PERIODIC MULTIPHASE NONLINEAR DIFFRACTIVE OPTICS WITH CURVED PHASES

Éric Dumas
Laboratoire de Mathématiques et Physique Théorique
Parc de Grandmont, 37200 TOURS - FRANCE
dumas@gargan.math.univ-tours.fr

Abstract: We describe diffraction for rapidly oscillating, periodically modulated nonlinear waves. This phenomenon arises for example when considering long-time propagation, or through perturbation of initial oscillations. We show existence and stability of solutions to *variable coefficient*, nonlinear hyperbolic systems, together with 3-scales multiphase infinite-order WKB asymptotics: the fast scale is that of oscillations, the slow one describes the modulation of the envelope, which is along rays for the oscillatory components, and the intermediate one corresponds to transverse diffraction. It gives rise to nonlinear *Schrödinger* equations on a torus for the profiles. The main difficulty resides in the fact that the coefficients in the original equations are variable: thus, phases are nonlinear, and rays are not parallel lines. This induces variable coefficients in the integro-differential system of profile equations, which in general is not solvable. We give sufficient (and, in general, necessary) geometrical *coherence* conditions on the phases for the formal asymptotics to be rigorously justified. Small divisors assumptions are also needed, which are generically satisfied.
MSC Classification: 34E20, 35B27, 35B40, 35L60, 35Q55.

Contents

Introduction

We deal with quasilinear (symmetric) hyperbolic systems,

$$L(x,u,\partial)u = \partial_t u + \sum_{j=1}^{d} A_j(x,u)\partial_j u = \sum_{j=0}^{d} A_j(x,u)\partial_j u = 0,$$

before the formation of shocks. Semilinear systems could be addressed by the same methods. Our smooth solution u takes the form of a highly oscillating (at frequency $1/\varepsilon$) function, with *3-scales* Wentzels-Krammers-Brillouin asymptotics,

$$u^\varepsilon \underset{\varepsilon\to 0}{\sim} \varepsilon^m \mathcal{U}^\varepsilon\left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon}\right),$$

for some smooth profile $\mathcal{U}^\varepsilon(x,\omega,\theta)$ *periodic* in ω and θ ($\omega \in \mathbb{T}^p, \theta \in \mathbb{T}^q$, where $\mathbb{T}$ is the torus $\mathbb{R}/\mathbb{Z}$). The amplitude ε^m is chosen so that nonlinear effects appear at leading order. The (*nonlinear*) vector-valued phases are $\phi = (\phi_1, \ldots, \phi_q)$ and $\psi = (\psi_1, \ldots, \psi_p)$. This Ansatz was introduced by J.K. Hunter ([16]) for the formal study of "singular rays" ($\mathcal{U}^\varepsilon$ is then a shock-type profile in ω). Such a formalism describes the propagation of q interacting waves which are *diffracted* in p directions.

Diffraction

In some cases, it is necessary to add corrections to the (linear or nonlinear) geometric optics approximations. For example, one may have to take diffractive effects into account: diffraction is a linear phenomenon, corresponding to variations of a wave packet in directions *transverse* to the rays of geometric optics.

The first rigorous results within this framework are due to Donnat, Joly, Métivier and Rauch ([9], [10]). They exhibit approximations of high frequency ($1/\varepsilon$, and $\varepsilon \to 0$) oscillating waves over time scales much larger than the ones for which geometric optics is valid. They consider initial-value problems associated with constant coefficient nonlinear hyperbolic systems:

$$\begin{cases} L(u^\varepsilon, \partial)u^\varepsilon = F(u^\varepsilon) \\ u^\varepsilon_{|T=0}(Y) = \varepsilon^m g\left(Y, \dfrac{\eta\cdot Y}{\varepsilon}\right),\ Y \in \Omega_0 \subset \mathbb{R}^d, \end{cases} \tag{0.1}$$

where $L(u,\partial) = \partial_T + \sum_j A_j(u)\partial_{Y_j}$ (the space-time variable is $X = (T,Y)$), and the $N \times N$ matrices $A_j(u)$ are symmetric. The initial data oscillate at frequency $1/\varepsilon$ according to a *linear* phase $\eta \cdot Y = \sum \eta_j Y_j$ ($g(Y,\theta)$ is periodic w.r.t. θ). The amplitude ε^m is chosen in order that nonlinearities have an influence at leading order at the same time as diffraction, that is the *Rayleigh time*, $T \sim 1/\varepsilon$. Even if the family of initial data is unbounded (in any Sobolev space $H^s, s > 0$) as ε goes to zero, the authors show the existence of a smooth solution to (0.1) for times $T \in [0, T_\star/\varepsilon]$ for all $\varepsilon \in]0,1]$ (and stability under perturbation of the data). These results are obtained after introduction of a new "variable" εX, and thanks to an approximate solution

$$u^\varepsilon_{app}(T,Y) = \varepsilon^m \sum_{n\in\mathbb{N}} \varepsilon^n u_n\left(\varepsilon X, X, \frac{\xi \cdot X}{\varepsilon}\right).$$

In [10], the profiles $u_n(\tilde{X}, X, \theta) \in \mathcal{C}_T\left([0, T_\star/\varepsilon], \mathcal{C}_{\tilde{T}}([0, T_\star], H^s(\mathbb{R}^d_{\tilde{Y}} \times \mathbb{R}^d_Y \times \mathbb{T}_\theta))\right)$ are purely oscillating ($\int_\mathbb{T} u_n d\theta = 0$, with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the 1-d torus), and satisfy:

$$\pi u_0 = u_0, \tag{0.2a}$$
$$V(\partial_X) u_0 = 0, \tag{0.2b}$$
$$V(\partial_{\tilde{X}}) u_0 + R(\partial_Y)\partial_\theta^{-1} u_0 + \pi[\Phi(u_0) + \Lambda(u_0)\partial_\theta u_0] = 0 \tag{0.2c}$$

for $n = 0$. The equations for higher order profiles are linear, and have the same structure. Equation (0.2a) expresses the polarization of u_0, and π is a (matrix) projector associated with L and η. The operator $V(\partial_X) = \partial_T + v \cdot \partial_Y$ is the transport field along rays, with group velocity v. These two equations are similar to the ones of usual geometric optics. Finally, (0.2c) –a nonlinear Schrödinger-type equation– represents transverse diffraction, at the time scale $\tilde{T}$, via the scalar second order operator $R(\partial_y) = \sum_{i,j} r_{i,j}\partial_{Y_i}\partial_{Y_j}$, whose coefficients are related to the curvature of the characteristic variety of L. The semilinear term $\Phi(u)$ and the quasilinear term $\Lambda(u)\partial_\theta u$ arise from the Taylor expansions (around 0) of $F(u)$ and of the $A_j(u)$'s, respectively.

This kind of asymptotics has also been studied by Joly, Métivier, Rauch in [23], when rectification effects are present, *i.e.* when interactions of oscillating modes can generate non-oscillatory waves. Equations (0.2) are then coupled with a hyperbolic system for the mean value (w.r.t. θ) of the profiles. In [24], D. Lannes considers the case of dispersive systems, with rectification. G. Schneider has treated the case of one equation, in space dimension one, by means of normal forms (see [28]). In [8], T. Colin has studied systems with a "transparency" property, allowing solutions with greater amplitude; the profiles are then solutions of Davey-Stewartson systems (see also [25]). Diffraction for pulses (*i.e.* when the profiles $u_n(\tilde{X}, X, \theta)$ have compact support in θ) leads to a somewhat different approximation, with a typical profile equation $2\partial_{\tilde{T}}\partial_\theta u_n - \Delta_Y u_n = \partial_\theta f(u_n)$; see [2], [1], and [5] for an approach via "continuous spectra'.

Variable coefficients and periodic profiles

All the results above are restricted to systems with *constant coefficients*, and involve a single *plane* phase. It seems a priori difficult to describe diffraction ruled by nonplanar phases:
1- Rays are then no longer parallel lines, and either they focus in finite time, leading to singularities of phases and profiles, or they spread out, and over large time scales, local energy becomes so weak that nonlinear effects are negligible (see [10]).
2- Systems with non-constant coefficients (heterogeneous media) generate curved phases, and also induce profile equations (0.2a)–(0.2c) with non-constant coefficients. Solvability of such a system is not at all obvious, since the equations may not commute.

However, our aim is to deal with non-constant coefficients and several nonplanar phases. We already know (from [15], [19], [21]) that in several space dimensions, this requires coherence properties of the phases. We shall see that coherence is also the key for solvability of the profile equations. Concerning the problem of focusing or spreading out of the rays, we turn the difficulty thanks to "weakly nonplanar" approximations (see Example 2 below). The rays are then order ε out of parallel (and for propagation over $1/\varepsilon$ distance, this rules out all approximate solutions based on planar phases).

Multiphase expansions allow one to consider resonant wave interactions. The paper [11] generalizes the results of [10] and [23] to the case of systems with variable coefficients; it also contains an attempt of rigorous justification of Hunter's approach to "singular rays" ([16]).

In the present paper, we are interested in asymptotics based on 3-scales profiles $u_n(x,\omega,\theta)$ *periodic* in ω and θ. The following examples motivate our study and illustrate the main theorems 8.2.1 and 8.3.1.

Example 1. *Phase perturbation*

Here, we give a first reason for dealing with *periodic* profiles (in the intermediate variable ω). Consider the linear problem:

$$\begin{cases} L(\partial)u^\varepsilon = (\partial_t + A_1\partial_{y_1} + A_2\partial_{y_2})\, u^\varepsilon = 0 \\ u^\varepsilon_{|t=0}(y) = g\left(y, \dfrac{\phi_0^\varepsilon(y)}{\varepsilon}\right), \end{cases}$$

where $A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $g \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{T})$.

When ϕ_0 is some phase (*i.e.* $\partial_y \phi_0$ does not vanish) which *does not depend* on ε, usual (linear) geometric optics yield existence of u^ε on some bounded domain $\Omega \subset \mathbb{R}^{1+d}$ independent of ε, as well as approximation by an envelope slowly modulated along rays ([26]).

Now, we assume that such a phase is perturbed by addition of a $\sqrt{\varepsilon}$ term: $\phi_0^\varepsilon(y) = \phi_0(y) + \sqrt{\varepsilon}\psi(y)$. Thus, 2-scales geometric optics fail in that context – the perturbation introduces a third scale, which we can't ignore. We propose a systematic treatment of this kind of asymptotics in the following way: set $h(y, \omega, \theta) := g(y, \theta + \omega)$, so that $h \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{T}^2)$. Theorem 8.3.1 then ensures existence of u^ε on a fixed domain Ω, thanks to an infinitely accurate approximate solution $\sum_n \varepsilon^{n/2} u_n(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$.

In order to compute explicit profile equations, we can choose $\phi(t, y) = t + y_1, \psi(t, y) = t + y_1 + \varphi(y_2)$ for some smooth φ (these phases satisfy the coherence assumptions needed). In this linear case, we can restrict to g of the form $g(y, \theta) = h(y)e^{i\theta}$ –and there are some polarization conditions on the vector-valued h, see Proposition 6.3.1. The oscillating part $u_0^\star = U_0(x, \omega)e^{i\theta}$ of the first profile is then determined by a *Schrödinger equation* on the torus $\mathbb{T}_\omega$:

$$i(\partial_t - \partial_{y_1})U_0 - \frac{1}{2}(\varphi'(y_2))^2 \partial_\omega^2 U_0 = 0,$$

$$\text{so that } U_0(x, \omega) = e^{it(\varphi'(y_2))^2/2} h(y_1 + t, y_2)e^{i\omega},$$

$$\text{and the exact solution: } u^\varepsilon(x) = e^{it(\varphi'(y_2))^2/2} h(y_1 + t, y_2)e^{i\psi/\sqrt{\varepsilon}} e^{i\phi/\varepsilon} + \mathcal{O}(\sqrt{\varepsilon}).$$

Example 2. *Long time propagation in heterogeneous media with initial periodic data*

Now, we emphasize the difference between constant and variable coefficients in the equations. Consider a wave initially oscillating at frequency $1/\epsilon$, and slowly modulated, at scale 1; let this wave propagate in a (more slowly varying) non-homogeneous medium (at scale $1/\epsilon$). This corresponds to a family $(v^\epsilon)_{\epsilon\in]0,1]}$ of solutions to

$$\begin{cases} L(\epsilon X, v^\epsilon, \partial)v^\epsilon = \partial_t v^\epsilon + \sum_{j=1}^{d} A_j(\epsilon X, v^\epsilon)\partial_{Y_j} v^\epsilon = 0 \\ v^\epsilon_{|T=0}(Y) = \epsilon^2 g\left(Y, \dfrac{k\cdot Y}{\epsilon}\right). \end{cases}$$

The A_j's are symmetric matrices, $k \in \mathbb{R}^d \setminus \{0\}$ is a given wavevector, and the smooth function $g(Y,\theta)$ is periodic in Y and θ.

For this quasilinear initial-value problem, we prove a long-time result of the same kind as in [10], [23] mentioned above, *i.e.* existence (and uniqueness) of each v^ϵ on a domain of the form Ω/ϵ, with $\Omega \subset \mathbb{R}^{1+d}$ independent of ϵ. Since L has variable (but slowly varying) coefficients, the linear phases analysis breaks down.

First, rescale the problem, setting $x = \epsilon X$ (and $\epsilon = \varepsilon$, in order to have frequency $1/\varepsilon$). Thus, we turn to

$$\begin{cases} L(x, u^\varepsilon, \partial)u^\varepsilon = 0 \\ u^\varepsilon_{|t=0}(y) = \varepsilon g\left(y, \dfrac{k\cdot y}{\varepsilon}\right). \end{cases}$$

We suppose that the coherence assumptions needed for the analysis are satisfied (see Remark 0.0.1 below). From Theorem 8.3.1, we get a unique solution $u^\varepsilon \in \mathcal{C}^1(\overline{\Omega})$ for all $\varepsilon \in]0,1]$, together with the infinite order asymptotics

$$\left\| (\varepsilon\partial)^\alpha \left[u^\varepsilon - \varepsilon \sum_{n<M} \varepsilon^{n/2} u_n\left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon}\right)\right]\right\|_{L^\infty} = \mathcal{O}(\varepsilon^{M/2+1})$$

–provided that the data admit such an asymptotics, which imposes some polarization conditions (see Proposition 6.3.1). The phases ϕ and $\psi = (\psi_1,\ldots,\psi_d)$ are defined by:

$$\begin{cases} \partial_t\phi + \lambda(x,\partial_y\phi) = 0 \\ \phi_{|t=0} = k\cdot y \end{cases}, \quad \begin{cases} \partial_t\psi_\mu + \partial_\eta\lambda(x,\partial_y\phi)\cdot\partial_y\psi_\mu = 0 \\ \psi_{\mu|t=0} = y_\mu \end{cases},$$

where $\lambda(x,\eta)$ is an eigenvalue of the (symmetric) matrix $\sum \eta_j A_j(x,0)$. When this matrix really depends on x, none of these phases is linear.

Conclusion : Coming back to the original scales, we have on $\dfrac{1}{\epsilon}\Omega$: $\forall \alpha \in$

$\mathbb{N}^{1+d}$,

$$\left\| (\varepsilon\partial)^\alpha \left[v^\epsilon - \epsilon^2 u_0 \left(\epsilon X, \frac{\psi(\epsilon X)}{\epsilon}, \frac{\phi(\epsilon X)}{\epsilon^2} \right) \right] \right\|_{L^\infty(\frac{1}{\epsilon}\Omega)} = o(\epsilon^2).$$

Since the phases ϕ and ψ depend on ϵX instead of X, this approximation is called *weakly nonplanar* (see [17]). Such a $o(\epsilon^2)$ approximation *cannot* be achieved using linear phases: $\psi(\epsilon X)/\epsilon$ and $\phi(\epsilon X)/\epsilon^2$ differ from their linear parts $\partial_x \psi_\mu(0) \cdot X$ and $\partial_x \phi_\mu(0) \cdot X/\epsilon$ by $\mathcal{O}(\epsilon |X|^2) = \mathcal{O}(1/\epsilon)$ and $\mathcal{O}(|X|^2) = \mathcal{O}(1/\epsilon^2)$ terms, respectively. The L^∞ error in the approximation of v^ϵ by $\epsilon^2 u_0(\epsilon X, \partial_x \psi_\mu(0) \cdot X, \partial_x \phi_\mu(0) \cdot X/\epsilon)$ then has size $O(\epsilon^2)$ (100% error), since $u_0(x, \omega, \theta)$ does not decay in ω and θ (furthermore, "errors" for the phases ϕ and ψ are much bigger than the period of $u_0(x, \cdot, \cdot)$, and have no algrebraic link with it, thus the error for the value of v^ϵ occurs at almost each point x, randomly in ϵ).

Remark 0.0.1. *In this example, in order to stress the qualitative difference between the variable and constant coefficient case, we have not discussed the validity of the coherence properties required on the phases. Checking Assumption 3.0.4 is immediate, since there is only one rapid phase ϕ, which satisfies an eikonal equation associated with L_1; the same is true for Assumption 4.2.2, since the characteristic variety of V is a hyperplane; without specifying more the form of the operator L, we cannot check Assumption 4.2.1.*

These coherence assumptions are needed to construct the profiles u_n of the approximate solution. Once this infinite order approximate solution is given, Theorem 8.3.1 shows the existence and stability of exact solutions having the corresponding asymptotics with no additional assumption. But coherence is not needed only to valid the WKB approach presented here: examples of explosive exact solution in [21] show that without coherence, the situation is qualitatively different.

Description of the paper

We begin (Paragraph 2) with the formal WKB expansion of Lu, setting the coefficients of all powers of ε equal to zero. This provides us with an infinite triangular system of equations, each involving three successive profiles u_n, u_{n+1}, u_{n+2}. So as to separate them and get equations for each profile, we use projections.

