NONLINEAR DIFFRACTIVE OPTICS WITH CURVED PHASES: BEAM DISPERSION AND TRANSITION BETWEEN LIGHT AND SHADOW.

E. Dumas
Laboratoire de Mathématiques et Physique Théorique
Parc de Grandmont
37200 Tours, FRANCE
tel: 02-47-36-73-14 fax: 02-47-36-70-68
dumas@gargan.math.univ-tours.fr

Abstract: We give asymptotic descriptions of smooth oscillating solutions of hyperbolic systems with variable coefficients, in the weakly nonlinear diffractive optics regime. The dependence of the coefficients of the system in the space-time variable (corresponding to propagation in a non-homogeneous medium) implies that the rays are *not* parallel lines –the same occurs with non-planar initial phases. Approximations are given by WKB asymptotics with 3-scales profiles and curved phases. The fastest scale concerns oscillations, while the slowest one describes the modulation of the envelope, which is along rays for the oscillatory components. We consider two kinds of behaviors at the intermediate scale: 'weakly decaying' (Sobolev), giving the transverse evolution of a 'ray packet', and 'shock-type' profiles describing a region of rapid transition for the amplitude.

Contents

Introduction

Long time propagation in homogeneous media

Geometric optics provides asymptotic approximations of waves in the limit of zero wavelength. These approximations are valid only for some propagation distances (see [22]). When looking at longer propagation scales, a diffractive correction is needed. The first rigorous works in this context are probably [7] and [8]. Under oddness assumptions on the nonlinearities, these authors give an approximation to the solution of the initial value problem associated with a nonlinear hyperbolic system $L(u, \partial)u = F(u)$, where $L(u, \partial) = \partial_t +$

$\sum_j A_j(u)\partial_j$. Here, the $A_j(u)$ are symmetric matrices possibly depending on u, but not on the coordinates (t, x): this means that the wave propagates in a homogeneous medium. The initial data oscillate at frequency $1/\varepsilon$, and the approximation is provided, on the Rayleigh distance (of order $1/\varepsilon$), by:

$$\varepsilon^m \sum_{n\in m\mathbb{N}} \varepsilon^n a_n \left(\varepsilon X, X, \frac{\beta \cdot X}{\varepsilon}\right), \; X = (T, Y) \in \mathbb{R}^{1+d}, \tag{0.0.1}$$

i.e. : There is $t^\star$ such that, for all $\varepsilon \in]0, 1]$, the exact solution u^ε is smooth on $[0, t^\star/\varepsilon] \times \mathbb{R}^d$, and admits the asymptotic expansion (0.0.1) as $\varepsilon \to 0$.

The amplitude ε^m is smaller than the one of geometric optics (for $\mathcal{O}(1)$ propagation), so that diffraction affects the principal term of the asymptotics at the same time as the accumulated effects of nonlinearities. The phase of the oscillating wave is $\beta \cdot x = \sum_j \beta_j x_j$. The profiles $a_n(\tilde{X}, X, \theta)$ are smooth, periodic in θ (with mean equal to zero). They are solutions to a coupled system of transport equation at the intermediate scale and Schrödinger equation with slow time. The system is nonlinear for the first profile:

$$\pi a_0 = a_0, \tag{0.0.2a}$$

$$V(\partial_X)a_0 = 0, \tag{0.0.2b}$$

$$V(\partial_{\tilde{X}})a_0 + R(\partial_Y)\partial_\theta^{-1}a_0 + \pi[\Phi(a_0) + \Lambda(a_0)\partial_\theta a_0] = 0. \tag{0.0.2c}$$

The next profiles are solutions to systems with the same structure, but linear.

Equation (0.0.2a) expresses the polarization of a_0, and π is a (matrix) projector associated to L and β. The operator $V(\partial_X) = \partial_T + v.\partial_Y$ is the transport field along rays, with group velocity v. These two equations are similar to the ones of usual geometric optics. Finally, (0.0.2c) represents transverse diffraction, at the time scale $\tilde{T}$, via the scalar operator $R(\partial_Y) = \sum_{i,j} r_{i,j}\partial_{Y_i}\partial_{Y_j}$, whose coefficients are related to the curvature of the characteristic variety of L. The nonlinear term is the same as the one arising in the weakly nonlinear geometric optics equations.

A qualitative difference between the approximate solution (0.0.1) and the geometric optic' one comes from Equation (0.0.2c), which implies non-conservation of supports: Even if the initial data have compact support, $a_0(\varepsilon X, X, \beta \cdot X/\varepsilon)$ does not, whereas the geometric optics approximation does, because it is transported along rays. This explains the spatial dispersion of a laser beam, for example.

This kind of asymptotics has also been studied by Joly, Métivier, Rauch in [19], when rectification effects are present, *i.e.* when interactions of oscillating modes can generate non-oscillatory waves. In [21], D. Lannes considers the case of dispersive systems, with rectification. G. Schneider has treated the case of one equation, in space dimension one, by means of normal forms (see [26]). In [6], T. Colin has studied systems with a 'transparency' property, allowing solutions with greater amplitude; the profiles are then solutions of Davey-Stewartson systems (see also [20]). Diffraction for pulses (*i.e.* when the profiles $a_n(\tilde{X}, X, \theta)$ have compact support in θ) leads to a somewhat different approximation, with a typical profile equation $2\partial_{\tilde{T}}\partial_\theta a_n - \Delta_Y a_n = \partial_\theta f(a_n)$; see [2], [1], and [3] for an approach via 'continuous spectra'.

All these results have been obtained in the general framework of 'long time' propagation (of order $1/\varepsilon$ when the wavelength is ε), and oscillations with respect to one linear phase (Ansatz $\varepsilon^m \sum \varepsilon^n u_n(\varepsilon X, X, \beta \cdot X/\varepsilon)$).

Variable coefficients

The previous results break down as soon as one considers equations with variable coefficients, for example the following wave equation with non-constant refractive index (see Example 0.1): $\left(\dfrac{n(\varepsilon X)^2}{c^2}\partial_T^2 - \Delta_Y\right) u^\varepsilon = 0$.

Here, we are interested in the case of curved phases, for which rays are no longer parallel lines –but before focusing (or caustics): Our study only concerns smooth ($\mathcal{C}^1$) phases. We begin with a change of scale, so that the propagation occurs for times of the order one. Using the slow variable $x = \varepsilon X$ ($= \mathcal{O}(1)$) instead of X, the approximate solution (0.0.1) reads

$$\varepsilon^m \sum_{n \in m\mathbb{N}} \varepsilon^n a_n\left(x, \frac{x}{\varepsilon}, \frac{\beta \cdot x}{\varepsilon^2}\right), \tag{0.0.3}$$

and setting $\varepsilon = \sqrt{\epsilon}$,

$$\epsilon^{m/2} \sum_{n \in m\mathbb{N}} \epsilon^{n/2} a_n\left(x, \frac{x}{\sqrt{\epsilon}}, \frac{\beta \cdot x}{\epsilon}\right).$$

Now, in the case of variable coefficient systems, nonlinear phases are involved, a priori defined on a bounded domain Ω (as $\varepsilon \to 0$) only. That's why we use the Ansatz

$$\epsilon^{m/2} \sum_{n \in m\mathbb{N}} \epsilon^{n/2} a_n\left(x, \frac{\psi(x)}{\sqrt{\epsilon}}, \frac{\phi(x)}{\epsilon}\right), \tag{0.0.4}$$

where $a_n = a_n(x, \omega, \theta) \in \cap_s H^s(\overline{\Omega} \times \mathbb{R}^p \times \mathbb{T}^q)$.

This Ansatz was introduced by J.K. Hunter in [16], and becomes

$$\varepsilon^m \sum_{n \in m\mathbb{N}} \varepsilon^n a_n \left(\varepsilon X, \frac{\psi(\varepsilon X)}{\varepsilon}, \frac{\phi(\varepsilon X)}{\varepsilon^2} \right) \tag{0.0.5}$$

in the scales of (0.0.1) (propagation on distances of order $1/\varepsilon$). The phases ϕ and ψ depend slowly on X, and this regime is called *weakly nonplanar*.

Example 0.1. Consider the linear wave equation

$$\left(\frac{n(\varepsilon X)^2}{c^2} \partial_T^2 - \Delta_Y \right) u^\varepsilon = 0,$$

with refractive index n smooth and bounded, and initial data

$$\begin{cases} u^\varepsilon_{|T=0} = \varepsilon^m g \left(Y, \dfrac{\eta \cdot Y}{\varepsilon} \right) \\ \partial_t u^\varepsilon_{|T=0} = \varepsilon^{m-1} h \left(Y, \dfrac{\eta \cdot Y}{\varepsilon} \right). \end{cases}$$

We choose $h = \frac{c}{n} |\eta| \partial_\theta g$ (polarized data), and $g \in \cap_s H^s(\mathbb{R}^d \times \mathbb{T})$ with $\int g d\theta = 0$ (purely oscillating profiles).

We are interested in the behavior of u^ε for times of the order $1/\varepsilon$. So as to apply the results of Paragraph 1.5, we change variables: $x = \varepsilon X$, $v^{\varepsilon^2}(x) = u^\varepsilon(X)$, and set $\varepsilon^2 = \epsilon$.

$$\begin{cases} \left(\dfrac{n(x)^2}{c^2} \partial_t^2 - \Delta_y \right) v^\epsilon = 0 \\ v^\epsilon_{|t=0} = \epsilon^{m/2} g \left(\dfrac{y}{\sqrt{\epsilon}}, \dfrac{\eta \cdot y}{\epsilon} \right) \\ \partial_t v^\epsilon_{|t=0} = \epsilon^{(m-1)/2} h \left(\dfrac{y}{\sqrt{\epsilon}}, \dfrac{\eta \cdot y}{\epsilon} \right). \end{cases}$$

Theorems 1.1 and 1.2 give an approximation of v^ϵ on the cone $\overline{\Omega} = \{x = (t, y) \in \mathbb{R}^{1+d} / 0 \leq t \leq t^\star, \delta t + |y| \leq \rho\}$: There are $\mathcal{V}^\epsilon, \mathcal{V}^\epsilon_{app} \in \cap_s H^s(\Omega \cap \{t \leq \underline{t}\} \times \mathbb{R}^d \times \mathbb{T})$ (for $\underline{t} < t^\star$) such that:

$$v^\epsilon(x) = \epsilon^{m/2} \mathcal{V}^\epsilon \left(x, \frac{\psi'}{\sqrt{\epsilon}}, \frac{\phi}{\epsilon} \right),$$
$$\forall s \in \mathbb{R}, \|\mathcal{V}^\epsilon_{app} - \mathcal{V}^\epsilon\|_{H^s} \underset{\epsilon \to 0}{\longrightarrow} 0.$$

The phases $\phi, \psi' = (\psi_1, \ldots, \psi_d)$ are defined by:
$$\begin{cases} \frac{n(x)}{c}\partial_t \phi = |\partial_y \phi| \\ \phi_{|t=0} = \eta \cdot y \end{cases}, \qquad \begin{cases} V_\phi(\partial_x)\psi_\mu = 0 \\ \psi_{\mu_{|t=0}} = y_\mu \end{cases},$$
where $V_\phi(x, \partial_x) = \dfrac{n(x)}{c}\partial_{\partial_t} - \dfrac{\partial_y \phi}{|\partial_y \phi|}.\partial_y$: ϕ satisfies an eikonal equation associated with $\frac{n^2}{c^2}\partial_t^2 - \Delta$, and each ψ_μ is annihilated by the associated tangent transport, *i.e.* for the vector field corresponding to the group velocity. When the refractive index really depends on $x = \varepsilon X$, none of these phases is linear.

The approximate profile $\mathcal{V}^\epsilon_{app}$ is given by $\mathcal{V}^\epsilon_{app} = v_0 + \sqrt{\epsilon}v_1 + \epsilon v_2$. The terms $v_n(x, \omega, \theta)$ are determined by equations (1.3.2) to (1.3.4c) and (1.2.5d) to (1.2.5g), which here restrict to (see Remark 1.4, *iii)*):

$$V_\phi(\partial_x)v_n - D(\partial_\omega)\partial_\theta^{-1}v_n = f_n,$$

with $D(\partial_\omega) = \dfrac{1}{2}\sum_\mu \left(\dfrac{n^2}{c^2}(\partial_t \psi_\mu)^2 - |\partial_y \psi_\mu|^2\right)\partial^2_{\omega_\mu}$, and f_n a function of v_{n-1}, v_{n-2} and their derivatives.

Coming back to the original scales and setting $\mathcal{U}^\varepsilon_{app} = \mathcal{V}^{\epsilon^2}_{app}$, we get an approximation of u^ε for times $T \sim 1/\varepsilon$:

$$\forall \underline{t} < t^\star, \forall \alpha \in \mathbb{N}^{1+d}, \text{ on } \left(\frac{1}{\varepsilon}\overline{\Omega}\right) \cap \{T \leq \underline{t}/\varepsilon\},$$
$$\left\| (\varepsilon\partial)^\alpha \left(u^\varepsilon - \varepsilon^m \mathcal{U}^\varepsilon_{app}\left(\varepsilon X, \frac{\psi'(\varepsilon X)}{\varepsilon}, \frac{\phi(\varepsilon X)}{\varepsilon^2}\right)\right)\right\|_{L^\infty} = o(\varepsilon^m).$$

One *cannot* obtain such an approximation (on $\Omega_{\underline{t}}/\varepsilon$) using plane phases: $\psi_\mu(\varepsilon X)/\varepsilon$ differs from its linear part $\partial_x \psi_\mu(0) \cdot X$ by a $\mathcal{O}(\varepsilon|X|^2) = \mathcal{O}(1/\varepsilon)$ term. Because of decaying of $\mathcal{U}^\varepsilon_{app}$, replacing $\psi_\mu(\varepsilon X)/\varepsilon$ by its linear part would generate a $\mathcal{O}(\varepsilon^m)$ error in the approximate solution.

Description of the paper

The main difficulties in the analysis are nonlinearities and variable coefficients. Nonlinearities allow interactions between propagating modes, and thus phase mixing, and induce coupling in the profile equations. These equations also have variable coefficients when the original system does. A solvable system determining the profiles can be obtained only when the operators involved commute. This is guaranteed by coherence assumptions on the phases (*cf.* Paragraph 1.3.2 and Paragraph 1.4). The asymptotic system finally inherits properties from the initial one, which provide energy estimates, in function spaces with different regularities for different variables; see Paragraph 1.4.

Our study deals with quasilinear hyperbolic systems, and the same methods apply to semilinear systems. The paper is organized as follows:

1-In the first part, we look at the weakly decaying case, when the initial profiles $g_n(y, Y, \theta) \in \cap_s H^s(\Omega_0 \times \mathbb{R}^{p-1} \times \mathbb{T})$. The Ansatz is given in Paragraph 1.1. We formally derive the first profile equations by the method of multiple scales in Paragraph 1.2.

We introduce an 'intermediate time' T $(X = (T, Y))$ in order to analyse and solve these equations. In particular, it is natural to require the profile u_1 to be sublinear with respect to T, so that the term $\sqrt{\varepsilon} u_1$ is a corrector of u_0 (Paragraph 1.3).

The next step consists in looking at the interactions between waves constituting u_0, so as to determine which ones have an influence at leading order: the others are considered as correctors. Thanks to our coherence assumptions on the phases, the conditions under which the profiles are sublinear can be derived via the techniques of Joly, Métivier, Rauch [19] or Lannes [21]. The size of the correctors cannot be more specified, because of interactions between oscillatory and non-oscillatory terms (rectification phenomenon). Hence, the asymptotics is based on a first term and two correctors only.

We show existence of u_0, u_1 and u_2 (Paragraph 1.4), and then, stability of exact solutions near the approximate one, via a singular system method and an additional coherence assumption involving both fast and slow phases (Paragraph 1.5).

We illustrate these results with the concrete example of isentropic Euler equations, exhibiting explicit coherent nonlinear phases and profile equations.

2-In the second part, we study oscillating waves whose amplitudes have a rapid variation across some hypersurface: they decay to zero on one side, whereas on the other side, they behave like usual slowly modulated oscillating waves. This describes transition between light and shadow (or sound and silence, in the case of sound waves). Instead of treating a 'matching problem' (such as in [16]), we construct waves with WKB asymptotics in terms of 'shock' profiles: they admit finite limits at $+\infty$ and $-\infty$ with respect to one of the intermediate variables, Y_1. The rays associated with ϕ are then tangent to the surface $\psi_1 = 0$. The profiles u_n split into $u_n = \chi(Y_1) a_n(Y_2, \dots, Y_p) + b_n(Y_1, \dots, Y_p)$, where χ is a fixed 'step' (function with finite limits at $+\infty$

and $-\infty$). The term a_n gives the behavior 'at infinity' (w.r.t. Y_1), while b_n represents the transition layer (of width $\sqrt{\varepsilon}$).

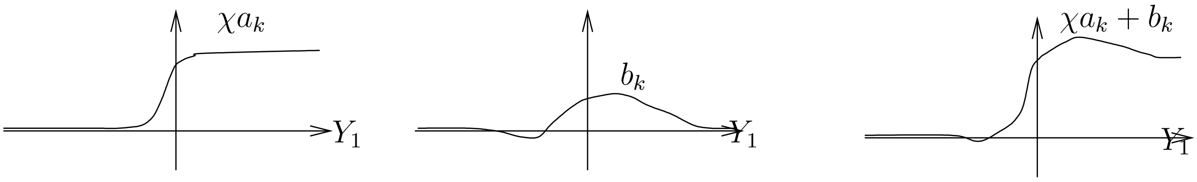

Figure 1: The profiles' shape.

The equations determining the a_n are independent of the b_n. In particular, in the model case when $p = 1$, a_n does not depend on Y, and satisfies the usual equations of weakly nonlinear optics in $\{\psi > 0\}$: see Remark 2.2.

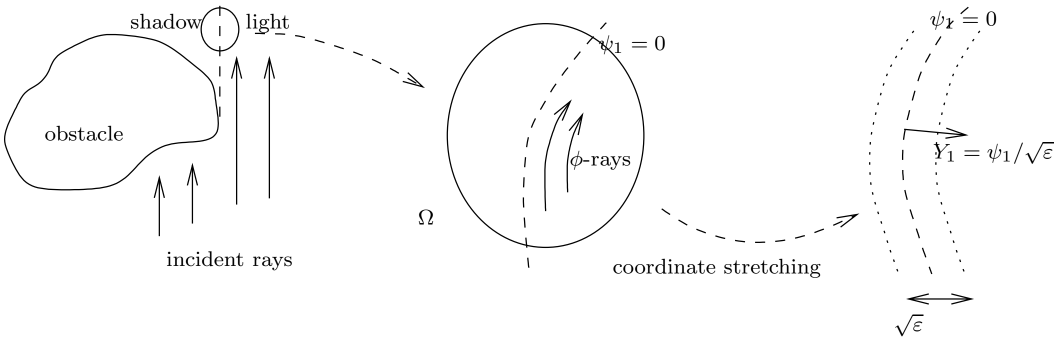

Figure 2: From 'macroscopic' to 'microscopic' description.

In Paragraphs 2.1 and 2.2, we give the Ansatz. We explain in Example 2.1 why only one intermediate phase ψ_1 can govern the rapid transition, and thus, why rectification effects must be avoided at each step of the asymptotics. The simplest way to do so consists in imposing a strong condition on the nonlinearities: their Taylor expansion (at zero) includes odd powers only. We are then able to construct infinite-order asymptotics, based on purely oscillating (or 'zero-mean') profiles.

We construct the profiles in Paragraph 2.4 (under the same coherence assumptions as in Section 1). They satisfy a stronger condition than T-sublinearity (they are bounded).

Finally, thanks to the infinite order asymptotics, we prove stability of the exact solutions via a smooth perturbation method (in the spirit of O. Guès, [13]).

3-The third part is devoted to another approach concerning this kind of rapid transitions, where we get rid of the previous oddness assumption on nonlinearities. We extend the singular system technique of Part 1 to the function spaces of Part 2, using semi-classical pseudo-differential calculus. This method simply requires non-generation of profile mean values at first order (which is for example satisfied in the case of systems of conservation laws), but as a consequence, we only obtain leading-order asymptotics. An additional geometrical 'coherence-type' assumption on the intermediate phase ψ is needed to ensure converge.

These assumptions are satisfied by the explicit example of acoustic waves in Paragraph 3.3.3.

Remark 0.1. *Our profiles depend on several intermediate phases, and only one rapid phase ϕ. One may prove the same existence and stability properties for multiphase asymptotics, and treat interactions of diffracted waves (adding a coherence assumption on the phases ϕ; see [10], [9]).*

1 Dispersion of beams

We study the solutions of a quasilinear symmetric hyperbolic system

$$L(x,u,\partial)u = \partial_t u + \sum_{j=1}^{d} A_j(x,u)\partial_j u = \sum_{j=0}^{d} A_j(x,u)\partial_j u = 0. \tag{1.0.6}$$

We denote by $x = (t,y)$ a point in Ω. Ω is a connected open subset of $\mathbb{R}^{1+d}$ on which the matrices A_j satisfy:

Assumption 1.1. *The matrices $A_j \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{C}^N, \mathcal{M}_N(\mathbb{C}))$ are Hermitian, and $A_0 \equiv I$.*

We are interested in the Cauchy problem associated to (1.0.6), for initial data of the form

$$\varepsilon g\left(y, \frac{\psi^0(y)}{\sqrt{\varepsilon}}, \frac{\phi^0(y)}{\varepsilon}\right), \text{ where } g \in \cap_s H^s(\overline{\Omega} \times \mathbb{R}^{p-1} \times \mathbb{T}).$$

1.1 The Ansatz

The profiles are 'weakly decaying' (*i.e.* H^s) with respect to the intermediate variable. We denote by Ψ the (real) vector space with generators ψ. We introduce an intermediate time $T = t/\sqrt{\varepsilon}$ (thus, a phase $\psi_0 \equiv t$), so as to treat Cauchy problems. The phases $\psi = (\psi_0, \ldots, \psi_{p-1})$ are $\mathbb{R}$-linearly independent. The size of correctors is also measured by T (see 1.3), so that we avoid ill-posed systems such as proposed in [16].

Remark 1.1. *In the sequel, the vector space Ψ will satisfy coherence assumptions. As explained in [18] (p. 56; see also [17]), such a space usually contains a timelike phase ψ_0. Changing variables, one can use this ψ_0 as time variable. But in this case, the matrix A_0 (coefficient of ∂_t in (1.0.6)) then depends on (t, y). For the sake of simplicity, we suppose that $\psi_0 \equiv t$.*

One of the features emphasized in [8] and [19] is rectification, *i.e.* the possibility of interaction between oscillating and non-oscillating modes (travelling at the same speed). This forces the use of an Ansatz with only one term and two correctors, which reads:

$$u^\varepsilon \sim \varepsilon \sum_{n=0}^{2} \varepsilon^{n/2} u_n \left(x, \frac{t}{\sqrt{\varepsilon}}, \frac{\psi'(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon} \right), \tag{1.1.1}$$

where $\psi = (t, \psi') \in \Psi^p$, with $u_n = u_n(x, X, \theta) = u_n(x, T, Y, \theta)$ periodic w.r.t. θ and smooth, and $u_n(x, T, ., \theta) \in \cap_s H^s(\mathbb{R}^{p-1})$.

