PROPAGATION OF OSCILLATIONS NEAR A DIFFRACTIVE POINT FOR A DISSIPATIVE AND SEMILINEAR KLEIN-GORDON EQUATION

Éric Dumas

IRMAR, Université Rennes 1, 35 042 Rennes Cédex, FRANCE

dumas@maths.univ-rennes1.fr

Abstract

We study the propagation of oscillating solutions for a semilinear dissipative Klein-Gordon equation,

$$(1+x)\partial_t^2 u^\varepsilon - \Delta_{x,z} u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon + |\partial_t u^\varepsilon|^{p-1} \partial_t u^\varepsilon = 0 \ (p > 1),$$

on a half space $\mathbb{R}^d_+ = \{(x, z) \ / \ x > 0, z \in \mathbb{R}^{d-1}\}$ near the origin, when incident rays reach the boundary tangentially at this point.

The solution u^ε is then given (*via* a H^1-asymptotics) as a sum of two oscillating profiles. These incident and reflected profiles are uniquely determined by a mixed system of coupled transport equations. Moreover, they are supported in a smooth open domain in $\mathbb{R}^{1+d}_+$, which is delimited by glancing rays: This emphasizes the 'shadow region'.

Introduction

We are interested in the propagation of an oscillatory wave, solution of a Cauchy-Dirichlet problem for a nonlinear and dissipative Klein-Gordon equation on a half space $\mathbb{R}^d_+$, and we look at the interaction with the reflected wave, when the incidence is supposed to be of diffractive type (see [3]). In this case, a "singular ray" separates the region reached by bicharacteristic curves and its complementary, the "shadow region".

Propagation of singularities for the linear wave equation was studied by F.G. Friedlander in [2], and R. Melrose gave microlocal parametrices for second order equations, by means of Fourier-Airy integral operators ([8],[9]).

Rigorous nonlinear geometric optics was initiated by A.J. Majda and M. Artola (see [7]), in the case of transverse reflection. In the diffractive case, C. Cheverry treated second order equations with globally Lipschitz nonlinearities: In [1], he gives first order asymptotics of the solution in H^1, using the facts that reflection at the boundary has defocusing properties, and that energy does not concentrate near the singular ray. In our case (dissipative nonlinearity), we must avoid his WKB method: It is based on Picard iteration, which destructs the dissipative structure (see also Remark 0.3).

Concerning the Cauchy problem for a hyperbolic system, Joly, Métivier and Rauch have studied the caustic crossing by oscillatory waves: At first, Lipschitz nonlinearities ([4]), and then, dissipative equations ([5]). Their method is based upon Lagrangian representation of the solutions, for which phases are globally defined, and on profiles extraction, relatively to each oscillating mode.

In the degenerate cases of gliding and contact of higher order, M. Williams obtains (see [11]) infinite order asymptotics, giving a precise description of interaction between elliptic, hyperbolic and glancing boundary layers.

We first explain the method we will use and describe the results. Our Klein-Gordon equation is a semilinear one ($p > 1$), with variable coefficients:

$$(1+x)\partial_t^2 u^\varepsilon - \Delta_{x,z} u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon + |\partial_t u^\varepsilon|^{p-1} \partial_t u^\varepsilon = 0, \tag{0.1}$$

on a half space $D := \{(t,x,z) \ / \ t \in \mathbb{R}, x > 0, z \in \mathbb{R}^{d-1}\}$, for initial data

$$\begin{aligned} u^\varepsilon_{|t=-T} &= u^\varepsilon_0 \sim \varepsilon g(x,z) e^{i\varphi_i(-T,x,z)/\varepsilon}, \\ \partial_t u^\varepsilon_{|t=-T} &= u^\varepsilon_1 \sim i\partial_t \varphi_i(-T,x,z) g(x,z) e^{i\varphi_i(-T,x,z)/\varepsilon}, \end{aligned} \tag{0.2}$$

and null boundary condition

$$u^\varepsilon_{|x=0} = 0. \tag{0.3}$$

Here, $g \in H^1_0(\mathbb{R}^d_+)$ (vanishing for $x = 0$ in order to satisfy the boundary condition (0.3)) has compact support, because we look at u^ε only on a neighbourhood of the origin. Equivalences in (0.2) have the following meaning (the difference of functions tends to zero in the appropriate space):

- for u_0 ($\sim$ in $H^1(\mathbb{R}^d_+)$):

(0.4)
$$\left\| \frac{u_0^\varepsilon}{\varepsilon} - g e^{i\varphi_i/\varepsilon} \right\|_{L^2} \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ and } \left\| \partial_{x,z} \left(u_0^\varepsilon - \varepsilon g e^{i\varphi_i/\varepsilon} \right) \right\|_{L^2} \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

the second condition being equivalent to $\left\| \partial_{x,z} u_0^\varepsilon - i(\partial_{x,z}\varphi_i) g e^{i\varphi_i/\varepsilon} \right\|_{L^2} \underset{\varepsilon \to 0}{\longrightarrow} 0$;

- for u_1 ($\sim$ in $L^2(\mathbb{R}^d_+)$):

$$\left\| u_1^\varepsilon - i(\partial_t \varphi_i) g e^{i\varphi_i/\varepsilon} \right\|_{L^2} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{0.5}$$

Remark 0.1.

i) The function φ_i is the incident phase, according to which the initial data (u_0, u_1) oscillate. These data have a particular form (the time derivative of φ_i is present): They are polarized, so that the nonvanishing oscillating part of the incident wave oscillates with φ_i, and not with the second characteristic phase generated by $\varphi_i(-T, .)$.

ii) The choice of an amplitude of size ε when the frequency is $1/\varepsilon$ corresponds to the so-called weakly nonlinear geometric optics *regime.*

Some notations:

