k^\varepsilon\wedge h_{\mathrm{ext}}(0)\Big)\Big)_{L^2(\Omega)}.$$

Concerning $II_{3,1}$, first write $m_k^\varepsilon=n_k^\varepsilon+\varphi_k^\varepsilon$, then integrate once by parts, so that $II_{3,1}$ takes the form of a L^2 scalar product between $\nabla\Delta\varphi_k^\varepsilon$ and a sum of terms $\nabla(abc)$, where a, b, c may be n_k^ε (or $h_{\mathrm{d}}(n_k^\varepsilon)$) or φ_k^ε (or $h_{\mathrm{d}}(\varphi_k^\varepsilon)$), and at least one of them is n_k^ε (or $h_{\mathrm{d}}(n_k^\varepsilon)$). Estimating each of a, b, c and their gradients in L^6, one gets: for all $\eta>0$, there is $C_\eta>0$ such that

$$II_{3,1}\leqslant\eta\|\nabla\Delta\varphi_k^\varepsilon\|^2_{L^2(\Omega)}+C_\eta\Big(1+\|n_k^\varepsilon\|^2_{H^2(\Omega)}+\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}\Big)^2\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}.\tag{4.20}$$

Then, split $II_{3,2}$,

$$II_{3,2} = -\alpha\Big(\Delta^2\varphi_k^\varepsilon \mid m_k^\varepsilon \wedge \Big(m_k^\varepsilon \wedge (h_{\rm ext}(t) - h_{\rm ext}(0))\Big)\Big)_{L^2(\Omega)}$$
$$- \alpha\Big(\Delta^2\varphi_k^\varepsilon \mid m_k^\varepsilon \wedge \Big(m_k^\varepsilon \wedge h_{\rm ext}(0)\Big) - n_k^\varepsilon \wedge \Big(n_k^\varepsilon \wedge h_{\rm ext}(0)\Big)\Big)_{L^2(\Omega)}.$$

The second term is estimated as $II_{3,1}$. The first one is split into a sum involving $n_k^\varepsilon \wedge \Big(n_k^\varepsilon \wedge (h_{\rm ext}(t) - h_{\rm ext}(0))\Big)$, and products of $h_{\rm ext}(t)$ with two terms, one of them being φ_k^ε, and the other, φ_k^ε or n_k^ε. This leads to: for all $\eta > 0$, there is $C_\eta > 0$ (also depending on $h_{\rm ext}$) such that

$$\begin{aligned} II_{3,2} \leqslant{}& \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta\Big(\Big(1 + \|n_k^\varepsilon\|_{H^2(\Omega)}^2 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2\Big)\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2 \\ & + t\|n_k^\varepsilon\|_{H^2(\Omega)}^2\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big). \end{aligned} \tag{4.21}$$

Finally, summing up (4.20) and (4.21), we have: for all $\eta > 0$, there is $C_\eta > 0$ (depending on $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$) such that

$$\begin{aligned} II_3 \leqslant{}& \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 \\ & + C_\eta\Big(\Big(1 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2\Big)^2\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2 + t\|\varphi_k^\varepsilon\|_{H^2(\Omega)}\Big). \end{aligned} \tag{4.22}$$

Estimating II_4***.*** Integrating once by parts, we get

$$II_4 = -\Big(\nabla\Delta\varphi_k^\varepsilon \mid \nabla[P_k, \mathcal{F}(0,\cdot)](n^\varepsilon)\Big)_{L^2(\Omega)}.$$

Thus, for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$II_4 \leqslant \eta\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 + C_\eta r_{k,1}^\varepsilon, \tag{4.23}$$

with

$$r_{k,1}^\varepsilon = \|\nabla[P_k, \mathcal{F}(0,\cdot)](n^\varepsilon)\|_{L^2(\Omega)}^2 \underset{k\to\infty}{\longrightarrow} 0 \text{ in } L^\infty(0,T) \text{ for all } T > 0, \text{ with } \varepsilon \text{ fixed,}$$

thanks to Lemma 4.1.

Gathering H^2 ***estimates.*** From (4.16), (4.19), (4.22) and (4.23), we deduce that there is a constant $C > 0$ (depending on $\|n_0\|_{L^\infty((0,\infty),H^3(\Omega))}$), and for all $\eta > 0$, there is $C_\eta > 0$ (depending on η, $\|n_0\|_{L^\infty((0,\infty),H^3(\Omega))}$, $\|h_{\rm ext}\|_{L_t^\infty W_x^{1,\infty}}$ and $\|\partial_t h_{\rm ext}\|_{L_t^\infty W_x^{2,\infty}}$), such that

$$\begin{aligned} & \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\|\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2\Big) \\ & \quad + \Big(\alpha - C(\eta + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}(1 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}))\Big)\|\nabla\Delta\varphi_k^\varepsilon\|_{L^2(\Omega)}^2 \\ & \quad \leqslant C_\eta\Big(\|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2(1 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2)(1 + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2 + \|\nabla\Delta n_k^\varepsilon\|_{L^2(\Omega)}^2) \\ & \quad\quad + t\|\varphi_k^\varepsilon\|_{H^2(\Omega)} + r_{k,1}^\varepsilon\Big). \end{aligned} \tag{4.24}$$