The first step concerns the fast variable θ (Paragraph 3), since the phases ψ don't appear in the first equation. The analysis is the same as that of geometric optics, mode by mode ($\alpha \in \mathbb{Z}^q$) for Fourier series, which leads to matrix operators $L(x, 0, d(\alpha \cdot \phi(x))$. Considering several phases ϕ ($q > 1$) and in the multi-d case, in order to avoid catastrophic focusing of rays, we then need a strong geometric assumption on the (real) vector space Φ generated by the phases ϕ: this is L_1-*coherence* (see Definition 3.0.4) introduced by Hunter, Majda and Rosales ([15]) and developed by Joly, Métivier and Rauch ([18], [19], [21]).

The next step (Paragraph 4.2) is the analogue, for the intermediate variable ω. In addition to the L_1-coherence of $\Psi = \mathrm{Vect}_{\mathbb{R}}(\psi_1, \ldots, \psi_p)$, we have to assume coherence of Ψ with respect to each transport field $V_{\alpha\cdot\phi}$ along the rays of the linear combination $\alpha \cdot \phi = \sum \alpha_\mu \phi_\mu$. In order to clarify the interplay of our three assumptions, we give (Paragraph 5) a simpler sufficient condition implying them all: L_1-coherence of $\Phi + \Psi$ (Proposition 5.2.1); some examples are also exhibited.

The profile equations obtained (Paragraph 6.3) are solvable on $\mathcal{C}^\infty$ and H^s only under Small Divisors assumptions on the phases (Assumption 6.2.1), which are generically satisfied.

These profile equations consist in a Schrödinger equation (which is nonlinear for the first profile) for each oscillating mode (w.r.t. θ), with "time" at scale x, measured along rays, and dispersion in the ω variable, these equations being coupled to a linear transport equation in ω. This first system for the oscillating part is also coupled to a symmetric system (again, nonlinear at first order) for the non-oscillating part, in variables x and ω.

We prove well-posedness for such nonlinear, non-constant coefficients systems (Paragraph 7) thanks to a standard Picard iterative scheme and energy estimates inherited from the hyperbolic structure of the original system. Variable coefficients actually force the introduction of anisotropic (Sobolev) spaces (with smoothness depending on the variable x, ω or θ considered). Then, solvability requires commutation of equations, which is a consequence of the coherence assumptions on phases.

Since, this way, we construct an infinite-order asymptotic solution to $Lu = 0$, existence and stability of exact solutions (Paragraph 8) follow from the "ε-derivatives" perturbative methods of O. Guès ([14]), for continuation or initial-value problems.

Finally, Paragraph 9 is devoted to some explicit examples from fluid dynamics.

1 The Ansatz

We study solutions of the following quasilinear hyperbolic system,

$$L(x,u,\partial)u = \partial_t u + \sum_{j=1}^{d} A_j(x,u)\partial_j u = \sum_{j=0}^{d} A_j(x,u)\partial_j u = 0. \tag{1.3}$$

We denote $x = (t, y)$ space-time coordinates in Ω, a bounded connected open subset of $\mathbb{R}^{1+d}$, for which:

Assumption 1.0.1. *Each matrix $A_j \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{C}^N, \mathcal{M}_N(\mathbb{C}))$ is Hermitian symmetric.*

Following [7], we look for solutions u, which are perturbations of a reference non-oscillating state $u^0 = u^0(x)$. The perturbation is supposed to admit WKB asymptotics,

$$u^\varepsilon \sim \varepsilon \sum_{n=0}^{+\infty} \varepsilon^{n/2} u_n \left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon} \right), \tag{1.4}$$

where the phases $\phi_1, \ldots, \phi_q$ ('rapid' phases) and $\psi_1, \ldots, \psi_p$ ('slow' phases) are given, smooth on $\overline{\Omega}$. The profiles $u_n = u_n(x, \omega, \theta)$ are **periodic** in $\omega \in \mathbb{R}^p$ and $\theta \in \mathbb{R}^q$, viewed as variables on $\mathbb{T}^p := (\mathbb{R}/2\pi\mathbb{Z})^p$ and $\mathbb{T}^q := (\mathbb{R}/2\pi\mathbb{Z})^q$, respectively. Finally, so as to ensure uniqueness of profile representations, phases are assumed independent (see Lemma 8.2.1):

Assumption 1.0.2. *The phases ψ_μ are $\mathbb{Q}$-linearly independent (as functions), as well as the phases ϕ_ν.*

Remark 1.0.2. *The amplitude ε^m is chosen so that nonlinearities affect the first term u_0 in the asymptotics, in finite time. Here, $m = 1$ because we implicitly assume that $\partial_u A_j(x, 0) \not\equiv 0$ for some j (if not, it is always possible to adjust the amplitude, matching the first non-vanishing terms in the Taylor expansions of the A_j's).*

2 WKB formal expansions

For the asymptotics of Lu to vanish, it is sufficient for the profiles to satisfy:

$$L_1(d\phi)\partial_\theta u_0 = 0, \tag{2.5}$$

$$L_1(d\phi)\partial_\theta u_1 + L_1(d\psi)\partial_\omega u_0 = 0, \tag{2.6}$$

$$L_1(d\phi)\partial_\theta u_2 + L_1(d\psi)\partial_\omega u_1 + L_1(\partial_x)u_0 + B(u_0, \partial_\theta)u_0 = 0, \tag{2.7}$$

$$\vdots$$

$$\begin{aligned} L_1(d\phi)\partial_\theta u_n + L_1(d\psi)\partial_\omega u_{n-1} + L_1(\partial_x)u_{n-2} + B(u_0, \partial_\theta)u_{n-2} \\ + B(u_{n-2}, \partial_\theta)u_0 + \mathcal{F}_n(x, u_0, \partial_{x,\omega,\theta}u_0, \ldots, u_{n-3}, \partial_{x,\omega,\theta}u_{n-3}) = 0, \end{aligned} \tag{2.8}$$

where we have set:

Notation 2.0.1. $L_1(x, \xi) := L(x, 0, \xi)$,
$L_1(d\phi)\partial_\theta := \sum_{\nu=1}^{q} L_1(x, d\phi_\nu(x))\partial_{\theta_\nu}$,
$L_1(d\psi)\partial_\omega := \sum_{\mu=1}^{p} L_1(x, d\psi_\mu(x))\partial_{\omega_\mu}$,
$B(u, \partial_\theta)v := \sum_{\nu=1}^{q} \sum_{j=0}^{d} \partial_j\phi_\nu(x)(\partial_u A_j(x, 0) \cdot u)\partial_{\theta_\nu} v$.

Furthermore, $\mathcal{F}_n$ is a smooth function of its arguments.

3 Analysis of $L_1(d\phi)$

Since our profiles are periodic in ω and θ, we use Fourier series. At a first level, we consider formal series:

$$u_n = \sum_{\alpha\in\mathbb{Z}^q} \sum_{\gamma\in\mathbb{Z}^p} u_n^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega},$$

with coefficients $u_n^{\alpha,\gamma} \in \mathcal{C}^\infty(\overline{\Omega})$. When the variable ω is considered as parameter, these expansions are also written:

$$u_n = \sum_{\alpha\in\mathbb{Z}^q} u_n^{\alpha}(x, \omega)e^{i\alpha\cdot\theta}.$$

Thus, Equation (2.5) is equivalent to:

$$\forall\alpha \in \mathbb{Z}^q,\ L_1(d(\alpha\cdot\phi))u_0^\alpha = 0. \tag{3.9}$$

So as to ensure geometric regularity, we assume that the characteristic variety of L_1 is indeed a differentiable manifold, away from the origin:

Assumption 3.0.3. *The matrix* $\mathcal{A}(x,\eta) := \sum_{j=1}^{d} \eta_j A_j(x,0)$ *has eigenvalues* $\lambda_1(x,\eta) < \cdots < \lambda_Z(x,\eta)$ *with constant multiplicity (on* $\overline{\Omega} \times (\mathbb{R}^d \setminus \{0\})$*).*

The eigenvalues $\lambda_k(x,\eta)$, as well as the associated spectral projectors, $\pi_k(x,\eta)$, are then smooth ($\mathcal{C}^\infty$ w.r.t. x and analytic w.r.t. η).

In addition, when the dimension N is greater than one, because of nonlinearities in Equation 2.7, resonances of phases may occur. Following the coherence method from [15], [18], [19] and [21], we impose:

Assumption 3.0.4. *The (real) vector space* Φ*, generated by the phases* ϕ_μ*, is* L_1*-coherent,* i.e. *:*

$$\forall \rho \in \Phi \setminus \{0\}, \text{ - either: } \forall x \in \overline{\Omega}, d\rho(x) \neq 0 \text{ and } detL_1(x, d\rho(x)) = 0,$$
$$\text{ - or: } \forall x \in \overline{\Omega}, detL_1(x, d\rho(x)) \neq 0.$$

Example 3.0.1. Some coherent spaces:
i) When one phase ϕ is characteristic for L_1 (*i.e.* $detL_1(x, d\phi(x)) \equiv 0$), the line generated ϕ is L_1-coherent.
ii) When Φ is a coherent space, every subspace of Φ is coherent.
iii) When L_1 has constant coefficients constants, plane phases $\phi_\mu(x) := \omega_\mu \cdot x$, $\omega_\mu \in \mathbb{R}^{1+d}$, generate a L_1-coherent space.
iv) For the Euler equations (symbol $L_1(\tau,\eta) = \tau(\tau^2 - |\eta|^2)$), phases $\phi_+ := t + |y|$, $\phi_- := t - |y|$ and $\phi_0 := |y|$ also generate a L_1-coherent space: for $\phi := \alpha_+\phi_+ + \alpha_-\phi_- + \alpha_0\phi_0$, the value of the determinant $detL_1(x, d\phi(x))$ only depends on the coefficients α_+, α_- and α_0.

Under this assumption, we can really perform a Fourier analysis of Equations (2.5)-(2.8)$_n$, since modes $\alpha \in \mathbb{Z}^q$ are now L_1-characteristic or not, independently of x:

Proposition 3.0.1. *Assume that* Φ *is* L_1*-coherent, and let* $(\varphi_1, \dots, \varphi_r)$ *be a* $(\mathbb{R}-)$*basis. Define* $\widetilde{\mathcal{C}^\varphi} := \{(x,\beta) \in \overline{\Omega} \times (\mathbb{R}^r \setminus \{0\}) \quad / \quad detL_1(x, d(\beta \cdot \varphi)(x)) = 0\}$*. Then,*
i) The set $\widetilde{\mathcal{C}^\varphi}$ *splits up into:*
$$\begin{aligned}\widetilde{\mathcal{C}^\varphi} &= \overline{\Omega} \times \mathcal{C}^\varphi \\ &= \widetilde{\mathcal{C}^\varphi_1} \sqcup \cdots \sqcup \widetilde{\mathcal{C}^\varphi_M} \\ &= (\overline{\Omega} \times \mathcal{C}^\varphi_1) \sqcup \cdots \sqcup (\overline{\Omega} \times \mathcal{C}^\varphi_M),\end{aligned}$$
where the cone $\mathcal{C}^\varphi_k$ *is given by the equation:* $\beta \cdot \partial_t \varphi(x) + \lambda_k(x, \beta \cdot \partial_y \varphi(x)) = 0$*.*
ii) Let $p^\varphi(x,\beta)$ *be the orthogonal projector on* $\ker L_1(x, \beta \cdot d\varphi(x))$*. For all* $\beta \in \mathbb{R}^r$*,* $p^\varphi(.,\beta)$ *is* $\mathcal{C}^\infty$ *on* $\overline{\Omega}$*, homogeneous w.r.t.* β *with degree zero, and*

takes the value $-$ 0 *if* $\beta \notin \mathcal{C}^\varphi \cup \{0\}$,
$-$ $\pi_k(x, \beta \cdot \partial_y \varphi(x))$ *if* $\beta \in \mathcal{C}^\varphi_k$,
$-$ Id *if* $\beta = 0$.

iii) There exists a Hermitian matrix $Q^\varphi(x,\beta) \in \mathcal{M}_N(\mathbb{C})$, *homogeneous w.r.t.* β *with degree* -1, *such that* $x \mapsto Q^\varphi(x,\beta)$ *is* $\mathcal{C}^\infty$ *on* $\overline{\Omega}$ *for all* $\beta \in \mathbb{R}^r$, *and:*

$$Q^\varphi(x,\beta) L_1(x, \beta \cdot d\varphi(x)) = Id - p^\varphi(x,\beta), \quad Q^\varphi(x,\beta) p^\varphi(x,\beta) = 0.$$

Remark 3.0.3. *Pay attention to the difference between phases* ϕ, *space* Φ, *and basis* φ.

Back to the phases ϕ, with φ a $\mathbb{R}$-basis of Vectϕ, there is $R \in \mathcal{M}_{q,r}(\mathbb{R})$ such that $\phi = R\varphi$, and we have the equality $L_1(x, \alpha \cdot d\phi(x)) = L_1(x, {}^tR\alpha \cdot d\varphi(x))$ for all $(x,\alpha) \in \overline{\Omega} \times \mathbb{Z}^q$. Linear $\mathbb{Q}$-independence of ϕ implies that ${}^tR$ is injective on $\mathbb{Z}^q$. Since we are only interested in combinations of ϕ_ν's with integer coefficients, we transport the previous objects through:

Notation 3.0.2. $\mathcal{C}^\phi := {}^tR^{-1}(\mathcal{C}^\varphi) \cap \mathbb{Z}^q$,
$\mathcal{C}^\phi_k := {}^tR^{-1}(\mathcal{C}^\varphi_k) \cap \mathbb{Z}^q$,
$p^\phi(x,\alpha) := p^\varphi(x, {}^tR\alpha)$,
$Q^\phi(x,\alpha) := Q^\varphi(x, {}^tR\alpha)$.

Hence, $p^\phi(x,\alpha)$ is the orthogonal projector on $\ker L_1(x, \alpha \cdot d\phi(x))$. For each Fourier mode, we express compatibility conditions on the equations, projecting via $p^\phi(x,\alpha)$; then, applying $Q^\phi(x,\alpha)$ corresponds to solving the equation. The set of equations (3.9) thus becomes:

$$\forall \alpha \in \mathbb{Z}^q,\ p^\phi_\alpha u_0^\alpha = u_0^\alpha, \tag{3.10}$$

which is the usual polarization condition of geometrical optics (*cf.* [26]).

Equation (2.6) is equivalent to:

$$\forall \alpha \in \mathbb{Z}^q,\ p^\phi_\alpha L_1(d\psi) \partial_\omega p^\phi_\alpha u_0^\alpha = 0, \tag{3.11a}$$

$$\forall \alpha \in \mathbb{Z}^q,\ (1 - p^\phi_\alpha) u_1^\alpha = i Q^\phi_\alpha L_1(d\psi) \partial_\omega u_0^\alpha. \tag{3.11b}$$

For oscillating modes, new operators appear: $p^\phi_\alpha L_1(d\psi)\partial_\omega p^\phi_\alpha$ in (3.11a), $p^\phi_\alpha L_1(\partial_x) p^\phi_\alpha$ and $p^\phi_\alpha L_1(d\psi)\partial_\omega Q^\phi_\alpha L_1(d\psi)\partial_\omega p^\phi_\alpha$ in (2.7), which becomes:

$$\begin{aligned}\forall \alpha \in \mathbb{Z}^q,\ p^\phi_\alpha L_1(d\psi)\partial_\omega p^\phi_\alpha u_1^\alpha + i p^\phi_\alpha L_1(d\psi)\partial_\omega Q^\phi_\alpha L_1(d\psi)\partial_\omega p^\phi_\alpha u_0^\alpha \\ + p^\phi_\alpha L_1(\partial_x) p^\phi_\alpha u_0^\alpha + p^\phi_\alpha (B(u_0, \partial_\theta) u_0)^\alpha = 0.\end{aligned} \tag{3.12}$$

4 Analysis w.r.t. the intermediate variables

4.1 Reductions to a scalar form

The principal parts of the operators above can be diagonalized in the Fourier β modes (for a proof, see [10]):

Proposition 4.1.1. *Under Assumptions 3.0.3 and 3.0.4, for all* $\beta \in \mathcal{C}_k^\varphi$,
i) $p_\beta^\varphi L_1(\partial_x) p_\beta^\varphi = p_\beta^\varphi [V_\beta^\varphi(x, \partial_x) + C_\beta^\varphi] = p_\beta^\varphi [(\partial_t + v_\beta^\varphi(x) \cdot \partial_y) + C_\beta^\varphi]$,

where $v_\beta^\varphi(x) := \partial_\eta \lambda_k(x, \beta \cdot \partial_y \varphi(x))$, *and* $C_\beta^\varphi(x) := \displaystyle\sum_{j=0}^{d} A_j(x,0)(\partial_j p_\beta^\varphi)(x)$ *;*

ii) $\forall \rho \in \mathcal{C}^\infty,\ p_\beta^\varphi L_1(d\rho) p_\beta^\varphi = p_\beta^\varphi V_\beta^\varphi(d\rho)$ *;*

iii) $p_\beta^\varphi L_1(d\psi)\partial_\omega Q_\beta^\varphi L_1(d\psi)\partial_\omega p_\beta^\varphi = -\dfrac{1}{2} p_\beta^\varphi \displaystyle\sum_{j,l=1}^{d} \frac{\partial^2 \lambda_k}{\partial \eta_j \partial \eta_l} (\beta \cdot \partial_y \varphi)(\partial_j \psi(x) \cdot \partial_\omega)(\partial_l \psi(x) \cdot \partial_\omega)$.
We write $D_\beta^\varphi(x, \partial_\omega) := -\frac{1}{2} \sum_{j,l=1}^{d} \frac{\partial^2 \lambda_k}{\partial \eta_j \partial \eta_l} (\beta \cdot \partial_y \varphi)(\partial_j \psi(x) \cdot \partial_\omega)(\partial_l \psi(x) \cdot \partial_\omega)$.