Notation 1.1. *We denote by Ψ' the (real) space generated by the phases ψ'; it is a dimension $p-1$ subspace of Ψ, such that $\Psi = \Psi' \oplus t\mathbb{R}$.*

1.2 First equations

One formally gets an asymptotic solution to (1.0.6) by plugging the Ansatz into the system and insisting that the coefficients of ε^0, $\varepsilon^{1/2}$ and ε in the residual all vanish. This yields :

$$L_1(d\phi)\partial_\theta u_0 = 0, \tag{1.2.1}$$

$$L_1(d\phi)\partial_\theta u_1 + L_1(d\psi)\partial_X u_0 = 0, \tag{1.2.2}$$

$$(1.2.3) \qquad L_1(d\phi)\partial_\theta u_2 + L_1(d\psi)\partial_X u_1 + L_1(\partial_x)u_0 + B(u_0)\partial_\theta u_0 = 0,$$

where we have set:

Notation 1.2. $L_1(x,\xi) := L(x,0,\xi)$,
$L_1(d\psi)\partial_X := \sum_{\mu=0}^{p-1} L_1(x, d\psi_\mu(x))\partial_{Y_\mu}$,
$B(u) := \sum_{j=1}^{d} \partial_j\phi(x)(\partial_u A_j(x,0).u)$.

Equation (1.2.1) has an oscillating solution only if the matrix $L_1(d\phi)$ is singular. In others words, we assume the phase ϕ satisfies an eikonal equation:

Assumption 1.2. *The quantity* $\det L_1(x, d\phi(x))$ *is identically zero on* $\overline{\Omega}$. *So, there is an eigenvalue* λ *of the matrix* $\mathcal{A}(x,\eta) := \sum_{j=1}^{d} \eta_j A_j(x,0)$ *such that:* $\partial_t\phi + \lambda(x, \partial_y\phi) = 0$. *In addition, we assume* $\partial_y\phi$ *does not vanish on* $\overline{\Omega}$.

Now, (1.2.1), (1.2.2), (1.2.3) are analyzed by means of projections, so that some parts of the profiles are immediately determined. First, we separate oscillations and mean values:

Notation 1.3. *Fourier series* $u = \sum_{\alpha\in\mathbb{Z}} u^\alpha(x,X)e^{i\alpha.\theta}$ *decompose into:* $u = \underline{u}(x,X) + u^\star(x,X,\theta) = \langle u\rangle + u^\star$.

Next, we perform matrix analysis. In order to get smoothness of geometric objects, we assume the characteristic variety of L_1 is a smooth manifold outside the origin:

Assumption 1.3. *The eigenvalues* $\lambda_1(x,\eta) < \cdots < \lambda_Z(x,\eta)$ *of the (Hermitian) matrix* $\mathcal{A}(x,\eta) := \sum_{j=1}^{d} \eta_j A_j(x,0)$ *have constant multiplicity (on* $\overline{\Omega} \times (\mathbb{R}^d \setminus \{0\})$*).*

Proposition 1.1. *Under Assumptions 1.1-1.3, the spectral projection of* $L_1(d\phi)$ *onto its Kernel is smooth on* Ω, *and* $L_1(x, d\phi(x))$ *admits a smooth (also symmetric) partial inverse* $Q(x)$*:*

$$(1.2.4a) \quad \pi(x)L_1(x,d\phi(x)) = L_1(x,d\phi(x))\pi(x) = 0,$$
$$(1.2.4b) \quad Q(x)L_1(x,d\phi(x)) = L_1(x,d\phi(x))Q(x) = 1 - \pi(x), \;\; Q(x)\pi(x) = 0.$$

Using projection of (1.2.1)-(1.2.3) on oscillating and non-oscillating modes, and then applying π and Q, we get:

$$
\begin{aligned}
&\pi u_0^\star = u_0^\star\\
&L_1(d\psi)\partial_X \underline{u}_0 = 0\\
&\pi L_1(d\psi)\partial_X \pi u_0^\star = 0\\
&(1-\pi)u_1^\star = -\partial_\theta^{-1} Q L_1(d\psi)\partial_X u_0^\star\\
&L_1(d\psi)\partial_X \underline{u}_1 + L_1(\partial_x)\underline{u}_0 + \langle B(u_0)\partial_\theta u_0\rangle = 0\\
&\pi L_1(d\psi)\partial_X \pi u_1^\star + \pi L_1(d\psi)\partial_X(1-\pi)u_1^\star + \pi L_1(\partial_x)\pi u_0^\star + \pi(B(u_0)\partial_\theta u_0)^\star = 0\\
&(1-\pi)u_2^\star = -\partial_\theta^{-1} Q\left[L_1(d\psi)\partial_X u_1^\star + L_1(\partial_x)u_0^\star + (B(u_0)\partial_\theta u_0)^\star\right].
\end{aligned}
$$

It is well-known (*cf.* [23], [7], [8]) that the new operators arising in these equations admit a simple diagonal structure:

Proposition 1.2. *Under assumptions 1.3 and 1.2,*
i) $\pi L_1(\partial_x)\pi = \pi[V(x,\partial_x) + C(x)] = \pi[(\partial_t + \mathbf{v}(x).\partial_y) + C(x)]$,
where $\mathbf{v}(x) := \partial_\eta \lambda(x, \partial_y\phi(x))$, *and* $C(x) := \sum_{j=0}^{d} A_j(x,0)(\partial_j\pi)(x)$ *;*
ii) $\forall \rho \in \mathcal{C}^\infty,\ \pi L_1(d\rho)\pi = \pi V(x, d\rho)$ *;*
iii)

$$
\begin{aligned}
\pi L_1(d\psi)\partial_X Q L_1(d\psi)\partial_X \pi &= -\frac{1}{2}\pi \sum_{j,k=1}^{d} \frac{\partial^2\lambda}{\partial\eta_j\partial\eta_k}(x,\partial_y\phi(x)).(\partial_j\psi(x).\partial_X)^2\\
&= -\frac{1}{2}\pi \sum_{j,k=1}^{d} \frac{\partial^2\lambda}{\partial\eta_j\partial\eta_k}(x,\partial_y\phi(x)).(\partial_j\psi'(x).\partial_Y)^2\\
&:= \pi D(x,\partial_Y).
\end{aligned}
$$

Finally, the profiles must be determined by:

(1.2.5a)
$$\pi u_0^\star = u_0^\star,$$
(1.2.5b)
$$L_1(x, d\psi)\partial_X \underline{u}_0 = 0,$$
(1.2.5c)
$$V(x, d\psi)\partial_X u_0^\star = 0,$$
(1.2.5d)
$$(1-\pi)u_1^\star = -\partial_\theta^{-1} Q(x) L_1(x, d\psi)\partial_X u_0^\star,$$
(1.2.5e)
$$L_1(x, d\psi)\partial_X \underline{u}_1 = -L_1(x, \partial_x)\underline{u}_0 - \langle B(x, u_0)\partial_\theta u_0\rangle,$$
(1.2.5f)
$$\pi V(x, d\psi)\partial_X u_1^\star = \partial_\theta^{-1} D(x, \partial_Y)u_0^\star - V(x, \partial_x)u_0^\star - \pi C(x)u_0^\star - \pi(B(x, u_0)\partial_\theta u_0)^\star,$$
(1.2.5g)
$$(1-\pi)u_2^\star = -\partial_\theta^{-1} Q(x)\left[L_1(x, d\psi)\partial_X u_1^\star + L_1(x, \partial_x)u_0^\star + (B(x, u_0)\partial_\theta u_0)^\star\right].$$

Now, equations (1.2.5e) and (1.2.5f) must be analysed.

1.3 The sublinearity condition

We proceed in the same way as in [19]: for $\sqrt{\varepsilon}u_1(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$ to be a corrector of $u_0(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$, it is sufficient that the profile $u_1(x, X, \theta)$ is sublinear with respect to T. This imposes constraints on the right-hand side of (1.2.5e) and (1.2.5f).

Simply setting these right-hand sides equal to zero leads to overdetermined systems of equations on $u_0^\star$ and $\underline{u}_0$. So, we have to look closely at the structure of the waves $u_0^\star$ and $\underline{u}_0$ given by equations (1.2.5b) and (1.2.5c), and to understand the role played by the possible resonances.

1.3.1 Function spaces

Our strategy for solving the profile equations is to use energy estimates. They are local, so from now on, $\overline{\Omega}$ is the cone

$$\overline{\Omega} := \{x = (t, y) \in \mathbb{R}^{1+d} / 0 \leq t \leq t_0, \delta t + |y| \leq \rho\},$$

where $\rho > 0$ is fixed, and δ is sufficiently large, so that on $\overline{\Omega}$,

$$\delta Id + \sum_{j=1}^{d} \frac{y_j}{|y|} A_j(x, 0) \text{ is positive definite and } \left(\delta + \sum_{j=1}^{d} \frac{y_j}{|y|} \mathbf{v}_j(x)\right) > 0.$$

Set $\omega_t := \{y \in \mathbb{R}^d / (t, y) \in \overline{\Omega}\}$, and $\overline{\Omega}_{t_1} := \overline{\Omega} \cap \{t \leq t_1\}$ for $0 < t_1 \leq t_0$.

The function spaces are of Sobolev type. Commutations between ∂_x and $D(\partial_Y)$ cause a loss of derivatives in Y, so we use anisotropic spaces:

Definition 1.1. *For $s \in \mathbb{N}/2$ and $0 < t_1 \leq t_0$, we consider functions $u(x, X, \theta)$ on $\overline{\Omega}_{t_1} \times \mathbb{R}^p \times \mathbb{T}$ such that $(\partial_{y,Y,\theta})^\gamma u$, extended by zero outside $\overline{\Omega}_{t_1}$, belongs to $\mathcal{C}^0\left(\mathbb{R}_T, \mathcal{C}^0([0, t_1], L^2(\mathbb{R}^d \times \mathbb{R}^{p-1} \times \mathbb{T}))\right)$, when $\gamma = (\gamma_y, \gamma_Y, \gamma_\theta) \in \mathbb{N}^{d+(p-1)+1}$ has weight $[\gamma] := |\gamma_y| + |\gamma_Y|/2 + \gamma_\theta/2 \leq s$. For such a u, when t and T are fixed, $u(t, T)$ belongs to the Hilbert space $M^s(\omega_t)$ equipped with the natural L^2 inner product. For T fixed, we say $u(T) \in \mathcal{E}^s(t_1)$ ($u \in \mathcal{C}^0(\mathbb{R}, \mathcal{E}^s(t_1))$), the Banach space with norm: $\| u \|_{\mathcal{E}^s(t_1)} := \sup_{t \in [0,t_1]} \| u(t) \|_{M^s(\omega_t)}$.*

Elements of $M^s(\omega_t)$ are restrictions of functions of Sobolev type on $\mathbb{R}^{d+p-1} \times \mathbb{T}$ (see for example [5]), so that they satisfy the classical properties (see [27] for the proof in the isotropic case):

Proposition 1.3 (Sobolev's embedding).
For $s \in \mathbb{N}/2$ and $s > \frac{2d+p}{4}$, $M^s(\omega_t)$ is a subspace of $L^\infty(\mathbb{R}^d \times \mathbb{R}^{p-1} \times \mathbb{T})$. The embedding has bounded norm for $t \in [0, t_0]$.

Proposition 1.4 (Gagliardo-Nirenberg's inequality). *If $k, s \in \mathbb{N}/2$, $k \leq s$, and $\alpha, \beta \in [1, +\infty], r \in [2, +\infty]$, satisfy $(1 - \frac{k}{s})\frac{1}{\alpha} + \frac{k}{s}\frac{1}{\beta} = \frac{1}{r}$, there is $C > 0$ such that, for all $t \in [0, t_0]$, $u \in \mathcal{S}(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})$ and $\gamma \in \mathbb{N}^{d+p}$ such that $[\gamma] = k$,*

$$\| \partial^\gamma u \|_{L^r} \leq C \| u \|_{L^\alpha}^{1-\frac{k}{s}} \quad \| \partial^s u \|_{L^\beta}^{\frac{k}{s}} .$$

Proposition 1.5 (Moser). *For $s > \frac{2d+p}{4}$, $\mathcal{E}^s(t_1)$ is a Banach algebra on which smooth functions act continuously,* i.e. *:*
If $G : \overline{\Omega}_{t_1} \times \mathbb{R}^{p-1} \times \mathbb{T} \times \mathbb{C}^N \to \mathbb{C}^n$ has regularity $\mathcal{C}^\infty$, for all $u, v \in \mathcal{E}^s(t_1)$,

$$\| G(x, Y, \theta, u+v) - G(x, Y, \theta, u) \|_{\mathcal{E}^s(t_1)} \leq C(\| u \|_{\mathcal{E}^s(t_1)}, \| v \|_{L^\infty(\Omega_{t_1})}) \| v \|_{\mathcal{E}^s(t_1)}$$

(If, in addition, $G(., ., ., 0) \equiv 0$, then $G(x, Y, \theta, u) \in \mathcal{E}^s(t_1)$).

We will also consider $M^s(\omega_t)$ as a set of functions of some variables (Y or (y, θ)), with values as functions of the other variables:

Lemma 1.1. *For every $t \in [0, t_0]$ and $0 \leq k \leq s$, $M^s(\omega_t)$ is a subspace of $H^{2k}(\mathbb{R}^{p-1}_Y, \tilde{M}^{s-k}(\omega_t \times \mathbb{T}))$ and $\tilde{M}^k(\omega_t \times \mathbb{T}, H^{2(s-k)}(\mathbb{R}^{p-1}_Y))$, with embedding bounded on $[0, t_0]$, where $\tilde{M}^k(\omega_t \times \mathbb{T})$ is the space of functions u on $\omega_t \times \mathbb{T}$ such that $\partial^\gamma u \in L^2(\omega_t \times \mathbb{T})$ for $|\gamma_y| + |\gamma_\theta|/2 \leq k$.*

Proof :
The proof is by induction on s and k. We give details only for the first inclusion, in the case $s = k = 1/2$: when $u \in M^{1/2}(\omega_t)$, show that $u \in H^1(\mathbb{R}^{p-1}_Y, L^2(\omega_t \times \mathbb{T}))$. By definition, $\|u\|_{L^2_{y,\theta}}$ belongs to L^2_Y; we have to prove $\partial_Y \|u\|_{L^2_{y,\theta}} \in L^2_Y$.

First chose u in the dense subspace C^∞_c. Let $\varphi \in \mathcal{S}(\mathbb{R}_Y)$ be a test function. Using the Dominated Convergence Theorem,

$$\begin{aligned}\left| \int \|u\|_{L^2_{y,\theta}} \partial_{Y_\mu} \varphi dY \right| &= \left| \lim_{Z_\mu \to 0} \int \|u\|_{L^2_{y,\theta}} \frac{\varphi(Y_\mu + Z_\mu) - \varphi(Y_\mu)}{Z_\mu} dY \right| \\ &= \lim_{Z_\mu \to 0} \left| \int \|u\|_{L^2_{y,\theta}} \frac{\varphi(Y_\mu + Z_\mu) - \varphi(Y_\mu)}{Z_\mu} dY \right|,\end{aligned}$$

and, dropping indices,

$$
\begin{aligned}
&= \underline{\lim}_{Z\to 0} \left| \int \|u\|_{L^2_{y,\theta}} \frac{\varphi(Y+Z)-\varphi(Y)}{Z} dY \right| \\
&= \underline{\lim}_{Z\to 0} \left| \int \frac{\|u\|_{L^2_{y,\theta}}(Y) - \|u\|_{L^2_{y,\theta}}(Y-Z)}{Z} \varphi(Y) dY \right| \\
&\leq \underline{\lim}_{Z\to 0} \int \left| \frac{\|u\|_{L^2_{y,\theta}}(Y) - \|u\|_{L^2_{y,\theta}}(Y-Z)}{Z} \right| |\varphi(Y)|\, dY \\
&\leq \underline{\lim}_{Z\to 0} \int \left\| \frac{u(Y)-u(Y-Z)}{Z} \right\|_{L^2_{y,\theta}} |\varphi(Y)|\, dY \\
&\leq \int \|\partial_Y u\|_{L^2_{y,\theta}}(Y)\, |\varphi(Y)|\, dY \\
&\leq \|u\|_{M^{1/2}(\omega_t)}\, \|\varphi\|_{L^2_Y} ,
\end{aligned}
$$

thanks to Fatou's Lemma and Cauchy-Schwarz inequality.

A density argument shows this is valid for $u \in M^{1/2}(\omega_t)$. □

1.3.2 Operators

Partial Fourier transform (in Y) is the appropriate tool for the study of the linear Cauchy problem associated with (1.2.5b) and (1.2.5c).

Notation 1.4. *Let $\tilde{\mathcal{D}}$ be the characteristic set*

$$\tilde{\mathcal{D}} := \{(x,\chi) \in \overline{\Omega} \times (\mathbb{R}^p \setminus \{0\}) \ / \ \det L_1(x, d(\chi.\psi(x))) = 0\}.$$

We write $\chi = (\sigma, \rho) \in \mathbb{R} \times \mathbb{R}^{p-1}$, so that $L_1(d(\chi.\psi)) = \sigma + L_1(d(\rho.\psi'))$, and we have the spectral decomposition of the symmetric matrix

$$L_1(d(\rho.\psi')) = \sum_{k=1}^{M} \sigma_k(x,\rho) E_k(x,\rho).$$

Once the evolution problem (w.r.t. the variable T) is solved for u_0, we plug the solution into equations (1.2.5e) and (1.2.5f). Now, we have to solve the associated Cauchy problem for u_1 (with vanishing data), expecting a T-sublinear solution. This requires vanishing of the resonances in the right-hand side of the equations. In order to identify clearly such phenomena, we make some coherence assumptions (see [15], [18]):

Assumption 1.4. *The (real) vector space* Ψ*, generated by the* ψ_ν*'s, is* L_1*-coherent,* i.e. *:* $\forall \varphi \in \Psi \setminus \{0\}$,

$$\text{- either: } \forall x \in \overline{\Omega}, d\varphi(x) \neq 0 \text{ and } det L_1(x, d\varphi(x)) = 0,$$
$$\text{- or: } \forall x \in \overline{\Omega}, det L_1(x, d\varphi(x)) \neq 0.$$

Assumption 1.5. *The space* Ψ *is* V*-coherent,* i.e. *:*

$$\forall \varphi \in \Psi \setminus \{0\}, \text{ - either: } \forall x \in \overline{\Omega}, d\varphi(x) \neq 0 \text{ and } V(x, d\varphi(x)) = 0,$$
$$\text{- or: } \forall x \in \overline{\Omega}, V(x, d\varphi(x)) \neq 0.$$

Lemma 1.2. *Under Assumption 1.3 of constant multiplicity, since* $\psi_0 \equiv t$*,*
i) If Ψ *is* L_1*-coherent (Assumption 1.4), the eigenvalues* σ_k *depend on* ρ *only. Hence,* $\tilde{\mathcal{D}}$ *splits up into* $\overline{\Omega} \times \mathcal{D}$*.*
ii) If Ψ *is* V*-coherent (Assumption 1.5), each* $V(x, d\psi_\mu)$ *is independent of* x*.*

Proof :
The proof is similar for the two items. For the first one, we decompose

$$L_1(d(\chi.\psi)) = \sigma + L_1(d(\rho.\psi')) = \sum_{k=1}^{M} (\sigma + \sigma_k(x, \rho))\, E_k(x, \rho),$$

so that $\det L_1(d(\chi.\psi)) = \prod_{k=1}^{M} (\sigma + \sigma_k(x, \rho))$. Coherence says that this determinant, when σ and ρ are fixed, is zero for every x, or does not vanish. Then, with ρ fixed, if we choose $\sigma := -\sigma_k(x_0, \rho)$ for a given x_0, we have $\det L_1(x, d(\chi.\psi)(x)) = 0$ for all x. Hence, for each x, there is k such that $\sigma_k(x, \rho) = -\sigma$.

But the index k does not depend on x, because of Assumption 1.3: for a given k, there is k' such that $\sigma_k(x, \rho) = \partial_t(\rho.\psi')(x) + \lambda_{k'}(x, \partial_y(\rho.\psi')(x))$. Now, when $k \neq l$, we have $k' \neq l'$ and $\sigma_k(x, \rho) - \sigma_l(x, \rho) = \lambda_{k'}(x, \partial_y(\rho.\psi')(x)) - \lambda_{l'}(x, \partial_y(\rho.\psi')(x))$, and this quantity can vanish only if $\partial_y(\rho.\psi')$ does. Finally, coherence implies that this is possible only if $\rho.\psi'$ identically vanishes:

Lemma 1.3. *Assuming coherence of* $\Psi = t\mathbb{R} \oplus \Psi'$*, if* $\varphi \in \Psi' \setminus \{0\}$*, then* $\partial_y \varphi$ *does not vanish on* $\overline{\Omega}$*.*

□

In the study of resonances, it is important to know whether waves travel with the same speed, or interact only briefly, with transverse directions of propagation.

Lemma 1.4. *Under Assumption 1.3 and Assumption 1.4, the functions σ_k are continuous on $\mathbb{R}^{p-1}$ and analytic on $\mathbb{R}^{p-1} \setminus \{0\}$, so that:*

$$\sigma_k(\rho) = c_k.\rho \text{ with } c_k \in \mathbb{R}^{p-1},$$
$$\text{or: for all } c \in \mathbb{R}^{p-1},\ \sigma_k(\rho) \neq c.\rho \text{ almost everywhere.}$$

Proof :

As in the proof of Lemma 1.2, we begin relating σ_k to λ_k:

$$\begin{aligned} L_1(d(\rho.\psi')) &= \partial_t(\rho.\psi') + \mathcal{A}(\partial_y(\rho.\psi')) \\ &= \sum_k \Big(\partial_t(\rho.\psi') + \lambda_k(\partial_y(\rho.\psi'))\Big)\pi_k(\partial_y(\rho.\psi')). \end{aligned} \tag{1.3.1}$$

Lemma 1.3 implies that $\partial_y(\rho.\psi')$ does not vanish if ρ does not, and the constant multiplicity assumption ensures analyticity of $\lambda_k(x,.)$ on $\mathbb{R}^{p-1} \setminus \{0\}$. Distinction of plane and curved modes is then a consequence of the analytic continuation principle. □

The profiles are decomposed by means of projections, thanks to the following Fourier multipliers:

Lemma 1.5. *Under Assumption 1.3 and Assumption 1.4,*
i) The operators $E_k(x, \partial_Y)$, defined by $\mathcal{F}_Y\left(E_k(x, \partial_Y)u\right) = E_k(x,\rho)\hat{u}(T,\rho)$, are projectors on $\mathcal{E}^s(t_1)$, which are orthogonal for $s = 0$.
ii) The operators $\sigma_k(\partial_Y)$, defined by $\mathcal{F}_Y\left(\sigma_k(\partial_Y)u\right) = i\sigma_k(\rho)\hat{u}(T,\rho)$, are continuous from $\mathcal{E}^s(t_1)$ to $\mathcal{E}^{s-1}(t_1)$.

Proof :

The proof of *ii*) is analogous and simpler than the proof of *i*), so we only give the proof of the first claim.

From (1.3.1), we express E_k as (changing numbering if necessary):

$$E_k(x,\rho) = \pi_k(\partial_y(\rho.\psi')),$$

with π_k the spectral projector associated to λ_k in Assumption 1.3. This gives smoothness of E_k on $\overline{\Omega} \times (\mathbb{R}^{p-1} \setminus \{0\})$.

Time variables t et T are only parameters, and continuity with respect to these variables is obtained via Lebesgue's dominated convergence theorem.