Sum up (4.10) and (4.24), to get: there is a constant $C > 0$ (depending on $\|n_0\|_{L^\infty((0,\infty),H^3(\Omega))}$), and for all $\eta > 0$, there is $C_\eta > 0$ (depending on η, $\|n_0\|_{L^\infty((0,\infty),H^2(\Omega))}$, $\|h_{\text{ext}}\|_{L^\infty_t W^{1,\infty}_x}$ and $\|\partial_t h_{\text{ext}}\|_{L^\infty_t W^{2,\infty}_x}$), such that

(4.25)

$$\begin{aligned}\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}\right) + \Big(\alpha - C(\eta + \|\varphi_k^\varepsilon\|_{H^2(\Omega)}(1+\|\varphi_k^\varepsilon\|_{H^2(\Omega)}))\Big)\|\nabla\Delta\varphi_k^\varepsilon\|^2_{H^2(\Omega)} \\ \leqslant C_\eta\Big(\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)}(1+\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)})(1+\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)} + \|\nabla\Delta n_k^\varepsilon\|^2_{L^2(\Omega)}) \\ + t\|\varphi_k^\varepsilon\|_{H^2(\Omega)} + \tilde{r}_k^\varepsilon\Big),\end{aligned}$$

with $\tilde{r}_k^\varepsilon = \|[P_k, \mathcal{F}(0,\cdot)](n^\varepsilon)\|^2_{H^1(\Omega)} \underset{k\to\infty}{\longrightarrow} 0$ in $L^\infty(0,T)$ for all $T > 0$, with ε fixed.

4.1.4 Conclusion

We take advantage of the H^2 estimate (4.25) by applying the following Gronwall lemma (the proof of which is postponed to Section 6.3).

Lemma 4.2. *There is a constant $K > 0$ (depending on n_0 and h_{ext}) such that, for all $c \in (0, 1/K)$, setting $t_\varepsilon = c\varepsilon\ln(1/\varepsilon)$, there is $\varepsilon_0 = \varepsilon_0(\alpha, c, K)$ such that (4.25) implies:*

$$\begin{aligned}&\forall\varepsilon\in(0,\varepsilon_0],\quad \exists\underline{k}(\varepsilon)\in\mathbb{N}^\star,\quad \forall k\geqslant\underline{k}(\varepsilon),\\ &\sup_{[0,t_\varepsilon]}\|\varphi_k^\varepsilon\|^2_{H^2(\Omega)} \leqslant \left(\frac{\varepsilon^{1-cK}}{K} + K\|\tilde{r}_k^\varepsilon\|_{L^1(0,t_{\varepsilon_0})}\varepsilon^{-1-cK}\right)e^{K\|\nabla\Delta P_k n_0\|^2_{L^2((0,\infty)\times\Omega)}}.\end{aligned}$$

4.1.5 Passing to the limit $k \to \infty$

For each $\varepsilon \in (0, \varepsilon_0]$ fixed, by Lemma 4.2, the sequence $(\varphi_k^\varepsilon)_{k\in\mathbb{N}^\star}$ is bounded in $L^\infty((0,t_\varepsilon), H^2(\Omega))$. Equation (4.5) then implies that the sequence $(\partial_t\varphi_k^\varepsilon)_{k\in\mathbb{N}^\star}$ is bounded in $L^\infty((0,t_\varepsilon), L^2(\Omega))$. Furthermore, (4.25) shows that $(\varphi_k^\varepsilon)_{k\in\mathbb{N}^\star}$ is also bounded in $L^2((0,t_\varepsilon), H^3(\Omega))$. Aubin's Lemma (see [2], [12]) then implies that there is a subsequence of $(\varphi_k^\varepsilon)_{k\in\mathbb{N}^\star}$ converging in $L^2((0,t_\varepsilon), H^2(\Omega))$ towards some φ^ε.

Up to a subsequence, we may assume that $(\partial_t\varphi_k^\varepsilon)_{k\in\mathbb{N}^\star}$ also converges weakly in $L^2((0,t_\varepsilon), L^2(\Omega))$ towards $\partial_t\varphi^\varepsilon$. As k goes to ∞, $P_k n^\varepsilon$ converges towards n^ε in $C([0,t_\varepsilon], H^2(\Omega))\cap H^1((0,t_\varepsilon), L^2(\Omega))$. Thus, $(m_k^\varepsilon)_{k\in\mathbb{N}^\star}$ converges towards some m^ε in $L^2((0,t_\varepsilon), H^2(\Omega))$, with $(\partial_t m_k^\varepsilon)_{k\in\mathbb{N}^\star}$ converging weakly in $L^2((0,t_\varepsilon), L^2(\Omega))$ towards $\partial_t m^\varepsilon$. This is enough to pass to the limit in (4.4), so that m^ε is solution to (4.2). With ε fixed, showing that m^ε is continuous in time with values in H^2 is standard, as well as uniqueness and stability properties: see [4], or [1].