Notation 0.1. *We will use the following functions and operators:*

$$F(v) := |v|^{p-1}v, \ \forall v \in \mathbb{C},$$

$$P^\varepsilon(\partial) := (1+x)\partial_t^2 - \Delta_{x,z} + \frac{1}{\varepsilon^2},$$

$$P(\partial) := (1+x)\partial_t^2 - \Delta_{x,z},$$

$$p(\tau,\xi,\zeta) := (1+x)\tau^2 - \xi^2 - |\zeta|^2 - 1, \ \forall(\tau,\xi,\zeta) \in \mathbb{R}^{1+d},$$

$$T_\varphi(\partial) := (1+x)(\partial_t\varphi)\partial_t - (\partial_{x,z}\varphi).\partial_{x,z} \ \textit{with} \ \varphi \in \mathcal{C}^1(\Omega).$$

Thanks to dissipativity of equation (0.1), we know (*cf.* [6], and Paragraph 2) that (0.1), (0.2) has a unique solution, satisfying (0.3). We prove the following result:

Theorem 0.1. *Let u^ε be the unique solution of (0.1), (0.2). Then, there exist unique $U_i, U_r \in L^{p+1}(\Omega_T) = L^{p+1}(]-T,T[\times]0,\underline{x}[\times B(0,\underline{r}))$, which satisfy*

$$u^\varepsilon \sim \varepsilon\left(U_i e^{i\varphi_i/\varepsilon} + U_r e^{i\varphi_r/\varepsilon}\right) \ \textit{in} \ H^1(\Omega_T), \quad \text{i.e.} \tag{0.6}$$

$$\left\|\frac{u^\varepsilon}{\varepsilon} - \left(U_i e^{i\varphi_i/\varepsilon} + U_r e^{i\varphi_r/\varepsilon}\right)\right\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

(0.7) *and*

$$\left\|\nabla u^\varepsilon - i\left((\nabla\varphi_i)U_i e^{i\varphi_i/\varepsilon} + (\nabla\varphi_r)U_r e^{i\varphi_r/\varepsilon}\right)\right\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The functions U_i and U_r have support in the "illuminated zone" delimited by S, the surface generated by glancing rays.

Remark 0.2.

i) Equivalences really mean, in L^q, that the difference between the two terms *goes to zero. The definition (given in the theorem) extends to H^1 in the following sense: First, the difference tends to zero in εL^2, but in addition, we only need a L^2 function to describe a function and its gradient.*

ii) Our approximation of the exact solution has only one term, because of the regularity of u^ε, which one can differentiate only once in L^2.

iii) The "shadow zone" makes sense here, even if the H^1 asymptotics allows one to neglect a (arbitrary small) neighbourhood of the singular ray S, because the approximation is given by profiles U_i and U_r vanishing "beyond S", where rays do not pass.

Our strategy is simple.

First, we define the incident and reflected phases φ_i and φ_r in the illuminated zone $\mathcal{T}$ (Paragraph 1). Then, we extract from u^ε incident and reflected profiles $U_i, U_r \in L^{p+1}(\mathcal{T})$, *i.e.* (Paragraph 2): There exists an extraction, still denoted by u^ε, such that

$$\tag{0.8} \begin{aligned} \forall a \in L^2(\mathcal{T}), \int \frac{u^\varepsilon}{\varepsilon}\overline{a}e^{-i\varphi_k/\varepsilon} &\underset{\varepsilon\to 0}{\longrightarrow} \int U_k\overline{a}, \\ \int \partial_{x,z}u^\varepsilon\overline{a}e^{-i\varphi_k/\varepsilon} &\underset{\varepsilon\to 0}{\longrightarrow} \int i(\partial_{x,z}\varphi_k)U_k\overline{a}, \text{ and} \\ \forall a \in L^{1+1/p}(\mathcal{T}), \int \partial_t u^\varepsilon\overline{a}e^{-i\varphi_k/\varepsilon} &\underset{\varepsilon\to 0}{\longrightarrow} \int i(\partial_t\varphi_k)U_k\overline{a}. \end{aligned}$$

In the same way, we extract profiles $F_i, F_r \in L^{1+1/p}(\mathcal{T})$ from $F(\partial_t u^\varepsilon)$. The problem (0.1), (0.2) imposes to these profiles to satisfy the following transport equations:

$$
(0.9) \qquad \begin{cases} 2T_{\varphi_i}(\partial)U_i - P(\varphi_i)U_i + \dfrac{1}{i}F_i = 0 \text{ on } \mathcal{T}_i \\ 2T_{\varphi_r}(\partial)U_r - P(\varphi_r)U_r + \dfrac{1}{i}F_r = 0 \text{ on } \mathcal{T}_r \\ U_{i|_{t=-T}} = g \\ (U_i + U_r)_{|x=0} = 0. \end{cases}
$$

In order to deduce the H^1-asymptotics of the theorem from this weak limits (Paragraph 5), we construct, by means of cut-off and regularization of profiles, an approximate solution v^ε (paragraph 4) so that its difference with u^ε converges strongly to zero: With $\Omega_t :=]-T,t[\times]0,\underline{x}[\times B(0,\underline{r})$,

(0.10)

$$
\begin{aligned}
&\left\|\sqrt{1+x}\partial_t(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(t) + \|\partial_{x,z}(u^\varepsilon - v^\varepsilon)\|^2_{L^2_{x,z}}(t) \\
&\qquad\qquad + \frac{1}{\varepsilon^2}\|u^\varepsilon - v^\varepsilon\|^2_{L^2_{x,z}}(t) + c\,\|\partial_t u^\varepsilon - \partial_t v^\varepsilon\|^{p+1}_{L^{p+1}(\Omega_t)} \\
&\le \left\|\sqrt{1+x}\partial_t(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(t) + \|\partial_{x,z}(u^\varepsilon - v^\varepsilon)\|^2_{L^2_{x,z}}(t) + \frac{1}{\varepsilon^2}\|u^\varepsilon - v^\varepsilon\|^2_{L^2_{x,z}}(t) \\
&\qquad\qquad + 2\mathrm{Re}\int_{\Omega_t}\left(F(\partial_t u^\varepsilon) - F(\partial_t v^\varepsilon)\right)\partial_t(\overline{u^\varepsilon - v^\varepsilon}) \\
&= \left\|\sqrt{1+x}\partial_t(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(-T) + \|\partial_{x,z}(u^\varepsilon - v^\varepsilon)\|^2_{L^2_{x,z}}(-T) + \frac{1}{\varepsilon^2}\|u^\varepsilon - v^\varepsilon\|^2_{L^2_{x,z}}(-T) \\
&\qquad\qquad + 2\mathrm{Re}\int_{\Omega_t}\left(P^\varepsilon(u^\varepsilon - v^\varepsilon) + F(\partial_t u^\varepsilon) - F(\partial_t v^\varepsilon)\right)\partial_t(\overline{u^\varepsilon - v^\varepsilon}),
\end{aligned}
$$