When $s = 0$, action on $\mathcal{E}^s(t_1)$ as orthogonal projector is clear, because π_k is such a projector on $\mathbb{C}^N$. Action on $\mathcal{E}^s(t_1)$ ($s \neq 0$) follows from the computation of the commutators $[\partial, E_k(\partial_Y)]$. From its definition, $E_k(\partial_Y)$ commutes with ∂_Y and ∂_θ. For ∂_y, we have:

Lemma 1.6. *The norm of $E_k(x,\rho)$ as linear mapping on $\mathbb{C}^N$, and the norms of the derivatives $\partial_y^\gamma(E_k(x,\rho))$, are bounded independently of $(x,\rho) \in \overline{\Omega} \times (\mathbb{R}^{p-1} \setminus \{0\})$.*

(This is a consequence of continuity with respect to (x,ρ), of degree zero homogeneity w.r.t. ρ and of the fact that x belongs to a compact set). □

Remark 1.2. *In the case $p = 1$, that is with a single phase ψ, the operators $E_k(x,\partial_Y)$ no more depend on ∂_Y, so that they commute with multiplication by Y. We can then define profiles with decay properties,* i.e. $(Y,\partial)^s u \in L^2$.

1.3.3 Profile equations

Our goal is to solve (1.2.5a)-(1.2.5f). We need to suppress u_1 from equations (1.2.5e) and (1.2.5f) so as to determine u_0, and then solve for u_1. The key is to use the constraint of T-sublinearity for u_1, and to eliminate the terms inducing secular parts.

In [19], the structure imposed on u_0 by equations (1.2.5a)-(1.2.5c) is analysed, in order to sort out the nonlinear interactions. In [21], D. Lannes gives a new tool for this derivation, 'average operators' G_W along the characteristic curves of $W(\partial_X) := \partial_T + \sigma(\partial_Y)$, when σ is a degree one homogeneous function:

$$G_W w(x,X,\theta) := \lim_{S\to+\infty} \frac{1}{S}\int_0^S \left(\int e^{i(Y\cdot\rho + S\rho(\sigma))} \hat{w}(x,T+S,\rho,\theta)d\rho \right) dS.$$

Here, the structure of u_0 is understood thanks to the decomposition $u_0 = u_0^\star + \sum_{k=1}^M \underline{u}_{0,k}$, where $\underline{u}_{0,k} := E_k(x,\partial_Y)\underline{u}$: equations (1.2.5b) and (1.2.5c) read

(1.3.2) $\quad (\partial_T + \sigma_k(\partial_Y))\underline{u}_{0,k} = 0$, with $E_k(x,\partial_Y)\underline{u}_{0,k} = \underline{u}_{0,k}$, $1 \le k \le M$,

(1.3.3) $\quad V(d\psi)\partial_X u_0^\star = 0$, with $\pi u_0^\star = u_0^\star$.

Thanks to Lemma 1.2, the symbols $\sigma_k(\rho)$ and $V(d(\rho\cdot\psi')) = V(d\psi')\cdot\rho$ are independent of x, and we can apply the results of [19], [21]. We must separate the case when $u_0^\star$ interacts with the $\underline{u}_{0,k}$'s (*i.e.* when the characteristic variety of $V(d\psi)\partial_X$, which is a hyperplane, is contained in the characteristic variety of $L_1(d\psi)\partial_X$):

Notation 1.5. *Set $\iota = 1$ if $\mathcal{E} := \{\chi = (\sigma, \rho) \in \mathbb{R}^p / V(d\psi) \cdot \chi = 0\}$ is contained in $\mathcal{D}$ (*cf.* Lemma 1.2), $\iota = 0$ if not. If $\iota = 1$ (*i.e.* $\exists k,\ (-\sigma_k(\rho), \rho) \in \mathcal{E},\ \forall \rho$), enumerate the eigenvalues σ_k so that $\sigma_1(\rho) = V(d\psi') \cdot \rho$.*

Finally, T-sublinearity of u_1 is equivalent to:

(1.3.4a) $\quad E_1(\partial_Y) L_1(\partial_x) \underline{u}_{0,1} + \iota E_1(\partial_Y) \langle B(u_0^\star + \underline{u}_{0,1}) \partial_\theta u_0^\star \rangle = 0;$

(1.3.4b) $\quad$ for $k \geq 2, \quad E_k(\partial_Y) L_1(\partial_x) \underline{u}_{0,k} = 0;$

(1.3.4c) $\quad \pi V(\partial_x) u_0^\star - D(\partial_Y) \partial_\theta^{-1} u_0^\star + \pi C u_0^\star + \pi (B(u_0^\star + \iota \underline{u}_{0,1}) \partial_\theta u_0^\star)^\star = 0.$

The first profile, u_0, satisfies equations (1.3.2) to (1.3.4c). The correctors u_1 and u_2 are given by equations (1.2.5d) to (1.2.5g).

1.4 Existence of profiles

In this section, we sketch the proof of:

Theorem 1.1. *For $s > \frac{2d+p}{4} + 1$ and under Assumptions 1.1 to 1.5, when $g \in M^s(\omega_0)$ is such that $\pi g^\star = g^\star$, there exist $t_\star \in]0, t_0]$ and a unique solution $u_0 \in \mathcal{E}^s(t),\ \forall t < t_\star$, to (1.3.2)-(1.3.4c) with initial value $u_{0|_{t=T=0}} = g$.*

In addition, given that u_0, $(1-\pi)u_1^\star$ is uniquely determined by Equation (1.2.5d). The remaining components $\pi u_1^\star$ and $\underline{u}_1$ are the unique solutions to Equations (1.2.5e), (1.2.5f) with polarized initial data in $M^s(\omega_0)$, respectively. Then, $(1-\pi)u_2^\star$ is defined by Equation (1.2.5g).

The maximal time of existence $t^\star := \sup\{t \ /\ u_0 \in \mathcal{E}^s(t)\}$ is in fact independent of s: when $s' \in]\frac{2d+p}{4} + 1, s]$, then $t^\star(s') = t^\star(s)$.

Finally, we have the following estimates, for all $t_1 < t^\star$ and all s:

$$\begin{aligned} &\lim_{T \to +\infty} \|u_0(T)\|_{\mathcal{E}^s(t_1)} < +\infty, \\ &\frac{1}{T} \|u_j(T)\|_{\mathcal{E}^s(t_1)} \underset{T \to +\infty}{\longrightarrow} 0,\ j = 1, 2. \end{aligned} \tag{1.4.1}$$

As a preliminary remark, we must insist on the fact that coherence implies the commutation of the operators $V(d\psi)\partial_X$ and $W_k(\partial_X)$ with ∂_x. This is a necessary condition to obtain an integrable system of equations for the profiles. Now, our strategy for solving the Cauchy problem associated to these equations (in $\mathcal{E}^s(t_1)$) consists in solving first (1.3.4a), (1.3.4b) and (1.3.4c) in $\{T = 0\}$, and then propagating the solutions thanks to (1.3.2) and (1.3.3).

In addition, Equations (1.2.5d) and (1.2.5g) determine completely the parts $(1-\pi)u_1$ and $(1-\pi)u_2$ of the correctors. The polarized parts πu_1 and πu_2 are solutions to linear inhomogeneous equations, and satisfy the estimates (1.4.1) thanks to the analysis of Paragraph 1.3.3, when u_0 solves (1.3.2)-(1.3.4c). As a consequence, we shall only solve (in $\{T=0\}$) the system constituted by Equations (1.3.4a) and (1.3.4c) with $\iota = 1$. We consider the linearized system:

$$(\mathcal{L}) \left\{ \begin{array}{l} E_1(\partial_Y)L_1(\partial_x)E_1(\partial_Y)v + E_1(\partial_Y)\langle B(v'+w',\partial_\theta)w\rangle = 0 \\ \pi V(\partial_x)w - D(\partial_Y)\partial_\theta^{-1}w + \pi C w + \pi(B(v'+w',\partial_\theta)w)^\star = 0. \end{array} \right.$$

The existence (and uniqueness) of solution follows from classical techniques based on energy estimates, such as:

Proposition 1.6. *Let $v', w' \in \mathcal{E}^s(t_1)$, and $s > \frac{2d+p}{4}+1$. If $(v,w)\in\mathcal{E}^s(t_1)^2$ is a solution of $(\mathcal{L})$, then*

$$\| v(t) \|_s^2 + \| w(t) \|_s^2 \leq e^{Ct}\left(\| v(0) \|_s^2 + \| w(0) \|_s^2\right),$$

with a constant C depending on the A_j's and on $\| v' \|_s, \| w' \|_s$ only.

They are consequence of L^2 estimates, and properties of the linear and nonlinear commutators:

Lemma 1.7. *Take $[\gamma]\leq s$.*
i) The operators $[\partial^\gamma, L_1(\partial_x)], [\partial^\gamma, V(\partial_x)]$ and $[\partial^\gamma, D(\partial_Y)\partial_\theta^{-1}]$ have weight less or equal to $[\gamma]$, and map continuously $\mathcal{E}^s(t_1)$ into $\mathcal{E}^0(t_1)$.
ii) The operators $[\partial^\gamma, E_1(\partial_Y)]$ and $[\partial^\gamma, \pi]$ have weight less or equal to $[\gamma]-1$, and map continuously $\mathcal{E}^s(t_1)$ into $\mathcal{E}^1(t_1)$.
iii) When $w, v', w' \in \mathcal{E}^s(t_1)$, then for all $t\in[0,t_1]$,

$$\| [\partial^\gamma, B(v'+w')\partial_\theta]w \|_{M^0(\omega_t)} \leq C(\| v'+w' \|_{\mathcal{E}^s(t_1)}) \| w \|_{M^s(\omega_t)} .$$

A duality argument then concludes for linear equations, and an iterative scheme (in $\mathcal{E}^s$, initialized for example at g_0 and h_0) is used for nonlinear ones:

$$\left\{ \begin{array}{l} E_1(\partial_Y)L_1(\partial_x)E_1(\partial_Y)v_{m+1} + E_1(\partial_Y)\langle B(v_m+w_m,\partial_\theta)w_{m+1}\rangle = 0 \\ \pi V(\partial_x)w_{m+1} - D(\partial_Y)\partial_\theta^{-1}w_{m+1} + \pi C w_{m+1} + \pi(B(v_m+w_m,\partial_\theta)w_{m+1})^\star = 0 \\ E_1(\partial_Y)\underline{v_{m+1}} = v_{m+1} \\ \pi {w_{m+1}}^\star = w_{m+1} \\ v_{m+1|_{t=0}} = g_0 \in \mathcal{E}^s \\ w_{m+1|_{t=0}} = h_0 \in \mathcal{E}^s \end{array} \right.$$

Convergence of this scheme is easily obtained in $\mathcal{E}^{s-1/2}(t_1)$ for some $t_1 > 0$ sufficiently small, and classical results show that the solution has the same regularity as the initial data (see for example [11]).

Finally, the existence time for maximal solutions only depends on the existence in a space $\mathcal{W}^{1,\infty}$ (and, naturally, on initial data). We make use of an 'ODE' argument (following A. Majda, [24]), relying on estimates of the same type as the previous ones:

Proposition 1.7. *If the maximal existence time $t^\star$ for solutions v_0 and w_0 to Equations (1.3.4a) and (1.3.4c) (with smooth initial data) belonging to $\mathcal{E}^s$ ($s > \frac{2d+p}{4} + 1$) is less than t_0, then*

$$\limsup_{t \to t^\star} \left(\| v_0(t) \|_{\mathcal{W}^{1,\infty}(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})} + \| w_0(t) \|_{\mathcal{W}^{1,\infty}(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})} \right) = +\infty,$$

where $\mathcal{W}^{1,\infty}(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})$ is the space of $u \in L^\infty(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})$ which derivatives $\partial^\gamma u$ w.r.t. y, Y and θ belong to $L^\infty(\omega_t \times \mathbb{R}^{p-1} \times \mathbb{T})$, when $[\gamma] \leq 1$.

1.5 Approximation of solutions

From our profiles (for smooth initial data g), we construct a function u^ε_{app}, and our aim is that it provides an asymptotic solution of (1.0.6), which is close to exact solutions: for $\underline{t} < t^\star$,

$$u^\varepsilon_{app}(x) := \varepsilon a^\varepsilon \left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon} \right), \tag{1.5.1}$$

$$a^\varepsilon(x, X, \theta) := (u_0 + \sqrt{\varepsilon} u_1 + \varepsilon u_2)(x, X, \theta) \in \cap_s \mathcal{E}^s(\underline{t}). \tag{1.5.2}$$

From now on, all previous assumptions are supposed to be verified.

1.5.1 Estimates on the residual

Proposition 1.8. *Define the residual $k^\varepsilon(x) := L(x, u^\varepsilon_{app}, \partial_x) u^\varepsilon_{app}$, which is the evaluation at $(x, X, \theta) = (x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$ of*

$$K^\varepsilon(x, X, \theta) := L(x, \varepsilon a^\varepsilon, \varepsilon \partial_x + \sqrt{\varepsilon} \partial\psi . \partial_X + \partial\phi \partial_\theta) a^\varepsilon. \tag{1.5.3}$$

Then, for all $\underline{t} < t^\star$:

$$\forall s, \quad \sup_{0 \leq T \leq \underline{t}/\sqrt{\varepsilon}} \| K^\varepsilon(T) \|_{\mathcal{E}^s(\underline{t})} = o(\varepsilon), \text{ and} \tag{1.5.4}$$

$$\forall \alpha \in \mathbb{N}^{1+d}, \| (\varepsilon \partial_x)^\alpha k^\varepsilon \|_{L^\infty(\Omega_{\underline{t}})} = o(\varepsilon). \tag{1.5.5}$$

Proof :

We expand (1.5.3), and subtract

$$L_1(d\phi)\partial_\theta u_0 + \sqrt{\varepsilon}[L_1(d\phi)\partial_\theta u_1 + L_1(d\psi)\partial_X u_0]$$
$$+ \varepsilon[L_1(d\phi)\partial_\theta u_2 + L_1(d\psi)\partial_X u_1 + L_1(\partial_x)u_0 + B(u_0)\partial_\theta u_0] = 0 :$$

$$\begin{aligned}
&L(x, \varepsilon a^\varepsilon, \varepsilon\partial_x + \sqrt{\varepsilon}\partial\psi\partial_X + \partial\phi\partial_\theta)a^\varepsilon = \\
&\quad \left[\sum_j (\partial_j\phi)A_j(\varepsilon a^\varepsilon) - L_1(d\phi) - \varepsilon\sum_j(\partial_j\phi)\partial_u A_j(0).u_0\right]\partial_\theta u_0 \\
&\quad +\sqrt{\varepsilon}\Bigg[\left(\sum_j(\partial_j\phi)A_j(\varepsilon a^\varepsilon) - L_1(d\phi)\right)\partial_\theta u_1 \\
&\qquad + \sum_\mu\left(\sum_j(\partial_j\psi_\mu)\left(A_j(\varepsilon a^\varepsilon) - A_j(0)\right)\right)\partial_{X_\mu}u_0\Bigg] \\
&\quad +\varepsilon\Bigg[\left(\sum_j(\partial_j\phi)A_j(\varepsilon a^\varepsilon) - L_1(d\phi)\right)\partial_\theta u_2 \\
&\qquad + \sum_\mu\left(\sum_j(\partial_j\psi_\mu)\left(A_j(\varepsilon a^\varepsilon) - A_j(0)\right)\right)\partial_{X_\mu}u_1 + \sum_j\left(A_j(\varepsilon a^\varepsilon) - A_j(0)\right)\partial_j u_0\Bigg] \\
&\quad +\varepsilon^{3/2}\sum_j A_j(\varepsilon a^\varepsilon)\left(\sum_\mu(\partial_j\psi_\mu)\partial_{X_\mu}u_2 + \partial_j u_1\right) \\
&\qquad + \varepsilon^2\sum_j A_j(\varepsilon a^\varepsilon)\partial_j u_2.
\end{aligned}$$

The first term reads

(1.5.6)
$$\left[\varepsilon\sum_j(\partial_j\phi)\partial_u A_j(0).(a^\varepsilon - u_0) + \varepsilon^2\left(\int_0^1 \partial_u^2 A_j(r\varepsilon a^\varepsilon)dr\right).(a^\varepsilon, a^\varepsilon)\right]\partial_\theta u_0.$$

According to (1.4.1), u_1 and u_2 are sublinear functions of T. This implies

$$\sup_{0\le T\le \underline{t}/\sqrt{\varepsilon}} \|u_j(T)\|_{\mathcal{E}^s(\underline{t})} = o(1/\sqrt{\varepsilon}),\ j = 1, 2.$$

This estimate gives the answer concerning (1.5.6). For others terms, we proceed in the same way, simply using the Taylor expansion at first order

$$A_j(\varepsilon a^\varepsilon) - A_j(0) = \varepsilon \left(\int_0^1 \partial_u A_j(r\varepsilon a^\varepsilon)dr \right) .a^\varepsilon.$$

Hence, (1.5.4) is satisfied, and (1.5.5) follows from substitution. □

1.5.2 Stability

We shall use a perturbation method for profiles: define $\mathcal{U}^\varepsilon \in \cap_s \mathcal{E}^s(\underline{t}) = \cap_s H^s(\Omega_{\underline{t}} \times \mathbb{R}^{p-1} \times \mathbb{T})$, $\underline{t} < t^\star$, by $\mathcal{U}^\varepsilon(x, Y, \theta) := a^\varepsilon \left(x, \frac{t}{\sqrt{\varepsilon}}, Y, \theta\right)$, so that

$$u^\varepsilon_{app}(x) = \varepsilon a^\varepsilon \left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon}\right) = \varepsilon \mathcal{U}^\varepsilon \left(x, \frac{\psi'}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon}\right). \tag{1.5.7}$$

Phase variables Y and θ play the same role, and we need a new assumption:

Assumption 1.6. *The space $\Phi + \Psi$ (generated by all phases) is L_1-coherent.*

Remark 1.3.
i) Under geometrical conditions on the characteristic variety of L_1, L_1-coherence of $\Phi + \Psi$ implies V-coherence of Ψ (Assumption 1.5); see [9].
ii) Since we deal with the space generated by all phases, we choose these so that $\psi_1 \equiv t \notin \Psi' + \Phi$.

Theorem 1.2. *Consider $g \in \cap_s M^s(\omega_0)$ as in Theorem 1.1, and f^ε, h^ε such that for all s, $\sup_T \|f^\varepsilon(T)\|_{\mathcal{E}^s(t^\star)} \underset{\varepsilon\to 0}{\longrightarrow} 0$ and $\|h^\varepsilon\|_{M^s(\omega_0)} \underset{\varepsilon\to 0}{\longrightarrow} 0$.*

Let ψ^0, ϕ^0 be initial values of the phases; choose any $\psi' \in \Psi^{p-1}$ such that $\psi'(0, y) = \psi^0$, and suppose that Assumptions 1.1 to 1.6 are satisfied (ϕ is then defined by the Eikonal Equation). Then, there exists $\underline{t} > 0$ such that

$$\begin{cases} L(v^\varepsilon, \partial)v^\varepsilon = \varepsilon f^\varepsilon(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) \\ v^\varepsilon_{|t=0} = \varepsilon(g^\varepsilon + h^\varepsilon)(y, \psi'(0, y)/\sqrt{\varepsilon}, \phi(0, y)/\varepsilon), \end{cases} \tag{1.5.8}$$

admits a unique solution $v^\varepsilon \in C^1(\Omega_{\underline{t}})$ for all $\varepsilon \in]0, 1]$.

In addition, v^ε has the form $v^\varepsilon(x) = \varepsilon \mathcal{V}^\varepsilon \left(x, \frac{\psi'}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon}\right)$, with $\mathcal{V}^\varepsilon \in \cap_s \mathcal{E}^s(\underline{t})$. The approximate solution u^ε_{app} from Equation (1.5.7) is accurate in the following sense:

$$\forall s, \ \|\mathcal{U}^\varepsilon - \mathcal{V}^\varepsilon\|_{\mathcal{E}^s(\underline{t})} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Remark 1.4. *Choosing phases:*
i) In order to define initial data for the profiles u_0*,* u_1 *and* u_2*, only the "matching" of* $u_0(x,\psi/\sqrt{\varepsilon},\phi/\varepsilon)_{|t=0}$ *with* $\frac{1}{\varepsilon}v^\varepsilon_{|t=0}$ *is required. In particular, only the corresponding phases are necessary.*
ii) Once the principal part $g(y,\psi^0(y)/\sqrt{\varepsilon},\phi^0(y)/\varepsilon)$ *of* $\frac{1}{\varepsilon}v^\varepsilon_{|t=0}$ *is known,* Ψ *can be built in the following way: From each* ψ^0_j*, construct all phases* $\psi_{j,k}$ *solutions to eikonal equations associated with* L_1 *and* V*. Because of the coherence assumptions, these phases must belong to* Ψ *(see [18]). Thus, define* Ψ *as the linear span of the* $\psi_{j,k}$*'s, and assume it is coherent.*
iii) In the absence of rectification (conservative systems, or odd nonlinearities), and when the initial data are purely oscillating at first order ($\underline{g}=0$*, implying* $\underline{u}_{0|_{t=T=0}}=0$*), it is possible to consider expansions with purely oscillating profiles* $u_n=u^\star_n(x,Y,\theta)$ *independent of* T *(and, for example, with a single intermediate phase* ψ*), Equations (1.2.5b) and (1.2.5e) for averages being trivially satisfied.*
iv) We could have considered profiles including parts involving different phases (e.g., initial oscillations $g^\star(y,\psi^1/\sqrt{\varepsilon},\phi^0/\varepsilon)$ *and initial averages* $\underline{g}(y,\psi^2/\sqrt{\varepsilon})$*), but this would lead to cumbersome notations.*

Proof :
For v^ε to be a solution to (1.5.8), it is sufficient for $\mathcal{V}^\varepsilon$ to satisfy:

$$\begin{cases} L\left(x,\varepsilon\mathcal{V}^\varepsilon,\partial_x+\dfrac{1}{\sqrt{\varepsilon}}\partial_x\psi'.\partial_Y+\dfrac{1}{\varepsilon}\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon=f^\varepsilon\left(x,\dfrac{t}{\sqrt{\varepsilon}},Y,\theta\right)\\ \mathcal{V}^\varepsilon_{|t=0}=\mathcal{U}^\varepsilon_{|t=0}+g^\varepsilon. \end{cases}\tag{1.5.9}$$

Here, $L\left(\varepsilon\mathcal{V}^\varepsilon,\frac{1}{\sqrt{\varepsilon}}\partial_x\psi'.\partial_Y+\frac{1}{\varepsilon}\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon$, which contains derivatives w.r.t. Y and θ, writes out

$$\begin{aligned}\frac{1}{\varepsilon}L\left(\varepsilon\mathcal{V}^\varepsilon,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon=&\frac{1}{\varepsilon}L_1\left(x,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon\\ &+\frac{1}{\varepsilon}\Bigg[L\left(\varepsilon\mathcal{V}^\varepsilon,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\\ &-L_1\left(x,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\Bigg]\mathcal{V}^\varepsilon\\ =&\frac{1}{\varepsilon}L_1\left(x,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon\\ &+T\left(\varepsilon,\mathcal{V}^\varepsilon,\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta\right)\mathcal{V}^\varepsilon,\end{aligned}$$

where $T(\varepsilon, \mathcal{V}, \eta) := \sum_{j=1}^{d} \eta_j T_j(\varepsilon, \mathcal{V})$, and

$$A_j(\varepsilon\mathcal{V}^\varepsilon) - A_j(0) = \varepsilon \left(\int_0^1 \partial_u A_j(\tau\varepsilon\mathcal{V}^\varepsilon) d\tau \right) .\mathcal{V}^\varepsilon := \varepsilon T_j(\varepsilon, \mathcal{V}^\varepsilon). \tag{1.5.10}$$

This leads to the following (symmetric hyperbolic) singular system:

$$\begin{aligned} &L\left(\varepsilon\mathcal{V}^\varepsilon, \partial_x\right) \mathcal{V}^\varepsilon + T\left(\varepsilon, \mathcal{V}^\varepsilon, \sqrt{\varepsilon}\partial_y\psi'.\partial_Y + \partial_y\phi\partial_\theta\right) \mathcal{V}^\varepsilon \\ &\qquad + \frac{1}{\varepsilon} L_1\left(x, \sqrt{\varepsilon}\partial_x\psi'.\partial_Y + \partial_x\phi\partial_\theta\right) \mathcal{V}^\varepsilon = f^\varepsilon(x, \frac{t}{\sqrt{\varepsilon}}, Y, \theta). \end{aligned} \tag{1.5.11}$$

Our strategy consists in establishing energy estimates that do not depend on ε. The key is the conjugation of $L_1(\sqrt{\varepsilon}\partial_x\psi'.\partial_Y + \partial_x\phi.\partial_\theta)$ to a constant coefficient operator:

Lemma 1.8. *Under Assumptions 1.3 and 1.6, there exists a family of elliptic symbols $V(x, \sqrt{\varepsilon}, \rho, \alpha) \in \mathcal{C}^\infty(\overline{\Omega}, U(N))$, homogeneous w.r.t. (ρ, α), with degree zero, such that: $\forall (x, \varepsilon, \rho, \alpha) \in \overline{\Omega} \times]0,1] \times \mathbb{R}^{p-1} \times \mathbb{Z}$,*

$$V(x, \sqrt{\varepsilon}, \rho, \alpha) L_1(x, d(\sqrt{\varepsilon}\rho.\psi' + \alpha\phi)(x)) V^\star(x, \sqrt{\varepsilon}, \rho, \alpha) = \Delta(\sqrt{\varepsilon}, \rho, \alpha),$$

with $\Delta(\sqrt{\varepsilon}, \rho, \alpha)$ diagonal, homogeneous w.r.t. (ρ, α), with degree one.