Finally, passing to the limit in Lemma 4.2 yields:

$$\sup_{[0,t_\varepsilon]}\|\varphi^\varepsilon\|^2_{H^2(\Omega)} \leqslant \frac{\varepsilon^{1-cK}}{K}e^{K\|\nabla\Delta n_0\|^2_{L^2((0,\infty)\times\Omega)}},$$

which we write

$$\sup_{[0,t_\varepsilon]} \|\varphi^\varepsilon\|^2_{H^2(\Omega)} \leqslant K'\varepsilon^{1-cK}. \tag{4.26}$$

4.2 Second step: following the slow dynamics after t_ε

From the local-in-time existence result, we know that, for each $\varepsilon \in (0, \varepsilon_0)$, there is $t^\varepsilon > t_\varepsilon$ such that m^ε exists, as a solution to (4.2), in $C([0,t^\varepsilon], H^2(\Omega)) \cap L^2((0,t^\varepsilon), H^3(\Omega))$. We shall show, *via a priori* estimates, that $t^\varepsilon \geqslant T$ (possibly reducing ε_0).

From (3.1), and with $\mathcal{F}$ from (4.3), we deduce that, on $[0,T]\times\Omega$,

$$\varepsilon\partial_t m_{\mathrm{eq}} - \alpha\Delta m_{\mathrm{eq}} = \mathcal{F}(t, m_{\mathrm{eq}}) + \varepsilon\partial_t m_{\mathrm{eq}}.$$

Substracting to (4.2), we get (on $[0,t^\varepsilon]\times\Omega$):

$$\begin{cases} (\varepsilon\partial_t - \alpha\Delta)(m^\varepsilon - m_{\mathrm{eq}}) = \Big(\mathcal{L}(m_{\mathrm{eq}}) + \mathcal{R}(m_{\mathrm{eq}})\Big)(m^\varepsilon - m_{\mathrm{eq}}) + \varepsilon\partial_t m_{\mathrm{eq}}, \\ \partial_\nu(m^\varepsilon - m_{\mathrm{eq}})_{|\partial\Omega} = 0, \end{cases} \tag{4.27}$$

and we consider the associated Cauchy problem with data given at time t_ε. The data at time $t_\varepsilon = c\varepsilon\ln(1/\varepsilon)$ satisfy (using (4.26) and (4.1)):

$$\begin{aligned} \|(m^\varepsilon - m_{\mathrm{eq}})(t_\varepsilon)\|_{H^2(\Omega)} &\leqslant \|(m^\varepsilon - n^\varepsilon)(t_\varepsilon)\|_{H^2(\Omega)} + \|n^\varepsilon(t_\varepsilon) - m_{\mathrm{eq}}(0)\|_{H^2(\Omega)} \\ &\qquad + \|m_{\mathrm{eq}}(0) - m_{\mathrm{eq}}(t_\varepsilon)\|_{H^2(\Omega)} \\ &\leqslant K'\varepsilon^{1-cK} + \|n_0(c\ln(1/\varepsilon)) - m_{\mathrm{eq}}(0)\|_{H^2(\Omega)} \\ &\qquad + \|m_{\mathrm{eq}}(0) - m_{\mathrm{eq}}(t_\varepsilon)\|_{H^2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{4.28}$$

Here, for all $\delta \in H^2(\Omega)$ and $t \in [0,T]$,

$$\begin{aligned} \mathcal{L}(t, m_{\mathrm{eq}}(t))\,\delta =& \alpha|\nabla m_{\mathrm{eq}}(t)|^2\delta + 2\alpha\Big(\nabla m_{\mathrm{eq}}(t)\cdot\nabla\delta\Big)m_{\mathrm{eq}}(t) \\ &+ \delta\wedge h(t, m_{\mathrm{eq}}(t)) + m_{\mathrm{eq}}(t)\wedge\Big(\Delta\delta + h_{\mathrm{d}}(\delta)\Big) \\ &- \alpha\delta\wedge\Big(m_{\mathrm{eq}}(t)\wedge\Big(h_{\mathrm{d}}(m_{\mathrm{eq}}(t)) + h_{\mathrm{ext}}(t)\Big)\Big) \\ &- \alpha m_{\mathrm{eq}}(t)\wedge\Big(\delta\wedge\Big(h_{\mathrm{d}}(m_{\mathrm{eq}}(t)) + h_{\mathrm{ext}}(t)\Big)\Big) \\ &- \alpha m_{\mathrm{eq}}(t)\wedge\Big(m_{\mathrm{eq}}(t)\wedge h_{\mathrm{d}}(\delta)\Big), \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \mathcal{R}(t, m_{\mathrm{eq}}(t))\,(\delta) = & 2\alpha\Big(\nabla m_{\mathrm{eq}}(t)\cdot\nabla\delta\Big)\delta + \alpha|\nabla\delta|^2\delta + \delta\wedge\Big(\Delta\delta + h_{\mathrm{d}}(\delta)\Big) \\ &- \alpha\delta\wedge\Big(\delta\wedge\Big(h_{\mathrm{d}}(m_{\mathrm{eq}}(t)) + h_{\mathrm{ext}}(t)\Big)\Big) - \alpha\delta\wedge\Big(m_{\mathrm{eq}}(t)\wedge h_{\mathrm{d}}(\delta)\Big) \\ &- \alpha m_{\mathrm{eq}}(t)\wedge\Big(\delta\wedge h_{\mathrm{d}}(\delta)\Big) - \alpha\delta\wedge\Big(\delta\wedge h_{\mathrm{d}}(\delta)\Big). \end{aligned} \tag{4.30}$$