We emphasize here the fact that D_β^φ has real coefficients. This is a consequence of hyperbolicity, crucial for the energy estimates in Paragraph 7.

Following the previous notations, the above objects are transported when one uses the phases ϕ instead of the base φ:

Notation 4.1.1. *When* $\alpha \in \mathcal{C}^\phi$, $V_\alpha^\phi := V_{{}^tR\alpha}^\varphi,\ C_\alpha^\phi := C_{{}^tR\alpha}^\varphi,\ D_\alpha^\phi := D_{{}^tR\alpha}^\varphi$.

4.2 Coherence, second step

We have to treat Equation (3.12) and possible resonances of Fourier modes in ω. When considering only one slow phase ψ, in order for the oscillating part of the profiles to depend effectively on the corresponding variable ω, Equation (3.11a) forces ψ to be characteristic for $p_\alpha^\phi L_1 p_\alpha^\phi$ (*i.e.* $V_\alpha^\phi(x, d\psi) = 0$) for some $\alpha \neq 0$, which means that ψ is constant along the rays associated with $\alpha \cdot \phi$. In the case of several slow phases (possibly generated by non-polarized initial data), this assumption is replaced by V-coherence of the vector space Ψ. The non-oscillating modes require L_1-coherence of Ψ (*cf.* Equations (3.11a) and (3.12)).

Assumption 4.2.1. *The vector space* $\Psi := Vect_{\mathbb{R}}(\psi_1, \ldots, \psi_p)$ *is* L_1*-coherent.*

Assumption 4.2.2. *The pair* (Φ, Ψ) *is V-coherent,* i.e. *:*

$$\forall \rho \in \Psi \setminus \{0\}, \forall \beta \in \mathcal{C}^\varphi, \text{ - either: } \forall x \in \overline{\Omega}, d\rho(x) \neq 0 \text{ and } V^\varphi_\beta(x, d\rho(x)) = 0,$$
$$\text{- or: } \forall x \in \overline{\Omega}, V^\varphi_\beta(x, d\rho(x)) \neq 0.$$

Proposition 4.2.1. *Suppose that all previous assumptions are satisfied. Let* $(\varphi_1, \dots, \varphi_r)$ *be a* $(\mathbb{R}-)$*basis of* Φ*, and* $(\chi_1, \dots, \chi_s)$ *a* $(\mathbb{R}-)$*basis of* Ψ*.*
Define $\widetilde{\mathcal{D}^\chi} := \{(x, \delta) \in \overline{\Omega} \times (\mathbb{R}^s \setminus \{0\}) \quad / \quad \det L_1(x, d(\delta \cdot \chi)(x)) = 0\},$
$\widetilde{\mathcal{E}^{\varphi,\chi}} := \{(x, \beta, \delta) \in \overline{\Omega} \times \mathcal{C}^\varphi \times (\mathbb{R}^s \setminus \{0\}) \quad / \quad V^\varphi_\beta(x, d(\delta \cdot \chi)(x)) = 0\}.$
Then,
i) The set $\widetilde{\mathcal{D}^\chi}$ *splits up into:*

$$\begin{aligned}\widetilde{\mathcal{D}^\chi} &= \overline{\Omega} \times \mathcal{D}^\chi \\ &= \widetilde{\mathcal{D}^\chi_1} \sqcup \dots \sqcup \widetilde{\mathcal{D}^\chi_{M'}} \\ &= (\overline{\Omega} \times \mathcal{D}^\chi_1) \sqcup \dots \sqcup (\overline{\Omega} \times \mathcal{D}^\chi_{M'}),\end{aligned}$$

where the cone $\mathcal{D}^\chi_k$ *is given by:* $\delta \cdot \partial_t \chi(x) + \lambda_k(x, \delta \cdot \partial_y \chi(x)) = 0.$
Furthermore,

$$\begin{aligned}\widetilde{\mathcal{E}^{\varphi,\chi}} &= \overline{\Omega} \times \mathcal{E}^{\varphi,\chi} \\ &= \widetilde{\mathcal{E}^{\varphi,\chi}_1} \sqcup \dots \sqcup \widetilde{\mathcal{E}^{\varphi,\chi}_M} \\ &= (\overline{\Omega} \times \mathcal{E}^{\varphi,\chi}_1) \sqcup \dots \sqcup (\overline{\Omega} \times \mathcal{E}^{\varphi,\chi}_M),\end{aligned}$$

with $\mathcal{E}^{\varphi,\chi}_k \cup \{0\}$ *a family of hyperplanes (or* $\mathbb{R}^s$ *itself), parametrized by* $\beta \in \mathcal{C}^\varphi_k$*, given by the equation:* $\sum_\mu V^\varphi_\beta(x, d\chi_\mu(x))\delta_\mu = 0.$
ii) Let $p^\chi(x, \delta)$ *be the orthogonal projector on* $\ker L_1(x, \delta \cdot d\chi(x))$*. For all* $\delta \in \mathbb{R}^s$*,* $p^\chi(\cdot, \delta)$ *is* $\mathcal{C}^\infty$ *on* $\overline{\Omega}$*, homogeneous w.r.t.* δ *with degree zero, and takes value* – 0 *if* $\delta \notin \mathcal{D}^\chi \cup \{0\}$*,*
– $\pi_k(x, \delta \cdot \partial_y \chi(x))$ *if* $\delta \in \mathcal{D}^\chi_k$*,*
– Id *if* $\delta = 0$*.*
iii) There exists a Hermitian matrix $S^\chi(x, \delta) \in \mathcal{M}_N(\mathbb{C})$*, homogeneous w.r.t.* δ *with degree* -1*, such that* $x \mapsto S^\chi(x, \delta)$ *is* $\mathcal{C}^\infty$ *on* $\overline{\Omega}$ *for all* $\delta \in \mathbb{R}^s$*, and such that:*

$$S^\chi(x, \delta) L_1(x, \delta \cdot d\chi(x)) = Id - p^\chi(x, \delta).$$

Remark 4.2.1. *Again, take care of the distinction between phases* ψ*, space* Ψ*, and base* χ*.*

Getting rid of basis of Φ and Ψ (since there is a matrix $R' \in \mathcal{M}_{p,s}(\mathbb{C})$ such that $\psi = R'\chi$):

Notation 4.2.1. $\mathcal{D}^\psi := {}^tR'^{-1}(\mathcal{D}^\chi) \cap \mathbb{Z}^p,$
$\mathcal{E}^{\phi,\psi} := ({}^tR \otimes {}^tR')^{-1}(\mathcal{E}^{\varphi,\chi}) \cap (\mathbb{Z}^q \times \mathbb{Z}^p),$
$p^\psi(x, \gamma) := p^\chi(x, {}^tR'\gamma),$
$S^\psi(x, \gamma) := S^\chi(x, {}^tR'\gamma).$

Remark 4.2.2. *The rectification phenomenon (see [10], [23], [24], [11]) corresponds to the interaction between oscillating and non-oscillating modes* u^α *(*$\alpha \neq 0$*) and* u^0*, allowing for example creation of a mean field, even if there isn't any in the initial data.*

In our context, the absence of rectification effect is equivalent to vanishing of all mean terms in nonlinearities $(B(u_0, \partial_\theta)u_0)$ *for Equations (3.12), as well as in all Equations* $(2.8)_n$ *corresponding to the next profiles. In that case, Assumption 4.2.1 (*L_1*-coherence of* Ψ*) is not needed.*

5 V-coherence and L_1-coherence

A precise study of coherence can be found in [21], as well as numerous examples. The aim of this section is rather to show the link between L_1-coherence of Φ and Ψ and V-coherence of (Φ, Ψ). We first consider only one phase ϕ, which is then L_1-characteristic. Furthermore, the vector field $V(\partial_x)$ no longer depends on modes β or α (see Proposition 4.1.1)).

Set some notations: define $\mathcal{C}_x := \{\xi/(x,\xi) \in \mathcal{C}\}$ and $\mathcal{E}_x := \{\xi/(x,\xi) \in \mathcal{E}\}$, projections of the characteristic varieties $\mathcal{C} := \{(x,\xi) \in \overline{\Omega} \times \mathbb{R}^{1+d} / \det L_1(x,\xi) = 0\}$ and $\mathcal{E} := \{(x,\xi) \in \overline{\Omega} \times \mathbb{R}^{1+d} / V(x,\xi) = 0\}$, respectively.

It is clear that $\mathcal{E}_x$ is the tangent plane to $\mathcal{C}_x$ at $d\phi(x)$. Now, coherence says that the gradient of a function in the space considered must either stay on the projected characteristic variety, or never touch it. That's why the link between $\mathcal{C}_x$ and $\mathcal{E}_x$ must induce a link between L_1- and V-coherence. In particular, when $\mathcal{C}_x$ is a hyperplane, $\mathcal{E}_x = \mathcal{C}_x$, and we have:

Proposition 5.0.2. *If for all* $x \in \overline{\Omega}$*,* $\mathcal{C}_x$ *is a hyperplane, then* V*-coherence of* Ψ *is equivalent to* L_1*-coherence of* Ψ *(and of* $\Phi + \Psi$*).*

5.1 V-coherence without L_1-coherence

Of course, V-coherence doesn't imply L_1-coherence: as an example, in space dimension $d = 2$, choose $\mathcal{C} = \overline{\Omega} \times \{\tau^2 = |\eta|^2\}$, and phases $\phi(t,x) := t + x_1$, $\psi(t,x) := t + x_1 + x_2^2$. Since $\nabla\psi$ belongs to the tangent plane to the cone, the line generated by ψ is V-coherent. But if $\Omega \cap \{x_2 = 0\} \neq \emptyset$, Vect$\psi$ is not L_1-coherent.

5.2 A stronger assumption

There are some cases where both L_1- and V-coherence of Vectψ follow from L_1-coherence of the space generated by all phases:

Definition 5.2.1. *The graph of a smooth function $\lambda : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ satisfies the (convexity) condition $(\mathcal{C})$ if it never crosses its tangent plane.*

Example 5.2.1. *The two sheets $\{\tau = \pm|\xi|\}$ of the usual light cone fulfill condition $(\mathcal{C})$.*

Proposition 5.2.1. *Under Assumption 3.0.3 (smoothness of the characteristic variety of L_1), suppose that for all $x \in \overline{\Omega}$, the sheet of $\mathcal{C}_x$ on which $\nabla\phi(x)$ lies satisfies condition $(\mathcal{C})$. Then, L_1-coherence of Vect(ϕ, ψ) implies V-coherence of Vect(ψ).*

Proof:
Write $\mathcal{G}_x$ for the graph of $\lambda(x, .)$ (when $\partial_t\phi + \lambda(\partial_y\phi) \equiv 0$). Hence, $\mathcal{G}_x$ satisfies condition $(\mathcal{C})$, and is 'on one side' of $\mathcal{E}_x$, for all x. Suppose that Vect(ϕ, ψ) is L_1-coherent, and that there exist $\rho \in$ Vect(ϕ, ψ), $x, y \in \Omega$ such that $\nabla\rho(x) \in \mathcal{E}_x$ and $\nabla\rho(y) \notin \mathcal{E}_y$ (see Figure 1).

The affine line $\nabla(\phi + \alpha\rho)(x)$ (parameterized by α) is then contained in $\mathcal{E}_x$, and say, under $\mathcal{G}_x$. On the contrary, for $|\alpha|$ small enough (and for the right sign of α), $\nabla(\phi + \alpha\rho)(y)$ is strictly above $\mathcal{G}_y$ (and does not belong to any other part of $\mathcal{C}_y$): since $\nabla\rho(y) \neq 0$, the line $(\nabla(\phi+\alpha\rho)(y))_\alpha$ is transverse to $\mathcal{E}_y$, and to $\mathcal{G}_y$, for α sufficiently small. This provides a $z \in \Omega$ such that $\nabla(\phi + \alpha\rho)(z)$ belonging to $\mathcal{G}_z$. We know that $\nabla(\phi + \alpha\rho)(y)$ doesn't belong to $\mathcal{C}_y$, so this contradicts L_1-coherence of Vect(ϕ, ψ). $\square$

In [16], we find the following example:

Example 5.2.2. (L_1- and V-coherence for curved phases)
In space dimension $d = 3$, consider an operator (of 'Euler' type) with symbol $\det(L_1) \equiv \tau^2(\tau^2 - |\eta|^2)$. In cylindrical coordinates (R, σ, x_3), define the phases:

$$\phi(t, R, \sigma, x_3) := R - t,\ \ \psi_1(t, R, \sigma, x_3) := \sigma,\ \ \psi_2(t, R, \sigma, x_3) := x_3.$$

The gradient of ϕ belongs to $\{\tau + |\eta| = 0\}$, and so $V(\xi) = (\partial_t\phi, -\partial_y\phi) \cdot \xi = -\tau - \dfrac{y \cdot \eta}{R}$. Next, when φ is the sum $\varphi = \alpha_0\phi + \alpha_1\psi_1 + \alpha_2\psi_2$, we

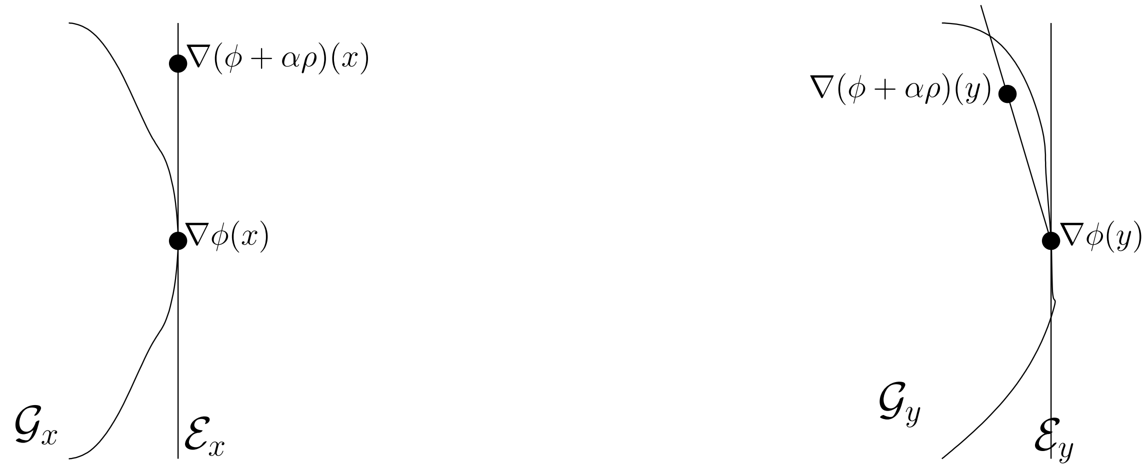

Figure 1: $\nabla(\phi + \alpha\rho)(y)$ is 'on one side' of $\mathcal{G}_y$.

have $\det(L_1(\nabla\varphi)) = -\alpha_0^2(\frac{\alpha_1^2}{R^2} + \alpha_2^2)$. Vanishing of this determinant does not depend on the point considered in $\overline{\Omega}$, so that Vect(ϕ, ψ) is L_1-coherent. Proposition 5.2.1 then ensures that Ψ is $\pi L_1 \pi$-coherent, since $\{\tau + |\eta| = 0\}$ is convex.

5.3 The case of several fast phases

When there are several phases ϕ_μ, V-coherence of (Φ, Ψ) is equivalent to V_α^ϕ-coherence of Ψ for all $\alpha \in \mathcal{C}^\phi$. This brings us back to the previous case, with one fast phase $\alpha \cdot \phi$.

Proposition 5.3.1. *Under Assumption 3.0.3 (smoothness of the characteristic variety of L_1), suppose that for all $x \in \overline{\Omega}$, each sheet of $\mathcal{C}_x$ satisfies condition ($\mathcal{C}$). Then, L_1-coherence of Vect(ϕ, ψ) implies V-coherence of Vect(ψ).*

6 Profile equations and mean operators

Throughout this section, we need Assumptions 3.0.4, 4.2.1 and 4.2.2 to be satisfied, for our definitions to make sense.