and we write out the last integral as

$$
\begin{aligned}
\int_{\Omega_t} (P^\varepsilon v^\varepsilon + F(\partial_t v^\varepsilon))\, \partial_t(\overline{u^\varepsilon - v^\varepsilon}) = &\int_{\Omega_t} (P^\varepsilon v^\varepsilon + \mathcal{F}(\varphi/\varepsilon))\, \partial_t(\overline{u^\varepsilon - v^\varepsilon}) \\
&+ \int_{\Omega_t} (\mathcal{E}(\varphi/\varepsilon) - \mathcal{F}(\varphi/\varepsilon))\, \partial_t(\overline{u^\varepsilon - v^\varepsilon}) \\
&+ \int_{\Omega_t} (F(\partial_t v^\varepsilon) - \mathcal{E}(\varphi/\varepsilon))\, \partial_t(\overline{u^\varepsilon - v^\varepsilon}),
\end{aligned}
\tag{0.11}
$$

each term going to zero by definition of v^ε:

- for the first one, $P^\varepsilon v^\varepsilon + \mathcal{F}(\varphi/\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ in $L^{1+1/p}$ and $\partial_t(u^\varepsilon - v^\varepsilon)$ is bounded in L^{p+1};

- for the second one, we recall that $\partial_t(u^\varepsilon - v^\varepsilon)$ has null profiles;

- for the third one, we show (Paragraph 3) that the difference $F(\partial_t v^\varepsilon) - \mathcal{E}(\varphi/\varepsilon)$ has no oscillation propagated by P^ε.

This strong convergence allows us to identify in the system (0.9) the profiles F_k with the nonlinearities $E_k(U_i, U_r)$, so that this system inherits the dissipativity properties of (0.1): We deduce from this uniqueness of the profiles, and validity of the asymptotics (0.6) for the whole sequence u^ε.

Remark 0.3.

i) In [1], Cheverry first defines the system of profile equations, and then solves it, quite easily, because of the sublinear property of the nonlinearities. Here, we cannot obtain existence of solutions to (0.9) with $F_k = E_k(U_i, U_r)$ through Picard iteration, which destructs the dissipative structure. This is one advantage of our method, where profiles are defined first, and then, nonlinearities are identified.

ii) This technique may also apply to the case of higher (finite) order tangential rays, when there are no gliding points in $T^\star(\partial D)$: Once the phases and domains are fixed (see Remark 1.1,iii)), we only use cut-offs and energy estimates to prove the approximation (0.6).

iii) In the non-dispersive case of a wave equation ($(1+x)\partial_t^2 u^\varepsilon - \Delta_{x,z} u^\varepsilon + F(\partial_t u^\varepsilon) = 0$), it is necessary to take care of harmonics in the approximate solution. The profiles then become functions $U_i(t,x,z,\theta)$, $U_r(t,x,z,\theta)$ on $\mathcal{T}\times\mathbb{T}$ (periodic w.r.t. θ) such that, for all $a \in \mathcal{C}_0^\infty(\mathcal{T}\times\mathbb{T}^2)$,

$$\int_{\mathcal{T}} \frac{u^\varepsilon}{\varepsilon} a\left(t,x,z,\frac{\varphi_i}{\varepsilon},\frac{\varphi_r}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{T}\times\mathbb{T}^2} (U_i(\theta_i)+U_r(\theta_r))a(\theta_i,\theta_r)$$

–see [10], [5] for extraction of such profiles.

Contents

1 Phases

The only propagated oscillations for the equation (0.1) are the characteristic ones (*cf.* Paragraph 3), *i.e.* the ones associated to phases satisfying the following eikonal equation:

$$p(d\varphi) = (1+x)(\partial_t \varphi)^2 - |\partial_{x,z}\varphi|^2 - 1 = 0. \tag{1.1}$$

The corresponding characteristic variety, $\{p(\tau, \xi, \zeta) = 0\}$, is shown on Figure 1 (with $(O\tau)$ a revolution axis).

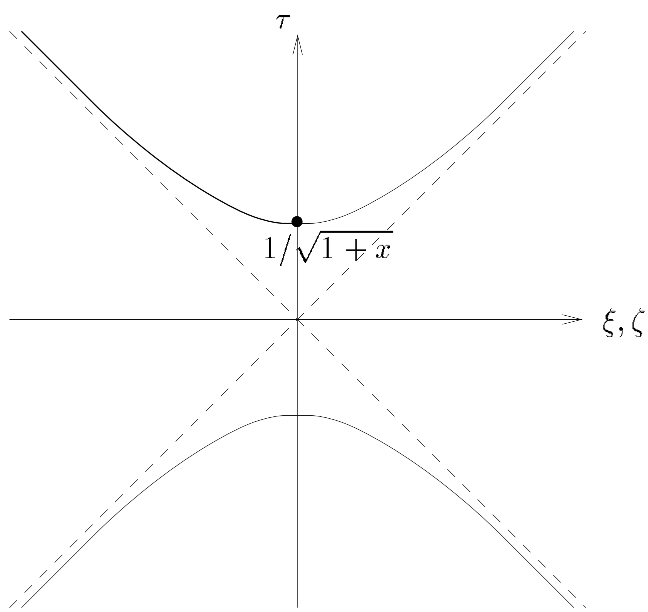

Figure 1: Characteristic variety of (0.1).

We will first consider the phases generated by the Cauchy-Dirichlet problem (0.1), (0.2), (0.3) by propagation of initial data and reflection at the boundary: Existence and regularity of such diffractive phases, as well as their transversality and nonresonance properties.