In the sequel, we consider

$$\delta^\varepsilon := m^\varepsilon - m_{\mathrm{eq}} \in C([0,t^\varepsilon], H^2(\Omega)) \cap L^2((0,t^\varepsilon), H^3(\Omega)),$$

and we simply prove that in (H^2) energy estimates, the term due to the residual $\mathcal{R}(m_{\mathrm{eq}})\delta^\varepsilon$ is dominated by the terms due to $\alpha\Delta\delta^\varepsilon$ and to the linear term $\mathcal{L}(m_{\mathrm{eq}})\delta^\varepsilon$. We thus come back to the Galerkin approximation δ_k^ε of δ^ε, as in Paragraph 4.1.1. Take the $L^2(\Omega)$ scalar product of the equations with δ_k^ε and $\Delta^2\delta_k^\varepsilon$ and integrate by parts. Estimating $(\delta_k^\varepsilon \mid \mathcal{R}(m_{\mathrm{eq}})(\delta_k^\varepsilon))_{L^2(\Omega)}$ is straightforward. Due to the continuity properties of h_{d} on Sobolev spaces, $(\Delta^2\delta_k^\varepsilon \mid \mathcal{R}(m_{\mathrm{eq}})(\delta_k^\varepsilon))_{L^2(\Omega)}$ produces three kinds of terms. Dropping the exponent ε and subscript k (and using the notation $L(v_1,\dots,v_n)$ for any n-linear application), we examine each of them.

From $\delta\wedge\Big(\delta\wedge h_{\mathrm{d}}(\delta)\Big)$***.*** We have

$$(\Delta^2\delta \mid L(\delta,\delta,\delta))_{L^2(\Omega)} = (\Delta\delta \mid \Delta L(\delta,\delta,\delta))_{L^2(\Omega)} \leqslant \|\Delta\delta\|_{L^2(\Omega)}\|\Delta L(\delta,\delta,\delta)\|_{L^2(\Omega)},$$

which is bounded from above by $C\|\delta\|^4_{H^2(\Omega)}$, since $H^2(\Omega)$ is an algebra.

In the same way, the terms of the form $(\Delta^2\delta \mid L(\delta,\delta))_{L^2(\Omega)}$ are controlled by $\|\delta\|^3_{H^2(\Omega)}$. This rules out the terms from $\delta\wedge h_{\mathrm{d}}(\delta)$, $\delta\wedge\Big(\delta\wedge\Big(h_{\mathrm{d}}(m_{\mathrm{eq}}(t))+h_{\mathrm{ext}}(t)\Big)\Big)$, $\delta\wedge\Big(m_{\mathrm{eq}}(t)\wedge h_{\mathrm{d}}(\delta)\Big)$ and $m_{\mathrm{eq}}(t)\wedge\Big(\delta\wedge h_{\mathrm{d}}(\delta)\Big)$.

From $|\nabla\delta|^2\delta$***.*** Write

$$\begin{aligned}(\Delta^2\delta \mid L(\nabla\delta,\nabla\delta,\delta))_{L^2(\Omega)} &= \\ &- (\nabla\Delta\delta \mid L(\nabla\delta,\nabla\delta,\nabla\delta))_{L^2(\Omega)} - (\nabla\Delta\delta \mid \tilde{L}(\Delta\delta,\nabla\delta,\delta))_{L^2(\Omega)}.\end{aligned}$$

Then,

$$\begin{aligned}|(\nabla\Delta\delta \mid L(\nabla\delta,\nabla\delta,\nabla\delta))_{L^2(\Omega)}| &\leqslant \|\nabla\Delta\delta\|_{L^2(\Omega)}\|L(\nabla\delta,\nabla\delta,\nabla\delta)\|_{L^2(\Omega)} \\ &\leqslant C\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\nabla\delta\|^3_{L^6(\Omega)},\end{aligned}$$

and by Sobolev's inequalities, $\|\nabla\delta\|_{L^6(\Omega)}$ is controlled by $\|\delta\|_{H^2(\Omega)}$.

Also,

$$|(\nabla\Delta\delta \mid \tilde{L}(\Delta\delta,\nabla\delta,\delta))_{L^2(\Omega)}| \leqslant C\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\Delta\delta\|_{L^2(\Omega)}\|\nabla\delta\|_{L^\infty(\Omega)}\|\delta\|_{L^\infty(\Omega)}.$$

From the estimate

$$\|\nabla\delta\|_{L^\infty(\Omega)} \lesssim \|\nabla\delta\|_{L^2(\Omega)} + \|\nabla\Delta\delta\|_{L^2(\Omega)},$$

we get

$$|(\nabla\Delta\delta \mid \tilde{L}(\Delta\delta,\nabla\delta,\delta))_{L^2(\Omega)}| \leqslant C(\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\delta\|^3_{H^2(\Omega)} + \|\nabla\Delta\delta\|^2_{L^2(\Omega)}\|\delta\|^2_{H^2(\Omega)}).$$

This leads to

$$(\Delta^2\delta \mid L(\nabla\delta,\nabla\delta,\delta))_{L^2(\Omega)} \leqslant C(\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\delta\|^3_{H^2(\Omega)} + \|\nabla\Delta\delta\|^2_{L^2(\Omega)}\|\delta\|^2_{H^2(\Omega)}).$$