6.1 Definitions in the context of formal series

The previous projections and algebraic resolutions of equations are equivalent to the action, on formal series, of the following Fourier multipliers:

$$\mathbf{L_1}(\partial_\theta)\left(\sum_{\alpha\in\mathbb{Z}^q} u^\alpha(x,\omega)e^{i\alpha\cdot\theta}\right) := i\sum_{\alpha\in\mathbb{Z}^q} L_1(x,d(\alpha\cdot\phi)(x))u^\alpha(x,\omega)e^{i\alpha\cdot\theta},$$

$$\mathbb{E}\left(\sum_{\alpha\in\mathbb{Z}^q} u^\alpha(x,\omega)e^{i\alpha\cdot\theta}\right) := \sum_{\alpha\in\mathcal{C}^\phi\cup\{0\}} p^\phi_\alpha(x)u^\alpha(x,\omega)e^{i\alpha\cdot\theta},$$

$$\mathbf{L_1}(\partial_\theta)^{(-1)}\left(\sum_{\alpha\in\mathbb{Z}^q} u^\alpha(x,\omega)e^{i\alpha\cdot\theta}\right) := -i\sum_{\alpha\in\mathbb{Z}^q\setminus\{0\}} Q^\phi_\alpha(x)u^\alpha(x,\omega)e^{i\alpha\cdot\theta},$$

$$\mathbf{L_1}(\partial_\omega)\left(\sum_{\gamma\in\mathbb{Z}^p} u^\gamma(x,\theta)e^{i\gamma\cdot\omega}\right) := i\sum_{\gamma\in\mathbb{Z}^p} L_1(x,d(\gamma\cdot\psi)(x))u^\gamma(x,\theta)e^{i\gamma\cdot\omega},$$

$$\mathbb{F}\left(\sum_{\gamma\in\mathbb{Z}^p\cup\{0\}} u^\gamma(x,\theta)e^{i\gamma\cdot\omega}\right) := \sum_{\gamma\in\mathcal{D}^\psi\cup\{0\}} p^\psi_\gamma(x)u^\gamma(x,\theta)e^{i\gamma\cdot\omega},$$

$$\mathbf{L_1}(\partial_\omega)^{(-1)}\left(\sum_{\gamma\in\mathbb{Z}^p} u^\gamma(x,\theta)e^{i\gamma\cdot\omega}\right) := -i\sum_{\gamma\in\mathbb{Z}^p\setminus\{0\}} S^\psi_\gamma(x)u^\gamma(x,\theta)e^{i\gamma\cdot\omega},$$

$$\mathbf{V}(d\psi)\cdot\partial_\omega\left(\sum_{\alpha,\gamma} u^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega}\right) := i\sum_{(\alpha,\gamma)\in\mathcal{C}^\phi\times\mathbb{Z}^p} V^\phi_\alpha(x,d(\gamma\cdot\psi)(x))u^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega},$$

$$\mathbb{G}\left(\sum_{\alpha,\gamma} u^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega}\right) := \sum_{(\alpha,\gamma)\in\overline{\mathcal{E}^{\phi,\psi}}} u^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega},$$

$$(\mathbf{V}(d\psi)\cdot\partial_\omega)^{(-1)}\left(\sum_{\alpha,\gamma} u^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega}\right) := \sum_{(\alpha,\gamma)\notin\overline{\mathcal{E}^{\phi,\psi}}} \frac{-iu^{\alpha,\gamma}(x)e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega}}{V^\phi_\alpha(x,d(\gamma\cdot\psi)(x))},$$

$$\mathbf{V}(\partial_x)\left(\sum_{\alpha\in\mathbb{Z}^q} u^\alpha(x,\omega)e^{i\alpha\cdot\theta}\right) := \left(\partial_t + \sum_j \mathbf{v_j}(x)\cdot\partial_j\right) u$$
$$:= \sum_{\alpha\in\mathcal{C}^\phi} V_\alpha^\phi(x,\partial_x)u^\alpha(x,\omega)e^{i\alpha\cdot\theta},$$

$$\mathbf{C}\left(\sum_{\alpha\in\mathbb{Z}^q} u^\alpha(x,\omega)e^{i\alpha\cdot\theta}\right) := \sum_{\alpha\in\mathcal{C}^\phi} C_\alpha^\phi(x)u^\alpha(x,\omega)e^{i\alpha\cdot\theta},$$

$$\mathbf{D}(\partial_\omega)\left(\sum_{\alpha,\gamma} u^{\alpha,\gamma}e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega}\right) := -\sum_{(\alpha,\gamma)\in\mathcal{E}^{\phi,\psi}} D_\alpha^\phi(d(\gamma\cdot\psi))u^{\alpha,\gamma}e^{i\alpha\cdot\theta}e^{i\gamma\cdot\omega},$$

where we have set: $\overline{\mathcal{E}^{\phi,\psi}} := \mathcal{E}^{\phi,\psi}\cup(\mathbb{Z}^q\times\{0\})\cup(\{0\}\times\mathbb{Z}^p)$.

Here, $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ are projectors on the kernels of $\mathbf{L_1}(\partial_\theta)$, $\mathbf{L_1}(\partial_\omega)$ and $\mathbb{E}\mathbf{V}(d\psi)\cdot\partial_\omega$, which are extensions to formal series of the principal parts of $L_1(d\phi)\partial_\theta$, $L_1(d\psi)\partial_\omega$ and $\mathbb{E}L_1(d\psi)\partial_\omega\mathbb{E}$, respectively. Next, $\mathbf{L_1}(\partial_\theta)^{(\mathbf{-1})}$, $\mathbf{L_1}(\partial_\omega)^{(\mathbf{-1})}$ and $(\mathbf{V}(d\psi)\cdot\partial_\omega)^{(\mathbf{-1})}$ provide pseudo-inverses of these operators. We sum up these properties as (see also [19]):

Proposition 6.1.1.
i) The equation $\mathbf{L_1}(\partial_\theta)U = F$ has formal solutions if and only if $\mathbb{E}F = 0$, and they are $U = \mathbf{L_1}(\partial_\theta)^{(\mathbf{-1})}F + W$, where $\mathbb{E}W = W$.
ii) The equation $\mathbf{L_1}(\partial_\omega)U = F$ has formal solutions if and only if $\mathbb{F}F = 0$, and they are $U = \mathbf{L_1}(\partial_\omega)^{(\mathbf{-1})}F + W$, where $\mathbb{F}W = W$.
iii) The equation $\mathbf{V}(d\psi)\cdot\partial_\omega\mathbb{E}U = \mathbb{E}F$ has formal solutions (in the range of $\mathbb{E}$) if and only if $\mathbb{G}\mathbb{E}F = 0$, and they are $\mathbb{E}U = (\mathbf{V}(d\psi)\cdot\partial_\omega)^{(\mathbf{-1})}\mathbb{E}F + \mathbb{E}W$, where $\mathbb{G}\mathbb{E}W = \mathbb{E}W$.

Note also the following commutation properties:

Lemma 6.1.1.
i) $\mathbb{G}$ commutes with $\mathbb{E}$ and $\mathbb{F}$.
ii) $\mathbb{E}$, $\mathbb{G}$, $\mathbf{V}(\partial_x)$, $\mathbf{C}$ and $\mathbf{D}(\partial_\omega)$ commute with ∂_θ; $\mathbb{F}$ and $\mathbf{D}(\partial_\omega)$, with ∂_ω.

Proof:
i) is immediate, since $\mathbb{G}$ simply 'selects' frequencies.

ii) follows from Proposition 4.1.1: for example, $\mathbb{F}$ and $\mathbf{D}(\partial_\omega)$ are Fourier multipliers in ∂_ω –thanks to coherence !–, and hence commute with multiplication by ∂_ω. □

6.2 Small divisors

So as to obtain existence of smooth profiles, we have to make some assumptions on the phases (and not only on the vector spaces they generate), in order to avoid 'small divisors' in the Fourier multipliers acting on our series (see also [3], [19], [4] and [12]). We use matrices R and R' from Notations 3.0.2 and 4.2.1.

Assumption 6.2.1. *There are $C > 0$ and $a \in \mathbb{R}$ such that:*
(i) $|{}^tR\alpha| \geq C|\alpha|^{-a}, \forall \alpha \in \mathbb{Z}^q \setminus \{0\}$,
(ii) $|\det L_1(x, \alpha \cdot d\phi(x))| \geq C|\alpha|^{-a}, \forall \alpha \notin \mathcal{C}^\phi \cup \{0\}, \forall x \in \overline{\Omega}$,
(iii) $|{}^tR'\gamma| \geq C|\gamma|^{-a}, \forall \gamma \in \mathbb{Z}^p \setminus \{0\}$,
(iv) $|\det L_1(x, \gamma \cdot d\psi(x))| \geq C|\gamma|^{-a}, \forall \gamma \notin \mathcal{D}^\psi \cup \{0\}, \forall x \in \overline{\Omega}$,
(v) $|V_\alpha^\phi(x, \gamma \cdot d\psi(x))| \geq C|\alpha|^{-a}|\gamma|^{-a}, \forall \alpha \in \mathcal{C}^\phi, (\alpha, \gamma) \in \left(\mathcal{C}^\phi \times \mathbb{R}^p\right) \setminus \overline{\mathcal{E}^{\phi,\psi}}, \forall x \in \overline{\Omega}$.

Proposition 6.2.1. *Under Assumption 6.2.1, the operators defined by (6.1)-(6.1) are bounded on $\mathcal{C}^\infty(\overline{\Omega} \times \mathbb{T}^p \times \mathbb{T}^q)$, and Proposition 6.1.1 is valid on this space.*

Proof:

We give the justification for $\mathbb{E}$ and $\mathbf{L_1}(\partial_\theta)^{(\mathbf{-1})}$, for example. We must prove that the action preserves the property of being rapidly decreasing, for formal series in α, as well as for their derivatives w.r.t. x.

Concerning $\mathbb{E}$, since $p_\alpha^\phi = p_{{}^tR\alpha}^\varphi$, the following lemma gives the answer:

Lemma 6.2.1. *The norms of p_β^φ (as linear transformation on $\mathbb{C}^N$), and of its derivatives $\partial_x^k(p_\beta^\varphi)$, are bounded independently of x and β.*

(This follows immediately from continuity w.r.t. x, which belongs to a compact set, and from degree zero homogeneity and continuity w.r.t. β.)

Concerning $\mathbf{L_1}(\partial_\theta)^{(\mathbf{-1})}$, when $\beta \in \mathcal{C}_k^\varphi$, according to the notations of Proposition 3.0.1,

$$Q_\beta^\varphi = \sum_{l \neq k} \frac{1}{m_l(x, \beta)} \pi_l(x, \beta).$$

Here, m_l is homogeneous with degree one, and Assumption 6.2.1(i) provides the desired bound. On the other hand, when $\beta \notin \mathcal{C}_k^\varphi$, writing

$$Q_\beta^\varphi = L_1(x, d(\beta \cdot \varphi))^{-1},$$

we have to bound terms as $\dfrac{|\beta|^{N-1}}{|\det L_1|}$. This is achieved thanks to Assumption 6.2.1(ii) (and (i)). Again, the same is valid for derivatives: one easily sees that $\mathbf{L_1}(\partial_\theta)^{(-\mathbf{1})}$ maps $H^s(\overline{\Omega} \times \mathbb{T}^p \times \mathbb{T}^q)$ to $H^{s(1-a)}(\overline{\Omega} \times \mathbb{T}^p \times \mathbb{T}^q)$. $\square$

In the case of 'strong coherence' (for example when a timelike phase belongs to the coherent space; *cf.* [19]), we show that small divisors are avoided with almost all choice of characteristic phases in $\Psi^p \times \Phi^q$:

Definition 6.2.1.
i) The space Φ is strongly L_1-coherent if it is L_1-coherent and if, in addition, for all $\underline{x} \in \overline{\Omega}$, there exist $C, b > 0$ such that $\forall (x, \alpha) \in \overline{\Omega} \times (\mathbb{Z}^q \setminus \{0\})$,

$$|\det L_1(x, d(\alpha \cdot \phi)(x))| \geq C \left(|\det L_1(\underline{x}, d(\alpha \cdot \phi)(\underline{x}))|\right)^b (1 + |\alpha|)^{N(1-b)}.$$

ii) The pair (Φ, Ψ) is strongly V-coherent if it is V-coherent and if, in addition, for all $\underline{x} \in \overline{\Omega}$, there exist $C', b' > 0$ such that

$$\forall (x, \alpha, \gamma) \in \overline{\Omega} \times \mathcal{C}^\phi \times (\mathbb{Z}^p \setminus \{0\}),$$
$$|V_\alpha^\phi(x, d(\gamma \cdot \psi)(x))| \geq C' \left(|V_\alpha^\phi(\underline{x}, d(\gamma \cdot \psi)(\underline{x}))|\right)^{b'} (1 + |\alpha| + |\gamma|)^{N(1-b')}.$$

Remark 6.2.1.
i) Such properties depend on the spaces Φ and Ψ only (in particular, they are independent of the numbers p and q).
ii) In the example 9.1, for Euler Equations, the spaces Φ (generated by the phase $R - c_0 t$) and Ψ (generated by σ and y_3) are strongly L_1-coherent, and the pair (Φ, Ψ) is strongly V-coherent.

The following Proposition is a direct application of [19], Section 7 –which follows ideas from [3]:

Proposition 6.2.2. *Assume that Φ and Ψ are strongly L_1-coherent phase spaces, and that (Φ, Ψ) is strongly V-coherent. Then, for almost all choice of L_1-characteristic phases $\phi \in \Phi^q$ and L_1- and V-characteristic phases $\psi \in \Psi^p$, Assumption 6.2.1 is satisfied.*

This statement is rather vague, and a more precise one requires some notations and explanations. Still denoting by φ a basis for Φ, each phase $\rho \in \Phi$ corresponds to a unique $C \in \mathbb{R}^r$: $\rho = C.\varphi$. Hence, the set $\phi = (\phi_1, \ldots, \phi_q)$ of L_1-characteristic phases corresponds to the set $C = (C_1, \ldots, C_q) \in \mathcal{C}^q$. The measure in Proposition 6.2.2 is the ($q(r-1)$-dimensional) Hausdorff measure of C's in $\mathcal{C}^q$.

Similarly, when $C = (C_1, \ldots, C_q) \in \mathcal{C}^q$ and $\beta \in \mathbb{Z}^q$ is such that $\beta.C \in \mathcal{C}$, we define $\mathcal{E}_{\beta.C} := \{D \in \mathbb{R}^s \setminus \{0\} / V(\beta.C, D) = 0\}$. It is either a hyperplane (with the origin removed), or the whole $\mathbb{R}^s \setminus \{0\}$. The corresponding Hausdorff measure is either $(s-1)$-, or s-dimensional, respectively.

When phases $\psi = (\psi_1, \ldots, \psi_p) \in \Psi^p$ are such that for all $\mu \leq p$, there is some $\beta^\mu \in \mathbb{Z}^q$ making ψ_μ $V^\varphi_{\beta^\mu.C}$-characteristic, these phases ψ correspond to $D = (D_1, \ldots, D_p) \in \prod_\mu \mathcal{E}_{\beta^\mu.C}$.

The more precise statement is then:

Proposition 6.2.3. *Let $a > \max\{p, q-N+1\}$. For almost all $C = (C_1, \ldots, C_q) \in \mathcal{C}^q$, when $\beta = (\beta^1, \ldots, \beta^p) \in (\mathbb{Z}^q)^p$ is such that $\beta^\mu.C \in \mathcal{C}$ for all μ, then for almost all $D = (D_1, \ldots, D_p) \in \prod_{\mu=1}^{p} \mathcal{E}_{\beta^\mu.C}$, there is $\kappa > 0$ such that:* $\forall (\alpha, \gamma) \in \mathbb{Z}^q \times \mathbb{Z}^p \ / \ \alpha.C \in \mathcal{C}$, *either* $V(\alpha.C, \gamma.D) = 0$,
or $|V(\alpha.C, \gamma.D)| \geq \kappa(|\alpha||\gamma|)^{-a}$.

6.3 Profile equations, endgame

Notation 6.3.1. *For a Fourier series $u = \sum_{\alpha \in \mathbb{Z}^q} u^\alpha(x, \omega) e^{i\alpha\cdot\theta}$, write*

$$u := \underline{u}(x, \omega) + u^\star(x, \omega, \theta) = \langle u \rangle(x, \omega) + u^\star(x, \omega, \theta) = u^0 + \sum_{\alpha \neq 0} u^\alpha e^{i\alpha\cdot\theta}.$$

Equation (3.10) then rewrites as:

$$\mathbb{E} u_0^\star = u_0^\star. \tag{6.1}$$

Separating oscillations and mean value in Equation (3.11a), we get:

$$\mathbb{F} \underline{u}_0 = \underline{u}_0, \tag{6.2}$$
$$\mathbb{G} u_0^\star = u_0^\star. \tag{6.3}$$

This last condition allows one to solve the next equation, (2.6), rephrasing Equation (3.11b):

$$(1-\mathbb{E})u_1^\star = -\mathbf{L_1}(\partial_\theta)^{(-\mathbf{1})}\mathbf{L_1}(\partial_\omega)u_0^\star.$$

The average of Equation (2.7) is of the form '$\mathbf{L_1}(\partial_\omega)U = F$'. This imposes the compatibility condition:

$$\mathbb{F}L_1(\partial_x)\underline{u}_0 + \mathbb{F}\langle B(u_0,\partial_\theta)u_0\rangle = 0. \tag{6.4}$$

Concerning the oscillating part (of the form $\mathbf{L_1}(\partial_\theta)u_2^\star = F$), one gets:

$$\mathbb{E}\mathbf{V}(d\psi)\cdot\partial_\omega u_1^\star + \mathbb{E}L_1(\partial_x)u_0^\star + \mathbb{E}\mathbf{L_1}(\partial_\omega)(1-\mathbb{E})u_1^\star + \mathbb{E}(B(u_0,\partial_\theta)u_0)^\star = 0,$$

and, projecting via $\mathbb{G}$ and plugging the expression of $(1-\mathbb{E})u_1^\star$ above,

$$\mathbb{G}\mathbb{E}L_1(\partial_x)u_0^\star - i\mathbf{D}(\partial_\omega)u_0^\star + \mathbb{G}\mathbb{E}(B(u_0,\partial_\theta)u_0)^\star = 0. \tag{6.5}$$

This set of conditions first determines u_0, and then solves Equations (2.5), (2.6) and (2.7), providing some parts of the other profiles:

$$\begin{aligned}
(1-\mathbb{E})u_1^\star =& -\mathbf{L_1}(\partial_\theta)^{(-\mathbf{1})}\mathbf{L_1}(\partial_\omega)u_0^\star \\
&:= \mathcal{G}_{1,1}(x,u_0,\partial_\omega u_0)^\star, \\[1ex]
(1-\mathbb{F})\underline{u}_1 =& -\mathbf{L_1}(\partial_\omega)^{(-\mathbf{1})}\left[L_1(\partial_x)\underline{u}_0 + \langle B(u_0,\partial_\theta)u_0\rangle\right] \\
&:= \langle\mathcal{G}_{1,2}(x,u_0,\partial_x u_0,\partial_\theta u_0)\rangle, \\
(1-\mathbb{G})\mathbb{E}u_1^\star =& -(\mathbf{V}(d\psi)\cdot\partial_\omega)^{(-\mathbf{1})}\mathbb{E}\left[L_1(\partial_x)u_0^\star - i\mathbf{D}(\partial_\omega)u_0^\star + (B(u_0,\partial_\theta)u_0)^\star\right] \\
&:= \mathcal{G}_{1,3}(x,u_0,\partial_x u_0,\partial_\omega^2 u_0,\partial_\theta u_0)^\star, \\
(1-\mathbb{E})u_2^\star =& -\mathbf{L_1}(\partial_\theta)^{(-\mathbf{1})}\left[\mathbf{L_1}(\partial_\omega)u_1^\star + L_1(\partial_x)u_0^\star + (B(u_0,\partial_\theta)u_0)^\star\right] \\
&:= \mathcal{G}_{2,1}(x,u_0,\partial_x u_0,\partial_\omega u_0,\partial_\theta u_0,\partial_\omega u_1)^\star.
\end{aligned}$$

According to Propositions 6.1.1 and 6.2.1, Equations (6.1) to (6.3) are equivalent to the three first terms of the expansion of $L(u,\partial)u = 0$, *i.e.* Equations (2.5), (2.6) and (2.7).