In our context, every glancing point in ∂T^*D (where D is the half space $\mathbb{R}\times\mathbb{R}^d_+$), *i.e.* every zero of p in $T^*(\partial D)$, is in fact a diffractive point (*cf.* [3]). Thus these points exactly compose the set of $((t,0,z),(\pm\langle\zeta\rangle,0,\zeta))$, $(t,z)\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^{d-1}$, using the usual notation $\langle\zeta\rangle := \sqrt{1+|\zeta|^2}$. We will see that only one such point can belong to each bicharacteristic curve (these curves foliate the graphs of our phases'differentials); in addition, the problem is invariant under (t,z)-translation, so we decide there is a diffractive point above the origin $(t,z=0)$, associated to a covector such that $\tau>0$. The bicharacteristic curve through this point is described by the Hamiltonian equations

$$\begin{cases} \dot{t} = 2(1+x)\tau \\ \dot{x} = -2\xi \\ \dot{z} = -2\zeta \\ \dot{\tau} = 0 \\ \dot{\xi} = -\tau^2 \\ \dot{\zeta} = 0 \end{cases} \tag{1.2}$$

with $(t,x,z,\tau,\xi,\zeta)_{|\sigma=0} = (0,0,0,\langle\underline{\zeta}\rangle,0,\underline{\zeta})$, so we have got the parametrization

$$
\begin{cases}
\zeta = \underline{\zeta} \\
\tau = \langle\underline{\zeta}\rangle \\
\xi = -\langle\underline{\zeta}\rangle^2\sigma \\
z = -2\underline{\zeta}\sigma \\
x = \langle\underline{\zeta}\rangle^2\sigma^2 \\
t = 2\langle\underline{\zeta}\rangle\sigma + \dfrac{2}{3}\langle\underline{\zeta}\rangle^3\sigma^3.
\end{cases}
\tag{1.3}
$$

We denote by S the set of glancing rays (rays are projections of bicharacteristic curves on the space of configurations (t,x,z)); we call it the "singular ray" by analogy with the one-dimensional case (Figure 2). The shadow region, $Schad$, is the open domain "beyond" S in D (see notation 1.1).

The phases are defined by:

Proposition 1.1. *There exist* $\underline{x},\underline{r},T>0$ *and phases* φ_i *and* φ_r *such that*

φ_i *- is defined on* $\mathcal{T}_i$*, open subset of* $D\setminus\overline{Schad}$*, whose closure is a neighbourhood of the origin in* $\overline{D\setminus Schad}$ *and contains* $\{-T\}\times[0,\underline{x}]\times B(0,\underline{r})$ *(*φ_i *is in fact regularly defined on a neighbourhood of the origin in* $\mathbb{R}^{1+d}$*);*

- is a $\mathcal{C}^\infty$ *solution of (1.1) on* $\overline{\mathcal{T}_i}$*;*

- satisfies $\nabla\varphi_i(0) = (\partial_z\varphi_i(0),0,\langle\partial_z\varphi_i(0)\rangle)$*;*

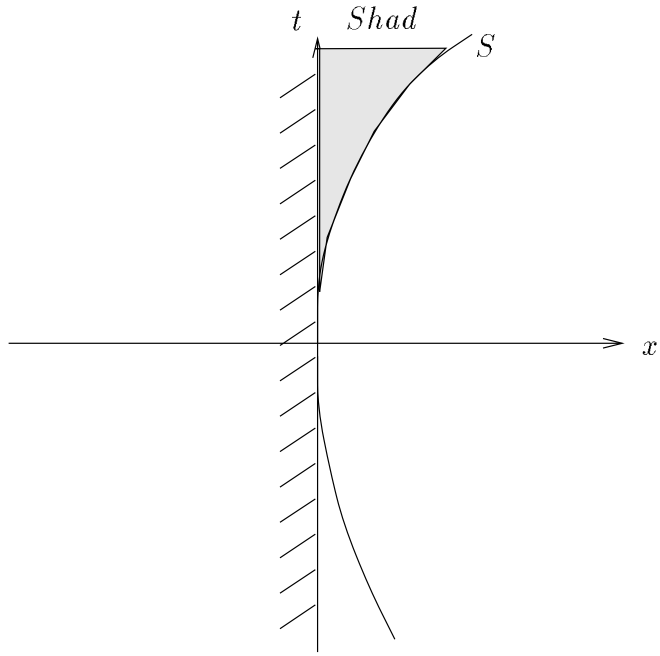

Figure 2: The singular ray near the origin.

φ_r *- is defined on* $\mathcal{T}_r$*, open subset of* $D \setminus \overline{Schad}$*, whose closure is a neighbourhood of the origin in* $\overline{D \setminus Schad}$ *and contains a neighbourhood of the origin in* $\overline{\{x=0\} \setminus Schad}$*;*

- is a $\mathcal{C}^1$ *solution of (1.1) on* $\overline{\mathcal{T}_r}$ *(*$\mathcal{C}^\infty$ *on* $\mathcal{T}_r$*);*

- satisfies on $\{x=0\}$*:* $\varphi_r = \varphi_i$*,* $\partial_x \varphi_r = -\partial_x \varphi_i$*.*

We denote by $\mathcal{T}$ *the union* $\mathcal{T}_i \cup \mathcal{T}_r$*.*

The form of domains $\mathcal{T}_i$ et $\mathcal{T}_r$ is given on Figure 3.