In the same way, we have

$$(\Delta^2\delta \mid (\nabla m_{\mathrm{eq}}(t)\cdot\nabla\delta)\delta) \leqslant C\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\delta\|^2_{H^2(\Omega)}.$$

The $\delta\wedge\Delta\delta$ term. Again,

$$(\Delta^2\delta \mid L(\delta,\Delta\delta))_{L^2(\Omega)} = -(\nabla\Delta\delta \mid L(\nabla\delta,\Delta\delta))_{L^2(\Omega)} - (\nabla\Delta\delta \mid L(\delta,\nabla\Delta\delta))_{L^2(\Omega)},$$

and as above, we get

$$(\Delta^2\delta \mid L(\delta,\Delta\delta))_{L^2(\Omega)} \leqslant C(\|\nabla\Delta\delta\|_{L^2(\Omega)}\|\delta\|^2_{H^2(\Omega)} + \|\nabla\Delta\delta\|^2_{L^2(\Omega)}\|\delta\|_{H^2(\Omega)}).$$

Finally, there is $C>0$, and for all $\eta>0$, there is $C_\eta>0$ such that

$$\begin{aligned}(\delta_k^\varepsilon \mid \mathcal{R}(m_{\mathrm{eq}})(\delta_k^\varepsilon))_{H^2(\Omega)} \leqslant &\Big(\eta + C\|\delta_k^\varepsilon\|_{H^2(\Omega)} + C\|\delta_k^\varepsilon\|^2_{H^2(\Omega)}\Big)\|\nabla\Delta\delta_k^\varepsilon\|^2_{L^2(\Omega)}\\ &+ C_\eta\Big(\|\delta_k^\varepsilon\|^3_{H^2(\Omega)} + \|\delta_k^\varepsilon\|^4_{H^2(\Omega)} + \|\delta_k^\varepsilon\|^6_{H^2(\Omega)}\Big).\end{aligned}$$

Let k go to infinity, so that the above estimate applies to δ^ε instead of δ_k^ε, up to the local existence time t^ε obtained *via* the convergence of the Galerkin scheme. Coming back to (4.27), still with $\delta^\varepsilon = m^\varepsilon - m_{\mathrm{eq}}$, we get, using (2.5): there is $C>0$, and for all $\eta>0$, there is $C_\eta>0$ such that

(4.31)
$$\begin{aligned}&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\|\delta^\varepsilon\|^2_{H^2(\Omega)}\Big) + \Big(\alpha - \eta - C(1+\|\delta^\varepsilon\|_{H^2(\Omega)})\|\delta^\varepsilon\|_{H^2(\Omega)}\Big)\|\nabla\delta^\varepsilon\|^2_{H^2(\Omega)}\\ &\qquad\leqslant \Big(C_\eta(1+\|\delta^\varepsilon\|_{H^2(\Omega)})^3\|\delta^\varepsilon\|_{H^2(\Omega)} - C_{\mathrm{lin}}\Big)\|\delta^\varepsilon\|^2_{H^2(\Omega)} + \varepsilon^2\|\partial_t m_{\mathrm{eq}}\|^2_{H^2(\Omega)}.\end{aligned}$$

As in the proof of Lemma 4.2, fix $\eta\in(0,\alpha)$, and consider the time $\tilde{t^\varepsilon}\leqslant t^\varepsilon$ up to which, in (4.31), the parenthesis in front of $\|\nabla\delta^\varepsilon\|^2_{H^2(\Omega)}$ (resp. $\|\delta^\varepsilon\|^2_{H^2(\Omega)}$) remains positive (resp. less than $-C_{\mathrm{lin}}/2$). We have, for $t\in(t_\varepsilon,\tilde{t^\varepsilon})$:

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\|\delta^\varepsilon\|^2_{H^2(\Omega)}\Big) \leqslant -\frac{C_{\mathrm{lin}}}{2}\|\delta^\varepsilon\|^2_{H^2(\Omega)} + \varepsilon^2\|\partial_t m_{\mathrm{eq}}\|^2_{H^2(\Omega)}.$$

Gronwall's lemma then implies that

$$\sup_{[t_\varepsilon,\tilde{t^\varepsilon}]}\|\delta^\varepsilon\|^2_{H^2(\Omega)} \leqslant \|\delta^\varepsilon(t_\varepsilon)\|^2_{H^2(\Omega)} + 2\varepsilon T\sup_{[0,T]}\|\partial_t m_{\mathrm{eq}}\|^2_{H^2(\Omega)},$$

so that, for ε small enough, we get $\tilde{t^\varepsilon}\geqslant T$, and $\sup_{[t_\varepsilon,T]}\|\delta^\varepsilon\|_{H^2(\Omega)} \xrightarrow[\varepsilon\to 0]{} 0$. This finishes the proof of Theorem 2.1. $\square$

5 Proof of Proposition 2.2 and Corollary 2.3

Proof of Proposition 2.2. For any $T > 0$ and $n \in C([0,T], H^2(\Omega))$, it is equivalent for n to be solution to (2.8) or to

$$(\partial_t - \alpha\Delta)(n - m_{\text{eq}}) = \Big(\mathcal{L}(0, m_{\text{eq}}) + \mathcal{R}(0, m_{\text{eq}})\Big)(n - m_{\text{eq}}),$$

with the same initial and boundary conditions. The operators $\mathcal{L}(0, m_{\text{eq}})$ and $\mathcal{R}(0, m_{\text{eq}})$ from (4.29) and (4.30) do not depend on time, now. Arguing as in Section 4.2, we get an estimate analogue to (4.31),