Furthermore, the family $\left(V_{\sqrt{\varepsilon}, \rho, \alpha}\right)$ is bounded in $\mathcal{C}^\infty(\overline{\Omega})$.

Proof :

We denote by χ' a basis for $\Psi' + \Phi$: there exists a (constant) matrix S' such that $(\psi', \phi) = S'\chi'$. So, we have the relation:

$$L_1(d(\sqrt{\varepsilon}\rho.\psi' + \alpha\phi)) = L_1(d({}^tS'(\sqrt{\varepsilon}\rho, \alpha).\chi')).$$

Now, the symmetric matrix $L_1(d(\gamma'.\chi'))$ has eigenvalues with constant multiplicity when $(x, \gamma') \in \overline{\Omega} \times (\mathbb{R}^s \setminus \{0\})$, since $\partial_y(\gamma'.\chi')$ does not vanish (from the coherence assumption, and because (t, χ') is a free family), and in view of the decomposition

$$L_1(d(\gamma'.\chi')) = \sum_k \left(\partial_t(\gamma'.\chi') + \lambda_k(\partial_y(\gamma'.\chi'))\right) \pi_k(\partial_y(\gamma'.\chi')).$$

This allows us to construct a family of unitary matrices $W(x, \gamma')$, smooth on $\overline{\Omega} \times (\mathbb{R}^s \setminus \{0\})$, homogeneous w.r.t. γ' with degree zero, such that

$$W(x, \gamma') L_1(d(\gamma'.\chi')) W(x, \gamma')^\star = D(x, \gamma'). \tag{1.5.12}$$

Here, $D(x,\gamma')$ is a diagonal matrix, which is smooth and homogeneous with degree one. We continue Identity (1.5.12) for $\gamma' = 0$ with $W(x,0) := Id$, $D(x,0) := 0$. Homogeneity then ensures that the family $(W_{\gamma'})_{\gamma'}$ is bounded in $\mathcal{C}^\infty(\overline{\Omega})$, and coherence implies (Lemma 1.2) that eigenvalues do not depend on x, so that D does not neither: $D(x,\gamma') = D(\gamma')$. Finally, set:

$$V(x,\sqrt{\varepsilon},\rho,\alpha) := W(x,{}^tS'(\sqrt{\varepsilon}\rho,\alpha))\ ,\ \Delta(\sqrt{\varepsilon},\rho,\alpha) := D({}^tS'(\sqrt{\varepsilon}\rho,\alpha)).$$

□

We now perform a change of functions:

$$\widetilde{\mathcal{V}^\varepsilon} := V_\varepsilon(x,\partial_Y,\partial_\theta)\mathcal{V}^\varepsilon, \tag{1.5.13}$$

and get a new system (equivalent to (1.5.11)):

$$\begin{aligned}(1.5.14)\quad & V_\varepsilon(\partial_Y,\partial_\theta)L\left(\varepsilon V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon},\partial_x\right)V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon}\\ &+V_\varepsilon(\partial_Y,\partial_\theta)T\left(\varepsilon,V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon},\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta\right)V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon}\\ &\qquad+\frac{1}{\varepsilon}\Delta_\varepsilon(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon}=V_\varepsilon(\partial_Y,\partial_\theta)f^\varepsilon(x,\frac{t}{\sqrt{\varepsilon}},Y,\theta).\end{aligned}$$

This is again a symmetric (non-differential) hyperbolic system, and energy estimates are consequences of the following ones for commutators:

Lemma 1.9. *Let $\mathcal{W}\in\mathcal{E}^s(t_1)$, for $s>\frac{2d+p}{4}$.*
i) $V_\varepsilon(x,\partial_Y,\partial_\theta)$ (Lemma 1.8) commutes with ∂_θ and ∂_Y, and $[\partial_{y_j},V_\varepsilon(x,\partial_Y,\partial_\theta)] = (\partial_{y_j}V)_\varepsilon(x,\partial_Y,\partial_\theta)$ is bounded on $\mathcal{E}^s(t_1)$ independently of $\varepsilon\in]0,1]$.
ii) For all $\mathcal{V}\in\mathcal{E}^s(t_1)$, there is $C=C(\|\mathcal{W}\|_{\mathcal{E}^s(t_1)})$ such that:

$$\forall\varepsilon\in]0,1],\left\|[\partial^s,T\left(\varepsilon,\mathcal{W},\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta\right)]\mathcal{V}\right\|_{\mathcal{E}^0(t_1)}\le C\left\|\mathcal{V}\right\|_{\mathcal{E}^s(t_1)}.$$

Proof :
The proof is the same as for Lemma 1.7, with ε fixed. We get uniform estimates just by remarking that ε is less than 1. □

Just as Proposition 1.6, we have:

Proposition 1.9. *Let $\mathcal{W}\in\mathcal{E}^s(t_1)$, for $s>\frac{2d+p}{4}+1$, and $\varepsilon\in]0,1]$.*

If $\widetilde{\mathcal{V}^\varepsilon} \in \mathcal{E}^1(t_1)$ *is a solution to*

$$(1.5.15)\quad V_\varepsilon(\partial_Y,\partial_\theta)L\left(\varepsilon\mathcal{W},\partial_x\right)V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon} + V_\varepsilon(\partial_Y,\partial_\theta)T\left(\varepsilon,\mathcal{W},\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta\right)V_\varepsilon^\star(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon} + \frac{1}{\varepsilon}\Delta_\varepsilon(\partial_Y,\partial_\theta)\widetilde{\mathcal{V}^\varepsilon} = V_\varepsilon(\partial_Y,\partial_\theta)f^\varepsilon(x,\frac{t}{\sqrt{\varepsilon}},Y,\theta),$$

$$(1.5.16)\qquad \widetilde{\mathcal{V}^\varepsilon}_{|t=0} = V_\varepsilon(\partial_Y,\partial_\theta)\mathcal{V}_{|t=0},$$

there is a constant C *depending on* $\|\mathcal{W},\partial_{y,Y,\theta}\mathcal{W}\|_{L^\infty}$ *only, such that:*

$$||\widetilde{\mathcal{V}^\varepsilon}(t)||^2_{M^0(\omega_t)} \leq e^{Ct}||\mathcal{V}(0)||^2_{M^0(\omega_0)} + \int_0^t e^{C(t-t')}||f^\varepsilon(t',y,\frac{t'}{\sqrt{\varepsilon}},Y,\theta)||^2_{M^0(\omega_{t'})}dt'.$$

Proposition 1.10. *Let* $\mathcal{W}\in\mathcal{E}^s(t_1)$*, where* $s>\frac{2d+p}{4}+1$*, and* $\varepsilon\in]0,1]$*.*

If $\widetilde{\mathcal{V}^\varepsilon}\in\mathcal{E}^s(t_1)$ *is a solution to (1.5.15), (1.5.16), then*

$$||\widetilde{\mathcal{V}^\varepsilon}(t)||^2_{M^s(\omega_t)} \leq C'e^{Ct}||\mathcal{V}(0)||^2_{M^s(\omega_0)} + \int_0^t e^{C(t-t')}||f^\varepsilon(t',y,\frac{t'}{\sqrt{\varepsilon}},Y,\theta)||^2_{M^s(\omega'_t)}dt',$$

where the constant C *depends on* $\|\mathcal{W}\|_{\mathcal{E}^s(t_1)}$ *only (and* C' *depends on the* $\mathcal{E}^s(t_1)$ *norm of* $\partial_j^s V_\varepsilon$*, bounded uniformly in* ε*).*

As in Paragraph 1.4, we obtain existence and uniqueness of the solution to the Cauchy problem via an iterative scheme.

So as to get the approximation by $\mathcal{U}^\varepsilon$, we use the same method for the perturbation $\mathcal{W}^\varepsilon := \mathcal{V}^\varepsilon - \mathcal{U}^\varepsilon$, which satisfies a system of the same type as (1.5.11):

$$\mathcal{L}\left(\varepsilon\mathcal{W}^\varepsilon,\partial_x\right)\mathcal{W}^\varepsilon + \mathcal{T}\left(\varepsilon,\mathcal{W}^\varepsilon,\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta\right)\mathcal{W}^\varepsilon + \varepsilon\mathcal{G}(\mathcal{W}^\varepsilon) + \frac{1}{\varepsilon}L_1\left(x,\sqrt{\varepsilon}\partial_x\psi'.\partial_Y+\partial_x\phi\partial_\theta\right)\mathcal{W}^\varepsilon = \mathcal{F}^\varepsilon,$$

with coefficients depending on x, Y and θ (via $\mathcal{U}^\varepsilon$):

$$\mathcal{L}(\partial_x) = \partial_t + \sum_{j=1}^d \mathcal{A}_j\partial_j,\ \text{with } \mathcal{A}_j(W) = A_j(x,\varepsilon\mathcal{U}^\varepsilon+W),$$

$$\mathcal{T}(W,\eta) = \sum_{j=1}^d \eta_j\mathcal{T}_j(W),\ \mathcal{T}_j(W) = T_j(\varepsilon,\mathcal{U}^\varepsilon+W)\ \text{(see (1.5.10))},$$

$$\mathcal{G}(W) = [T(\varepsilon,\mathcal{U}^\varepsilon+W)-T(\varepsilon,\mathcal{U}^\varepsilon)](\sqrt{\varepsilon}\partial_y\psi'.\partial_Y+\partial_y\phi\partial_\theta)\mathcal{U}^\varepsilon + [L(\varepsilon(\mathcal{U}^\varepsilon+W),\partial_x)-L(\varepsilon\mathcal{U}^\varepsilon,\partial_x)]\mathcal{U}^\varepsilon.$$

Here, in view of Estimate (1.5.4) for the residual K^ε, the right-hand side

$$\mathcal{F}^\varepsilon\left(x, \frac{t}{\sqrt{\varepsilon}}, Y, \theta\right) = f^\varepsilon\left(x, \frac{t}{\sqrt{\varepsilon}}, Y, \theta\right) - \frac{1}{\varepsilon}K^\varepsilon\left(x, \frac{t}{\sqrt{\varepsilon}}, Y, \theta\right)$$

and the initial data g^ε are 'small'. Again, we have

$$\|\mathcal{W}^\varepsilon(t)\|^2_{M^s(\omega_t)} \leq C' e^{Ct} \|g^\varepsilon\|^2_{M^s(\omega_0)} + \int_0^t e^{C(t-t')} \left\|\mathcal{F}^\varepsilon\left(t', \frac{t'}{\sqrt{\varepsilon}}\right)\right\|^2_{M^s(\omega_{t'})} dt',$$

which supremum w.r.t. t goes to zero with ε. □

1.6 Diffraction for the weakly compressible, isentropic 3-d Euler equations

Setting: The system of 3-dimensional (compressible) Euler equations reads:

$$\begin{cases} \partial_t\rho + div_y(\rho v) = 0 \\ \partial_t v + (v.\nabla_y)v + \dfrac{\nabla_y p}{\rho} = 0. \end{cases} \tag{1.6.1}$$

Here variables are $x = (t, y) = (t, x_1, x_2, x_3) \in \mathbb{R}^4$, and unknowns are the density ρ and the fluid velocity $v = (v_1, v_2, v_3)$.

In the isentropic case, the pressure p is a function of ρ. We note $f(\rho) := p'(\rho)/\rho$. Weak compressibility means that ρ is near a constant state $\rho_0 \neq 0$: $\rho = \rho_0 + \rho'$, with $\rho' << 1$. We assume that $p'(\rho_0) > 0$, and denote by $c = \sqrt{p'(\rho_0)}$ the sound speed.

We set $\tilde{u} := (\rho', v) = (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \mathbb{R}^4$ and symmetrize (1.6.1), taking the product on the left with $S(\tilde{u}) := Diag(f, \rho, \rho, \rho)$, so that it becomes

$$\tilde{L}(\tilde{u}, \partial)\tilde{u} := S(\tilde{u})\partial_t\tilde{u} + \sum_{j=1}^{3} \tilde{A}_j(\tilde{u})\partial_j\tilde{u} = 0, \ \tilde{A}_j(\tilde{u}) = \begin{pmatrix} f\tilde{u}_j & & f\rho & \\ & \rho\tilde{u}_j & \vdots & \\ f\rho & \dots & \rho\tilde{u}_j & \dots \\ & & \vdots & \rho\tilde{u}_j \end{pmatrix}, \tag{1.6.2}$$

where the doted lines are the $(j+1)$-th.

Now, for $\xi = (\tau, \eta) \in \mathbb{R}^{1+3}$, setting as a new unknown $u := S(0)^{1/2}\tilde{u}$, we conjugate the linearized $(u = 0)$ operator $\tilde{L}_1$ by $S(0)^{-1/2}$ to $L_1(\xi) =$

$\tau Id + \sum_{j=1}^{3} \eta_j A_j(0)$, with symbol $\det L_1(\xi) = \tau^2(\tau^2 - c^2|\eta|^2)$.

Initial data: This linearized operator has constant coefficients, but we consider initial data oscillating with respect to a non-planar phase ϕ^0,

$$u_{|t=0} = \varepsilon h^\varepsilon \left(y, \frac{y_3}{\sqrt{\varepsilon}}, \frac{R}{\varepsilon} \right), \tag{1.6.3}$$

with $R = (y_1^2 + y_2^2)^{1/2}$ the polar radius in the plane (y_1, y_2). There are two possible interpretations of the problem:

Finite time diffraction: We look at a wave propagating in the horizontal directions (according to a cylindrical phase ϕ), and describe the diffraction transversally to this plane. Can we give such a description on a domain independent of ε? What is the qualitative influence of the diffractive perturbation?

Long time propagation: Changing scales as in (0.0.3)- (0.0.5), let us consider the initial value problem

$$\tilde{L}(\tilde{u}, \partial)\tilde{u} = 0, \quad \tilde{u}_{|t=0} = \epsilon^2 g^\epsilon(Y_3, |Y_1, Y_2|/\epsilon) \tag{1.6.4}$$

(case when $h^\varepsilon(y, \omega, \theta)$ does not depend on y). Such initial data correspond to an oscillating wave, modulated in the direction Y_3. Since it has small amplitude, we can ask the question of long-time ($\sim 1/\epsilon$) existence of a smooth solution to these nonlinear conservation laws.

More precisely, the function $h^\varepsilon(y, Y, \theta)$ splits into $h^\varepsilon = h_0 + \sqrt{\varepsilon} h_1 + \varepsilon h_2$, with each $h_n \in H^s(\Omega_0 \times \mathbb{R} \times \mathbb{T})$ periodic w.r.t. θ, with mean value zero. Furthermore, we assume that h_0 is polarized, so that the only eikonal phase generated by R is $\phi_- = R - ct$:

$$\pi_-(0, y) h_0(y) = h_0(y), \ \ i.e.\ \ h_0 \in \ker L_1(d\phi_-) = r_- \mathbb{R}, \tag{1.6.5}$$

where $r_- := (1, y_1/R, y_2/R, 0)/\sqrt{2}$.

The approximate solution: u^ε_{app} takes the form:

$$u^\varepsilon_{app}(x) = \varepsilon \left[u_0 + \sqrt{\varepsilon} u_1 + \varepsilon u_2 \right] \left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi_-}{\varepsilon} \right),$$

with purely oscillating profiles $u_n \in \cap_s H^s(\Omega \times \mathbb{R} \times \mathbb{T})$ and a scalar phase ψ (no intermediate time is needed: see Remark 1.4*iii)*) determined by

$$V(d\psi) = \left[\partial_t + \frac{c}{R} y'.\partial_{y'} \right] \psi = 0 \text{ and } \psi_{|t=0} = y_3,$$

so that: $\psi(x) = y_3$. Since they satisfy the corresponding eikonal equations, ϕ_- and ψ respectively generate L_1- and V-coherent spaces $\Phi = \mathbb{R}\phi_-$ and $\Psi = \mathbb{R}\psi$. In order to apply Theorem 1.1, we only need to check that $Vect(t, \phi, \psi)$ is L_1-coherent. Now, if $\varphi := \alpha t + \beta\phi + \gamma\psi$, $\det L_1(d\varphi) = (\alpha - \beta)^2[(\alpha - \beta)^2 - c^2(\beta^2 - \gamma^2)]$, which is constant, and Assumption 1.6 is fulfilled.

We write the profile equations ((1.2.5d) to (1.2.5g), and (1.3.2) to (1.3.4c) for the principal part u_0. When $u = ar_-$ is polarized, the nonlinear term $B(u)\partial_\theta u$ is given by the auto-interaction coefficient c_-:

$$B(ar_-).r_- = c_- a, \text{ where } c_- = \frac{1+h}{\sqrt{2}}, \; h = (\sqrt{p'})'_{|\rho=\rho_0}.$$

Hence, we have for the amplitude $a_0(x, \omega, \theta)$ of $u_0 = a_0 r_-$:

$$\begin{cases} \displaystyle\int_0^{2\pi} a_0(x, Y, \theta) d\theta = 0 \\ \displaystyle\left(\partial_t + \frac{c}{R}(y_1\partial_{y_1} + y_2\partial_{y_2})\right) a_0 + \frac{c}{2}\partial_\omega^2\partial_\theta^{-1} a_0 + \frac{c}{2R} a_0 + \frac{1+h}{\sqrt{2}}\partial_\theta(a_0^2) = 0. \end{cases}$$

Finally, Theorem 1.1 ensures existence of $u_0, u_1, u_2 \in \cup_s H^s(\Omega_{\underline{t}} \times \mathbb{R} \times \mathbb{T})$ on some $\Omega_{\underline{t}} := \overline{\Omega} \cap \{t \leq \underline{t}\}$. The cone $\overline{\Omega}$ avoids the origin:

$$\overline{\Omega} := (0, \underline{y}) + \{x = (t, y) \in \mathbb{R}^4 \; / \; 0 \leq t \leq t_0, \delta t + |y| \leq \rho\}.$$

Some polarization conditions are required for the data, and we may choose $u_{0|t=0} = h_0$. Then, we apply Theorem 1.2, taking as initial data $u^\varepsilon_{app|t=0} + [\sqrt{\varepsilon}(h_1 - u_{1|t=0}) + \varepsilon(h_2 - u_{2|t=0})(x, y_3/\sqrt{\varepsilon}, R/\varepsilon)$: This provides existence and uniqueness of $u^\varepsilon \in C^1(\Omega_{\underline{t}})$, solution to the Cauchy problem (1.6.2), (1.6.3) for all $\varepsilon \in]0, 1]$, together with the approximation (where $\mathcal{U}^\varepsilon_{app} = u_0 + \sqrt{\varepsilon} u_1 + \varepsilon u_2$):

$$u^\varepsilon(x) = \varepsilon\mathcal{U}^\varepsilon\left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi_-}{\varepsilon}\right),$$
$$\|\mathcal{U}^\varepsilon - \mathcal{U}^\varepsilon_{app}\|_{H^s(\Omega_{\underline{t}} \times \mathbb{R} \times \mathbb{T})} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Conclusion:

Finite time diffraction: We have obtained a solution on a domain independent of ε. This solution differs from the geometric optics' one (approximating the solution of the Cauchy problem with initial data $\varepsilon h^\varepsilon(y, R/\varepsilon)$) by the diffusion

term $\frac{c}{2}\partial_\omega^2\partial_\theta^{-1}$ in the profile equation, which induces dispersion of u^ε in the y_3 direction. In particular, there is decay and non-preservation of compact supports in this direction.
Long time propagation: We have proved existence of the (small) solution $\tilde{u}^\epsilon$ to the Cauchy problem (1.6.4) with lifespan at least $\underline{t}/\epsilon$ for some $\underline{t} > 0$: Even if it strongly oscillates, $\tilde{u}^\epsilon = \epsilon^2 \mathcal{U}^{\epsilon^2}(\epsilon X, Y_3, |Y_1, Y_2|/\epsilon)$ remains smooth, *i.e.* does not develop any shock on this time interval.

2 Transition between light and shadow (for odd nonlinearities)

We now wish to give an asymptotic description of waves for which the amplitude has finite limits at $+\infty$ and $-\infty$ in a given direction. The previous 3-scale asymptotics look appropriate for this purpose, the intermediate scale corresponding to a transition layer (of width $\sqrt{\varepsilon}$).

As shows the following example, rectification is an obstacle to the construction of appropriate smooth profiles.

Example 2.1. The trouble comes from the generation of non-oscillatory terms propagating in different directions, so we consider a non-oscillating problem.

Let χ be a $\mathcal{C}^\infty$ and nondecreasing function on $\mathbb{R}$, with value 0 on $]-\infty, -1]$ and 1 on $[1, +\infty[$. We consider the maximal solution of the system

$$\begin{cases} (\partial_t + \partial_y) v_+^\varepsilon = 0 \\ (\partial_t - \partial_y) v_-^\varepsilon = v_+^\varepsilon v_-^\varepsilon \end{cases} \tag{2.0.6}$$

with initial data $v^\varepsilon_{+|t=0} = \alpha_+ \chi\left(\frac{y}{\sqrt{\varepsilon}}\right)$, $v^\varepsilon_{-|t=0} = \alpha_- \chi\left(\frac{y}{\sqrt{\varepsilon}}\right)$, where α_+ and α_- are two fixed complex numbers, and $\varepsilon \in]0, 1]$.

The solution is globally defined, and reads:

$$\begin{cases} v_+^\varepsilon(t, y) = \alpha_+ \chi\left(\dfrac{y - t}{\sqrt{\varepsilon}}\right), \\ v_-^\varepsilon(t, y) = \alpha_- \chi\left(\dfrac{y + t}{\sqrt{\varepsilon}}\right) e^{\alpha_+ \int_0^t \chi\left(\frac{y+t-2t'}{\sqrt{\varepsilon}}\right) dt'} \\ \qquad\qquad = \alpha_- \chi\left(\dfrac{y + t}{\sqrt{\varepsilon}}\right) exp\left(\dfrac{1}{2}\sqrt{\varepsilon}\alpha_+ \displaystyle\int_{(y-t)/\sqrt{\varepsilon}}^{(y+t)/\sqrt{\varepsilon}} \chi(r) dr\right). \end{cases} \tag{2.0.7}$$

The shape of the graph of $v_-^\varepsilon(t,.)$ is represented on Figure 3.