Next, we decompose successively all equations in the same way, so as to obtain a triangular integro-differential system. We put together the 'solved' part of the oscillations of Equations $(2.8)_n$ and $(2.8)_{n+1}$, and the compatibility conditions from Equations $(2.8)_{n+1}$ and $(2.8)_{n+2}$. This leads to the following system for the new unknowns $(1-\mathbb{E})u_n^\star$ ((6.1) also writes '$(1-\mathbb{E})u_0^\star = 0$'), $(1-\mathbb{G})\mathbb{E}u_n^\star$, $\mathbb{G}\mathbb{E}u_n^\star$, $\mathbb{F}\underline{u}_n$ and $(1-\mathbb{F})\underline{u}_n$:

$$\begin{aligned}(1-\mathbb{E})u_n^\star = &- \mathbf{L_1}(\partial_\theta)^{(-\mathbf{1})}\left[\mathbf{L_1}(\partial_\omega)u_{n-1}^\star + L_1(\partial_x)u_{n-2}^\star\right.\\ &\left.+(B(u_{n-2},\partial_\theta)u_0)^\star + (B(u_0,\partial_\theta)u_{n-2})^\star + \mathcal{F}_n^\star\right]\\ &:= \mathcal{G}_{n,1}(x,\partial^2 u_0,\dots,\partial^2 u_{n-1})^\star,\end{aligned}$$

$$\begin{aligned}(1-\mathbb{F})\underline{u}_n = &- \mathbf{L_1}(\partial_\omega)^{(-\mathbf{1})}\left[L_1(\partial_x)\underline{u}_{n-1}\right.\\ &\left.+\langle B(u_{n-1},\partial_\theta)u_0\rangle + \langle B(u_0,\partial_\theta)u_{n-1}\rangle + \langle \mathcal{F}_{n+1}\rangle\right]\\ &:= \langle \mathcal{G}_{n,2}(x,\partial^2 u_0,\dots,\partial^2 u_{n-1})\rangle,\end{aligned}$$

$$\begin{aligned}(1-\mathbb{G})\mathbb{E}u_n^\star = &- (\mathbf{V}(d\psi)\cdot\partial_\omega)^{(-\mathbf{1})}\mathbb{E}\left[L_1(\partial_x)u_{n-1}^\star - \mathbf{D}(\partial_\omega)u_{n-1}^\star\right.\\ &\left.+(B(u_{n-1},\partial_\theta)u_0)^\star + (B(u_0,\partial_\theta)u_{n-1})^\star + \mathcal{F}_{n+1}^\star\right]\\ &:= \mathcal{G}_{n,3}(x,\partial^2 u_0,\dots,\partial^2 u_{n-1})^\star,\end{aligned}$$

$$\begin{aligned}&\mathbb{F}L_1(\partial_x)\mathbb{F}\underline{u}_n + \mathbb{F}\langle B(u_n,\partial_\theta)u_0\rangle + \mathbb{F}\langle B(u_0,\partial_\theta)u_n\rangle\\ &\qquad = -\mathbb{F}\left[L_1(\partial_x)(1-\mathbb{F})\underline{u}_n + \langle\mathcal{F}_{n+2}\rangle\right] := \langle \mathcal{G}_{n,4}(x,\partial^2 u_0,\dots,\partial^2 u_{n-1})\rangle,\end{aligned}$$

$$\begin{aligned}&\mathbb{G}\mathbb{E}L_1(\partial_x)\mathbb{G}\mathbb{E}u_n^\star - i\mathbb{G}\mathbb{E}\mathbf{D}(\partial_\omega)u_n^\star + \mathbb{G}\mathbb{E}(B(u_0,\partial_\theta)u_n)^\star + \mathbb{G}\mathbb{E}(B(u_n,\partial_\theta)u_0)^\star\\ &\qquad +\mathbb{G}\mathbb{E}\mathbf{C}u_n^\star = -\mathbb{G}\mathbb{E}\left[L_1(\partial_x)(1-\mathbb{E})u_n^\star + L_1(\partial_x)(1-\mathbb{G})\mathbb{E}u_n^\star + \mathcal{F}_{n+2}^\star\right]\\ &\qquad\qquad := \mathcal{G}_{n,5}(x,\partial^2 u_0,\dots,\partial^2 u_{n-1})^\star.\end{aligned} \tag{6.6}$$

The $\mathcal{F}_n$'s are defined at 2, the $\mathcal{G}_{n,i}$'s are also smooth functions, and we set $\mathcal{G}_{0,i} := 0$ for $i = 0,\dots,5$.

Hence, we have a system of equations equivalent to (2.5),...,$(2.8)_n$,... Furthermore, we only need to determine the $\mathbb{F}\underline{u}_n$'s and $\mathbb{G}\mathbb{E}u_n^\star$'s, since other terms are explicit functions of them. Finally, we recognize, in Equations (6.5) and (6.6), the operator $\mathbb{G}\mathbb{E}L_1(\partial_x)\mathbb{G}\mathbb{E}$, and rewrite it $\mathbb{G}\mathbb{E}(\mathbf{V}(\partial_x)+\mathbf{C})$, thanks to Proposition 4.1.1. These results are summarized in:

Proposition 6.3.1. *If the profiles $u_n \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{T}^p \times \mathbb{T}^q)$ are solutions to Equations (2.8)$_n$ for $n \in \mathbb{N}$, then $v_n := \mathbb{F}\underline{u}_n$ and $w_n := \mathbb{G}\mathbb{E}u_n^\star$ are solutions to:*

$$\mathbb{F}\underline{v}_0 = v_0 \tag{6.7}$$
$$\mathbb{G}\mathbb{E}w_0^\star = w_0 \tag{6.8}$$
$$\mathbb{F}L_1(\partial_x)\mathbb{F}v_0 + \mathbb{F}\langle B(v_0 + w_0, \partial_\theta)w_0\rangle = 0 \tag{6.9}$$
$$\mathbb{G}\mathbb{E}\left[\mathbf{V}(\partial_x)w_0 - i\mathbf{D}(\partial_\omega)w_0 + \mathbf{C}w_0 + (B(v_0 + w_0, \partial_\theta)w_0)^\star\right] = 0, \tag{6.10}$$

and for $n \geq 0$,

$$\mathbb{F}\underline{v}_n = v_n \tag{6.11}$$
$$\mathbb{G}\mathbb{E}w_n^\star = w_n \tag{6.12}$$
$$\mathbb{F}L_1(\partial_x)\mathbb{F}v_n + \mathbb{F}\langle B(v_n + w_n, \partial_\theta)w_0 + B(v_0 + w_0, \partial_\theta)w_n\rangle = \langle\mathcal{H}_n\rangle \tag{6.13}$$
$$\begin{aligned}&\mathbb{G}\mathbb{E}[\mathbf{V}(\partial_x)w_n - i\mathbf{D}(\partial_\omega)w_n + \mathbf{C}w_n \\ &\qquad +(B(v_n + w_n, \partial_\theta)w_0)^\star + (B(v_0 + w_0, \partial_\theta)w_n)^\star] = \mathcal{I}_n^\star,\end{aligned} \tag{6.14}$$

where we have set, using the functions $\mathcal{G}_{n,j}$ above:

$$\langle\mathcal{H}_n\rangle(x, \partial^2 v_0, \partial^2 w_0, \ldots, \partial^2 v_{n-1}, \partial^2 w_{n-1}) := \langle\mathcal{G}_{n,4}(x, \partial^2 u_0, \ldots, \partial^2 u_{n-1})\rangle,$$
$$\mathcal{I}_n^\star(x, \partial^2 v_0, \partial^2 w_0, \ldots, \partial^2 v_{n-1}, \partial^2 w_{n-1}) := \mathcal{G}_{n,5}(x, \partial^2 u_0, \ldots, \partial^2 u_{n-1})^\star.$$

Conversely, assume that the (v_n, w_n)'s solve this system. Then, we recursively define solutions u_n to Equations (2.8)$_n$ by:

$$(1 - \mathbb{E})u_n^\star := \mathcal{G}_{n,1}^\star \tag{6.15}$$
$$(1 - \mathbb{G})\mathbb{E}u_n^\star := \mathcal{G}_{n,2}^\star \tag{6.16}$$
$$\mathbb{G}\mathbb{E}u_n^\star := w_n \tag{6.17}$$
$$\mathbb{F}\underline{u}_n := v_n \tag{6.18}$$
$$(1 - \mathbb{F})\underline{u}_n := \langle\mathcal{G}_{n,2}\rangle. \tag{6.19}$$

Remark 6.3.1.
i) The equations for v_0 and w_0 are nonlinear, but the equations for the other profiles are linear.
ii) Each set of equations contains three parts: polarizations, transport equations and Schrödinger equations. Polarizations are given in Equations (6.7)

and (6.8) (or (6.7) and (6.8)) by the projectors in $\mathbb{E}$ and $\mathbb{F}$. The relation $\mathbb{G}w_n = w_n$ is better understood as a transport equation in the variables ω, from Proposition 6.1.1, iii): $\mathbf{V}(d\psi) \cdot \partial_\omega w_n = 0$. As well, Equation (6.9) (or (6.13)) constitutes a hyperbolic system, which can be viewed as a transport equation in each ω-Fourier mode (the operator $\mathbb{F}L_1(\partial_x)\mathbb{F}$, restricted at one such mode, is scalar, according to Proposition 4.1.1). Finally, Equation (6.10) (or (6.14)) is a Schrödinger equation, with 'time' measured, for each θ-Fourier mode α, along the ray associated with $\alpha \cdot \phi$. The second order part is the operator $\mathbf{D}(\partial_\omega)$.

7 Existence of profiles

In this section, we prove the –local in time and space– existence of smooth profiles u_n, solutions to the Cauchy problem associated with Equations $(2.8)_n$. From Proposition 6.3.1, it is equivalent to prove existence of the pairs (v_n, w_n), solutions to the Cauchy problem associated with Equations $(6.11)_n$-$(6.14)_n$. The structure of these equations is explained in Remark 6.3.1; modulo polarization conditions, they are: a hyperbolic system (in x) for the mean profile v_n, possibly coupled with a transport equation (in ω) and a Schrödinger Equation (with time $t = x_0$) for each mode of the oscillations w_n.

Two remarks will be useful. First, if one wants the system to be solvable, it is necessary that the equations for the oscillations w_n commute. This is the role of the 'second coherence' Assumptions 4.2.1 and 4.2.2, necessary for the commutations (and definitions) in Lemma 6.1.1. Second, these equations were derived from a hyperbolic system, and that is why energy estimates will be available for them.

7.1 Function spaces

So as to take advantage of the finite propagation speed property of hyperbolic systems, we fix $\overline{\Omega}$ as the cone

$$\overline{\Omega} := \{x = (t, y) \in \mathbb{R}^{1+d} / 0 \le t \le t_0, \delta t + |y| \le \rho\},$$

with $\rho > 0$ fixed, and δ big enough, so as to get on the whole $\overline{\Omega}$:

$$\delta Id + \sum_{j=1}^{d} \frac{y_j}{|y|} A_j(x, 0) \text{ positive definite, and } \left(\delta + \sum_{j=1}^{d} \frac{y_j}{|y|} \mathbf{v}_j(x)\right) > 0. \quad (7.1)$$

We choose t_0 small enough, so that $\delta t_0 < \rho$ (and suppose that all phases are defined on the whole $\overline{\Omega}$). Set

$$\omega_t := \{y \in \mathbb{R}^d / (t, y) \in \overline{\Omega}\}, \quad \overline{\Omega}_{t_1} := \overline{\Omega} \cap \{t \leq t_1\} \text{ when } 0 < t_1 \leq t_0.$$

We work with function spaces of Sobolev type in the variables y, ω and θ. Since the equations have non-constant coefficients, commutators appear in energy estimates. In particular, the commutator between ∂_x and $\mathbf{D}(\partial_\omega)$ is of order one in ∂_ω. For example, seeking a L^2-estimate for $\partial_{y,\omega,\theta} w_n$, we differentiate Equation $(6.14)_n$ with respect to y, ω and θ. From differentiation w.r.t. y, we get for the linear part:

$$\mathbf{V}(\partial_x)\partial_y w_n - i\mathbf{D}(\partial_\omega)\partial_y w_n + \mathbb{E}\mathbf{C}\partial_y w_n + [\partial_y, \mathbf{V}(\partial_x) + \mathbb{E}\mathbf{C}]w_n - i[\partial_y, \mathbf{D}(\partial_\omega)]w_n, \tag{7.2}$$

and the last commutator has order two in ∂_ω. Thus, taking the scalar product of (7.2) with $\partial_y w_n$ (and integrating in y, ω, θ) gives a term $\int \partial_y w_n . [\partial_y, \mathbf{D}(\partial_\omega)]w_n$, which is not controlled by $\|w_n\|_{H^1}$. In order to balance this loss of derivatives, we consider 'anisotropic' regularities:

Notation 7.1.1. *Consider a multi-index $\gamma = (\gamma_y, \gamma_\omega, \gamma_\theta) \in \mathbb{N}^{d+p+q}$. We call 'length' the usual quantity, $|\gamma| := \sum_j |\gamma_{y_j}| + \sum_\mu |\gamma_{\omega_\mu}| + \sum_\nu |\gamma_{\theta_\nu}|$, and 'weight' the quantity, $[\gamma] := |\gamma_y| + |\gamma_\omega|/2 + |\gamma_\theta|/2$.*

Definition 7.1.1. *Let $s \in \mathbb{N}/2$ and $0 < t_1 \leq t_0$. We define $\mathcal{E}^s(t_1)$ as the space of functions $u(x, \omega, \theta)$ on $\overline{\Omega}_{t_1} \times \mathbb{T}^p \times \mathbb{T}^q$ which derivatives $\partial^\gamma u$ (w.r.t. y, ω, θ), continued by zero outside $\overline{\Omega}_{t_1}$, belong to $\mathcal{C}^0([0, t_1], L^2(\mathbb{R}^d \times \mathbb{T}^p \times \mathbb{T}^q))$, for all $\gamma \in \mathbb{N}^{d+p+q}$ such that $[\gamma] \leq s$. When $u \in \mathcal{E}^s(t_1)$, and t fixed, $u(t)$ belongs to the Hilbert space $K^s(\omega_t)$ equipped with the scalar product:*

$$(u, v)_s := \sum_{[\gamma] \leq s} (\partial^\gamma u, \partial^\gamma v)_{L^2(\omega_t \times \mathbb{T}^p \times \mathbb{T}^q)}.$$

We denote by $\| \cdot \|_s$ the associated norm. We endow $\mathcal{E}^s(t_1)$ with the norm: $\| u \|_{\mathcal{E}^s(t_1)} := \sup_{t \in [0, t_1]} \| u(t) \|_s$, so that it becomes a Banach space.

Since elements of $K^s(\omega_t)$ are restrictions of Sobolev type functions on $\mathbb{R}^d \times \mathbb{T}^p \times \mathbb{T}^q$ (see for example [6]), we have the classical properties:

Proposition 7.1.1 (Sobolev's Injection)**.** *Let $s \in \mathbb{N}/2$ and $s > \frac{2d+p+q}{4}$. Then, $K^s(\omega_t)$ is a subspace of $L^\infty(\mathbb{R}^d \times \mathbb{T}^p \times \mathbb{T}^q)$, and the norm of the injection is bounded uniformly in $t \in [0, t_0]$.*

Proposition 7.1.2 (Gagliardo-Nirenberg's Inequality). *Let $k, s \in \mathbb{N}/2$, $\alpha, \beta \in [1, +\infty]$, and $r \in [2, +\infty]$, satisfy $k \leq s$ and $(1 - \frac{k}{s})\frac{1}{\alpha} + \frac{k}{s}\frac{1}{\beta} = \frac{1}{r}$.*
Then, there exists $C > 0$ such that, for all $t \in [0, t_0]$, all $u \in \mathcal{S}(\omega_t \times \mathbb{T}^p \times \mathbb{T}^q)$:

$$\| \partial^k u \|_{L^r} \leq C \| u \|_{L^\alpha}^{1-\frac{k}{s}} \quad \| \partial^s u \|_{L^\beta}^{\frac{k}{s}} .$$

Proof:

We won't give a proof of the classical (isotropic) case of these inequalities (see [29]), but see how one deduces the anisotropic case.

First, the case $[\gamma] \leq k$ and $\gamma_y = 0$ is similar to the isotropic case for variables ω and θ, with derivations of length less than $2k$.