(5.1)
$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\|n - m_{\text{eq}}\|^2_{H^2(\Omega)}\Big) \\
&\quad + \Big(\alpha - \eta - C(1 + \|n - m_{\text{eq}}\|_{H^2(\Omega)})\|n - m_{\text{eq}}\|_{H^2(\Omega)}\Big)\|\nabla(n - m_{\text{eq}})\|^2_{H^2(\Omega)} \\
&\quad \leqslant \Big(C_\eta(1 + \|n - m_{\text{eq}}\|_{H^2(\Omega)})^3\|n - m_{\text{eq}}\|_{H^2(\Omega)} - C_{\text{lin}}\Big)\|n - m_{\text{eq}}\|^2_{H^2(\Omega)}.
\end{aligned}$$

Once $\eta \in (0, \alpha/2)$ is chosen, take $\eta_0 > 0$ such that, when $\|m_0 - m_{\text{eq}}\|_{H^2(\Omega)} \leqslant \eta_0$, the parentheses in front of $\|\nabla(n - m_{\text{eq}})\|^2_{H^2(\Omega)}$ and in front of $\|n - m_{\text{eq}}\|^2_{H^2(\Omega)}$ are positive and negative at $t = 0$, respectively. The bootstrap argument then shows that $n \in C([0, \infty), H^2(\Omega))$, and that $n(t)$ converges in $H^2(\Omega, S^2)$, as t goes to ∞, towards $m_{\text{eq}}(t_0)$:

$$\|n(t) - m_{\text{eq}}\|_{H^2(\Omega)} \leq \eta_0 e^{-Ct}, \tag{5.2}$$

for some $C \in (0, C_{\text{lin}})$ depending on η_0. Coming back to (5.1), we see also that $\nabla(n - m_{\text{eq}}) \in L^2((0, \infty), H^2(\Omega))$. $\square$

Proof of Corollary 2.3. When $m_{\text{eq}}(0)$ is constant over Ω, Proposition 2.2 ensures there exists some $\eta_0 > 0$ such that for all $m_0 \in H^2_N(\Omega, S^2)$ satisfying

$$\|m_0 - m_{\text{eq}}\|_{H^2(\Omega)} \leqslant \eta_0,$$

Asumption (ii) in Theorem 2.1 holds true. Furthermore, estimations (5.1) and (5.2) show that the corresponding function n_0 has norms in $L^\infty((0, \infty), H^2(\Omega))$ and in $L^2((0, \infty) \times \Omega))$ controlled in terms of $m_{\text{eq}}(0)$ and η_0 only. Thus, ε_0 in the proof of Theorem 2.1 may also be chosen depending on $m_{\text{eq}}(0)$ and η_0 only, uniformly with respect to m_0. $\square$

6 Appendix

6.1 About the dissipation property (2.5): proof of Lemmas 3.5 and 3.6

For $\delta \in C([0,T], H^\infty(\Omega))$ such that $|m_{\text{eq}} + \delta| \equiv 1$ and $\partial_\nu \delta_{|\partial\Omega} = \partial_\nu \Delta\delta_{|\partial\Omega} = 0$,

(6.1)
$$\mathcal{L}(m^\pm_{\text{eq}})\,\delta = (\lambda \mp d)\delta \wedge u \pm u \wedge (\Delta\delta + h_{\text{d}}(\delta)) + \alpha(d \mp \lambda)u \wedge (\delta \wedge u) - \alpha u \wedge (u \wedge h_{\text{d}}(\delta)).$$

6.1.1 L^2 estimates

Take the $L^2(\Omega)$ scalar product of (6.1) with δ. This yields

$$\begin{aligned}(6.2)\qquad \big(\mathcal{L}(m_{\mathrm{eq}}^{\pm})\,\delta \mid \delta\big)_{L^2(\Omega)} &= \pm\int_\Omega \delta\cdot(u\wedge\Delta\delta) \pm \int_\Omega \delta\cdot(u\wedge h_{\mathrm{d}}(\delta)) \\ &\quad + \alpha(d\mp\lambda)\int_\Omega |\delta\wedge u|^2 - \alpha\int_\Omega (u\cdot h_{\mathrm{d}}(\delta))(u\cdot\delta) + \alpha\int_\Omega \delta\cdot h_{\mathrm{d}}(\delta).\end{aligned}$$

First consider the case of m_{eq}^{+}. Denoting n the exterior normal vector to Ω, the first term in the right-hand side of (6.2) is equal to

$$(6.3)\qquad \sum_{i=1}^{3}\int_\Omega \delta\cdot\partial_i(u\wedge\partial_i\delta) = \sum_{i=1}^{3}\int_{\partial\Omega}\delta\cdot(u\wedge\partial_i\delta)n_i = \int_{\partial\Omega}\delta\cdot(u\wedge\partial_n\delta) = 0.$$