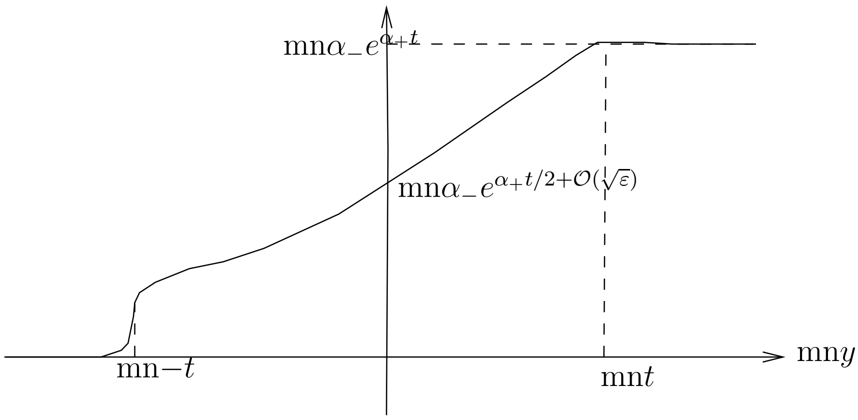

Figure 3: The graph of $v_-^\varepsilon(t,.)$

The scale for the variations of $v_-^\varepsilon(t,.)$ near t and $-t$ is $\sqrt{\varepsilon}$, so one must use variables $(y-t)/\sqrt{\varepsilon}$ and $(y+t)/\sqrt{\varepsilon}$ (the function $v_-^\varepsilon(t,.)$ of the 'slow' variable y converges to a non smooth function, discontinuous at $-t$, and with discontinuous derivative at t). In fact, we can try to decompose v_-^ε as a sum of profiles,

$$v_-^\varepsilon(t,y) = (v_{-,0} + \sqrt{\varepsilon} v_{-,1})\left(t, y, \frac{y+t}{\sqrt{\varepsilon}}, \frac{y-t}{\sqrt{\varepsilon}}\right). \tag{2.0.8}$$

Roughly speaking, if $v_-^\varepsilon(t,.)$ is a function of y near t, $v_{-,0}$ is non smooth, and if $v_-^\varepsilon(t,.)$ depends on $Y_- = (y-t)/\sqrt{\varepsilon}$, it does not look like a 'step': at $y=0$, $(y-t)/\sqrt{\varepsilon} \to -\infty$ as $\sqrt{\varepsilon} \to 0$, but $v_-^\varepsilon(t,0) = e^{\alpha_+ t/2} \neq 0$.

More precisely, if we look for v_-^ε under the form (2.0.8) with profiles $v_{-,n}(x, Y_+, Y_-)$ admitting limits as $Y_\pm \to +\infty$ and going to 0 as $Y_\pm \to -\infty$, we get ($v_+^\varepsilon = v_{+,0}((y-t)/\sqrt{\varepsilon})$, $v_{+,0}(Y_-) = \alpha_+ \chi(Y_-)$):

$$(\partial_t - \partial_y) v_-^\varepsilon(x) = \left(\frac{-2}{\sqrt{\varepsilon}} \partial_{Y_-} v_{-,0} + (\partial_t - \partial_y) v_{-,0} - 2\partial_{Y_-} v_{-,1}\right)\left(t, y, \frac{y+t}{\sqrt{\varepsilon}}, \frac{y-t}{\sqrt{\varepsilon}}\right),$$

$$\begin{cases} v_{-,0} = v_{-,0}(x, Y_+) \\ 2\partial_{Y_-} v_{-,1} = (\partial_t - \partial_y) v_{-,0}(x, Y_+) - v_{+,0}(Y_-) v_{-,0}(x, Y_+). \end{cases} \tag{2.0.9}$$

Taking the limit $(Y_- \to +\infty)$ of the last equation, we obtain:

$$(\partial_t - \partial_y)v_{-,0} = \alpha_+ v_{-,0}, \text{ so } v_{-,0} = \alpha_- \chi(Y_+)e^{\alpha_+ t}. \tag{2.0.10}$$

Now, if we want to integrate the derivative $\partial_{Y_-} v_{-,1}$, we must impose, as Y_- goes to $-\infty$: $(\partial_t - \partial_y)v_{-,0} = 0$. This is in contradiction with (2.0.10). The system (2.0.9) is overdetermined, and the Ansatz (2.0.8) is not valid.

Conclusion: The rapid transition is impossible as soon as the system generates several intermediate phases governing this transition. This is typically the case when rectification occurs, because the non-oscillating profiles are solutions to a hyperbolic system (see Equation (1.2.5b)). As shown in Paragraph 2.3.3, the only admissible intermediate phase is constant along the ϕ-rays (eikonal equation in Assumption 2.2).

2.1 Framework, notations

We consider the same system as in the first part:

$$L(x, u, \partial)u = \partial_t u + \sum_{j=1}^{d} A_j(x, u)\partial_j u = 0.$$

We assume again symmetric hyperbolicity (Assumption 1.1) and constant multiplicity (Assumption 1.3). Furthermore, nonlinearities will have an 'oddness' property:

Assumption 2.1. *For all x, the Taylor expansions of the matrices $A_j(x, u)$ at $u = 0$ contain only even powers. We denote the lowest order term of this expansion by $\Lambda_j(x, .)$, $(K_j - 1)$-linear and symmetric ($K_j - 1 \geq 2$):*

$$A_j(x, u) - A_j(x, 0) = \Lambda_j(x, u, \dots, u) + \mathcal{O}(|u|^{K_j}).$$

Example 2.2. *This kind of nonlinearity arises in optics models, such as Maxwell equations coupled with an anharmonic oscillator (see [4]),*

$$\begin{cases} \partial_t E = -\text{curl } B - \partial_t P, \\ \partial_t B = \text{curl } E, \\ \partial_t^2 P + \dfrac{1}{T}\partial_t P + \nabla_P V(P) = \gamma E, \end{cases}$$

when the potential V has a Taylor expansion at $P=0$ containing only even powers.

Again, the phase ϕ is defined as solution to an eikonal equation asssociated with L_1 (Assumption 1.2).

Now, the nonlinear term which will appear at leading order in the formal computations is $\Lambda(x,u_1,\dots,u_{K-1})\partial_\theta v$:

Notation 2.1. *We define $K := \min_{j\geq 1} K_j$, the smallest order for nonlinear terms, and $\Lambda(x,u_1,\dots,u_{K-1}) := \sum\limits_{K_j=K} \partial_j\phi(x)\Lambda_j(x,u_1,\dots,u_{K-1})$, modulo permutations of arguments*
*(*i.e. *$\Lambda(x,u_1,\dots,u_{K-1})$ stands for $\frac{1}{(K-1)!}\sum\Lambda(x,u_{j_1},\dots,u_{j_{K-1}})$).*

Actually, our Ansatz has no mean (non-oscillating) term; its spectrum is odd, so that such non-oscillating terms do not appear through nonlinear interaction. In addition, we have to modify the amplitude (in comparison with the weakly decaying case of Section 1), because of the change in the order of the nonlinearity. So, our WKB expansion reads

$$u^\varepsilon \sim \varepsilon^m \sum_{n\in m\mathbb{N}} \varepsilon^n u_{2n}\left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon}\right),$$

where m is linked to the order K of nonlinear terms: $m := \dfrac{1}{K-1}$, and the profiles $u_k = u_k(x,X,\theta)$ split into:

$$u_k = \chi(Y_1)a_k(x,\hat{X},\theta) + b_k(x,X,\theta).$$

Notation 2.2. *From now on, $X=(T,Y)=(T,Y_1,\tilde{Y})\in\mathbb{R}^{1+p}$, and $(T,\tilde{Y})=\hat{X}$ (corresponding to $\psi=(\psi_0,\psi')=(\psi_0,\psi_1,\tilde{\psi})$ and $\hat{\psi}=(\psi_0,\tilde{\psi})$). Since we are dealing with Cauchy problems again, we suppose that one of the intermediate phases is timelike, and more precisely, we suppose it is t: $\psi_0\equiv t$.*

As in [14] and [12], the term b_k represents the transition layer. In fact, we first fix the function χ ('step-function'):

$$\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \text{ with } \chi'\in\mathcal{S},\ \chi(Z)=\int_{-\infty}^{Z}\chi'(Z')dZ',\ \int_{-\infty}^{+\infty}\chi'(Z')dZ'=1.$$

Our goal is to construct

$$a_k \in \mathcal{C}^\infty \left(\mathbb{R}_T \times \Omega_{t_1}, \cap_s H^s(\mathbb{R}^{p-1}_{\tilde{Y}} \times \mathbb{T}_\theta)\right), \ b_k \in \mathcal{C}^\infty \left(\mathbb{R}_T \times \Omega_{t_1}, \cap_s H^s(\mathbb{R}^p_Y \times \mathbb{T}_\theta)\right),$$

with odd spectrum (w.r.t. the periodic variable θ). The domain $\overline{\Omega}$ is still the cone $\{x = (t, y) \in \mathbb{R}^{1+d}/0 \leq t \leq t_0, \delta t + |y| \leq \rho\}$.

2.2 Function spaces, and the mean operator $\mathcal{M}$

Definition 2.1. *We denote by $\mathcal{H}^s(t_1)$ the space $\mathcal{E}^s_{p-1}(t_1) \oplus \mathcal{E}^s_p(t_1)$ of functions v on $\Omega_{t_1} \times \mathbb{R}^p \times \mathbb{T}$ which split into $v = \chi a + b$, where*

$$a \in \mathcal{E}^s_{x,\tilde{Y},\theta}(t_1) := \mathcal{E}^s_{p-1}(t_1), \ b \in \mathcal{E}^s_{x,Y,\theta}(t_1) := \mathcal{E}^s_p(t_1).$$

Equipped with the following norm, $\mathcal{H}^s(t_1)$ becomes a Banach space:

$$\|v\|_s := \|a\|_{\mathcal{E}^s_{p-1}} + \|b\|_{\mathcal{E}^s_p}.$$

Such a decomposition is unique, since the 'mean value' b is given by the following operator, $\mathcal{M}$:

Lemma 2.1. *Let $u \in \mathcal{H}^s(t_1)$, $s > 0$ be given as $u = \chi a + b$. Then,*

$$\mathcal{M}u(x, \hat{X}, \theta) := \lim_{Y_1 \to +\infty} u(x, X, \theta) \text{ exists, and } a = \mathcal{M}u, \ b = u - \chi \mathcal{M}u.$$

In this way, we define a linear and bounded operator $\mathcal{M} : \mathcal{H}^s(t_1) \to \mathcal{E}^s_{p-1}(t_1)$ ($\subset \mathcal{H}^s(t_1)$).

We use $\mathcal{M}$ to analyse profile equations, with the following calculus:

Lemma 2.2. *On $\mathcal{H}^s(t_1)$, for $s > 1/2$, we have:*

$$[\mathcal{M}, \partial_x] = [\mathcal{M}, \partial_{\hat{X}}] = [\mathcal{M}, \partial_\theta] = 0, \text{ and } \mathcal{M}\partial_{Y_1} = \partial_{Y_1}\mathcal{M} = 0.$$

2.3 Formal derivation of profile equations

Again, we plug the Ansatz into the equation, and let the (formal) asymptotics vanish. Because of our choice of amplitude, which is a non-integer power ε^m of ε ($m = 1/(K-1)$, from Notation 2.1), u_0 is no longer given by the three lowest powers in the asymptotics, but by coefficients of ε^{m-1}, $\varepsilon^{m-1/2}$ and ε^m.

We will solve sets of equations corresponding to ε^{n+m-1}, $\varepsilon^{n+m-1/2}$ and ε^{n+m}, with increasing $n \in m\mathbb{N}$:

$$L_1(d\phi)\partial_\theta u_0 = 0, \tag{2.3.1}$$

$$L_1(d\phi)\partial_\theta u_1 + L_1(d\psi)\partial_X u_0 = 0, \tag{2.3.2}$$

$$L_1(d\phi)\partial_\theta u_2 + L_1(d\psi)\partial_X u_1 + L_1(\partial_x)u_0 + \Lambda(u_0^{K-1})\partial_\theta u_0 = 0, \tag{2.3.3}$$

$$\tag{2.3.4}$$

$$\vdots$$

$$\begin{aligned} &L_1(d\phi)\partial_\theta u_{n+2} + L_1(d\psi)\partial_X u_{n+1} + L_1(\partial_x)u_n + \Lambda(u_0^{K-1})\partial_\theta u_n \\ &\qquad + \Lambda(u_0^{K-2}, u_n)\partial_\theta u_0 + \mathcal{F}_n(x, u_k, \partial_x u_k, \partial_X u_k, \partial_\theta u_k, k < n) = 0. \end{aligned} \tag{2.3.5}$$

Notations are the same as in Section 1, and we set $u_k := 0$ for $k < 0$.

2.3.1 Using the mean operator $\mathcal{M}$

Each equation (E) is equivalent to $\mathcal{M}(E)$ and $(E) - \chi\mathcal{M}(E)$, and this separates (partially) a_n and b_n:

$$\begin{cases} L_1(d\phi)\partial_\theta a_0 = 0 \\ L_1(d\phi)\partial_\theta b_0 = 0, \end{cases}$$

$$\begin{cases} L_1(d\phi)\partial_\theta a_1 + L_1(d\hat{\psi})\partial_{\hat{X}} a_0 = 0 \\ L_1(d\phi)\partial_\theta b_1 + L_1(d\psi)\partial_X b_0 + \chi' L_1(d\psi_1)a_0 = 0, \end{cases}$$

$$\begin{cases} L_1(d\phi)\partial_\theta a_2 + L_1(d\hat{\psi})\partial_{\hat{X}} a_1 + L_1(\partial_x)a_0 + \Lambda(a_0^{K-1})\partial_\theta a_0 = 0 \\ L_1(d\phi)\partial_\theta b_2 + L_1(d\psi)\partial_X b_1 \\ \qquad + \chi' L_1(d\psi_1)a_1 + L_1(\partial_x)b_0 \\ \qquad + \Lambda\left((\chi a_0 + b_0)^{K-1}\right)\partial_\theta(\chi a_0 + b_0) - \chi\Lambda(a_0^{K-1})\partial_\theta a_0 = 0, \end{cases}$$

$$\vdots$$
$$\begin{cases} L_1(d\phi)\partial_\theta a_{n+2} + L_1(d\hat{\psi})\partial_{\hat{X}} a_{n+1} + L_1(\partial_x)a_n + \Lambda(a_0^{K-1})\partial_\theta a_n \\ \qquad\qquad + \Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 + \mathcal{F}_n(x, a_k, \partial_X a_k, \partial_\theta a_k, k < n) = 0 \\ L_1(d\phi)\partial_\theta b_{n+2} + L_1(d\psi)\partial_X b_{n+1} + \chi' L_1(d\psi_1)a_{n+1} + L_1(\partial_x)b_n \\ \qquad\qquad + \Lambda\left((\chi a_0 + b_0)^{K-1}\right)\partial_\theta(\chi a_n + b_n) - \chi\Lambda(a_0^{K-1})\partial_\theta a_n \\ \qquad\qquad + \Lambda\left((\chi a_0 + b_0)^{K-2}, (\chi a_n + b_n)\right)\partial_\theta u_0 - \chi\Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 \\ \qquad\qquad\qquad\qquad + \mathcal{F}_n[\chi a_k + b_k, k < n] - \chi\mathcal{F}_n[a_k, k < n] = 0. \end{cases}$$

2.3.2 Fast scale analysis

Analysis of the operator $L_1(d\phi)$ is performed mode by mode (in Fourier series). This leads to matrix analysis, and we assume as previously that ϕ satisfies an eikonal equation associated with L_1, and that $\partial_y\phi$ does not vanish (Assumption 1.2). This is a sufficient condition for smoothness of π and Q (Paragraph 1.2), so that:

$$\pi a_0 = a_0, \quad \pi b_0 = b_0,$$

$$\begin{cases} \pi L_1(d\hat{\psi})\partial_{\hat{X}}\pi a_0 = 0 \\ (1-\pi)a_1 = -Q\partial_\theta^{-1}L_1(d\hat{\psi})\partial_{\hat{X}}a_0, \end{cases} \tag{2.3.6}$$

$$\begin{cases} \pi L_1(d\psi)\partial_X\pi b_0 = -\chi'\pi L_1(d\psi_1)\pi a_0 \\ (1-\pi)b_1 = -Q\partial_\theta^{-1}[L_1(d\psi)\partial_X b_0 + \chi' L_1(d\psi_1)a_0], \end{cases} \tag{2.3.7}$$

$$\begin{cases} \pi L_1(d\hat{\psi})\partial_{\hat{X}}\pi a_1 = \pi L_1(d\hat{\psi})\partial_{\hat{X}}QL_1(d\hat{\psi})\partial_{\hat{X}}\pi\partial_\theta^{-1}a_0 \\ \qquad\qquad - \pi L_1(\partial_x)\pi a_0 - \pi\Lambda(a_0^{K-1})\partial_\theta a_0 \\ (1-\pi)a_2 = -Q\partial_\theta^{-1}[L_1(d\hat{\psi})\partial_{\hat{X}}a_1 + L_1(\partial_x)a_0 + \Lambda(a_0^{K-1})\partial_\theta a_0], \end{cases}$$

$$\begin{cases} \pi L_1(d\psi)\partial_X\pi b_1 = \pi L_1(d\psi)\partial_X QL_1(d\psi)\partial_X\pi\partial_\theta^{-1}b_0 - \pi L_1(\partial_x)\pi b_0 \\ \qquad\qquad - \pi\Lambda\left((\chi a_0 + b_0)^{K-1}\right)\partial_\theta(\chi a_0 + b_0) + \chi\pi\Lambda(a_0^{K-1})\partial_\theta a_0 \\ \qquad\qquad - \chi'\pi L_1(d\psi_1)a_1 + \pi L_1(d\psi)\partial_X QL_1(d\psi_1)\chi'\partial_\theta^{-1}a_0 \\ (1-\pi)b_2 = -Q\partial_\theta^{-1}[L_1(d\psi)\partial_X b_1 + L_1(\partial_x)b_0 + \Lambda\left((\chi a_0 + b_0)^{K-1}\right)\times \\ \qquad\qquad \times\,\partial_\theta(\chi a_0 + b_0) - \chi\Lambda(a_0^{K-1})\partial_\theta a_0 + \chi' L_1(d\psi_1)a_1], \end{cases}$$

$$\vdots$$

$$\begin{cases} \pi L_1(d\hat{\psi})\partial_{\hat{X}}\pi a_{n+1} = \pi L_1(d\hat{\psi})\partial_{\hat{X}} Q L_1(d\hat{\psi})\partial_{\hat{X}}\pi\partial_\theta^{-1} a_n - \pi L_1(\partial_x)\pi a_n \\ \qquad - \pi\Lambda(a_0^{K-1})\partial_\theta a_n - \pi\Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 \\ \qquad - \pi\mathcal{F}_n(x, a_k, \partial_{\hat{X}} a_k, \partial_\theta a_k, k < n) \\ (1-\pi)a_{n+2} = -Q\partial_\theta^{-1}[L_1(d\hat{\psi})\partial_{\hat{X}} a_{n+1} + L_1(\partial_x)a_n + \Lambda(a_0^{K-1})\partial_\theta a_n \\ \qquad + \Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 + \mathcal{F}_n(x, a_k, \partial_{\hat{X}} a_k, \partial_\theta a_k, k < n)], \end{cases}$$

$$\begin{cases} \pi L_1(d\psi)\partial_X \pi b_{n+1} = \pi L_1(d\psi)\partial_X Q L_1(d\psi)\partial_X \pi\partial_\theta^{-1} b_n - L_1(\partial_x)b_n \\ \qquad - \pi\Lambda\left((\chi a_0 + b_0)^{K-1}\right)\partial_\theta(\chi a_n + b_n) + \chi\pi\Lambda(a_0^{K-1})\partial_\theta a_n \\ \qquad - \pi\Lambda\left((\chi a_0 + b_0)^{K-2}, (\chi a_n + b_n)\right)\partial_\theta u_0 + \chi\pi\Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 \\ \qquad - \pi\mathcal{F}_n[\chi a_k + b_k, k < n] + \chi\pi\mathcal{F}_n[a_k, k < n] - \chi'\pi L_1(d\psi_1)a_{n+1} \\ (1-\pi)b_{n+2} = -Q\partial_\theta^{-1}\Bigg(L_1(d\psi)\partial_X b_{n+1} + \chi' L_1(d\psi_1)a_{n+1} + L_1(\partial_x)b_n \\ \qquad + \Lambda\left((\chi a_0 + b_0)^{K-1}\right)\partial_\theta(\chi a_n + b_n) - \chi\Lambda(a_0^{K-1})\partial_\theta a_n \\ \qquad + \Lambda\left((\chi a_0 + b_0)^{K-2}, (\chi a_n + b_n)\right)\partial_\theta u_0 - \chi\Lambda(a_0^{K-2}, a_n)\partial_\theta a_0 \\ \qquad + \mathcal{F}_n[\chi a_k + b_k, k < n] - \chi\mathcal{F}_n[a_k, k < n]\Bigg). \end{cases}$$

(Arguments between brackets indicate that $\mathcal{F}_n$ is a functional).

Remark 2.1. *Remark that no b_n appears in the equations determining the a_n's: Behavior 'at infinity' does not depend on the transition layer.*

2.3.3 Analysis w.r.t. the (remaining) intermediate variables

Of course, we take advantage of the reductions to the scalar form in Proposition 1.2, which let the vector fields $V(d\psi)\partial_X$ and $V(d\hat{\psi})\partial_{\hat{X}}$ appear. So as to avoid the creation of 'steps' in the b_n's (see (2.3.7i), for example), we assume that ψ_1 is constant along the ϕ-rays:

Assumption 2.2. *On $\overline{\Omega}$, $V(d\psi_1) \equiv 0$.*

This simplifies the equations: only one transport operator persists, $V(d\hat{\psi})\partial_{\hat{X}}$. As a consequence, we have to deal with only one linear problem, namely:

$$\begin{cases} V(d\hat{\psi})\partial_{\hat{X}} u = f \\ u_{|T=0} = g, \end{cases}$$

with $V(d\hat{\psi})\partial_{\hat{X}} f = 0$. Under Assumption 1.5 ($V$-coherence of Ψ), it is equivalent to require u to be T-sublinear and to say that u is bounded (in this case, f vanishes). This shows that for all n, we have $V(d\hat{\psi})\partial_{\hat{X}} a_n = 0$ and $V(d\hat{\psi})\partial_{\hat{X}} b_n = 0$.