When the multi-index γ involves y coordinates as well as ω or θ, we permute derivations, and separate them as follows: set $v := \partial_y^{\gamma_y} u$ (and suppose $\gamma_\theta = 0$). Then,

$$\begin{aligned} \| \partial^\gamma u \|_{L^r} &\leq C \| v \|_{L^a}^{1-\frac{|\gamma_y|}{s-|\gamma_\omega|/2}} \quad \| \partial_y^{s-|\gamma_\omega|/2} v \|_{L^b}^{\frac{|\gamma_y|}{s-|\gamma_\omega|/2}} \\ &\leq C \| v \|_{L^a}^{1-\frac{|\gamma_y|}{s-|\gamma_\omega|/2}} \quad \| \partial^s u \|_{L^b}^{\frac{|\gamma_y|}{s-|\gamma_\omega|/2}}, \\ \text{for } &\left(1 - \frac{2|\gamma_y|}{2s - |\gamma_\omega|}\right)\frac{1}{a} + \frac{2|\gamma_y|}{2s - |\gamma_\omega|}\frac{1}{b} = \frac{1}{r}. \end{aligned} \tag{7.3}$$

Furthermore, $\| v \|_{L^a}$ is controlled in the same way:

$$\begin{aligned} \| \partial^{\gamma_\omega} u \|_{L^a} &\leq C \| u \|_{L^c}^{1-\frac{|\gamma_\omega|}{2s}} \quad \| \partial_\omega^{2s} u \|_{L^d}^{\frac{|\gamma_\omega|}{2s}}, \\ \text{with } &\left(1 - \frac{|\gamma_\omega|}{2s}\right)\frac{1}{c} + \frac{|\gamma_\omega|}{2s}\frac{1}{d} = \frac{1}{a}. \end{aligned} \tag{7.4}$$

Choose $b = d = \beta$ and $c = \alpha$; this determines a, and the relation between $1/r$, $1/a$ and $1/b$, deduced from (7.3) and (7.4), exactly becomes the assumption linking $1/r$, $1/\alpha$ and $1/\beta$. □

Proposition 7.1.3 (Moser). *When $s > \frac{2d+p+q}{4}$, $\mathcal{E}^s(t_1)$ is a Banach algebra, on which $\mathcal{C}^\infty$ functions act continuously,* i.e. *:*
Let $G : \overline{\Omega}_{t_1} \times \mathbb{T}^p \times \mathbb{T}^q \times \mathbb{C}^N \to \mathbb{C}^n$ be $\mathcal{C}^\infty$. Then,

$$\forall u, v \in \mathcal{E}^s(t_1), G(x, \omega, \theta, u) \in \mathcal{E}^s(t_1) \text{ and}$$

$$\| G(x, \omega, \theta, u + v) - G(x, \omega, \theta, u) \|_{\mathcal{E}^s(t_1)} \leq C(\| u \|_{\mathcal{E}^s(t_1)}, \| v \|_{L^\infty}) \| v \|_{\mathcal{E}^s(t_1)} .$$

(See [29] for a proof.)

Proposition 7.1.4. *Under Assumptions 3.0.4, 4.2.1 and 4.2.2:*
i) For all $t \in [0, t_1[$ and $s \in \mathbb{N}/2$, $\mathbb{E}(t)$ and $\mathbb{F}(t)$ are projectors on $K^s(\omega_t)$, and they depend continuously on t. They are self-adjoint on $K^0(\omega_t) = L^2(\omega_t)$. The operator $\mathbb{G}$ is a self-adjoint projector on $K^s(\omega_t)$ for all s.
ii) Adding Assumption 6.2.1 (no small divisors), $\mathbf{C}$ acts continuously on $\mathcal{E}^s(t_1)$, and so do $\mathbf{V}(\partial_x)$ and $\mathbf{D}(\partial_\omega)$ from $\mathcal{E}^s(t_1)$ to $\mathcal{E}^{s-1}(t_1)$.

Proof:

Proceed as for Proposition 6.2.1: use the Fourier expansion of u, and Lemma 6.2.1, which says that the coefficients of $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ (and their derivatives) are bounded uniformly in x, α and γ. Continuity with respect to time follows from dominated convergence for series.

Proceed in the same way for $\mathbf{C}, \mathbf{V}(\partial_x)$ et $\mathbf{D}(\partial_\omega)$, taking into account homogeneity of the coefficients. □

7.2 Solving the profile equations

Theorem 7.2.1. *Under all previous (explicitly numbered) assumptions, consider $s > \frac{2d+p+q}{4} + 1$, $g_0, h_0 \in K^s(\omega_0)$, and for all $n \geq 1$, $g_n \in K^s(\omega_0)$ and $h_n \in K^s(\omega_0)$ such that $\mathbb{F} g_n = g_n$ et $\mathbb{E}\mathbb{G} h_n^\star = h_n$ for all $n \in \mathbb{N}$.*

Then, there exist $t_\star \in]0, t_0]$ and unique (maximal) solutions $v_0, w_0 \in \mathcal{E}^s(t), \forall t < t_\star$, to Equations (6.7)- (6.10) with initial data $v_{0|_{t=0}} = g_n$ and $w_{0|_{t=0}} = h_n$. Furthermore, when $s > \frac{2d+p+q}{4} + n + 1$, there exist unique solutions $v_n, w_n \in \mathcal{E}^{s-n}(t)$ $(t < t_\star)$ to Equations (6.11)-(6.14) with initial data g_n et h_n.

In addition, $t_\star$ is bounded from below independently from s: if $g_n \in \cap_{s\in\mathbb{N}/2} K^s(\omega_0)$ and $h_n \in \cap_{s\in\mathbb{N}/2} K^s(\omega_0)$, setting $t^\star(s) := \{t_\star \ / \ \forall n,\ v_n, w_n \in \mathcal{E}^s(t_\star)\}$, then $t^\star$ is the same for all $s > \frac{2d+p+q}{4} + 1$.

Sketch of the proof:

Existence of solutions to linear equations is classical, and based on energy estimates. Solutions are obtained in the nonlinear case via an iterative scheme and a standard fixed-point argument. Thus, we consider the system:

$$
(\mathcal{L})\begin{cases}
\mathbb{F}L_1(\partial_x)\mathbb{F}v + \mathbb{F}\langle B(v'+w',\partial_\theta)w\rangle = \mathbb{F}\langle\mathcal{H}\rangle \\
\mathbb{GE}[\mathbf{V}(\partial_x)w - i\mathbf{D}(\partial_\omega)w + \mathbf{C}w + (B(v'+w',\partial_\theta)w)^\star] = \mathbb{GE}\mathcal{I}^\star \\
\mathbb{F}\underline{v} = v \\
\mathbb{GE}w^\star = w \\
v_{|t=0} = v^0 \\
w_{|t=0} = w^0
\end{cases}
$$

where $v', w', \mathcal{H}$ and $\mathcal{I}$ are given functions in $\mathcal{E}^s(t_1)$ (for some $t_1 \in [0, t_0]$) satisfying $\mathbb{F}\underline{v'} = v'$, $\mathbb{GE}w'^\star = w'$.

Proposition 7.2.1. *Let $v', w' \in \mathcal{E}^s(t_1)$, $s > \frac{2d+p+q}{4} + 1$, and $\mathcal{H}, \mathcal{I} \in \mathcal{E}^s(t_1)$. If $(v, w) \in \mathcal{E}^s(t_1)^2$ is a solution to $(\mathcal{L})$ with these data, then*

$$
\begin{aligned}
\| v(t) \|_s^2 + \| w(t) \|_s^2 \leq e^{Ct}\left(\| v(0) \|_s^2 + \| w(0) \|_s^2\right) \\
+ \int_0^t e^{C(t-t')}(\| \mathcal{H}(t') \|_s^2 + \| \mathcal{I}(t') \|_s^2)dt'.
\end{aligned}
$$

The constant C depends only on the A_j's and on $\| v' \|_s, \| w' \|_s$.

This is a consequence of easy L^2 estimates, together with the following properties of linear and nonlinear commutators:

Lemma 7.2.1. *Let $[\gamma] \leq s$.*
i) The operators $[\partial^\gamma, L_1(\partial_x)], [\partial^\gamma, \mathbf{V}(\partial_x)], [\partial^\gamma, \mathbf{D}(\partial_\omega)]$ map $\mathcal{E}^s(t_1)$ into $\mathcal{E}^0(t_1)$.
ii) The operators $[\partial^\gamma, \mathbb{E}], [\partial^\gamma, \mathbb{F}]$ and $[\partial^\gamma, \mathbb{G}]$ map $\mathcal{E}^s(t_1)$ into $\mathcal{E}^1(t_1)$.
iii) Let $w, v', w' \in \mathcal{E}^s(t_1)$. Then, for all $t \in [0, t_1]$,

$$
\| [\partial^\gamma, B(v'+w', \partial_\theta)]w \|_{K^0(\omega_t)} \leq C(\| v'+w' \|_{\mathcal{E}^s(t_1)}) \| w \|_{K^s(\omega_t)} .
$$

Proof:
i),ii) Simply count the number of derivatives on the coefficients of the Fourier multipliers, and use the bounds given in Proposition 7.1.4.
iii) Decompose first $[\partial^\gamma, B(v'+w',\partial_\theta)] = \sum_\nu [\partial^\gamma, B_\nu(v'+w')\partial_{\theta\nu}]$, and for each term $[\partial^\gamma, B_\nu(v'+w')\partial_{\theta\nu}] = \sum_{0<\gamma'\leq\gamma} \partial^{\gamma'}(B_\nu(v'+w'))\,\partial^{\gamma-\gamma'}\partial_{\theta\nu}$. Thus, we now try to estimate $\| \partial^{\gamma'}(B_\nu(v'+w'))\,\partial^{\gamma-\gamma'}\partial_{\theta\nu}w \|_{L^2}$. From Hölder's Inequality, for all p, q such that $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{2}$,

$$
\| \partial^{\gamma'}(B_\nu(v'+w'))\,\partial^{\gamma-\gamma'}\partial_{\theta\nu}w \|_{L^2} \leq \| \partial^{\gamma'}(B_\nu(v'+w')) \|_{L^p} \| \partial^{\gamma-\gamma'}\partial_{\theta\nu}w \|_{L^q} . \tag{7.1}
$$

Moser's Theorem implies that $B_\nu(v'+w')$ belongs to $\mathcal{E}^s(t_1)$, and

$$\| B_\nu(v'+w') \|_{\mathcal{E}^s(t_1)} \leq C(\| v'+w' \|_{L^\infty}) \| v'+w' \|_{\mathcal{E}^s(t_1)} .$$

Hence, we bound each term in the right-hand side of (7.1) thanks to Gagliardo-Nirenberg's Inequality: since $[\gamma'] > 0$, with $e < \gamma'$ and $|e| = 1$,

$$\begin{aligned}
\| \partial^{\gamma'} (B_\nu(v'+w')) \|_{L^p} \leq & C \| \partial^e (B_\nu(v'+w')) \|_{L^a}^{1-\frac{[\gamma'-e]}{s-[e]}} \| (B_\nu(v'+w')) \|_s^{\frac{[\gamma'-e]}{s-[e]}} \\
\leq & C \left(\| \partial^e (B_\nu(v'+w')) \|_{L^\infty}^{1-\frac{2}{a}} \| \partial^e (B_\nu(v'+w')) \|_{L^2}^{\frac{2}{a}} \right)^{1-\frac{[\gamma'-e]}{s-[e]}} \\
& \times \| (B_\nu(v'+w')) \|_s^{\frac{[\gamma'-e]}{s-[e]}}, \\
& \text{where } \frac{s-[\gamma']}{a} + \frac{[\gamma'-e]}{2} = \frac{s-[e]}{p},
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
\text{and : } \| \partial^{\gamma-\gamma'} \partial_\theta w \|_{L^q} \leq & C \| \partial_\theta w \|_{L^b}^{1-\frac{[\gamma-\gamma']}{s-1/2}} \| w \|_s^{\frac{[\gamma-\gamma']}{s-1/2}} \\
\leq & C \left(\| \partial_\theta w \|_{L^\infty}^{1-\frac{2}{b}} \| \partial_\theta w \|_{L^2}^{\frac{2}{b}} \right)^{1-\frac{[\gamma-\gamma']}{s-1/2}} \| w \|_s^{\frac{[\gamma-\gamma']}{s-1/2}}, \\
& \text{where } \frac{s-1/2-[\gamma-\gamma']}{b} + \frac{[\gamma-\gamma']}{2} = \frac{s-1/2}{q}.
\end{aligned} \tag{7.3}$$

Each norm involved in (7.2) and (7.3) is controlled by K^s-norms of $v'+w'$ and w. Looking at exponents, we can adjust a and b in order for p and q to satisfy the constraint from (7.1):

$$\frac{1}{p} + \frac{1}{q} \in \left[\frac{1}{2} \left(\frac{[\gamma'-e]}{s-[e]} + \frac{[\gamma-\gamma']}{s-1/2} \right), +\infty \right[, \text{ from (7.2) and (7.3).}$$

$$\begin{aligned}
\text{But: for } [e] = \frac{1}{2},\ & \frac{[\gamma'-e]}{s-[e]} + \frac{[\gamma-\gamma']}{s-1/2} = \frac{[\gamma-e]}{s-1/2} = \frac{[\gamma]-1/2}{s-1/2} < 1 ; \\
\text{for } [e] = 1,\ & \frac{[\gamma'-e]}{s-[e]} + \frac{[\gamma-\gamma']}{s-1/2} < \frac{[\gamma]-[\gamma-\gamma']-1}{s-1} + \frac{[\gamma-\gamma']}{s-1} < 1.
\end{aligned}$$

□

Convergence of the iterative scheme is easily obtained in $\mathcal{E}^{s-1/2}(t_1)$ for some $t_1 > 0$ sufficiently small, and classical results show that the solution has the same regularity as the initial data (see for example [13]).

Finally, the existence time for maximal solutions only depends on the existence in the space $\mathcal{W}^{1,\infty}$ (and, naturally, on initial data). We make use of an 'ODE' argument (following A. Majda, [27]), relying on estimates of the same type as the previous ones:

Proposition 7.2.2. *Suppose that the maximal existence time $t^\star$ of the solutions v_0 and w_0 in $\mathcal{E}^s$ ($s > \frac{2d+p+q}{4}+1$) to Equations (6.7) – (6.10), for smooth initial data, is less than t_0. Then,*

$$\limsup_{t\to t^\star} \left(\| v_0(t) \|_{\mathcal{W}^{1,\infty}(\omega_t\times\mathbb{T}^p\times\mathbb{T}^q)} + \| w_0(t) \|_{\mathcal{W}^{1,\infty}(\omega_t\times\mathbb{T}^p\times\mathbb{T}^q)}\right) = +\infty.$$

Here, $\mathcal{W}^{1,\infty}(\omega_t \times \mathbb{T}^p \times \mathbb{T}^q)$ is the space of functions $u \in L^\infty(\omega_t \times \mathbb{T}^p \times \mathbb{T}^q)$ which derivatives $\partial^\gamma u$ w.r.t. y, ω and θ belong to $L^\infty(\omega_t \times \mathbb{T}^p \times \mathbb{T}^q)$, when $[\gamma] \leq 1$.

Proof:

Rewrite the estimates in the proof of Proposition 7.2.1, with $v' = v$ et $w' = w$. Simply, in (7.2) and (7.3), don't control the L^∞-norms by K^s-norms:

$$||v(t)||_s^2 + ||w(t)||_s^2 \leq ||v(0)||_s^2 + ||w(0)||_s^2 + C(\| v, w \|_{\mathcal{W}^{1,\infty}}) \int_0^t \left(\| v(t') \|_s^2 + \| w(t') \|_s^2\right) dt',$$

and Gronwall's Lemma gives the result. □

8 Approximation of exact solutions

8.1 Asymptotic solutions

We suppose that the profiles from Theorem 7.2.1 are given. We denote by $t^\star \leq t_0$ the maximal existence time (in K^∞) for these profiles, fixing initial data belonging to $\mathcal{C}^\infty$. This provides asymptotic solutions to the hyperbolic system (1.3) up to arbitrary orders: for all $M \in \mathbb{N}$, set

$$u_M^\varepsilon(x) := \varepsilon \sum_{n=0}^{M-1} \varepsilon^{n/2} u_n \left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon}\right) := \varepsilon U_M^\varepsilon(x). \tag{8.1}$$

We construct approximations of the same type as in [14], introducing:

Definition 8.1.1. *Let $t_1 \in [0,t_0]$ and $s \in \mathbb{N}$. We denote by $E^s(t_1)$ the space of functions u on $\overline{\Omega}_{t_1}$ such that, for $|\alpha| \leq s$, $\partial_y^\alpha u$ belongs to $\mathcal{C}^0([0,t_1], L^2(\mathbb{R}^d))$, when continued by zero out of $\overline{\Omega}_{t_1}$. We endow $E^s(t_1)$ with the family of norms $\| u \|_{s,\varepsilon} := \sup_{[0,t_1]} \left(\sum_{|\alpha|\leq s} \| (\varepsilon\partial_y)^\alpha u \|_{L^2}^2 \right)^{1/2}$ and set, when $\rho > 0$: $B_\rho^s(t_1) := \{(u^\varepsilon)_{\varepsilon\in]0,1]} \ / \ \forall\varepsilon, \| u \|_{s,\varepsilon} \leq \rho\}$.*

As a consequence of the very construction of the profiles:

Proposition 8.1.1. *Let $s, M \in \mathbb{N}$, $t < t^\star$.*
i) $\forall \alpha \in \mathbb{N}^{1+d}$, $\sup_{[0,t]} \| (\varepsilon\partial_x)^\alpha U_M^\varepsilon(t) \|_{L^\infty} < \infty$;
ii) There are $\rho > 0$ and $r_M^\varepsilon \in B_\rho^s(t)$ such that: $L(u_M^\varepsilon, \partial)u_M^\varepsilon = \varepsilon^{M/2} r_M^\varepsilon$.

Next, we follow Olivier Guès' techniques: perturbation methods and fixed-point arguments show the existence of an exact solution to (1.3) close to the asymptotic solutions above.