Since h_{d} is continuous on L^2 with norm 1, the second term is bounded from above by $\|\delta\|_{L^2(\Omega)}^2$. Similarly, due to the non-positivity of h_{d}, the last term is non-positive. In the two other terms, we inject the identities

$$(6.4)\qquad |\delta|^2 = -2u\cdot\delta \quad\text{and}\quad |\delta\wedge u|^2 = |\delta|^2 - \frac{1}{4}|\delta|^4,$$

which stem from the equality $|u+\delta|\equiv 1$. This leads to

$$\begin{aligned}(6.5)\qquad \big(\mathcal{L}(m_{\mathrm{eq}}^{+})\,\delta \mid \delta\big)_{L^2(\Omega)} &\leqslant \|\delta\|_{L^2(\Omega)}^2 + \alpha(d-\lambda)\int_\Omega \big(|\delta|^2 - |\delta|^4/4\big) \\ &\qquad + \frac{\alpha}{2}\int_\Omega (u\cdot h_{\mathrm{d}}(\delta))|\delta|^2 \\ &= (1+\alpha(d-\lambda))\|\delta\|_{L^2(\Omega)}^2 + \mathcal{O}(\|\delta\|_{L^2(\Omega)}^3).\end{aligned}$$

In the case of m_{eq}^{-}, we obtain in the same way

$$(6.6)\qquad \big(\mathcal{L}(m_{\mathrm{eq}}^{-})\,\delta \mid \delta\big)_{L^2(\Omega)} \geqslant (\alpha(\lambda+d)-c)\,\|\delta\|_{L^2(\Omega)}^2 + \mathcal{O}(\|\delta\|_{L^2(\Omega)}^3),$$

for some constant c depending on Ω and α only.

6.1.2 H^2 estimates

Take the $L^2(\Omega)$ scalar product of the Laplacian of each term in (6.1) with $\Delta\delta$. This yields

$$\begin{aligned}(6.7)\qquad \big(\Delta\mathcal{L}(m_{\mathrm{eq}}^{\pm})\,\delta \mid \Delta\delta\big)_{L^2(\Omega)} =&(\lambda\mp d)\int_\Omega \Delta\delta\cdot\Delta(\delta\wedge u) \pm \int_\Omega \Delta\delta\cdot\Delta(u\wedge\Delta\delta) \\ &\pm\int_\Omega \delta\cdot\Delta(u\wedge h_{\mathrm{d}}(\delta)) + \alpha(d\mp\lambda)\int_\Omega |\Delta\delta\wedge u|^2 \\ &-\alpha\int_\Omega \Delta\delta\cdot\Delta(u\wedge(u\wedge h_{\mathrm{d}}(\delta))).\end{aligned}$$

Since $\Delta(\delta\wedge u) = (\Delta\delta)\wedge u = 0$, the first term on the right-hand side vanishes. So does the second one, by the same argument as in (6.3). The equality $|u+\delta| \equiv 1$ implies

$$|\Delta\delta \wedge u|^2 = |\Delta\delta|^2 - \left(|\nabla\delta|^2 - (\delta\cdot\Delta\delta)^2\right)^2,$$

so that (6.7) gives, for m_{eq}^+:

(6.8)
$$\left(\Delta\mathcal{L}(m_{\mathrm{eq}}^+)\,\delta \mid \Delta\delta\right)_{L^2(\Omega)} \leqslant -\alpha(\lambda-d)\|\Delta\delta\|_{L^2(\Omega)}^2 + c\|\delta\|_{H^2(\Omega)}^2 + \mathcal{O}\left(\|\delta\|_{H^2(\Omega)}^3\right),$$

for some constant c depending on Ω and α only. Together with (6.5), we get

$$\left(\mathcal{L}(m_{\mathrm{eq}}^+)\,\delta \mid \delta\right)_{H^2(\Omega)} \leqslant -\left(\alpha(\lambda-d)-c\right)\|\delta\|_{H^2(\Omega)}^2 + \mathcal{O}\left(\|\delta\|_{H^2(\Omega)}^3\right), \tag{6.9}$$

which concludes the proof of Lemma 3.5. □

In the case of m_{eq}^-, we have

(6.10)
$$\begin{aligned}\left(\Delta\mathcal{L}(m_{\mathrm{eq}}^+)\,\delta \mid \Delta\delta\right)_{L^2(\Omega)} = &-\int_\Omega \Delta\delta\cdot\Delta(u\wedge h_{\mathrm{d}}(\delta)) + \alpha(\lambda+d)\|\Delta\delta\|_{L^2(\Omega)}^2\\ &-\alpha\int_\Omega \Delta\delta\cdot\Delta(u\wedge(u\wedge h_{\mathrm{d}}(\delta))) + \mathcal{O}\left(\|\delta\|_{H^2(\Omega)}^3\right),\end{aligned}$$

which, together with (6.6), leads to Lemma 3.6. □

6.2 Proof of the commutator lemma 4.1

Writting

$$[P_k, \mathcal{F}(0,\cdot)](n) = (P_k-1)\mathcal{F}(0,n) + \mathcal{F}(0,n) - \mathcal{F}(0,P_k n),$$

the result follows from the convergence of P_k towards 1 pointwise as an operator on $H^1(\Omega)$ (which rules out the term $(P_k-1)\mathcal{F}(0,n)$) as well as on $H_N^2(\Omega)$, combined (to deal with $\mathcal{F}(0,n)-\mathcal{F}(0,P_k n)$) with the continuity of $\mathcal{F}(0,\cdot)$ from $C([0,T],H^2(\Omega))\cap L^2((0,T),H^3)$ to $L^2((0,T),H^1)$.