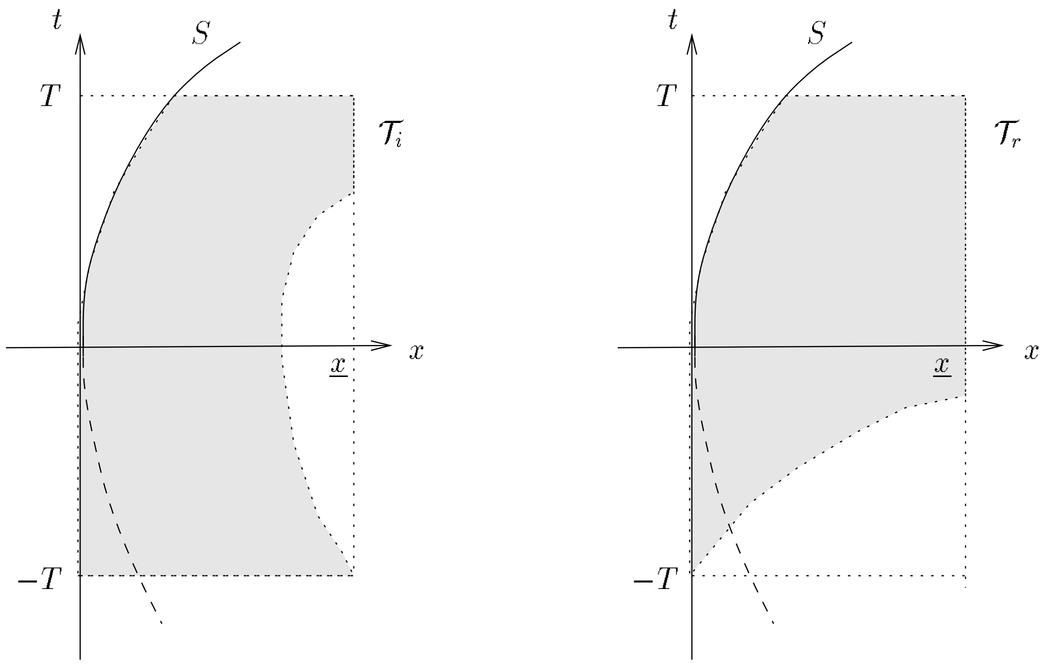

Figure 3: The domains $\mathcal{T}_i$ and $\mathcal{T}_r$.

Proof.

In order to define φ_i, we set its values at $t = 0$, on a neighbourhood of 0 in $\mathbb{R}^d_{x,z}$, requiring only regularity and a null x-derivative at the origin. The classical theory of Hamilton-Jacobi equations then allows us (if necessary restricting the neighbourhood, *i.e.* $\underline{x}$ and $\underline{r}$) to construct a unique (smooth) solution of (1.1) corresponding to this initial data, with $\partial_t\varphi_i(0) = \langle \partial_z\varphi_i(0)\rangle$, thanks to the flow of (1.2). For T small enough, all projections of integral curves (rays) reach $\{t = -T\}$ and foliate a neighbourhood of 0 in $\mathbb{R}^{1+d}$.

The equation (parametrized by σ) of the bicharacteristic curve through

$(t_0, x_0, z_0, \underline{\tau}, \xi_0, \underline{\zeta})$, with $\underline{\tau} > 0$ and $(1+x_0)\underline{\tau}^2 - \xi_0^2 - |\underline{\zeta}|^2 - 1 = 0$, is

$$(1.4) \qquad \begin{cases} \zeta = \underline{\zeta} \\ \tau = \underline{\tau} \\ \xi = \xi_0 - \underline{\tau}^2 \sigma \\ z = z_0 - 2\underline{\zeta}\sigma \\ x = x_0 - 2\xi_0 \sigma + \underline{\tau}^2 \sigma^2 \\ t = t_0 + 2(1+x_0)\underline{\tau}\sigma - 2\xi_0 \underline{\tau}\sigma^2 + \dfrac{2}{3}\underline{\tau}^3\sigma^3. \end{cases}$$

When σ is close to zero, t increases with σ; the bicharacteristic curve is then oriented with increasing $\sigma \in \mathbb{R}_+$. Restricting if necessary the (x, z)-domain, we have, in a neighbourhood of the bicharacteristic curve through the origin, and for $t = -T$: $\xi_0 > 0$, and x decreasing with σ.

Notation 1.1. *There are three cases for rays: They do not reach the boundary, cross it transversally, or touch it at a glancing point. The set of glancing rays form S. For rays crossing $\{x = 0\}$, we only consider the part of the ray from $\{t = -T\}$ and "before" the boundary (according to the orientation of the ray). Those form, together with the nonreflecting rays, the domain $\mathcal{T}_i$. Transverse rays meet the boundary again and foliate, beyond S, the open set $Schad$.*

Differentiating the eikonal equation with respect to x, and using $\partial_t\varphi(0) = \langle \partial_z \varphi(0) \rangle \neq 0$, we see that the gradient $\partial_{t,z}\partial_x\varphi$ does not vanish at the origin, so that the trace of S on $\{x = 0\}$, i.e. *$\{(t,z) \ / \ \partial_x\varphi(t,0,z) = 0\}$, is a (codimen-*

sion one) submanifold of $\{x = 0\}$*, which we denote by* S_0*. The surface* S *is parametrized by this submanifold, and transversally (see (1.8)) by rays spreading out of it; except at this point, they are contained in* $\{x > 0\}$*,* x *being a polynomial of degree two in the variable* σ*, with minimum value on* S_0*. Hence,* S *locally separates (in* $\mathbb{R}_+^{1+d}$*)* $Schad$ *and the enlighted zone. The boundary is separated by* S_0 *into a part included in* $Schad$*, and another one, "*$Light_0$*", defined by* $\partial_x \varphi_i > 0$*.*

The reflected phase φ_r is also constructed thanks to the flow of (1.2), for initial data $(s, 0, z, \partial_t \varphi_i(s, 0, z), -\partial_x \varphi_i(s, 0, z), \partial_z \varphi_i(s, 0, z))$, $(s, z) \in Light_0$. The reflected rays fold in the half-space $\{x < 0\}$, but they have defocusing properties on $\{x > 0\}$ (see (1.8) below); this allows us to define the following change of variables:

Lemma 1.1. *For* T*,* $\underline{x}$ *and* $\underline{r}$ *small enough, reflected rays define an application*

$$\begin{array}{rcll} Z_r : & Light_0 \times \mathbb{R}_+ & \longrightarrow & \overline{D} \\ & (s, y, \eta) & \longmapsto & Z_r(s, y, \eta) = (t, x, z) \end{array}$$

which - is a local diffeomorphism on $Light_0 \times]0, \underline{\eta}[$*;*

- is injective and continuous on $Z_r^{-1}(\{0 \leq x \leq \underline{x}\}) := \omega$*.*

Hence, there is a diffeomorphism from the interior of ω *onto its image* $\mathcal{T}_r$*, which extends to a homeomorphism from* ω *onto* $Z_r(\omega) = \overline{\mathcal{T}_r}$*, neighbourhood of the origin in* $\overline{D} \setminus Schad$*.*

Proof.