Finally, we only need to solve the equations for πa_n and πb_n: the other part of profiles is obtained immediately. We set $c_n := \pi a_n$, $d_n := \pi b_n$:

$$\begin{cases} \pi c_0 = c_0 \\ V(d\hat{\psi})\partial_{\hat{X}} c_0 = 0 \\ \pi V(\partial_x) c_0 - D(\partial_{\tilde{Y}})\partial_\theta^{-1} c_0 + \pi C c_0 + \pi \Lambda(c_0^{K-1})\partial_\theta c_0 = 0 \end{cases} \tag{2.3.8}$$

$$\begin{cases} \pi d_0 = d_0 \\ V(d\hat{\psi})\partial_{\hat{X}} d_0 = 0 \\ \pi V(\partial_x) d_0 - D(\partial_Y)\partial_\theta^{-1} d_0 + \pi C d_0 \\ \qquad + \pi \Lambda\left((\chi c_0 + d_0)^{K-1}\right)\partial_\theta(\chi c_0 + d_0) - \chi \pi \Lambda(c_0^{K-1})\partial_\theta c_0 \\ \qquad\qquad + \mathcal{G}_0\left(\chi^{(k)}, \partial_{\hat{X}}^l c_0, k = 1,2, l = 0,1\right) = 0 \end{cases} \tag{2.3.9}$$

$$\vdots$$

$$\begin{cases} \pi c_n = c_n \\ V(d\hat{\psi})\partial_{\hat{X}} c_n = 0 \\ \pi V(\partial_x) c_n - D(\partial_{\tilde{Y}})\partial_\theta^{-1} c_n + \pi C c_n \\ \qquad + \pi \Lambda[c_0, \ldots, c_{n-1}]\partial_\theta c_n + \pi \mathcal{G}_n[c_0, \ldots, c_{n-1}] = 0 \end{cases} \tag{2.3.10}$$

$$\begin{cases} \pi d_n = d_n \\ V(d\hat{\psi})\partial_{\hat{X}} d_n = 0 \\ \pi V(\partial_x) d_n - D(\partial_Y)\partial_\theta^{-1} d_n + \pi C d_n \\ \qquad + \pi \Lambda\left((\chi c_0 + d_0)^{K-1}\right)\partial_\theta(\chi c_n + d_n) - \chi \pi \Lambda(c_0^{K-1})\partial_\theta c_n \\ \qquad + \mathcal{G}_n[\chi c_k + d_k, k < n].(\chi c_n + d_n) - \chi \mathcal{G}_n[c_k, k < n].c_n \\ \qquad\qquad + \mathcal{H}_n[\chi c_k + d_k, k < n] - \chi \mathcal{H}_n[c_k, k < n] = 0. \end{cases} \tag{2.3.11}$$

2.4 Existence of profiles and approximation of exact solutions

2.4.1 Solving the profile equations

We look for $(c_n, d_n) \in \mathcal{C}(\mathbb{R}_T, \mathcal{H}^s(t_1))$. The only nonlinear equations are (2.3.8) and (2.3.9). In the case of $(2.3.10)_n$, $(2.3.11)_n$ ($n \in m\mathbb{N}\backslash\{0\}$), there are right-hand sides, functions of the preceding profiles. For (2.3.9) and $(2.3.11)_n$, nonlinearities and right-hand sides are of the form:

Lemma 2.3. *Let $H : \overline{\Omega} \times \mathbb{C}^N \to \mathbb{C}^N$ be smooth, and $H(.,0) \equiv 0$. If $c \in \mathcal{E}^s_{p-1}(t_1)$ and $d \in \mathcal{E}^s_p(t_1)$ for $s > \frac{2d+p+1}{4}$, then*

$$H(\chi c + d) - \chi H(c) \in \mathcal{E}^s_p(t_1),$$

$$\|H(\chi c + d) - \chi H(c)\|_{\mathcal{E}^s_p} \leq C\left(\|c\|_{\mathcal{E}^s_{p-1}}, \|d\|_{\mathcal{E}^s_p}\right)\left(\|d\|_{\mathcal{E}^s_{p-1}} + 1\right).$$

Proof : First, decompose

$$H(\chi c + d) - \chi H(c) = [H(\chi c + d) - H(\chi c)] + [H(\chi c) - \chi H(c)].$$

The first term is equal to $\displaystyle\int_0^1 \partial_v H(\chi c + \nu d).d\ d\nu$, and by differentiation,

$$\partial^\gamma_{y,Y,\theta}\left(\int_0^1 \partial_v H(\chi c + \nu d).d\ d\nu\right) = \sum_{\alpha^1+\cdots+\alpha^k\leq\gamma} \int_0^1 \partial^{\alpha^{k+1}}_{y,v} H(\chi c + \nu d).\left(\partial^{\alpha^1} d, \partial^{\alpha^2}(\chi c + d), \ldots, \partial^{\alpha^k}(\chi c + d)\right) d\nu.$$

We then come back to the usual technique of the proof of Moser's theorem, using Hölder's and Gagliardo-Nirenberg's inequalities (in $\mathcal{E}^s_p$). Indeed, $\chi c \in L^\infty(\mathbb{R}_{Y_1}, \mathcal{E}^s_{p-1}(t_1))$, and $\chi' c \in \mathcal{S}(\mathbb{R}_{Y_1}, \mathcal{E}^s_{p-1}(t_1))$.

For the second term, first notice that for all Y_1, $H(\chi c)$ and $\chi H(c)$ are elements of $\mathcal{E}^s_{p-1}$. So, we consider their difference as a $\mathcal{E}^s_{p-1}$- valued function of Y_1. Taylor expanding shows:

$$\begin{aligned} H(\chi c) - \chi H(c) &= \chi \int_0^1 \left[\partial_v H(\nu\chi c) - \partial_v H(\nu c)\right].c\ d\nu \\ &= \chi(1-\chi)\int_0^1 \nu\left[\int_0^1 \partial_v^2 H(\nu c + \mu\nu(\chi - 1)c).c\ d\mu\right].c\ d\nu. \end{aligned}$$

We can apply Moser's theorem in $\mathcal{E}^s_{p-1}$ inside the integral, with $\chi(Y_1)$ bounded. Finally, the $\chi(1-\chi)$ factor implies that $H(\chi c) - \chi H(c)$ belongs to $\mathcal{E}^s_p$. $\square$

The classical iterative schemes in $\mathcal{E}^s_p$ give existence and uniqueness of c_0 and d_0, and the linear profile equations are then solved successively.

Proposition 2.1. *Let $g_n \in \cap_s \mathcal{H}^s(0)$ be polarized: $\forall n \in m\mathbb{N}$, $\pi g_n = g_n$.*

Then, under the preceding assumptions (coherence, and ψ_1 constant along the ϕ-rays: Assumptions 1.1, 1.2, 1.3, 1.5, 2.1, 2.2), there exist $t^\star > 0$ and a unique maximal solution $v_0 = \chi c_0 + d_0 \in \mathcal{C}(\mathbb{R}_T, \cap_s \mathcal{H}^s(t))$, $t < t^\star$, to (2.3.8), (2.3.9) with initial data g_0. In addition, for all $n > 0$, there exists a unique $v_n = \chi c_n + d_n \in \mathcal{C}(\mathbb{R}_T, \cap_s \mathcal{H}^s(t))$, $t < t^\star$, solution to (2.3.10)$_n$, (2.3.11)$_n$ with initial data g_n.

2.4.2 Stability

Once we have obtained profiles, Borel's theorem exhibits an approximate solution $u^\varepsilon_{app}(x)$: $\forall M \in m\mathbb{N}, \forall \alpha \in \mathbb{N}^{1+d}$,

$$\left\| (\varepsilon\partial)^\alpha \left[u^\varepsilon_{app}(x) - \varepsilon^m \sum_{n<M} \varepsilon^n u_{2n}\left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon}\right)\right] \right\|_{L^\infty} = \mathcal{O}(\varepsilon^{m+M}).$$

This is an asymptotic solution to the system, and exact solutions with initial data close to $u^\varepsilon_{app|_{t=0}}$ stay close to u^ε_{app}:

Theorem 2.1. *For all $\underline{t} < t^\star$,*
i) $L(u^\varepsilon_{app}, \partial) u^\varepsilon_{app} \sim 0$ in $\mathcal{C}^\infty(\Omega_{\underline{t}})$.
ii) If $f^\varepsilon \sim 0$ in $\mathcal{C}^\infty(\overline{\Omega})$ and $v^{0,\varepsilon}(y) \sim \sum_{n \in m\mathbb{N}} \varepsilon^n u_{2n|_{t=0}}\left(y, \frac{\psi^0}{\sqrt{\varepsilon}}, \frac{\phi^0}{\varepsilon}\right)$, $\varepsilon \in]0,1]$, there exists $\varepsilon_{\underline{t}}$ such that the solution to the Cauchy problem

$$\begin{cases} L(v^\varepsilon, \partial) v^\varepsilon = f^\varepsilon \\ v^\varepsilon_{|_{t=0}}(y) = \varepsilon^m v^{0,\varepsilon}(y) \end{cases} \tag{2.4.1}$$

exists on $\Omega_{\underline{t}}$, for $\varepsilon \le \varepsilon_{\underline{t}}$. In addition, $v^\varepsilon - u^\varepsilon_{app} \sim 0$ in $\mathcal{C}^\infty(\Omega_{\underline{t}})$.

Proof :
We seek v^ε as a perturbation, $v^\varepsilon = u^\varepsilon_{app} + w^\varepsilon$. First, rescale amplitudes to 1: set $v^\varepsilon = \varepsilon^m V^\varepsilon$, $u^\varepsilon_{app} = \varepsilon^m U^\varepsilon$, ...Now, v^ε is solution to (2.4.1) if and

only if W^ε satisfies

$$\begin{cases} L\left(\varepsilon^m(U^\varepsilon + W^\varepsilon), \partial\right)(U^\varepsilon + W^\varepsilon) = F^\varepsilon \sim 0 \\ W^\varepsilon_{|t=0}(y) = G^\varepsilon \sim 0. \end{cases} \tag{2.4.2}$$

We adopt the same strategy as in [13], and introduce the following spaces:

Definition 2.2. *We define $H^s_\varepsilon(\omega)$ (resp. $C^s_\varepsilon(\omega)$) as the set of (families of) functions u^ε on ω such that*

$$\forall \alpha \in \mathbb{N}^d,\ \varepsilon > 0, \|(\varepsilon\partial_y)^\alpha u^\varepsilon\|_{L^2} \leq C_\alpha (resp.\ \|(\varepsilon\partial_y)^\alpha u^\varepsilon\|_{L^\infty} \leq C_\alpha).$$

It is evident that U^ε belongs to $H^\infty_\varepsilon = C^\infty_\varepsilon$. These ε-derivatives will allow us to show existence of W^ε in H^s_ε. The following properties deduce from the classical ones ($\varepsilon = 1$) by change of scales:

Proposition 2.2. *Let H be smooth on $\mathbb{R}^N \times \mathbb{R}^N$ and such that $H(.,0) \equiv 0$, and $s > d/2$. For all $U^\varepsilon \in C^s_\varepsilon(\Omega)$, $V^\varepsilon \in H^s_\varepsilon(\Omega)$,*

$$\begin{aligned} &\|V^\varepsilon\|_{L^\infty} \leq C_s \varepsilon^{-d/2} \|V^\varepsilon\|_{H^s_\varepsilon}, \\ &\|H(U^\varepsilon, V^\varepsilon)\|_{H^s_\varepsilon} \leq C_s\left(\|U^\varepsilon\|_{C^s_\varepsilon}, \|V^\varepsilon\|_{L^\infty}\right)\|V^\varepsilon\|_{H^s_\varepsilon}. \end{aligned}$$

When ε is fixed, in a classical way, we have existence of a smooth solution W^ε to the quasilinear hyperbolic system (2.4.2) on $[0, T_\varepsilon]$, with

$$\begin{aligned} \|W^\varepsilon(t)\|^2_{H^s_\varepsilon} \leq C_s \Bigg[\|G^\varepsilon\|^2_{H^s_\varepsilon} &+ C\left(\|W^\varepsilon, \partial W^\varepsilon\|_{L^\infty}\right) \int_0^t \|W^\varepsilon(t')\|^2_{H^s_\varepsilon} dt' \\ &+ \int_0^t \|F^\varepsilon(t')\|^2_{H^s_\varepsilon} dt' \Bigg]. \end{aligned}$$

Gronwall's lemma then implies:

$$\begin{aligned} \|W^\varepsilon(t)\|^2_{H^s_\varepsilon} \leq C_s \Bigg[&e^{Ct}\left(\|G^\varepsilon\|^2_{H^s_\varepsilon} + \int_0^t \|F^\varepsilon(t')\|^2_{H^s_\varepsilon} dt'\right) \\ &+ \int_0^t e^{C(t-t')}\left(\|G^\varepsilon\|^2_{H^s_\varepsilon} + \int_0^{t'} \|F^\varepsilon(t'')\|^2_{H^s_\varepsilon} dt''\right) dt' \Bigg]. \end{aligned}$$

Hence, if $T_\varepsilon \leq \underline{t}$ is a time until which $\|W^\varepsilon\|_{L^\infty}, \|\partial W^\varepsilon\|_{L^\infty} \leq R$ for a given $R > \|G^\varepsilon\|$, then, for $t \leq T_\varepsilon$ and $M \in \mathbb{N}$:

$$\|W^\varepsilon(t)\|^2_{H^s_\varepsilon} \leq C_{M,s}\varepsilon^M\left(1 + C(R)\underline{t}e^{C\underline{t}}\right),$$

and by Sobolev's embedding,

$$\|W^\varepsilon\|_{L^\infty}, \|\partial W^\varepsilon\|_{L^\infty} \leq C(M,s,R)\varepsilon^{(M-d)/2}.$$

When $M > d$ is fixed, for $\varepsilon \leq \varepsilon_R$, this quantity is lower than R. This proves existence of W^ε up to $\underline{t}$.

Since M is as big as desired, we have the asymptotics $W^\varepsilon \sim 0$. $\square$

Remark 2.2. *In the simplest case, with only one phase ψ ($X = Y \in \mathbb{R}$, see Remark 1.4iii)), Equations (2.3.8)-(2.3.11) become, for the first profile:*

$$\begin{cases} \pi c_0 = c_0(x,\theta) \\ V(\partial_x)c_0 + \pi C c_0 + \pi\Lambda(c_0^{K-1})\partial_\theta c_0 = 0 \end{cases}$$

$$\begin{cases} \pi d_0 = d_0(x,Y,\theta) \\ V(\partial_x)d_0 - D(x)\partial_Y^2\partial_\theta^{-1}d_0 + \pi C d_0 \\ \qquad + \pi\Lambda\left((\chi c_0 + d_0)^{K-1}\right)\partial_\theta(\chi c_0 + d_0) - \chi\pi\Lambda(c_0^{K-1})\partial_\theta c_0 \\ \qquad\qquad + \mathcal{G}_0\left(\chi', \chi'', c_0, k=1,2\right) = 0 \end{cases}$$

The profile $c_0 = a_0$ is exactly the one from (2 scales-)geometric optics on $\overline{\Omega}$, and the term b_0 can be seen as a corrector of the profile $\chi(Y)a_0(x,\theta)$: If we test the solution v^ε of (2.4.1) against $\varphi \in \mathcal{C}(\overline{\Omega}\times\mathbb{T})$, we get:

$$\varepsilon^{-m}\int_{\overline{\Omega}} v^\varepsilon(x)\varphi\left(x,\frac{\phi}{\varepsilon}\right)dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{\{\psi>0\}\times\mathbb{T}} a_0(x,\theta)\varphi(x,\theta)dxd\theta.$$

The 'main part' of v^ε is effectively given by geometric optics on $\{\psi > 0\}$.

3 Wave transitions for systems of conservation laws

In this section, we give an alternative method for the treatment of systems which do not satisfy the Oddness Hypothesis 2.1. In fact, we restrict to *conservative* systems. In that case, the mean value $\underline{u}_0$ of the first profile vanishes, and we can define an approximate solution $u^\varepsilon_{app}(x) =$

$\varepsilon(u_0+\sqrt{\varepsilon}u_1+\varepsilon u_2)(x,\psi/\sqrt{\varepsilon},\phi/\varepsilon)$ whose profiles have mean zero. The singular system technique of Paragraph 1.5.2 must be modified. If one wants to solve Equation (1.5.9),

$$L\left(x,\varepsilon\mathcal{V}^\varepsilon,\partial_x+\frac{1}{\sqrt{\varepsilon}}\partial_x\psi\partial_Y+\frac{1}{\varepsilon}\partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon=f^\varepsilon\left(x,Y,\theta\right), \tag{3.0.3}$$

for profiles $\mathcal{V}^\varepsilon=\chi a^\varepsilon+b^\varepsilon\in\mathcal{H}^s(\underline{t})$, projecting via $\mathcal{M}$ decouples the equation into

$$\begin{aligned} &L\left(\varepsilon a^\varepsilon,\partial_x\right)a^\varepsilon+T\left(\varepsilon,a^\varepsilon,\partial_y\phi\partial_\theta\right)a^\varepsilon\\ &\qquad+\frac{1}{\varepsilon}L_1\left(x,\partial_x\phi\partial_\theta\right)a^\varepsilon=\mathcal{M}f^\varepsilon(x,\theta)\end{aligned} \tag{3.0.4}$$

and

$$\begin{aligned} L\left(\varepsilon\mathcal{V}^\varepsilon,\partial_x\right)\mathcal{V}^\varepsilon&-\chi L\left(\varepsilon a^\varepsilon,\partial_x\right)a^\varepsilon\\ &+T\left(\varepsilon,\mathcal{V}^\varepsilon,\sqrt{\varepsilon}\partial_y\psi\partial_Y+\partial_y\phi\partial_\theta\right)\mathcal{V}^\varepsilon-\chi T\left(\varepsilon,a^\varepsilon,\partial_y\phi\partial_\theta\right)a^\varepsilon\\ &+\frac{1}{\varepsilon}L_1\left(x,\sqrt{\varepsilon}\partial_x\psi.\partial_Y+\partial_x\phi\partial_\theta\right)b^\varepsilon\\ &=f^\varepsilon(x,Y,\theta)-\chi\mathcal{M}f^\varepsilon(x,\theta)-\frac{1}{\sqrt{\varepsilon}}\chi' L_1(\partial_x\psi)a^\varepsilon.\end{aligned} \tag{3.0.5}$$

Once Equation (3.0.4) is solved for a^ε, even if the singular term in Equation (3.0.5), $\frac{1}{\varepsilon}L_1\left(x,\sqrt{\varepsilon}\partial_x\psi.\partial_Y+\partial_x\phi\partial_\theta\right)b^\varepsilon$, can be treated in the same way as in Paragraph 1.5.2 (by coherence), the forcing term $\frac{1}{\sqrt{\varepsilon}}\chi' L_1(\partial_x\psi)a^\varepsilon$ remains, and leads to blow-up of b^ε as ε goes to zero.

Our strategy consists in performing a change of (dependent) variables (see Lemma 1.8) that leads to a decomposition of Equation (3.0.3) different from Equations (3.0.4)-(3.0.5), which reveals that the singular forcing terms vanish (under some additional assumption on the phases, Assumption 3.4; Paragraph 3.3.3 gives an example from nonlinear acoustics for which it is satisfied).

3.1 The system and the Ansatz

Our system will now take the form of conservation laws:

$$M(v,\partial) = \partial_t v + \sum_{j=1}^{d} \partial_j F_j(v) = 0, \tag{3.1.1}$$

with given smooth functions $F_j : \mathbb{R}^{2N} \simeq \mathbb{C}^N \to \mathbb{C}^N$. The system (3.1.1) is assumed symmetrizable (with constant multiplicity):

Assumption 3.1. *There exists a positive definite matrix $S(v)$, smooth function of v on a neighbourhood of zero in $\mathbb{C}^N$, such that, for all j, the matrix $S(v)\partial_v F_j(v)$ is symmetric, and the eigenvalues $\lambda_1(\eta) < \cdots < \lambda_Z(\eta)$ of $\mathcal{A}(\eta) := \sum_{j=1}^d \eta_j S(0)^{1/2} \partial_v F_j(0) S(0)^{-1/2}$ have constant multiplicity (on $\mathbb{R}^d \setminus \{0\}$).*

So as to depart from the Oddness Hypothesis 2.1, and in view of the example at Paragraph 3.3.3, we fix

The order of nonlinearities: We assume $\partial_v^2 F_j(0) \neq 0$ for some j.

(But in fact, the following is valid for any order of nonlinearity, as well as for flux functions F_j depending explicitly on x).

The approximate solution we seek takes the form:

$$v^\varepsilon_{app}(x) = \varepsilon\left(v_0 + \sqrt{\varepsilon} v_1 + \varepsilon v_2\right)\left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon}\right)$$

with profiles $v_n \in \mathcal{H}^s(t_1) = \mathcal{E}_0^s(t_1) \oplus \mathcal{E}_1^s(t_1)$ (see Definition 2.1) and *independent of T* (we still denote by $\mathcal{H}^s(t_1)$ this subspace of $\mathcal{H}^s(t_1)$). Just like before, we consider the case of only one fast phase ϕ, and now take only one intermediate phase ψ too, but the same methods apply to the multi-ψ case, as in the previous section.

3.2 The approximate solution

The (formal) profile equations are obtained as in Paragraph 2.3, by a Taylor expansion. So as to recover the same notations as before (and a coefficient Id for ∂_t), we change variables, setting $v = S(0)^{-1/2}u$, and then multiply (on the left) all equations ($M(v,\partial)$) by $S(0)^{1/2}$.

Notation 3.1. *In agreement with the previous notations, set* $L_1(\partial) := \partial_t + \sum_{j=1}^d A_j(0)\partial_j$*, where* $A_j(v) := S(0)^{1/2}\partial_v F_j(v) S(0)^{-1/2}$*:* $A_j(0)$ *is symmetric. Set* $\Lambda(u) := S(0)^{1/2}\sum_{j=1}^d \partial_j\phi(x)(\partial_v^2 F_j(0).S(0)^{-1/2}u)S(0)^{-1/2}u$.