8.2 The continuation problem

The easiest case is when an exact solution v^ε to (1.3) is given on $\overline{\Omega}_{\underline{t}}$, for $\underline{t} < t_0$, admitting an asymptotic expansion:

Theorem 8.2.1. *Consider phase spaces Φ and Ψ satisfying the previous L_1- and V-coherence Assumptions 3.0.4, 4.2.1 and 4.2.2 (phases are defined on the whole $\overline{\Omega}$, up to time t_0).*

Let $M/2 \geq s > d/2+1$, $\rho > 0$, $\underline{t} \in]0, t_0[$, and $f^\varepsilon \in \varepsilon^{M/2} B_\rho^{s-1}(t_0)$. Suppose that $v^\varepsilon \in E^s(\underline{t})$ is an exact solution to the system

$$L(v^\varepsilon, \partial)v^\varepsilon = f^\varepsilon \ \text{on} \ \overline{\Omega}_{\underline{t}}, \tag{8.1}$$

and $v^\varepsilon \in \varepsilon \sum_{n=0}^{M-1} \varepsilon^{n/2} v_n \left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon} \right) + \varepsilon^{M/2+1} B_\rho^s(\underline{t})$, $v_n \in \mathcal{C}^\infty(\overline{\Omega}_{\underline{t}} \times \mathbb{T}^p \times \mathbb{T}^q)$.

Then, there is a time $t_\star > \underline{t}$, independent of $\varepsilon \leq \varepsilon_\rho$, and a unique continuation of v^ε on $\overline{\Omega}_{t_\star}$ as a solution of (1.3) in $E^s(t_\star)$. Furthermore, this continuation admits an asymptotic expansion of the same type (with the same phases, and with residual in $\varepsilon^{M/2+1} B_\sigma^s(t_\star)$ for some $\sigma \geq \rho$).

Proof:
1-We first prove that the profiles v_n satisfy Equations (2.5)-(2.8)$_{M-1}$ on $\overline{\Omega}_{\underline{t}}$.

Plugging the asymptotic expansion of v^ε into Equation (8.1), we get:

$$f^\varepsilon = L(v^\varepsilon, \partial)v^\varepsilon = \sum_{n=0}^{M-1} \varepsilon^{n/2} E_n(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) + \varepsilon^{M/2} R_M^\varepsilon. \tag{8.2}$$

Here, $R_M^\varepsilon \in B_{\rho'}^{s-1}(\underline{t})$ and $E_n = L_1(d\phi)\partial_\theta v_n + L_1(d\psi)\partial_\omega v_{n-1} + L_1(\partial_x)v_{n-2} + B(v_0, \partial_\theta)v_{n-2} + B(v_{n-2}, \partial_\theta)v_0 + \mathcal{F}_n$, so that Equation (2.8) is equivalent to $E_n = 0$.

Now, since $f^\varepsilon \in \varepsilon^{M/2} B_\rho^{s-1}(\underline{t}) \subset L^\infty$, a recursive argument shows that $E_n(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$ in L^∞, for all $n < M$. Applying the following lemma gives the desired conclusion ($\mathbb{Q}$-independence and coherence imply the assumptions in the lemma):

Lemma 8.2.1. *Consider phases ϕ such that the gradient of any non-vanishing entire linear combination does not vanish on any open set. Consider phases ψ with the same property.*

If $E \in \mathcal{C}^0(\overline{\Omega}_{\underline{t}} \times \mathbb{T}^p \times \mathbb{T}^q)$ and $E(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$ in L^∞, then $E = 0$.

2-Proposition 6.3.1 then asserts that $\mathbb{F}\underline{v}_n$ and $\mathbb{GE}v_n^\star$ solve Equations (6.7)-(6.14) on $\overline{\Omega}_{\underline{t}}$. This provides initial data required in Theorem 7.2.1, which in turn ensures existence of a unique continuation for each of these functions (and thus, for the profiles v_n, thanks to Proposition 6.3.1). This continuation is defined on $\overline{\Omega}_{t^\star}$, with $t^\star > \underline{t}$. In order to obtain an asymptotic solution, with residual in $\varepsilon^{M/2} B_\sigma^s(t^\star)$, make use of Proposition 8.1.1*ii*). Taking Estimate *i*) into account, one can finally apply Theorem 1.1 from [14], and finish the proof. ☐

Remark 8.2.1. *As emphasized in [14], the exact solution is not defined on a 'small' time interval: as long as the asymptotic solution u_M^ε exists, so does the exact one, v^ε, provided that ε is small enough. We give a simple proof of this fact in the next paragraph.*

Proof of Lemma 8.2.1:
Actually, we prove the following asymptotic equivalence:

$$\text{For } x \text{ in a dense open set, } \|E\|_{L^\infty} \leq \varliminf_{\varepsilon \to 0} |E(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)| \tag{8.3}$$

(the reversed inequality being obvious).

To do so, we adapt the proof of Theorem 4.5.1 of [20]. In this paper, Lemma 4.5.2 says that, for phases $\varphi = (\varphi_1, \ldots, \varphi_r)$ satisfying our assumptions, there is a dense open subset of Ω such that, for any x in this set, $l_0 \in \mathbb{Z}$ and $\beta^1, \ldots, \beta^M \in \mathbb{Z}^r \setminus \{0\}$, the set $(l(\beta^m \cdot \varphi(x))_m)_{l \geq l_0}$ is dense in $\mathbb{T}^M$. Here, to show that for some sequence ε, for x in a dense open subset of Ω and for all $\alpha^1, \ldots, \alpha^M \in \mathbb{Z}^q \setminus \{0\}$, $\gamma^1, \ldots, \gamma^N \in \mathbb{Z}^p \setminus \{0\}$, the sequences $(\alpha^j \cdot \phi/\varepsilon)_j$ and $(\gamma^k \cdot \psi/\sqrt{\varepsilon})_k$ are *simultaneously* dense in $\mathbb{T}^M$ and $\mathbb{T}^N$ (respectively), we need a more precise (uniform w.r.t. l) statement; see *iii)* below.

First, remark that it is sufficient to restrict to a trigonometric polynomial: given a challenging $\delta > 0$, there is a trigonometric polynomial E^δ such that $\|E - E^\delta\|_{L^\infty} < \delta$, so that $\|E\|_{L^\infty} \leq \|E^\delta\|_{L^\infty} + \delta$, and $|E^\delta(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)| < |E(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)| + \delta$. Proving (8.3) for E^δ and letting δ go to zero gives the result for E. Thus, in the sequel,

$$E(x, \omega, \theta) = \sum_{j,k=1}^{Z} c^{\alpha^j, \gamma^k = 1} e^{i(\gamma^k \cdot \omega + \alpha^j \cdot \theta)}, \tag{8.4}$$

and we show that for some sequence ε, for any ω and θ, and for x in a dense open subset of Ω, the differences $\inf_k |\alpha^j \cdot (\theta - \phi/\varepsilon) + 2k\pi|$ and $\inf_k |\gamma^k \cdot (\omega - \psi/\sqrt{\varepsilon}) + 2k\pi|$ are less that δ.

A- Begin with some general properties for phases $\varphi = (\varphi_1, \ldots, \varphi_r)$ (satisfying the assumptions above).

i) The set $\{x \in \Omega \ / \ \forall \beta \in \mathbb{Z}^r \setminus \{0\}, \ \beta \cdot \varphi(x) \neq 0 \bmod 2\pi\}$ is a dense open subset of Ω. Indeed, its complementary set is the countable union of $\{x \in \Omega \ / \ \beta \cdot \varphi(x) = 2k\pi\}$, which (for each k and β) are closed and have empty interior. Thus, Baire's Theorem concludes.

ii) For any x in this set, and any trigonometric polynomial $P(x, \zeta)$,

$$\frac{1}{L} \sum_{l_0 \leq l < l_0 + L} P(x, l\varphi(x)) \underset{L \to \infty}{\longrightarrow} \int_{\mathbb{T}^r} P(x, \zeta) \text{ uniformly w.r.t. } l_0 \in \mathbb{Z}.$$

This is shown by direct summation of geometric series.

iii) The desired uniform density result is: for any x in this set, $\beta^1, \ldots, \beta^M \in \mathbb{Z}^r \setminus \{0\}$ and $\zeta_0 \in \mathbb{T}^M$,

$$\inf_{l_0 \leq l < l_0 + L} \mathrm{dist}\,(l(\beta^m \cdot \varphi(x))_m, \zeta_0) \underset{L \to \infty}{\longrightarrow} 0 \text{ uniformly w.r.t. } l_0 \in \mathbb{Z}.$$

The proof of the case $M = 1$ suffices, since M is finite. Then, the characteristic function of any smooth open set A in the torus can be approximated by a trigonometric polynomial. Step *ii)* and a Cantor diagonal argument finally show that the averaged number of points $l(\beta^m \cdot \varphi(x))_m$ in A tends to the measure of A, uniformly w.r.t. l_0.

B- Now, apply these results to $\varphi = \phi$ and $\varphi = \psi$. The intersection of the corresponding sets from *i)* is still a dense open subset of Ω. Choose an x in this intersection, as well as $\omega_0 \in \mathbb{T}^p$, $\theta_0 \in \mathbb{T}^q$ and $\delta > 0$.

iv) The idea is that $\phi(x)/\varepsilon$ 'turns faster' as $\psi(x)/\sqrt{\varepsilon}$. Indeed, for all $l_0 \in \mathbb{Z} \setminus \{0\}$, *iii)* shows that there are $\sqrt{\varepsilon} = 1/l$ (for some $l \in \mathbb{Z}$, $l \geq l_0$) and $n_k \in \mathbb{Z}$ such that:

$$\begin{array}{l} \forall k \in \{1, \ldots, Z\}, \ |\gamma^k \cdot (\omega_0 - \psi(x)/\sqrt{\varepsilon}) + 2n_k\pi| < \delta, \\ \text{where the } \gamma^k \text{ are given in (8.4),} \end{array} \tag{8.5}$$

and in fact, the maximal interval I_l containing l such that these inequalities are satisfied for all $1/\sqrt{\varepsilon} \in I_l$ has the form $I_l = [l/\sqrt{\varepsilon_0}, l/\sqrt{\varepsilon_0} + e]$, with length $e > 0$, depending on δ, ω_0 and the integers n_k *only* (not on l_0 and l).

v) Finally, we apply *iii)* to ϕ: when $1/\sqrt{\varepsilon} \in I_l$, $1/\varepsilon$ runs over $[1/\varepsilon_0, 1/\varepsilon_0 + e^2 + 2e/\varepsilon_0]$, whose length goes to infinity with l_0. Thus, it contains an interval of the form $[l_1, l_1 + L[$, with l_1 and L going to infinity with l_0. With $\zeta_0 = \theta_0$ and $\beta^j = \alpha^j$, *iii)* says that for any l_0, there is l such that some $1/\varepsilon \in [l_1(l), l_1(l) + L(l)[$ (with $l_1(l) \geq L_0$) satisfies $\inf_k |\alpha^j \cdot (\theta - \phi/\varepsilon) + 2k\pi| < \delta$ for all j. From *iv)*, (8.5) is also satisfied. □

8.3 The initial-value problem

Before stating the theorem, we must understand the new difficulty arising here. It mainly comes from *compatibility conditions*: as shown in Proposition 6.3.1, the polarized parts $\mathbb{F}\underline{u}_n$ and $\mathbb{G}\mathbb{E}u_n^\star$ determine the whole profile u_n. Thus, in this setting, there are restrictions in the choice of initial data.

So as to avoid these restrictions, we could assume that our coherent phase spaces contain 'timelike phases' ϕ_1 and ψ_1 (see [21] for a definition and Remark 2.3.3 in this reference for the genericity of this assumption). Then, we consider general initial data depending on $\omega' = (\omega_2, \ldots, \omega_p)$ and $\theta' = (\theta_2, \ldots, \theta_q)$, and construct profiles depending on $\omega = (\omega_1, \omega')$ and $\theta = (\theta_1, \theta')$: from Proposition 6.1.1, the needed compatibility conditions, *e.g.* $\mathbb{E}F_{n+1} = 0$, are equivalent to equations of the type $L_1(\partial_\theta)U_n = F_{n+1}$, which can be seen as an evolution problem on the torus, with initial data $u_{n|_{\omega_1=\theta_1=0}} = v_n^0$. They provide correct (polarized) initial data (at $t = 0$). But the corresponding profile u_0, defined on $\Omega_{t_\star} \times (\mathbb{R} \times \mathbb{T}^{p-1}) \times (\mathbb{R} \times \mathbb{T}^{q-1})$, is a priori only almost periodic w.r.t. θ and ω (see [21], p 57).

For simplicity, we restrict to the periodic case, and thus, to polarized data. So as to select the generated phases, polarization must be checked at each step of the asymptotic expansion. That's why our data are constructed from the profiles: once solutions to (6.7)-(6.14) are constructed up to $t < t^\star$, Borel's Lemma ensures existence of a smooth asymptotic solution $u^\varepsilon_{app}(x)$ to (1.3): $\forall M \in \mathbb{N}, \alpha \in \mathbb{N}^{1+d}$,

$$\left\| (\varepsilon\partial)^\alpha \left[u^\varepsilon_{app}(x) - \varepsilon \sum_{n<M} \varepsilon^{n/2} u_n \left(x, \frac{\psi(x)}{\sqrt{\varepsilon}}, \frac{\phi(x)}{\varepsilon} \right) \right] \right\|_{L^\infty} = \mathcal{O}(\varepsilon^{M/2+1}).$$

Finally, O. Guès' stability theorem [14] leads to:

Theorem 8.3.1. *Consider the approximate solution u^ε_{app} above on $[0, t^\star[$, under Assumptions 3.0.4, 4.2.1 and 4.2.2. Let $\underline{t} < t^\star$, $f^\varepsilon \sim 0$ in $\mathcal{C}^\infty(\overline{\Omega})$, and $v^{0,\varepsilon}(y) \sim \sum_{n\in\mathbb{N}} \varepsilon^{n/2} u_{n|t=0} \left(y, \psi^0_{|t=0}/\sqrt{\varepsilon}, \phi^0_{|t=0}/\varepsilon \right)$, $\varepsilon \in]0,1]$. Then, there is $\varepsilon_{\underline{t}}$ such that the solution v^ε to the Cauchy problem*

$$\begin{cases} L(v^\varepsilon, \partial) v^\varepsilon = f^\varepsilon \\ v^\varepsilon_{|t=0}(y) = \varepsilon v^{0,\varepsilon}(y) \end{cases} \tag{8.1}$$

exists on $\Omega_{\underline{t}}$, for all $\varepsilon \leq \varepsilon_{\underline{t}}$. Furthermore, it admits an infinite-order asymptotic expansion: $v^\varepsilon - u^\varepsilon_{app} \sim 0$.

9 Examples

9.1 Diffraction of a single wave, for slightly compressible isentropic Euler equations

Presentation: The isentropic 3-d Euler equations for a compressible fluid are:

$$\begin{cases} \partial_t \rho + div_y(\rho v) = 0 \\ \partial_t v + (v \cdot \nabla_y) v + \dfrac{\nabla_y p}{\rho} = 0. \end{cases} \tag{9.1}$$

The space-time variables are $x = (t, y) = (t, x_1, x_2, x_3) \in \mathbb{R}^4$, and the unknowns are the density ρ and the velocity $v = (v_1, v_2, v_3)$. In this isentropic case, the pressure p is a function of ρ. We set $f(\rho) := p'(\rho)/\rho$. Weak compressibility means that ρ belongs to a neighbourhood of a constant state $\rho_0 \neq 0$: $\rho = \rho_0 + \rho'$, with $\rho' << 1$. We assume $p'(\rho_0) > 0$, and denote by $c = \sqrt{p'(\rho_0)}$ the sound velocity.

We are interested in the Cauchy problem associated with (9.1) for initial data of the form

$$(\rho', v)_{|t=0} = \varepsilon g\left(y, \frac{y}{\sqrt{\varepsilon}}, \frac{R}{\varepsilon}\right), \text{ with } R = (y_1^2 + y_2^2)^{1/2}. \tag{9.2}$$

As emphasized in the Introduction, this has two possible interpretations:
- *Finite time diffraction*: a highly oscillating wave, carried by the phase $R \pm ct$ (generating rays parallel to the horizontal plane $y_3 = 0$), and slowly modulated (y dependence), is diffracted transversally to the rays. We derive the envelope equations, and prove validity of diffractive optics (on a space-time domain independent of ε).
- *Long time propagation* (for simplicity, assume $g(y, \omega, \theta) = \tilde{g}(\omega, \theta)$): changing scales as in Example 2 (Introduction), we get initial data

$$(\rho', v)_{|T=0} = \epsilon^2 \tilde{g}\left(Y, \frac{(Y_1^2 + Y_2^2)^{1/2}}{\epsilon}\right),$$

which have ϵ^2 amplitude, smaller than the one of usual weakly nonlinear optics. We are then able to prove existence of the solution to the nonlinear system of conservation laws (9.1) over times of order $1/\epsilon$.
WKB setting: We set $\tilde{u} := (\rho', v) = (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \mathbb{R}^4$ and symmetrize (9.1), taking the product on the left with $S(\tilde{u}) := Diag(f, \rho, \rho, \rho)$, so that it becomes

$$\tilde{L}(\tilde{u}, \partial)\tilde{u} := S(\tilde{u})\partial_t \tilde{u} + \sum_{j=1}^{3} \tilde{A}_j(\tilde{u})\partial_j \tilde{u} = 0, \; \tilde{A}_j(\tilde{u}) = \begin{pmatrix} f\tilde{u}_j & & f\rho & \\ & \rho\tilde{u}_j & \vdots & \\ f\rho & \dots & \rho\tilde{u}_j & \dots \\ & & \vdots & \rho\tilde{u}_j \end{pmatrix},$$

where doted lines are the $(j+1)$-th. Now, for $\xi = (\tau, \eta) \in \mathbb{R}^{1+3}$, setting as a new unknown $u := S(0)^{1/2}\tilde{u}$, we conjugate by $S(0)^{-1/2}$ the linearized operator $\tilde{L}_1$ $(u = 0)$ to $L_1(\xi) := S(0)^{-1/2}\tilde{L}_1(\xi)S(0)^{-1/2} = \tau Id + \sum_{j=1}^{3} \eta_j A_j(0)$, with symbol $\det L_1(\xi) = \tau^2(\tau^2 - c^2|\eta|^2)$.