The latter is a consequence of the continuity properties of h_{d} and of Sobolev's embeddings, implying that $H^2(\Omega)$ is an algebra (so that all applications $n\mapsto n\wedge h_{\mathrm{d}}(n)$, $n\mapsto n\wedge(n\wedge h_{\mathrm{d}}(n))$, $n\mapsto n\wedge h_{\mathrm{ext}}(0)$, $n\mapsto n\wedge(n\wedge h_{\mathrm{ext}}(0))$ are continuous on $L^\infty((0,T),H^2(\Omega))$), and that the product operation maps $H^2\times H^1$ to H^1, so that $n\mapsto n\wedge\Delta n$ and $n\mapsto|\nabla n|^2 n$ are continuous from $L^\infty((0,T),H^2(\Omega))\cap L^2((0,T),H^3)$ to $L^2((0,T),H^1)$. □

6.3 Proof of Gronwall's lemma 4.2

First, consider $k\in\mathbb{N}^\star$ and $\varepsilon>0$ fixed. Set $\phi^\varepsilon(t) = \|\varphi_k^\varepsilon\|_{H^2(\Omega)}^2$, $r(t)=\tilde{r}_k^\varepsilon(t)$, $N_0(t) = \|\nabla\Delta P_k n_0(t)\|_{L^2(\Omega)}^2$ and $N^\varepsilon(t) = \|\nabla\Delta n_k^\varepsilon(t)\|_{L^2(\Omega)}^2$, so that

$$N^\varepsilon(t) = N_0(t/\varepsilon) \quad\text{and}\quad N_0\in L^1(0,\infty).$$

With C from (4.25), choose $\eta \in (0, \alpha/(2C))$. Hence, there exists $\kappa_\eta \in (0,1)$ such that

$$\forall \varphi \in [0, \kappa_\eta], \quad C(\eta + \varphi(1+\varphi)) < \alpha/2.$$

Set $K = 8C_\eta$ (also from (4.25)), $c \in (0, 1/K)$ and $t_\varepsilon = c\varepsilon \ln(1/\varepsilon)$. Then, with

$$t_k^\varepsilon = \sup\{t \in [0, t_\varepsilon] \mid \forall t' \in [0,t],\, \phi^\varepsilon(t') \leqslant \kappa_\eta\}$$

($t_k^\varepsilon > 0$ since $\phi^\varepsilon(0) = 0$), we have:

$$\forall t \in [0, t_k^\varepsilon], \quad \varepsilon {\phi^\varepsilon}'(t) \leqslant K((1 + N^\varepsilon(t))\phi^\varepsilon(t) + t + r(t)).$$

From this, we deduce:

$$\begin{aligned}
\forall t \in [0, t_k^\varepsilon], \quad \phi^\varepsilon(t) &\leqslant \int_0^t \frac{K}{\varepsilon}\Big(t' + r(t')\Big) \exp\left(\frac{K}{\varepsilon}\int_{t'}^t (1 + N^\varepsilon(t''))dt''\right) dt' \\
&\leqslant \left(\int_0^t \frac{K}{\varepsilon}\Big(t' + r(t')\Big) \exp\left(K\frac{t-t'}{\varepsilon}\right) dt'\right) e^{K\|N_0\|_{L^1(0,\infty)}} \\
&\leqslant \left(\frac{\varepsilon}{K} + \frac{K}{\varepsilon}\|r\|_{L^1(0,T_0)}\right) e^{Kt/\varepsilon} e^{K\|N_0\|_{L^1(0,\infty)}},
\end{aligned}$$

with $T_0 = c\varepsilon_0 \ln(1/\varepsilon_0)$ (and ε_0 is chosen below). Now, since $c \in (0, 1/K)$ and $t_\varepsilon = c\varepsilon \ln(1/\varepsilon)$,

$$\forall t \in [0, \min(t_k^\varepsilon, t_\varepsilon)], \quad \phi^\varepsilon(t) \leqslant \left(\frac{\varepsilon^{1-cK}}{K} + K\|r\|_{L^1(0,T_0)}\varepsilon^{-1-cK}\right) e^{K\|N_0\|_{L^1(0,\infty)}},$$

which is less or equal to κ_η as soon as ε belongs to $(0, \varepsilon_0]$, for

$$\varepsilon_0 = \left(\frac{1}{2}\kappa_\eta K e^{-K\|N_0\|_{L^1(0,\infty)}}\right)^{1/(1-cK)},$$

and k greater than $K(\varepsilon)$ such that

$$\forall k \geqslant K(\varepsilon), \quad \|\tilde{r}_k^\varepsilon\|_{L^1(0,T_0)} \leqslant \frac{1}{2}\kappa_\eta \frac{\varepsilon^{1+cK}}{K} e^{-K\|N_0\|_{L^1(0,\infty)}}$$

(which is possible by Lemma 4.1). For this choice of ε and k, we thus have $t_k^\varepsilon \geqslant t_\varepsilon$, and the result follows. □

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