As in Section 2, the phases ϕ and ψ have to satisfy eikonal equations (see the conclusion of Example 2.1):

Assumption 3.2. *The phase* ϕ *is solution to an eikonal equation associated with* L_1 *(Assumption 1.2), and* ψ *is solution to the eikonal equation associated with* V *(see Assumption 2.2).*

Thus, for $u_n(x, Y, \theta) = \chi(Y)a_n(x, \theta) + b_n(x, Y, \theta)$, the profile equations are (with notations of Proposition 1.2):

$$\begin{cases} \pi a_0 = a_0 \\ V(\partial_x)a_0 + \pi C a_0 + \pi\Lambda(a_0)\partial_\theta a_0 = 0 \end{cases} \tag{3.2.1}$$

$$\begin{cases} \pi b_0 = b_0 \\ V(\partial_x)b_0 + \pi C b_0 - D(\partial_Y)\partial_\theta^{-1} b_0 + \pi\Lambda\left(\chi a_0 + b_0\right)\partial_\theta(\chi a_0 + b_0) \\ \qquad - \chi\pi\Lambda(a_0)\partial_\theta a_0 - \pi\left[D(\partial_Y)\chi\right]\partial_\theta^{-1} a_0 = 0 \end{cases} \tag{3.2.2}$$

$$(1-\pi)a_1 = 0 \tag{3.2.3}$$

$$(1-\pi)b_1 = -Q\partial_\theta^{-1} L_1(d\psi)\left[\partial_Y b_0 + \chi' a_0\right] \tag{3.2.4}$$

$$(1-\pi)a_2 = -Q\partial_\theta^{-1}\left[L_1(\partial_x)a_0 + \Lambda\left(a_0\right)\partial_\theta a_0\right] \tag{3.2.5}$$

$$\begin{aligned} (1-\pi)b_2 = -Q\partial_\theta^{-1}\big[&L_1(d\psi)\partial_Y b_1 + L_1(\partial_x)b_0 \\ &+\Lambda\left(\chi a_0 + b_0\right)\partial_\theta(\chi a_0 + b_0) - \chi\Lambda(a_0)\partial_\theta a_0\big] \end{aligned} \tag{3.2.6}$$

These equations are solved in Paragraph 2.4 (Proposition 2.1), with maximal existence time $t^\star$. Define the approximate solution v^ε_{app} (for $\underline{t} < t^\star$) by

$$v^\varepsilon_{app}(x) := \varepsilon\mathcal{V}^\varepsilon_{app}\left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon}\right),$$

$$\mathcal{V}^\varepsilon_{app}(x, Y, \theta) := S(0)^{-1/2}(u_0 + \sqrt{\varepsilon}u_1 + \varepsilon u_2)(x, Y, \theta) \in \cap_s \mathcal{H}^s(\underline{t}).$$

Proposition 3.1. *Define the residual* $k^\varepsilon(x) := M(v^\varepsilon_{app}, \partial_x)$*, which is the evaluation at* $(x, Y, \theta) = (x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$ *of*

$$K^\varepsilon(x, Y, \theta) := \frac{1}{\varepsilon} M(\varepsilon \mathcal{V}^\varepsilon_{app}, \varepsilon\partial_x + \sqrt{\varepsilon}\partial\psi\partial_Y + \partial\phi\partial_\theta). \tag{3.2.7}$$

Then, for all $\underline{t} < t^\star$*:*

$$\forall s,\ \|K^\varepsilon\|_{\mathcal{H}^s(\underline{t})} = \mathcal{O}(\varepsilon^{3/2}),\ \text{and}$$
$$\forall \alpha \in \mathbb{N}^{1+d}, \|(\varepsilon\partial_x)^\alpha k^\varepsilon\|_{L^\infty(\Omega_{\underline{t}})} = \mathcal{O}(\varepsilon^{3/2}).$$

Proof :
Proceed as for Proposition 1.8, but now, profiles do not depend on T. □

3.3 Stability

Notation 3.2. *From now on,* $L(v, \partial)v = \sum_{j=0}^d A_j(v)\partial_j v$ *simply denotes* $S(v)M(v, \partial)$ *(the coefficient of* ∂_t *no more needs to be* Id*).*

Our approximate solution v^ε_{app} is based upon three profiles, so we prove a stability theorem of the kind of Theorem 1.2, using a singular system. We know we have to make some coherence assumption:

Assumption 3.3. *The real vector space generated by* t*,* ψ *and* ϕ *is* L_1*-coherent. The functions* ϕ *and* ψ *are linearly independent.*

(The second part is an addition to Assumption 1.6; see Lemma 3.1). Another assumption is necessary so as to deal with the singular terms (Assumption 3.4). We shall explain it in the sequel. Our aim is to prove the following:

Theorem 3.1. *Under Assumptions 3.1, 3.2, 3.3 and 3.4 (see Section 3.3.2 below), consider* f^ε *and* g^ε *such that for all* s*,* $\|f^\varepsilon\|_{\mathcal{H}^s(t^\star)} \underset{\varepsilon\to 0}{\longrightarrow} 0$*,* $\|g^\varepsilon\|_{\mathcal{H}^s(0)} \underset{\varepsilon\to 0}{\longrightarrow} 0$*, and in addition,* $\|\mathcal{M}\langle f^\varepsilon\rangle\|_{\mathcal{H}^s(t^\star)} = o(\sqrt{\varepsilon})$*,* $\|\mathcal{M}\langle g^\varepsilon\rangle\|_{\mathcal{H}^s(0)} = o(\sqrt{\varepsilon})$*. Then, there exist* $\varepsilon_0 > 0$ *and* $\underline{t} > 0$ *such that the Cauchy problem*

$$\begin{cases} M(v^\varepsilon, \partial) = \varepsilon f^\varepsilon(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) \\ v^\varepsilon_{|t=0} = v^\varepsilon_{app_{|t=0}} + \varepsilon g^\varepsilon(y, \psi(0, y)/\sqrt{\varepsilon}, \phi(0, y)/\varepsilon) \end{cases} \tag{3.3.1}$$

admits a unique solution $v^\varepsilon \in C^1(\Omega_{\underline{t}})$ *for all* $\varepsilon \le \varepsilon_0$.

In addition, there is $\mathcal{V}^\varepsilon \in \cap_s \mathcal{H}^s(\underline{t})$ *such that* v^ε *has the form*

$$v^\varepsilon(x) = \varepsilon \mathcal{V}^\varepsilon \left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon} \right), \tag{3.3.2}$$

$$\text{and } \forall s,\ \|\mathcal{V}^\varepsilon_{app} - \mathcal{V}^\varepsilon\|_{\mathcal{H}^s(\underline{t})} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

We try to mimic the proof of Paragraph 1.5.2, so we seek $\mathcal{V}^\varepsilon \in \mathcal{H}^s$ (and independent of T) satisfying

$$\begin{aligned} L\left(\varepsilon \mathcal{V}^\varepsilon, \partial_x\right) \mathcal{V}^\varepsilon + T^\varepsilon\left(\mathcal{V}^\varepsilon, \sqrt{\varepsilon}\partial_y \psi \partial_Y + \partial_y \phi \partial_\theta\right) \mathcal{V}^\varepsilon \\ + \frac{1}{\varepsilon} L_1\left(x, \sqrt{\varepsilon}\partial_x \psi \partial_Y + \partial_x \phi \partial_\theta\right) \mathcal{V}^\varepsilon = S(\varepsilon \mathcal{V}^\varepsilon) f^\varepsilon(x, Y, \theta), \end{aligned}$$

with $T^\varepsilon(\mathcal{V}, \eta) = \sum_{j=0}^{d} \eta_j (\int_0^1 \partial_u A_j(\tau \varepsilon \mathcal{V}) d\tau).\mathcal{V}$.

The key point is the conjugation of $\frac{1}{\varepsilon} L_1\left(x, \sqrt{\varepsilon}\partial_x \psi \partial_Y + \partial_x \phi \partial_\theta\right)$ to a constant coefficient (singular) operator.

3.3.1 The conjugation operator $\mathbb{V}_\varepsilon$

From Lemma 1.8, there is a bounded family $(V_{\rho,\alpha})$ of unitary matrix valued smooth functions on $\overline{\Omega}$, homogeneous in (ρ, α) (degree zero), such that

$$V(x, \rho, \alpha) L_1(x, \partial_x(\rho\psi + \alpha\phi)(x)) V^\star(x, \rho, \alpha) = \Delta(\rho, \alpha).$$

The corresponding Fourier multipliers $V(\sqrt{\varepsilon} D_Y, D_\theta)$ ($\mathcal{C}^\infty$ parametrized by $x \in \overline{\Omega}$) map $\mathcal{S}_{Y,\theta}$ into itself. This section is devoted to the proof of the following proposition:

Proposition 3.2. *Under Assumptions 3.1, 3.2, 3.3, for every* t_1*, the Fourier multipliers* $V(\sqrt{\varepsilon} D_Y, D_\theta)$ *above define a bounded family* $(\mathbb{V}_\varepsilon)_\varepsilon$ *of unitary operators on* $\mathcal{H}^s(t_1)$ *for all* $s \ge 0$*. Separating oscillations and mean value for* $\chi c + d \in \mathcal{H}^s(t_1)$ *(*$c = c^0(x) + c^\star(x, \theta)$ *and* $d = d^0(x, Y) + d^\star(x, Y, \theta)$*),*

$$\begin{aligned} &\mathbb{V}_\varepsilon L_1\left(x, \sqrt{\varepsilon}\partial_x \psi D_Y + \partial_x \phi D_\theta\right) \mathbb{V}_\varepsilon{}^\star(\chi c + d) = \\ &\quad \chi \Delta(0, D_\theta) c^\star \\ &\quad + \sqrt{\varepsilon} \Delta(D_Y, 0)\left(\chi c^0 + d^0\right) \\ &\quad + \Delta(\sqrt{\varepsilon} D_Y, D_\theta) d^\star - i\sqrt{\varepsilon} \chi' \partial_\rho \Delta(0,1) c^\star + r_\varepsilon(c^\star), \end{aligned} \tag{3.3.3}$$

where $\Delta(\rho,\alpha)$ is the (diagonal) matrix of eigenvalues of $L_1(\rho d\psi + \alpha d\phi)$, independent of x, and r_ε is a (linear) continuous operator from $\mathcal{E}_0^s(t_1)$ to $\mathcal{E}_1^{s-1/2}(t_1)$, with norm $\mathcal{O}(\varepsilon)$.

First, this action is mode by mode on formal series (in θ):

$$\mathbb{V}_\varepsilon \sum u^\alpha(x,Y)e^{i\alpha\theta} = \sum {\mathbb{V}_\varepsilon}^\alpha u^\alpha(x,Y)e^{i\alpha\theta} :$$

we now deal with functions in $\mathcal{H}^s(t_1) = \mathcal{E}_0^s(t_1) \oplus \mathcal{E}_1^s(t_1)$ (cst w.r.t. θ).

• When $\alpha = 0$, because of homogeneity,

$$\mathbb{V}_\varepsilon^0 = V(1,0) \text{ (the matrix } V(\rho,\alpha) \text{ for } \alpha = 0,\ \rho = 1).$$

• The definition of $\mathbb{V}_\varepsilon^\alpha$ ($\alpha \neq 0$) as operator on $\mathcal{E}_1^s(t_1)$ is the same as in 1.5.2: it is a Fourier multiplier (in $\sqrt{\varepsilon}D_Y$), as well as $[\partial_x, \mathbb{V}_\varepsilon^\alpha] = \partial_x V(x, \sqrt{\varepsilon}D_Y, \alpha)$.

• The trouble comes from the action on the 'step part' χa^α of u^α. To identify 'step' and decaying parts of $\mathbb{V}_\varepsilon^\alpha \chi a^\alpha$, first perform the commutation:

$$\mathbb{V}_\varepsilon^\alpha \chi a^\alpha = \chi \mathbb{V}_\varepsilon^\alpha a^\alpha + [\mathbb{V}_\varepsilon^\alpha, \chi] a^\alpha.$$

From now on, we will consider $V(D_Y,\alpha)$ and $\chi(Y)$ as pseudo-differential operators, computing the commutator $[\mathbb{V}_\varepsilon^\alpha, \chi]$ by means of symbol calculus. But first, if we want $V(D_Y,\alpha)$ ($\alpha \neq 0$) to act on the constant (w.r.t. Y) a^α, using the formula

$$V(D_Y,\alpha)a^\alpha = \frac{1}{2\pi}\int e^{i\rho Y} V(\rho,\alpha)\widehat{a^\alpha}(\rho)d\rho, \quad \text{with } \widehat{a^\alpha} = 2\pi\delta_0,$$

$V(\rho,\alpha)$ must be continuous at $\rho = 0$. This was the aim of the independence assumption (Assumption 3.3):

Lemma 3.1. *Under Assumptions 3.1, 3.2, 3.3, the family $(V^\alpha)_{\alpha\in\mathbb{Z}^\star}$ is bounded in $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R})$, and for all $\gamma \in \mathbb{N}^{1+d}$, $\delta \in \mathbb{N}$, there is a constant C independent of $\alpha \in \mathbb{Z}^\star$ such that:*

$$\left\|\partial_x^\gamma \partial_\rho^\delta V(.,\alpha,\rho)\right\|_{L^\infty} \leq C < \rho >^{-\delta} . \tag{3.3.4}$$

Proof :

Because of the constant multiplicity assumption 3.1, V^α is smooth on the open subset $\mathcal{O}$ of $\overline{\Omega} \times \mathbb{R}_\rho$ where $\partial_y(\alpha\phi + \rho\psi)(x)$ does not vanish (see Lemma 1.8 and (1.5.12)). Now, if this derivative vanishes at a point x, since ϕ and ψ satisfy eikonal equations (respectively associated with L_1 and $V(\partial_x)$), $\partial_t(\alpha\phi+\rho\psi)(x)$ also vanishes. Coherence of $Vect(\phi,\psi)$ then implies that $\alpha\phi=\rho\psi$, contradicting the independence from Assumption 3.3. Thus, $\mathcal{O}=\overline{\Omega}\times\mathbb{R}_\rho$.

Boundedness of the family $(V^\alpha)_{\alpha\in\mathbb{Z}^\star}$ and of the constant $C_{\alpha,\rho}$ in (3.3.4) follows from this smoothness property and from degree zero homogeneity of $V(\rho,\alpha)$. □

Hence, the symbols $V^\alpha(\rho)$ and $\chi(Y)$ define (smooth functions of x, with values) semi-classical operators ($V^\alpha(\sqrt{\varepsilon}D_Y)$ and $\chi(Y)$) on $\mathcal{S}'_Y$. So as to take into account the decay in Y, we could consider them as symbols $S^{0,0}_{\infty,1}$ and $S^{0,0}_{\infty,\infty}$, respectively, where $p(Y,\rho)\in S^{m,\mu}_{r,\delta}$ if $|\partial_Y^a\partial_\rho^\beta p(Y,\rho)|\leq C_{a,\beta} < Y >^{m-r|a|}< \rho >^{\mu-\delta|\beta|}$. Uniformity in α is immediate, and the role played by $\sqrt{\varepsilon}$ in the asymptotic calculus is well-known (see for instance [25]). But here, we are only interested in the action on constant functions (w.r.t. Y) $a^\alpha(x)\in\mathcal{E}^s_0(t_1)$, so we prove the following lemma 'by hand':

Lemma 3.2. *For $\alpha\in\mathbb{Z}^\star$ and $s\geq 0$, the family of operators $V(\sqrt{\varepsilon}D_Y,\alpha)\chi(Y)$, $\varepsilon\in]0,1]$, maps $\mathcal{E}^s_0(t_1)$ to $\mathcal{H}^s(t_1)$. Precisely, for all $a(x)\in\mathcal{E}^s_0(t_1)$,*

$$\begin{aligned}V(\sqrt{\varepsilon}D_Y,\alpha)\chi(Y)a &= \chi V(0,1)a+[V^\alpha(\sqrt{\varepsilon}D_Y),\chi]a,\\ &=\chi(Y)V(0,1)a(x)-i\sqrt{\varepsilon}\chi'(Y)\partial_\rho V(0,\alpha)a(x)+\tilde r^\alpha_\varepsilon(a),\end{aligned}$$

and $\tilde r^\alpha_\varepsilon$ maps $\mathcal{E}^s_0(t_1)$ to $\mathcal{E}^s_1(t_1)$, with norm $\mathcal{O}(\varepsilon)$ (uniformly in α).

Proof :

The operators $V(\sqrt{\varepsilon}D_Y,\alpha)$ and (multiplication by) $\chi(Y)$ are associated with the symbols $V^\alpha_\varepsilon(\rho):=V^\alpha(\sqrt{\varepsilon}\rho)$ and $\chi(Y)$, respectively, through the formula

$$Op(\sigma)(Y,D_Y)u=\frac{1}{2\pi}\iint e^{i(Y-Z)\rho}\sigma(Y,\rho)u(Z)dZd\rho.$$

The resulting composed operator $V(\sqrt{\varepsilon}D_Y,\alpha)\chi(Y)$ then admits a symbol σ_ε, given by

$$\sigma_\varepsilon(Y,\rho)=J\left(V^\alpha_\varepsilon\chi\right),\ \text{where } J=e^{iD_YD_\rho}.$$

Thus, the Taylor expansion $J=1+iD_YD_\rho-\left(\int_0^1(1-r)J^rdr\right)(D_YD_\rho)^2$ provides the asymptotic expansion (where $J^r=e^{irD_YD_\rho}$):

$$\sigma_\varepsilon(Y,\rho)=\chi(Y)V^\alpha_\varepsilon(\rho)-i\chi'\partial_\rho V^\alpha_\varepsilon(\rho)-\int_0^1(1-r)J^r\left(\chi''\partial_\rho^2V^\alpha_\varepsilon\right)dr.$$

Then, the derivatives in ρ provide the expected factor ε, since $\partial_\rho^2 V_\varepsilon^\alpha(\rho) = \varepsilon\left(\partial_\rho^2 V^\alpha\right)(\sqrt{\varepsilon}\rho)$, denoted as before: $\varepsilon\left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\rho)$.

Finally, recall that the action of $J^r\left(\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon\right)$ on $\mathcal{E}_0^s$ is through multiplication by $\left(J^r\left(\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon\right)\right)(Y,0)$; the following lemma ends the proof:

Lemma 3.3. *The family* $\left(J^r\left(\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon\right)\right)_{r,\alpha,\varepsilon}(Y,0)$, $r \in [0,1]$, $\alpha \in \mathbb{Z}^\star$, $\varepsilon \in]0,1]$, *is bounded in* $\mathcal{C}^\infty(\overline{\Omega}, \mathcal{S}(\mathbb{R}_Y))$.

Proof :

The action of J^r is convolution (in Y,ρ) by $\mathcal{F}^{-1}\left(e^{ir\hat{Y}\hat{\rho}}\right) = (4\pi^2|r|)^{-1}e^{ir^{-1}Y\rho}$. Hence, $J^r\left(\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon\right)(x,Y,\rho)$ is smooth. We prove estimates uniform w.r.t. x. The same are valid for derivatives, simply replacing V by $\partial_x^\gamma V$, which has the same regularity, and homogeneity in ρ: $\forall k \in 2\mathbb{N}$,

$$J^r\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon(Y,0) = \frac{1}{4\pi^2|r|}\iint \chi''(Y+Z)\left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\zeta)e^{-ir^{-1}Z\zeta}dZd\zeta$$
$$= \frac{1}{4\pi^2|r|}\iint \chi''(Y+Z)\left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\zeta) < r^{-1}Z >^{-k} < D_\zeta >^k e^{-ir^{-1}Z\zeta}dZd\zeta$$
$$= \frac{1}{4\pi^2|r|}\iint < r^{-1}Z >^{-k} \chi''(Y+Z) < D_\rho >^k \left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\zeta)e^{-ir^{-1}Z\zeta}dZd\zeta,$$

after integration by parts.

Now, according to Peetre's inequality,

$$|\chi''(Y+Z)| \le C_l < Y+Z >^{-l} \le C_l < Y >^{-l} < Z >^l .$$

Taking $k = l+2$ ensures, for all $l \in 2\mathbb{N}$ ($r \in]0,1]$):

$$|< r^{-1}Z >^{-k} \chi''(Y+Z)| \le C < r^{-1}Z >^{-2} < Y >^{-l} . \tag{3.3.5}$$

The second term, $< D_\rho >^k \left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\zeta)$, is bounded thanks to homogeneity:

$$|< D_\rho >^k \left(\partial_\rho^2 V^\alpha\right)_\varepsilon(\zeta)| \le C_k \varepsilon^{k/2} < \sqrt{\varepsilon}\zeta >^{-2-k} \le C_k < \zeta >^{-k} . \tag{3.3.6}$$

Finally, thanks to (3.3.5) and (3.3.5), performing the change of variable $r^{-1}Z = Z'$, the quantity $\left|J^r\chi''\left(\partial_\rho^2 V^\alpha\right)_\varepsilon(Y,0)\right|$ is estimated, for all $l \in 2\mathbb{N}$, by:

$$C_l < Y >^{-l} \iint < \zeta >^{-l-2} < Z' >^{-2} dZ'd\zeta = C_l' < Y >^{-l} .\square$$

We can summarize the definition of $\mathbb{V}_\varepsilon$ as: for $\chi a + b \in \mathcal{H}^s(t_1)$,

$$\begin{aligned}\mathbb{V}_\varepsilon(\chi a + b) =& \chi\left(V(1,0)a^0 + V(0,1)a^\star\right)\\ &+ V(1,0)b^0 + V(\sqrt{\varepsilon}D_Y, D_\theta)b^\star + [V(\sqrt{\varepsilon}D_Y, D_\theta), \chi]a^\star,\end{aligned}$$

$$\begin{aligned}\mathbb{V}_\varepsilon{}^\star(\chi c + d) =& \chi\left(V(1,0)^\star c^0 + V(0,1)^\star c^\star\right)\\ &+ V(1,0)^\star d^0 + V^\star(\sqrt{\varepsilon}D_Y, D_\theta)\left(d^\star - [V(\sqrt{\varepsilon}D_Y, D_\theta), \chi]V(0,1)^\star c^\star\right)\\ =& \chi\left(V(1,0)^\star c^0 + V(0,1)^\star c^\star\right)\\ &+ V(1,0)^\star d^0 + V^\star(\sqrt{\varepsilon}D_Y, D_\theta)d^\star\\ &+ i\sqrt{\varepsilon}\chi' V^\star(0,1)\partial_\rho V(0, D_\theta)V^\star(0,1)c^\star + \tilde{r}_\varepsilon(c^\star).\end{aligned}$$

The proof of (3.3.3) is then straightforward:

$$\begin{aligned}&L_1\left(x, \sqrt{\varepsilon}\partial_x\psi D_Y + \partial_x\phi D_\theta\right)\mathbb{V}_\varepsilon{}^\star(\chi c + d) =\\ &\quad \chi L_1(\partial_x\phi D_\theta)V(0,1)^\star c^\star\\ &\quad + \sqrt{\varepsilon}L_1(\partial_x\psi D_Y)V(0,1)^\star\left(\chi c^0 + d^0\right)\\ &\quad + L_1(\partial_x\psi D_Y)\chi V(0,1)^\star c^\star + L_1\left(\sqrt{\varepsilon}\partial_x\psi D_Y + \partial_x\phi D_\theta\right)V^\star(\sqrt{\varepsilon}D_Y, D_\theta)d^\star\\ &\quad + i\sqrt{\varepsilon}L_1(\partial_x\phi D_\theta)V(0,1)^\star\partial_\rho V(0, D_\theta)V(0,1)^\star c^\star\\ &\quad + \varepsilon R_\varepsilon(c^\star),\end{aligned}$$

where the residual R_ε writes

$$-\chi'' L_1(\partial_x\psi)V(0,1)^\star\partial_\rho V(0, D_\theta)V(0,1)^\star c^\star + L_1\left(\sqrt{\varepsilon}\partial_x\psi D_Y + \partial_x\phi D_\theta\right)\frac{\tilde{r}_\varepsilon(c^\star)}{\varepsilon},$$

which is $\mathcal{O}(1)$ in $\mathcal{E}_1^{s-1/2}$, because of the derivatives in $L_1\left(\sqrt{\varepsilon}\partial_x\psi D_Y + \partial_x\phi D_\theta\right)$.