Begin with a change of coordinates: We know $\partial_{t,z}\partial_x\varphi_i(0) \neq 0$ (Notation 1.1), so there are local coordinates $b = (b_1, \ldots, b_d)$ such that $\partial_x\varphi_i(s, 0, y) = -b_1$. The surface S_0 is then represented by $b_1 = 0$, and the boundary enlighted zone $Light_0$, by $b_1 < 0$. The equation for the characteristic variety is

$$q(x, b, \beta) - \xi^2 = 0, \tag{1.5}$$

or (we set $\varphi_i(s, 0, y) := \varphi_0(b)$):

$$(1 + x)(\sum_k \beta_k \partial_t b_k)^2 - |\sum_k \beta_k \partial_z b_k|^2 - 1 - \xi^2 = 0. \tag{1.6}$$

The reflected rays are given as projections of the integral curves of

$$\begin{cases} \dot{x} = -2\xi \\ \dot{b} = \partial_\beta q(x, b, \beta) \\ \dot{\xi} = -\partial_x q(x, b, \beta) \\ \dot{\beta} = -\partial_b q(x, b, \beta), \end{cases} \tag{1.7}$$

where initially ($\eta = 0$): $x_0 = 0$, $b_{1,0} (= b_1(\eta = 0)) < 0$, $\xi_0 = -q(0, b_0, \beta_0)^{1/2} = -|b_{1,0}|$, $\beta_0 = \partial_b\varphi_0(b_0)$. Moreover, these rays, coming from $\{b_1 < 0\}$, go to $\{b_1 > 0\}$: Differentiating $q(x, b, \partial_b\varphi) - \partial_x\varphi^2 = 0$ with respect to x and evaluating the result at $x = 0$, $b = 0$, we get

$$\partial_{\beta_1} q(0, 0, \partial_b\varphi_0(0)) = \partial_x q(0, 0, \partial_b\varphi_0(0)) = \langle \beta . \partial_z b\rangle^2 > 0,$$

and in a neighbourhood of the origin,

$$\partial_{\beta_1} q_0 := \partial_{\beta_1} q(0, b_0, \beta_0) > 0 \text{ (so, using (1.7), } b_1(\eta) \text{ is increasing).} \tag{1.8}$$

The property of being a local diffeomorphism is induced by nonvanishing of the Jacobian determinant of Z_r. We write the Taylor expansion of (1.7) up to the order k (with a $\mathcal{O}_k$ residual with respect to b, η):

$$\begin{cases} x = 2\eta|b_{1,0}| + \mathcal{O}_3 \\ b = b_0 + \eta\partial_\beta q_0 + \mathcal{O}_2. \end{cases} \tag{1.9}$$

Consequently, the Jacobian determinant of Z_r is equivalent to $2(|b_{1,0}|+\eta\partial_{\beta_1}q_0) > 0$ thanks to (1.8).

Continuity on the closure is inherited from the one of the Hamiltonian bicharacteristic flow.

So as to prove injectivity on ω, we assume $Z_r(b_0, \eta) = Z_r(b_0', \eta') = (x, b)$. The lower order terms of b_1 read

$$b_{1,0}' \sim b_{1,0} + (\eta - \eta')\partial_{\beta_1}q_0,$$

and we plug this into the expression for x:

$$(\eta - \eta')b_{1,0} \sim \eta'(\eta - \eta')\partial_{\beta_1}q_0.$$

Knowing that $\eta, \eta' > 0$, $b_{1,0} < 0$ and $\partial_{\beta_1}q_0 > 0$, this is possible only if $\eta = \eta'$, which implies $b_0 = b_0'$. □

The reflected phase is defined by use of the inverse of Z_r on ω: $\varphi_r(t, x, z) := \varphi_r(s, 0, y) + 2\eta$. Its gradient is also transported, so φ_r has regularity $\mathcal{C}^1$ on $\overline{\mathcal{T}_r}$. □

Remark 1.1.

i) This construction clearly shows that the "broken flow" constituted by rays from $\{t<0\}$ and reflecting on the boundary does not reach the shadow region $Schad$.

ii) Even if the gradient of φ_r continuously extends to $\overline{\mathcal{T}_r}$, the second order derivatives do not. We recall the form of the singularity of $P\varphi_r = (1+x)\partial_t^2\varphi_r - \Delta_{x,z}\varphi_r$ at the origin, which plays a defocusing role in the equation for the reflected profile. This singularity is computed in [1]: Near the origin, in reflected flow coordinates,

$$P\varphi_r \sim \frac{1}{2\eta+\gamma|b_1|}, \tag{1.10}$$

with $\gamma = 1/\partial_{\beta_1}q_0 > 0$.

iii) In the case of higher finite order glancing (but not gliding) points in $T^\star(\partial D)$, Theorem 24.3.9 in [3] shows that each ray meets the boundary ∂D at most at one point. This monotonicity property is the analogue to (1.8) needed in the construction of reflected rays. Again, the set of tangential rays foliate a smooth hypersurface S.

So as to describe influence of nonlinearities, we have to look at interactions, or more precisely, noninteraction between phases:

Lemma 1.2.

i) The phases φ_i and φ_r have noncolinear gradients everywhere in $\mathcal{T}_i \cap \mathcal{T}_r$.

ii) More generally, three waves resonance does not occur for (0.1): If φ_1 *and* φ_2 *are two characteristic phases, for all* $k \in \mathbb{Z}^2$, $|k| \neq 1$ *implies* $k.\varphi := k_1\varphi_1 + k_2\varphi_2$ *is noncharacteristic.*

Proof.

i) If the gradients of φ_i and φ_r were colinear at one point of $\mathcal{T}_i \cap \mathcal{T}_r$, they would be on the whole associated ray, and hence at $x = 0$, which is not true.

ii) Let's assume the linear combination $k.\varphi$ is nontrivial ($k_1k_2 \neq 0$) and characteristic; we call it φ_3:

$$k_1\varphi_1 + k_2\varphi_2 = \varphi_3, \tag{1.11}$$

or equivalently,

$$k_1\varphi_1 + k_2\varphi_2 + k_3\varphi_3 = 0, \tag{1.12}$$

with $k_1, k_2, k_3 > 0$ (replacing φ_j by $-\varphi_j$ if necessary).