Initial data take a slightly more restrictive form than in (9.2):

$$u_{|t=0} = \varepsilon h^\varepsilon \left(y, \frac{\sigma}{\sqrt{\varepsilon}}, \frac{y_3}{\sqrt{\varepsilon}}, \frac{R}{\varepsilon} \right), \tag{9.3}$$

with $R = (y_1^2 + y_2^2)^{1/2}$ the polar radius in the plane (y_1, y_2), and the $y/\sqrt{\varepsilon}$ dependence is only through the polar angle σ and the third coordinate y_3. The function $h^\varepsilon(y, \omega, \theta)$ admits an asymptotic expansion $\sum_{n\in\mathbb{N}} \varepsilon^{n/2} h_n$ in $\mathcal{C}^\infty$, with smooth h_n, 2π-periodic w.r.t. the last three arguments:

$$h_n(y, \omega_1, \omega_2, \theta) = \sum_{(\alpha,\gamma)\in\mathbb{Z}^3} h_n^{\alpha,\gamma}(y) e^{i(\alpha\theta + \gamma\cdot\omega)}.$$

Since we are only interested in a qualitative comprehension of the evolution of such a wave, we write in the sequel only equations for the first term u_0 of the expansion $u_{app}(x) = \varepsilon \sum \varepsilon^{n/2} u_n \left(x, \frac{\psi}{\sqrt{\varepsilon}}, \frac{\phi}{\varepsilon} \right)$.

Phases: So as to generate at first order a single wave, oscillating w.r.t. one rapid phase $\phi_- = R - ct$, we consider purely oscillating, polarized initial data:

$$\pi_-(0, y) h_0^\alpha = h_0^\alpha, \ \forall \alpha \neq 0. \tag{9.4}$$

This writes: $h_0^\alpha \in \ker L_1(d(\alpha\phi_-)) = \ker L_1(d\phi_-) = \begin{pmatrix} 1 \\ y_1/R \\ y_2/R \\ 0 \end{pmatrix} \mathbb{R}$.

The vector field determining the intermediate phases is (Proposition 4.1.1):

$$V(x, \partial) = \partial_t + c\frac{y' \cdot \partial_{y'}}{R}, \text{ with } x = (t, y) = (t, y', z) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}.$$

Since $\psi_{|t=0} = (\sigma, z)$, V-characteristic generated phases are independent of t, and we set $\psi = (\sigma, z)$.

Profile equations: For some polarized $u = ar_-$ (a scalar function , $r_- = (1, y_1/R, y_2/R)/\sqrt{2})$), the nonlinear term $B(x, u) = \sum_j \partial_j \phi_-(x)(\partial_u A_j(0).u)$ in the profile equations is fully described by the self-interaction coefficient c_-:

$$B(ar_-) \cdot r_- = c_- a, \text{ where } c_- = \frac{1 + h}{\sqrt{2}}, \ h = (\sqrt{p'})'_{|\rho=\rho_0}. \tag{9.5}$$

As quoted by P. Donnat in [9], this coefficient is the same as for usual weakly nonlinear geometric optics. In particular, $B(u, \partial_\theta)u$ is the derivative of a function, which corresponds to the fact that the system (9.1) is *conservative* (this explains why only oscillations are propagated).

Consequently, the amplitude a_0 of u_0 satisfies:

$$\begin{cases} \int_0^{2\pi} a_0(x,\omega,\theta)d\theta = 0 \ , \ a_{0|t=0} = g_0 \ (h_0 = g_0 r_-) \\ \left(\partial_t + \frac{c}{R} y' \cdot \partial_{y'}\right) a_0 + \frac{c}{2}(\partial^2_{\omega_1} + R^2 \partial^2_{\omega_2})\partial_\theta^{-1} a_0 + \frac{c}{2R} a_0 + \frac{1+h}{\sqrt{2}} \partial_\theta(a_0^2) = 0. \end{cases}$$

Theorem 7.2.1 ensures local existence of smooth profiles, on the cone $\overline{\Omega} := (0, \underline{y}) + \{x = (t,y) \in \mathbb{R}^4 \ / \ 0 \leq t \leq t_0, \delta t + |y| \leq \rho\}$, when $\underline{y}$ is chosen 'sufficiently far' from planes $y_1 = 0$ and $y_2 = 0$.

Conclusion: Theorem 8.3.1 shows existence on $\overline{\Omega}$ (and uniqueness) of an exact solution u^ε to (9.1) with initial value (9.3). The function u^ε admits infinite order asymptotics, whose first term is $\varepsilon u_0(x, \psi/\sqrt{\varepsilon}, \phi/\varepsilon)$. In the long-time propagation setting ($u_{|t=0} = \epsilon^2 h(\sigma, y_3, R/\epsilon)$), this reads $u^\epsilon \sim \epsilon^2 u_0(\epsilon t, \sigma, y_3, (R-ct)/\epsilon)$ for $t \in [0, \underline{t}/\epsilon]$.

The approximate solution oscillates at scale ε along rays sweeping out of (and orthogonally to) the surfaces $R - ct =$ cst, and the self-interaction coefficient of this simple wave is exactly the one of usual nonlinear optics (9.5). Finally, variations in the directions transverse to the rays are described, at scale $\sqrt{\varepsilon}$, by the –variable coefficients– diffusion operator $\partial^2_{\omega_1} + R^2 \partial^2_{\omega_2}$: initial data even slightly departing from constant in $\omega = (\sigma, y_3)/\sqrt{\varepsilon}$ generate in finite time solutions radically different from unperturbed ones.

9.2 Interactions of diffracted waves

We study in this paragraph the influence of diffraction on an example from [15]. We consider two waves satisfying (9.1), propagating in a plane and diffracting in the third space direction.

Initial data

$$u^\varepsilon_{|t=0} = \varepsilon h^\star \left(y, \frac{y_3}{\sqrt{\varepsilon}}, \frac{-2\cos(\sigma_i) y_1}{\varepsilon}, \frac{\cos(\sigma_i) y_1 + \sin(\sigma_i) y_2}{\varepsilon}\right)$$

are polarized, so that the characteristic phases involved be exactly:

$$\phi_0 = -2\cos(\sigma_i) y_1 \text{ and } \phi_i = \cos(\sigma_i) y_1 + \sin(\sigma_i) y_2 - ct.$$

Precisely, h is periodic w.r.t. θ_0 and θ_i (with the same period, say 2π), with mean value zero:

$$h^\star(y,\omega,\theta_0,\theta_i) = \sum_{\mathbb{Z}^2\backslash\{0\}} h^\alpha(y,\omega)e^{i\alpha\cdot\theta},$$

with

$$\begin{aligned}
&h^\alpha = \pi_0 h^\alpha \text{ when } \alpha\in\mathcal{Z}_0 = \mathbb{Z}^\star\times\{0\},\\
&h^\alpha = \pi_i h^\alpha \text{ when } \alpha\in\mathcal{Z}_i = \{0\}\times\mathbb{Z}^\star,\\
&h^\alpha = 0 \text{ else.}
\end{aligned}$$

As before, π_0 et π_i are spectral projectors on the kernels of $L_1(d\phi_0)$ and $L_1(d\phi_i)$, respectively.

Hence, initial data have spectrum $\mathcal{Z}_0\cup\mathcal{Z}_i$. From nonlinear interactions, the characteristic phases generated by ϕ_0 and ϕ_i are only multiples of:

$$\phi_r = \phi_0+\phi_i = \cos(\sigma_r)y_1+\sin(\sigma_r)y_2 - ct, \text{ where } \sigma_i+\sigma_r=\pi.$$

We can interpret this computation as the reflection of the incident sound wave (oscillating according to ϕ_i) on the entropy wave (oscillating according to ϕ_0), which gives rise to a reflected wave.

This time, the first term of the Ansatz is:

$$\begin{aligned}
u_0 &= \sum_{\alpha\in\mathcal{Z}_0\cup\mathcal{Z}_i\cup\mathcal{Z}_r} u_0^\alpha(x,\omega)e^{i\alpha\cdot\theta}\\
&= u_{0,0}(x,\omega,\theta_0)+u_{0,i}(x,\omega,\theta_i)+u_{0,r}(x,\omega,\theta_0+\theta_i),
\end{aligned}$$

where modes from the same vector line in $\mathbb{Z}^2$ are gathered together (setting $\mathcal{Z}_r=\mathbb{Z}^\star(1,1)$). Each component $u_{0,*}$ is polarized:

$$\begin{aligned}
&u_{0,*} = a_* r_* \text{ for } * = i,r, \text{ with } r_* = (1,\cos\sigma_*,\sin\sigma_*,0)/\sqrt{2}\\
&u_{0,0} = a_{0,1}r_1+a_{0,2}r_2, \text{ with } r_1=(0,0,1,0),\ r_2=(0,0,0,1).
\end{aligned}$$

For each mode $\gamma\in\mathcal{Z}_*$, the amplitude a^γ ($a_*=\sum_{\gamma\in\mathcal{Z}_*}a^\gamma e^{i\gamma\cdot\theta}$) satisfies an equation of the form

$$V_*(\partial_x)a^\gamma + \partial_{\theta_*}^{-1}D_*(\partial_\omega)a^\gamma + \pi_*(B(u_0,\partial_\theta)u_0)^\gamma = 0.$$

Transport operators V_* and second order operators D_* are defined at Paragraph 9.1. We now compute nonlinear B terms.

When $\gamma \in \mathcal{Z}_*$:

$$(B(u_0, \partial_\theta)u_0)^\gamma = \sum_{\alpha+\beta=\gamma} i(\beta \cdot \phi) \sum_{j=0}^{d} (\partial_u A_j(0) \cdot u^\alpha) u^\beta.$$

The term $\pi_*(B(u_0, \partial_\theta)u_0)^\gamma$ writes

$$\begin{aligned}
\pi_*(B(u_0, \partial_\theta)u_0)^\gamma &= \pi_* \left(\sum_{\alpha+\beta=\gamma} i(\beta \cdot \phi) \sum_{j=0}^{d} (\partial_u A_j(0) \cdot r^\alpha) r^\beta \right) a^\alpha a^\beta \\
&= i \sum_{\alpha+\beta=\gamma} \Gamma_\gamma^{\alpha,\beta} a^\alpha a^\beta r_*,
\end{aligned}$$

where, as in [22], the coefficient $\Gamma_\gamma^{\alpha,\beta}$ describes the creation of the mode γ from interaction of modes α and β; the symmetric expression of this coefficient is: $c_\gamma^{\alpha,\beta} := \Gamma_\gamma^{\alpha,\beta} + \Gamma_\gamma^{\beta,\alpha}$.

We finally get the system:

$$\partial_t a_0 = 0, \tag{9.1a}$$

$$\begin{aligned}
(\partial_t + c\cos\sigma_i \partial_{y_1} + c\sin\sigma_i \partial_{y_2}) a_i + \frac{c}{2}\partial_\theta^{-1}\partial_\omega^2 a_i + \frac{1+h}{\sqrt{2}}\partial_\theta(a_i^2) \\
- 2\sin\sigma_i \cos(2\sigma_i)\partial_\theta(a_r *_\theta a_{0,1}) = 0,
\end{aligned} \tag{9.1b}$$

$$\begin{aligned}
(\partial_t - c\cos\sigma_i \partial_{y_1} + c\sin\sigma_i \partial_{y_2}) a_r + \frac{c}{2}\partial_\theta^{-1}\partial_\omega^2 a_r + \frac{1+h}{\sqrt{2}}\partial_\theta(a_r^2) \\
- 2\sin\sigma_i \cos(2\sigma_i)\partial_\theta(a_i *_\theta a_{0,1}) = 0,
\end{aligned} \tag{9.1c}$$

The amplitude $a_{0,2}$ corresponds to the third component of the velocity (for $r_{0,2} = (0,0,0,1)$). Since it vanishes at time $t = 0$, it vanishes for every time. Equation (9.1a) is particularly simple, and there are several reasons to this fact. First, the absence of self -interaction terms comes from linear degeneracy of the entropy mode. Vanishing of other coupling terms is due to the particular form of nonlinearities (*cf.* [15]): two sound waves cannot generate any entropy wave. Finally, the gradient of ϕ_0 belongs to a flat part of the characteristic variety, so that the corresponding diffraction coefficients also vanish.

The two other equations are similar to the ones from nonlinear geometric optics for 2-d Euler equations, with simply diffraction terms $\partial_\theta^{-1} D(\partial_\omega)$ added. The interaction of these diffracted waves, the incident and the reflected one, is thus obtained via linear coupling through the kernel $a_{0,1}$ (independently defined).

References

[1] Deborah Alterman and Jeffrey Rauch. Diffractive short pulse asymptotics for nonlinear wave equations. *Phys. Lett. A*, 264(5):390–395, 2000.

[2] Deborah Alterman and Jeffrey Rauch. The linear diffractive pulse equation. *Methods Appl. Anal.*, 7(2):263–274, 2000. Cathleen Morawetz: a great mathematician.

[3] V. Arnol'd. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*. Mir, Moscow, 1984. Translated from the Russian by Djilali Embarek, Reprint of the 1980 edition.

[4] Anatoli V. Babin, Alex Mahalov, and Basil Nicolaenko. Resonances and regularity for Boussinesq equations. *Russian J. Math. Phys.*, 4(4):417–428, 1996.

[5] Karen Barrailh and David Lannes. A general framework for diffractive optics and its applications to lasers with large spectrum and short pulses. *To appear*.

[6] J. Chazarain and A. Piriou. *Introduction à la théorie des équations aux dérivées partielles linéaires*. Gauthier-Villars, 1981.

[7] Y. Choquet-Bruhat. Ondes asymptotiques et approchées pour des systèmes d'équations aux dérivées partielles non linéaires. *J. Math. Pures Appl. (9)*, 48:117–158, 1969.

[8] T. Colin. Rigorous derivation of the nonlinear schrödinger equation and davey-stewartson systems from quadratic hyperbolic systems. *Asymptotic Analysis*, 31(1):69–91, 2002.

[9] P. Donnat. Quelques contributions mathématiques en optique non linéaire. pdd thesis, 1994.

[10] P. Donnat, J.L. Joly, G. Métivier, and J. Rauch. Diffractive nonlinear geometric optics i. *Séminaire Équations aux Dérivées Partielles, École Polytechnique, Palaiseau*, pages XVII1–XVII23, 1995-1996.

[11] E. Dumas. Nonlinear diffractive optics with curved phases: beam dispersion and transitions between light and shadow. *To appear*.

[12] I. Gallagher. Asymptotic of the solutions of hyperbolic equations with a skew-symmetric perturbation. *J. Differential Equations*, 150(2):363–384, 1998.

[13] L. Gårding. Problème de cauchy pour les systèmes quasi-linéaires d'ordre un strictement hyperboliques. In *Colloques du CNRS*, volume 117, pages 33–40, 1962.

[14] O. Guès. Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires. *Asymptotic Analysis*, pages 241–269, 1993.

[15] J. K. Hunter, A. Majda, and R. Rosales. Resonantly interacting, weakly nonlinear hyperbolic waves. II. Several space variables. *Stud. Appl. Math.*, 75(3):187–226, 1986.

[16] J.K. Hunter. Transverse diffraction of nonlinear waves and singular rays. *SIAM J. Appl. Math.*, 48:1–37, 1988.

[17] J.K. Hunter. Asymptotic equations for nonlinear hyperbolic waves. In J.B. Keller, D.W. McLaughlin, and G.C. Papanicolaou, editors, *Surveys in Applied Mathematics*, volume 2, pages 167–276. Plenum Press, 1995.

[18] J.-L. Joly, G. Métivier, and J. Rauch. Remarques sur l'optique géométrique non linéaire multidimensionnelle. In *Séminaire sur les Équations aux Dérivées Partielles, 1990–1991*, pages I1–I17. École Polytech., Palaiseau, 1991.

[19] J.-L. Joly, G. Métivier, and J. Rauch. Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.*, 70:373–404, 1993.

[20] J.-L. Joly, G. Métivier, and J. Rauch. Resonant one-dimensional nonlinear geometric optics. *J. Funct. Anal.*, 114(1):106–231, 1993.

[21] J.-L. Joly, G. Métivier, and J. Rauch. Coherent and focusing multidimensional nonlinear geometric optics. *Ann. scient. Ec. Norm. Sup.*, 28:51–113, 1995.

[22] J.-L. Joly, G. Métivier, and J. Rauch. Diffractive nonlinear geometric optics with rectification. *Indiana Univ. Math. J.*, 47(4):1167–1241, 1998.

[23] J.L. Joly, G. Mtivier, and J. Rauch. Diffractive nonlinear geometric optics with rectification. *Indiana Univ. Math. J.*, 47(4), 1998.

[24] D. Lannes. Dispersive effects for nonlinear geometrical optics with rectification. *Asymptot. Anal.*, 18(1-2):111–146, 1998.

[25] D. Lannes. Quelques phénomènes d'interaction d'ondes en optique non linéaire. phd thesis, 1999.

[26] P.D. Lax. Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24:627–646, 1957.

[27] A. Majda. *Compressible fluid flow of conservation laws in several space variables.* Springer Verlag, 1984.

[28] G. Schneider. Justification of modulation equations for hyperbolic systems via normal forms. *Nonlinear Differential Equations and Applications*, 5:69–82, 1998.

[29] Michael E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.