Applying $\mathbb{V}_\varepsilon$ gives:

$$\begin{aligned}&\chi\Delta(0, D_\theta)c^\star + \sqrt{\varepsilon}\Delta(D_Y, 0)\left(\chi c^0 + d^0\right) + \Delta(\sqrt{\varepsilon}D_Y, D_\theta)d^\star\\ &\quad + \sqrt{\varepsilon}V(\sqrt{\varepsilon}D_Y, D_\theta)\left[L_1(\partial_x\psi D_Y)\chi V(0,1)^\star c^\star\right.\\ &\qquad\qquad \left. + i\chi' L_1(\partial_x\phi D_\theta)V(0,1)^\star\partial_\rho V(0, D_\theta)V(0,1)^\star c^\star\right]\\ &\quad - i\sqrt{\varepsilon}\chi'\partial_\rho V(0, D_\theta)L_1(\partial_x\phi D_\theta)V(0,1)^\star c^\star + \varepsilon\tilde{R}_\varepsilon(c^\star).\end{aligned}$$

The residual $\tilde{R}_\varepsilon(c^\star)$ is now bounded in $\mathcal{E}_1^{s-1/2}$ (when $c^\star$ is in $\mathcal{E}_0^s$). The first line above contains terms in (3.3.3), and there remain three terms, with factor $\sqrt{\varepsilon}$:

- $-i\chi'\partial_\rho V(0, D_\theta)L_1(\partial_x\phi D_\theta)V(0,1)^\star c^\star = -i\chi'\partial_\rho V(0,1)L_1(\partial_x\phi)V(0,1)^\star c^\star$;

• $V\ (\sqrt{\varepsilon}D_Y, D_\theta)L_1(\partial_x\psi D_Y)\chi V(0,1)^\star c^\star$
$$= \frac{1}{i}\left(\chi' V(\sqrt{\varepsilon}D_Y, D_\theta) + [V(\sqrt{\varepsilon}D_Y, D_\theta), \chi']\right) L_1(\partial_x\psi)V(0,1)^\star c^\star.$$
As in Lemma 3.2, the commutator is $\mathcal{O}(\sqrt{\varepsilon})$ in $\mathcal{E}_1^s$, and this term finally writes $\frac{1}{i}\chi' V(0,1)L_1(\partial_x\psi)V(0,1)^\star c^\star + \mathcal{O}(\sqrt{\varepsilon})$;

• $i\chi' V(\sqrt{\varepsilon}D_Y, D_\theta)\ L_1(\partial_x\phi D_\theta)V(0,1)^\star\partial_\rho V(0, D_\theta)V(0,1)^\star c^\star$
$$= i\chi' V(0,1)L_1(\partial_x\phi)V(0,1)^\star\partial_\rho V(0,1)V(0,1)^\star c^\star.$$

Summing these three terms, we have, applied to $(-i\sqrt{\varepsilon}\chi' c^\star)$, up to a $\mathcal{O}(\sqrt{\varepsilon})$ in $\mathcal{E}_1^s$:

$$\partial_\rho V(0,1)L_1(\partial_x\phi)V(0,1)^\star + V(0,1)L_1(\partial_x\psi)V(0,1)^\star + V(0,1)L_1(\partial_x\phi)\partial_\rho V(0,1)^\star,$$

using $\partial_\rho V(0,1)V(0,1)^\star = -V(0,1)\partial_\rho V(0,1)^\star$ for the third one. This is exactly $\partial_\rho\Delta(0,1)$, as can be seen differentiating the identity $V(\rho,1)L_1(\rho\partial_x\psi + \partial_x\phi)V(\rho,1)^\star = \Delta(\rho,1)$. Thus, the proof of (3.3.3) is complete.

3.3.2 The singular system

The full system

We now solve the Cauchy problem (3.3.1). First, using the symmetrizor S, it becomes

$$\begin{cases} L(v^\varepsilon, \partial)v^\varepsilon = \varepsilon S(v^\varepsilon)f^\varepsilon(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) \\ v^\varepsilon_{|t=0} = v^\varepsilon_{app|_{t=0}} + \varepsilon g^\varepsilon(y, \psi(0,y)/\sqrt{\varepsilon}, \phi(0,y)/\varepsilon), \end{cases}$$

where $L(v,\partial) = \sum_{j=0}^d A_j(v)\partial_j$ is hyperbolic symmetric, with $A_0 = S$, $A_j = S\partial_v F_j$, $j > 0$ (and there are semilinear as well as quasilinear terms).

We look for v^ε of the form $v^\varepsilon(x) = \varepsilon\mathcal{V}^\varepsilon(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$, with $\mathcal{V}^\varepsilon \in \mathcal{H}^\infty(\underline{t})$ solving the singular system

$$(3.3.7)\qquad \begin{cases} L\left(\varepsilon\mathcal{V}^\varepsilon, \partial_x\right)\mathcal{V}^\varepsilon + T\left(\varepsilon, \mathcal{V}^\varepsilon, \sqrt{\varepsilon}\partial_x\psi\partial_Y + \partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon \\ \qquad\qquad + \frac{1}{\varepsilon}L_1\left(\sqrt{\varepsilon}\partial_x\psi\partial_Y + \partial_x\phi\partial_\theta\right)\mathcal{V}^\varepsilon = S(\varepsilon\mathcal{V}^\varepsilon)f^\varepsilon \\ \mathcal{V}^\varepsilon_{|t=0} = \mathcal{V}^\varepsilon_{app|_{t=0}} + g^\varepsilon \end{cases}$$

where

$$T\left(\varepsilon, \mathcal{V}, \sqrt{\varepsilon}\partial_x\psi\partial_Y + \partial_x\phi\partial_\theta\right) = \sum_{j=0}^d \left[\left(\int_0^1 \partial_v A_j(r\varepsilon\mathcal{V})\right).\mathcal{V}\right](\sqrt{\varepsilon}\partial_j\psi\partial_Y + \partial_j\phi\partial_\theta).$$

We first perform the change of variable $\mathcal{V}^\varepsilon = \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon$, which leads to the hyperbolic symmetric pseudo-differential system

(3.3.8)
$$\begin{cases} \mathbb{V}_\varepsilon L\left(\varepsilon \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon, \partial_x\right) \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon \\ \qquad\qquad + \mathbb{V}_\varepsilon T\left(\varepsilon, \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon, \sqrt{\varepsilon}\partial_x\psi\partial_Y + \partial_x\phi\partial_\theta\right) \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon \\ \qquad\qquad + \dfrac{i}{\varepsilon}\mathbb{V}_\varepsilon L_1\left(\sqrt{\varepsilon}\partial_x\psi D_Y + \partial_x\phi D_\theta\right) \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon = \mathbb{V}_\varepsilon S(\varepsilon \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon) f^\varepsilon \\ \mathcal{W}^\varepsilon_{|t=0} = \mathbb{V}_\varepsilon \mathcal{V}^\varepsilon_{app_{|t=0}} + \mathbb{V}_\varepsilon g^\varepsilon \end{cases}$$

and the singular term is given by (3.3.3).

The 'step' part

The strategy consists in projecting via $\mathcal{M}$, and solving for $\mathcal{M}\mathcal{W}^\varepsilon = c$ first. We know there could remain singular forcing terms in the equation for $(1-\chi\mathcal{M})\mathcal{W}^\varepsilon$, such as $\dfrac{1}{\sqrt{\varepsilon}}\chi'\Delta(1,0)\underline{c}$. This one is in fact not singular, according to the following estimates:

Proposition 3.3. *Let $\mathcal{M}f^\varepsilon \in \mathcal{E}_0^\infty(t^\star)$ and $\mathcal{M}g^\varepsilon \in \mathcal{E}_0^\infty(0)$ with $\|\mathcal{M}f^\varepsilon\|_{\mathcal{E}_0^s(t^\star)}$ and $\|\mathcal{M}g^\varepsilon\|_{\mathcal{E}_0^s(0)}$ bounded (resp. $\to 0$) for all s, as $\varepsilon \to 0$.*

Then, there are $\varepsilon_0 > 0$ and $\underline{t} \leq t^\star$ such that (3.3.8) determines a unique $\mathcal{M}\mathcal{W}^\varepsilon \in \mathcal{E}_0^\infty(\underline{t})$ for all $\varepsilon \leq \varepsilon_0$, and for all s, $\|\mathcal{M}\mathcal{W}^\varepsilon\|_{\mathcal{E}_0^s(\underline{t})}$ is bounded (resp. goes to zero) as $\varepsilon \to 0$.

Furthermore, if $\|\mathcal{M}\langle f^\varepsilon\rangle\|_{\mathcal{E}_0^s(t^\star)}, \|\mathcal{M}\langle g^\varepsilon\rangle\|_{\mathcal{E}_0^s(0)} = \mathcal{O}(\sqrt{\varepsilon})$ (resp. $o(\sqrt{\varepsilon})$), then $\|\mathcal{M}\langle \mathcal{W}^\varepsilon\rangle\|_{\mathcal{E}_0^s(\underline{t})} = \mathcal{O}(\sqrt{\varepsilon})$ (resp. $o(\sqrt{\varepsilon})$).

Proof :

We decompose $\mathcal{W}^\varepsilon = \chi c^\varepsilon + d^\varepsilon$, and write $\mathbb{V}_\varepsilon{}^\star c^\varepsilon$ for $\mathcal{M}\mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon)$ (identifying c^ε with χc^ε). Applying $\mathcal{M}$ to (3.3.8) gives

$$\begin{cases} \mathbb{V}_\varepsilon L\left(\varepsilon \mathbb{V}_\varepsilon{}^\star c^\varepsilon, \partial_x\right) \mathbb{V}_\varepsilon{}^\star c^\varepsilon + \mathbb{V}_\varepsilon T\left(\varepsilon, \mathbb{V}_\varepsilon{}^\star c^\varepsilon, \partial_x\phi\partial_\theta\right) \mathbb{V}_\varepsilon{}^\star c^\varepsilon \\ \qquad\qquad + \dfrac{1}{\varepsilon}\Delta(0,1)\partial_\theta c^\varepsilon = \mathbb{V}_\varepsilon S(\varepsilon \mathbb{V}_\varepsilon{}^\star c^\varepsilon)\mathcal{M}f^\varepsilon \\ c^\varepsilon_{|t=0} = \mathbb{V}_\varepsilon \mathcal{M}\mathcal{V}^\varepsilon_{app_{|t=0}} + \mathbb{V}_\varepsilon \mathcal{M}g^\varepsilon \end{cases} \tag{3.3.9}$$

This is a symmetric pseudo-differential system with both semilinear and quasilinear terms. Energy estimates in $\mathcal{E}_0^s$ with ε fixed are standard (as in Paragraph 1.5.2; commutators estimates for $\mathbb{V}_\varepsilon$ are given in Lemma 1.9). The singular term is antisymmetric, and does not appear; one finds only positive powers of ε, smaller than one. Hence, the estimates are uniform in ε; ε_0 is chosen small enough, so that all iterates from the iterative scheme stay in the domain of definition of the symmetrizor S. Finally, the maximal existence time in $\mathcal{E}_0^s$ is the same for all $s \geq 1$, as shows Proposition 1.7.

The estimates on the non-oscillating part $\underline{c}^\varepsilon$ (for the corresponding assumptions on the data) come from the conservative properties of the initial system $M(v, \partial)$. Coming back to the function $\mathcal{V}^\varepsilon = \mathbb{V}_\varepsilon{}^\star \mathcal{W}^\varepsilon$, its 'step' part $a^\varepsilon = \mathcal{M}\mathcal{V}^\varepsilon \in \mathcal{E}_0^\infty(\underline{t})$ satisfies, in view of (3.3.9):

$$M\left(\varepsilon a^\varepsilon, \partial_x + \frac{1}{\varepsilon}\partial_x\phi\partial_\theta\right) = \varepsilon \mathcal{M} f^\varepsilon.$$

Take the mean value of this equation to suppress the θ-derivatives:

$$\varepsilon\partial_t \underline{a}^\varepsilon + \sum_{j>0} \partial_j \langle F_j(\varepsilon a^\varepsilon)\rangle = \varepsilon\langle \mathcal{M} f^\varepsilon\rangle.$$

We now take advantage of the fact that a^ε is bounded (as ε goes to zero) in $\mathcal{E}_0^s(\underline{t})$: a Taylor expansion gives

$$\partial_t \underline{a} + \sum_{j>0} \partial_u F_j(0)\partial_j \underline{a}^\varepsilon = \langle \mathcal{M} f^\varepsilon\rangle + \varepsilon G^\varepsilon, \tag{3.3.10}$$

where $G^\varepsilon(x) = -\sum_{j=1}^{d} \left\langle \left[\left(\int_0^1 \partial_u^2 F_j(r\varepsilon a^\varepsilon)dr\right).a^\varepsilon \right] \partial_j a^\varepsilon \right\rangle$ is bounded in $\mathcal{E}_0^s(\underline{t})$ for all s. The result then follows from standard linear estimates. $\square$

The transition layer

We now look at the difference (3.3.8)-(3.3.9), which constitutes the system

for $d^\varepsilon = (1-\chi\mathcal{M})\mathcal{W}^\varepsilon$:
(3.3.11)

$$\begin{cases} \mathbb{V}_\varepsilon L\left(\varepsilon \mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon + d^\varepsilon), \partial_x\right) \mathbb{V}_\varepsilon{}^\star d^\varepsilon \\ \quad + \mathbb{V}_\varepsilon L\left(\varepsilon \mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon + d^\varepsilon), \partial_x\right) \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon - \chi\mathcal{M}\mathbb{V}_\varepsilon L\left(\varepsilon \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon, \partial_x\right) \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon \\ \quad + \mathbb{V}_\varepsilon T\left(\varepsilon, \mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon + d^\varepsilon), \sqrt{\varepsilon}\partial_x\psi\partial_Y + \partial_x\phi\partial_\theta\right) \mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon + d^\varepsilon) \\ \qquad - \chi\mathcal{M}\mathbb{V}_\varepsilon T\left(\varepsilon, \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon, \partial_x\phi\partial_\theta\right) \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon \\ \quad + \dfrac{1}{\sqrt{\varepsilon}}\Delta(\partial_Y, 0)\underline{d}^\varepsilon + \dfrac{1}{\varepsilon}\Delta(\sqrt{\varepsilon}\partial_Y, \partial_\theta)(d^\varepsilon)^\star \\ \quad = \mathbb{V}_\varepsilon S(\varepsilon \mathbb{V}_\varepsilon{}^\star(\chi c^\varepsilon + d^\varepsilon)) f^\varepsilon - \chi\mathcal{M}\mathbb{V}_\varepsilon S(\varepsilon \mathbb{V}_\varepsilon{}^\star \chi c^\varepsilon)\chi\mathcal{M} f^\varepsilon \\ \quad - \dfrac{1}{\sqrt{\varepsilon}}\chi'\left(\Delta(1,0)\underline{c}^\varepsilon + \partial_\rho\Delta(0,1)(c^\varepsilon)^\star\right) - \dfrac{i}{\varepsilon} r_\varepsilon((c^\varepsilon)^\star) \\ d^\varepsilon_{|t=0} = (1-\chi\mathcal{M})\mathbb{V}_\varepsilon\left(\mathcal{V}^\varepsilon_{app_{|t=0}} + g^\varepsilon\right). \end{cases}$$

As for c^ε, energy estimates (with ε fixed) follow from symmetry: c^ε is a smooth coefficient, and linear estimates of nonlinear terms are provided by Lemma 2.3.

The singular antisymmetric terms don't appear in these estimates. There remain forcing terms on the right-hand side. The residual $r_\varepsilon((c^\varepsilon)^\star)/\varepsilon$ is bounded in $\mathcal{E}^s_1(t_1)$ uniformly in ε, according to Propositions 3.3 and 3.2. From Proposition 3.3, $\chi'\Delta(1,0)\underline{c}^\varepsilon/\sqrt{\varepsilon}$ is $\mathcal{O}(1)$ as soon as $\mathcal{M}\langle f^\varepsilon\rangle$ and $\mathcal{M}\langle g^\varepsilon\rangle$ are $\mathcal{O}(\sqrt{\varepsilon})$. In order to eliminate $\chi'\partial_\rho\Delta(0,1)(c^\varepsilon)^\star/\sqrt{\varepsilon}$, we finally assume (see the example at Paragraph 3.3.3 for a concrete situation where this occurs):

Assumption 3.4. *The matrix $\partial_\rho\Delta(0,1)$ vanishes.*

Diminishing if necessary the time $\underline{t}$ for c^ε, we have:

Proposition 3.4. *Assume $\partial_\rho\Delta(0,1) = 0$, and let $f^\varepsilon \in \mathcal{E}^\infty_0(t^\star)$ and $g^\varepsilon \in \mathcal{E}^\infty_0(0)$ be such that for all s,*
- $\|f^\varepsilon\|_{\mathcal{E}^s_0(t^\star)}$ and $\|g^\varepsilon\|_{\mathcal{E}^s_0(0)}$ are bounded (resp. go to zero) as $\varepsilon \to 0$,
- $\|\mathcal{M}\langle f^\varepsilon\rangle\|_{\mathcal{E}^s_0(t^\star)}, \|\mathcal{M}\langle g^\varepsilon\rangle\|_{\mathcal{E}^s_0(0)} = \mathcal{O}(\sqrt{\varepsilon})$ (resp. $o(\sqrt{\varepsilon})$).

Then, there exist $\varepsilon_0 > 0$ and $\underline{t} \le t^\star$ such that (3.3.11) determines a unique $d^\varepsilon = (1-\chi\mathcal{M})\mathcal{W}^\varepsilon \in \mathcal{E}^\infty_1(\underline{t})$ for all $\varepsilon \le \varepsilon_0$. In addition, for all s, $\|d^\varepsilon\|_{\mathcal{E}^s_1(\underline{t})}$ is bounded (resp. goes to zero) as $\varepsilon \to 0$.

Remark 3.1. *We can interpret Assumption 3.4 geometrically: the coefficients of the diagonal matrix $\Delta(\rho,1)$ are the eigenvalues of $L_1(\rho\partial_x\psi + \partial_x\phi)$.*

These are exactly $\sigma_k(\rho) := \partial_t(\rho\psi+\phi)+\lambda_k(\rho\partial_y\psi+\partial_y\phi)$, $1 \leq k \leq Z$, following notations of Assumption 3.1. This assumption ensures that $\mathcal{C}_k := \{(\tau,\eta) \in \mathbb{R}^{1+d} \setminus \{0\} \ / \ \tau + \lambda_k(\eta) = 0\}$ is a smooth manifold. Differentiating σ_k, we get $\sigma_k'(0) = \partial_t\psi + \partial_\eta\lambda_k(\partial_y\phi).\partial_y\psi$, so that $\partial_\rho\Delta(0,1) = 0$ says that, for all k, $\nabla\psi$ belongs to the tangent plane $T_{(\tau_k,\partial_y\phi)}\mathcal{C}_k$ to $\mathcal{C}_k$ at the point above $\partial_y\phi$, $(\tau_k,\partial_y\phi) = (-\lambda_k(\partial_y\phi),\partial_y\phi)$. In particular, this contains the assumption of V-coherence of ψ.

The approximation

The perturbation result (3.3.2) is proven as in Paragraph 1.5.2, considering the Cauchy problem for $\mathcal{V}_r^\varepsilon = \mathcal{V}^\varepsilon - \mathcal{V}_{app}^\varepsilon$: it is of the same kind as (3.3.7), but the initial data is g^ε, assumed small, and the source term in the equation is $S(\varepsilon\mathcal{V}^\varepsilon)\,(f^\varepsilon - K^\varepsilon)$ (with K^ε the residual from (3.2.7), $\mathcal{O}(\varepsilon^{3/2})$ in $\mathcal{H}^s(\underline{t})$). We simply make use of the second part of Proposition 3.3 with $o(\sqrt{\varepsilon})$ instead of $\mathcal{O}(\sqrt{\varepsilon})$, in order to get solutions going to zero with ε.

3.3.3 Example of phases for Euler equations

We come back to the example of Paragraph 1.6, for which we prove that Assumption 3.4 is satisfied. The symmetrizable system (1.6.1) of 3-d isentropic Euler equations is written in the conservative form

$$\begin{cases} \partial_t\rho + div_y(\rho v) = 0 \\ \partial_t(\rho v_j) + div_y(\rho v_j v + p e_j) = 0, 1 \leq j \leq 3. \end{cases} \tag{3.3.12}$$

The characteristic variety of L_1 is $\{\tau^2(\tau^2 - c^2|\eta|^2) = 0\}$, and we consider again the phases

$$\phi = R - ct = (y_1^2 + y_2^2)^{1/2} - ct \ , \ \psi = y_3,$$

for which $Vect(t,\phi,\psi)$ is L_1-coherent, and $V(d\psi) = \partial_t\psi + \dfrac{c}{R}y'.\partial_y\psi = 0$.

Now, after Remark 3.1, it is equivalent, for Assumption 3.4 to be satisfied, to verify that the two remaining characteristic fields of L_1, $V_0 = \partial_t$ and $V_+ = \partial_t - \dfrac{c}{R}y'.\partial_y$, vanish on ψ, which is immediate.

References

[1] Deborah Alterman and Jeffrey Rauch. *The linear diffractive pulse equation.* Methods and Applications of Analysis 7 (2000), no. 2, 263–274.

[2] Deborah Alterman and Jeffrey Rauch. *Diffractive short pulse asymptotics for nonlinear wave equations.* Phys. Lett. A 264 (2000), no. 5, 390–395.

[3] Karen Barrailh and David Lannes. *A general framework for diffractive optics and its applications to lasers with large spectrum and short pulses.* SIAM J. on Math. Analysis 34 (2003), no. 3, 636–674.

[4] N. Bloembergen, *Nonlinear optics*, W.A. Benjamin Inc., New York, 1965.

[5] J. Chazarain and A. Piriou. *Introduction à la théorie des équations aux dérivées partielles linéaires.* Gauthier-Villars, 1981.

[6] T. Colin. *Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems.* Asymptot. Anal. 31 (2002), no. 1, 69–91.

[7] P. Donnat. *Quelques contributions mathématiques en optique non linéaire.* PhD Thesis, 1994.

[8] P. Donnat, J.L. Joly, G. Métivier, and J. Rauch. *Diffractive nonlinear geometric optics.* Sminaire Equations aux Drives Partielles, Ecole Polytechnique, Palaiseau, 1995-1996.

[9] E. Dumas. *Periodic diffractive optics with curved phases.* To appear in Indiana Univ. Math. J.

[10] E. Dumas. *Un peu d'optique diffractive non linéaire à phases courbes.* PhD Thesis, 2000.

[11] L. Gårding. *Problème de cauchy pour les systèmes quasi-linéaires d'ordre un strictement hyperboliques.* In Colloques du CNRS 117 (1962), 33–40.

[12] E. Grenier and O. Guès. Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems. J. Differential Equations 143 (1998), no. 1, 110–146.

[13] O. Guès. *Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires.* Asymptotic Analysis 6 (1993), no. 3, 401–446.

[14] O. Guès. *Viscous boundary layers and high frequency oscillations. singularities and oscillations.* IMA Vol. Math. Appl. 91 (1997), Springer, New York, 61–77.

[15] J. K. Hunter, A. Majda, and R. Rosales. *Resonantly interacting, weakly nonlinear hyperbolic waves II. Several space variables.* Stud. Appl. Math. 75 (1986), no. 3, 187–226.

[16] J.K. Hunter. *Transverse diffraction of nonlinear waves and singular rays.* Siam J. Appl. Math. 48 (1988), 1–37.

[17] J.L. Joly, G. Métivier, and J. Rauch. *Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves.* Duke Math. J. 70 (1993), no. 2, 373–404.

[18] J.L. Joly, G. Métivier, and J. Rauch. *Coherent and focusing multidimensional nonlinear geometric optics.* Ann. scient. Ec. Norm. Sup. 28 (1995), .

[19] J.L. Joly, G. Métivier, and J. Rauch. *Diffractive nonlinear geometric optics with rectification.* Indiana Univ. Math. J. 47 (1998), no. 4, .

[20] D. Lannes. *Quelques phénomènes d'interaction d'ondes en optique non linéaire.* PhD Thesis, 1999.

[21] D. Lannes. *Dispersive effects for nonlinear geometrical optics with rectification.* Asymptot. Anal. 18 (1998), no. 1-2, 111–146.

[22] D. Lannes, J. Rauch. *Validity of nonlinear geometric optics with times growing logarithmically.* Proc. Amer. Math. Soc. 129 (2001), 1087–1096.

[23] P.D. Lax. *Asymptotic solutions of oscillatory initial value problems.* Duke Math. J. 24 (1957), 627–646.

[24] A. Majda. *Compressible fluid flow of conservation laws in several space variables.* Springer Verlag, 1984.

[25] D. Robert. *Autour de l'approximation semi-classique.* Birkhäuser Boston Inc., Boston, MA, 1987.

[26] G. Schneider. *Justification of modulation equations for hyperbolic systems via normal forms.* Nonlinear Differential Equations and Applications 5 (1998), no. 1, 69–82.

[27] M. Taylor. *Partial Differential Equations III.* Appl. Math. Sc. 117, Springer Verlag, 1996.

[28] M. Williams. *Boundary layers and glancing blow-up in nonlinear geometric optics.* Ann. Scient. de l'Ecole Norm. Sup. 33 (2000), n. 3, 383-432.