Two gradients $\nabla\varphi_j$ are then on the same part of the characteristic variety –say, $\nabla\varphi_1$ and $\nabla\varphi_2$. Hence,

$$-\nabla\varphi_3 = \frac{k_1}{k_3}\nabla\varphi_1 + \frac{k_2}{k_3}\nabla\varphi_2 \tag{1.13}$$

belongs to the line through $\nabla\varphi_1$ and $\nabla\varphi_2$, and to the same part of the characteristic variety, according to the signs $k_j/k_3 > 0$. This is inconsistent, because of strict convexity of this surface. $\square$

2 Definition of profiles and first equations

We know existence and uniqueness of solutions for (0.1) and (0.2), together with continuous dependence with respect to the data ([6]):

Theorem 2.1. *If $u_0 \in H_0^1(\mathbb{R}_+^d)$ and $u_1 \in L^2(\mathbb{R}_+^d)$, there exists a unique solution $u^\varepsilon \in \mathcal{C}([-T, +\infty[, H_0^1(\mathbb{R}_+^d))$, with $\partial_t u^\varepsilon \in \mathcal{C}([-T, +\infty[, L^2(\mathbb{R}_+^d)) \cap L^{p+1}(D)$, to*

$$\text{(2.1)} \qquad \begin{cases} P^\varepsilon u^\varepsilon + F(\partial_t u^\varepsilon) = 0 \\ u^\varepsilon_{|t=-T} = u_0, \ \ \partial_t u^\varepsilon_{|t=-T} = u_1. \end{cases}$$

In addition, if (v_0, v_1) are other initial data, we have:

(2.2)

$$\left\| \sqrt{1+x}\,\partial_t (u-v)(t) \right\|_{L^2}^2 + \|\partial_{x,z}(u-v)(t)\|_{L^2}^2 + \frac{1}{\varepsilon^2}\|u(t)-v(t)\|_{L^2}^2 + c\|\partial_t(u-v)\|_{L^{p+1}_{t,x,z}}^{p+1}$$
$$\leq \|\partial_{x,z}(u_0-v_0)\|_{L^2}^2 + \frac{1}{\varepsilon^2}\|u_0-v_0\|_{L^2}^2 + \left\|\sqrt{1+x}(u_1-v_1)\right\|_{L^2}^2 .$$

Recall that for this result, we only need the following properties of F:

$$\text{(2.3)} \qquad \begin{aligned} &(i)\ F \text{ is continuous from } L^{p+1}(D) \text{ to } L^{1+1/p}(D). \\ &(ii)\ \forall u, u' \in \mathbb{C},\ \ \mathrm{Re}\left((F(u)-F(u'))(\overline{u-u'})\right) \geq c|u-u'|^{p+1}. \end{aligned}$$

For initial data of the form (0.2), the sequence u^ε is bounded in the spaces given by (2.2), and for Ω bounded in D, u^ε is bounded in $H^1(\Omega)$, with $u^\varepsilon/\varepsilon$ bounded in $L^2(\Omega)$.

We introduce the following notion of profiles:

Definition 2.1. *The index k means i or r.*

i) Let u^ε be a bounded sequence in $H^1(\mathcal{T}_k)$, with $u^\varepsilon/\varepsilon$ bounded in $L^2(\mathcal{T}_k)$ and $\partial_t u^\varepsilon$ bounded in $L^{p+1}(\mathcal{T}_k)$. We say that $U_k \in L^{p+1}(\mathcal{T}_k)$ is a profile of u^ε associated to φ_k if (extracting a subsequence if necessary –still denoted by u^ε),

1- $u^\varepsilon e^{-i\varphi_k/\varepsilon}/\varepsilon$ weakly converges to U_k in L^2, and

2- $\partial_t u^\varepsilon e^{-i\varphi_k/\varepsilon}$ weakly converges to $i(\partial_t\varphi_k)U_k$ in L^{p+1}, and $\partial_{x,z}u^\varepsilon e^{-i\varphi_k/\varepsilon}$ to $i(\partial_{x,z}\varphi_k)U_k$ in L^2.

ii) When F^ε is bounded in $L^{1+1/p}(\mathcal{T}_k)$, $F_k \in L^{1+1/p}(\mathcal{T}_k)$ is a profile of F^ε associated to φ_k if $F^\varepsilon e^{-i\varphi_k/\varepsilon}$ (or a subsequence) weakly converges to F_k in $L^{1+1/p}$.

Remark 2.1. *Existence and uniqueness of a profile implies convergence of the whole sequence.*

Notation 2.1. *In the case of existence of incident and reflected profiles (U_i and U_r), we also call profile the function*

$$\mathcal{U}(t,x,\theta) := U_i e^{i\theta_i} + U_r e^{i\theta_r} \in L^{p+1}(\mathcal{T}\times\mathbb{T}^2) \text{ (with } U_r \text{ extended by } 0). \tag{2.4}$$

In the same way, we set

$$\mathcal{F}(t,x,\theta) := F_i e^{i\theta_i} + F_r e^{i\theta_r} \in L^{1+1/p}(\mathcal{T}\times\mathbb{T}^2) \text{ (with } F_r \text{ extended by } 0). \tag{2.5}$$

We also write $(\nabla\varphi)\mathcal{U}$ for $(\nabla\varphi_i)U_i e^{i\theta_i} + (\nabla\varphi_r)U_r e^{i\theta_r}$.

In the case we are interested in, thanks to (2.2), we immediately have existence of profiles for u^ε and $F(\partial_t u^\varepsilon)$ (the idea in this definition by duality is to truncate an –arbitrary small– neighbourhood of the singular ray), but in addition, these profiles must satisfy a system of transport equations:

Proposition 2.1. *Let u^ε be the solution of (0.1), (0.2). For $k = i, r$, u^ε and $F(\partial_t u^\varepsilon)$ admit profiles $U_k \in L^{p+1}(\mathcal{T}_k)$ and $F_k \in L^{1+1/p}(\mathcal{T}_k)$ which satisfy (following Notations 0.1):*

$$(2.6)\qquad \begin{aligned} &(i) && 2T_{\varphi_i}(\partial)U_i - P(\varphi_i)U_i + \frac{1}{i}F_i = 0 \text{ on } \mathcal{T}_i \\ & && \textit{(and, in fact, on a neighbourhood of the origin in } D\textit{)} \\ &(ii) && 2T_{\varphi_r}(\partial)U_r - P(\varphi_r)U_r + \frac{1}{i}F_r = 0 \text{ on } \mathcal{T}_r \\ &(iii) && U_{i|_{t=-T}} = g \\ &(iv) && (U_i + U_r)_{|_{x=0}} = 0. \end{aligned}$$

Proof.

Begin by testing equation (0.1) against a function $\overline{a}e^{-i\varphi_i/\varepsilon}$, with $a \in \mathcal{C}_0^\infty(\mathcal{T}_i \cup S \cup Schad \cup \{t = -T\})$. The boundary term at $\{t = -T\}$ is well defined because

u^ε and $\partial_t u^\varepsilon$ are continuous with respect to time:

(2.7)

$$
\begin{aligned}
&\int P^\varepsilon u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a} = \\
&\quad -\int \left[(1+x)\partial_t u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a}\right]_{|t=-T} dxdz \\
&\quad +\int (1+x)\partial_t u^\varepsilon \left(\frac{i}{\varepsilon}(\partial_t\varphi_i)\overline{a} - \partial_t\overline{a}\right) e^{-i\varphi_i/\varepsilon} \\
&\quad -\int \partial_{x,z} u^\varepsilon . \left(\frac{i}{\varepsilon}(\partial_{x,z}\varphi_i)\overline{a} - \partial_{x,z}\overline{a}\right) e^{-i\varphi_i/\varepsilon} + \frac{1}{\varepsilon^2}\int u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a} \\
&= -\int \left[(1+x)\partial_t u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a}\right]_{|t=-T} dxdz \\
&\quad -\int \left[(1+x)u^\varepsilon \left(\frac{i}{\varepsilon}(\partial_t\varphi_i)\overline{a} - \partial_t\overline{a}\right) e^{-i\varphi_i/\varepsilon}\right]_{|t=-T} dx \\
&\quad -\int (1+x)u^\varepsilon \left(\frac{(\partial_t\varphi_i)^2}{\varepsilon^2}\overline{a} + \frac{i}{\varepsilon}\left(2(\partial_t\varphi_i)\partial_t\overline{a} + (\partial_t^2\varphi_i)\overline{a}\right) - \partial_t^2\overline{a}\right) e^{-i\varphi_i/\varepsilon} \\
&\quad +\int u^\varepsilon \left(\frac{|\partial_{x,z}\varphi_i|^2}{\varepsilon^2}\overline{a} + \frac{i}{\varepsilon}\left(2(\partial_{x,z}\varphi_i).\partial_{x,z}\overline{a} + (\Delta_{x,z}\varphi_i)\overline{a}\right) - \Delta_{x,z}\overline{a}\right) e^{-i\varphi_i/\varepsilon} \\
&\quad +\frac{1}{\varepsilon^2}\int u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a} \\
&= -\int (1+x)\left[\partial_t u^\varepsilon e^{-i\varphi_i/\varepsilon}\overline{a} + u^\varepsilon\left(\frac{i}{\varepsilon}(\partial_t\varphi_i)\overline{a} - \partial_t\overline{a}\right) e^{-i\varphi_i/\varepsilon}\right]_{|t=-T} dxdz \\
&\quad - i\int \frac{u^\varepsilon}{\varepsilon} e^{-i\varphi_i/\varepsilon}\left(2T_{\varphi_i}(\partial)\overline{a} + P(\varphi_i)\overline{a}\right) + \varepsilon\int \frac{u^\varepsilon}{\varepsilon} e^{-i\varphi_i/\varepsilon} P\overline{a}.
\end{aligned}
$$

We can then extract from ε a subsequence such that this sum of terms converges to

$$
-2i\int (1+x) g\left((\partial_t\varphi_i)\overline{a}\right)_{|t=-T} dxdz - i\int U_i\left(2T_{\varphi_i}(\partial)\overline{a} + P(\varphi_i)\overline{a}\right). \tag{2.8}
$$

On the other hand, extracting a new subsequence if necessary, we obtain

$$
\int F(\partial_t u^\varepsilon) e^{-i\varphi_i/\varepsilon}\overline{a} \underset{\varepsilon\to 0}{\longrightarrow} \int F_i\overline{a}, \tag{2.9}
$$

so that

(2.10)

$$\int \left(U_i \left(2T_{\varphi_i}(\partial)\overline{a} + P(\varphi_i)\overline{a}\right) + iF_i\overline{a}\right) + 2\int (1+x) g \left((\partial_t\varphi_i)\overline{a}\right)_{|t=-T} dxdz = 0,$$

which implies equation (2.6*(i)*) when $Supp\ a \cap \{t=-T\} = \emptyset$.

This equation suffices to give sense to traces of U_i at $t=-T$ and at $x=0$:

Lemma 2.1. *If $U \in L^{p+1}(D)$ satisfies (2.6(i)), then $U \in \mathcal{C}([-T,T], L^2(\mathbb{R}^d))$, U admits a trace at $\{x=0\}$, $U_{|x=0} \in L^2(\partial_x\varphi_i(t,0,z)dt\,dz)$, and satisfies the energy estimate*

(2.11)

$$\begin{aligned} \int (1+x)(\partial_t\varphi_i)|U|^2(t)dxdz &+ \int_{x=0} (\partial_x\varphi_i)|U|^2(s,0,z)dsdz \\ &- \int_{\mathcal{T}_i\cap\{s\le t\}} P(\varphi_i)|U|^2 dsdxdz \\ &= Re\int iF_i\overline{U} + \int (1+x)(\partial_t\varphi_i)_{|s=-T}|g|^2 dxdz. \end{aligned}$$

Proof.

Continuity follows from integration of a vector field $(T_{\varphi_i}(\partial))$ for which surfaces $\{t=c^{st}\}$ are noncharacteristic:

First, regularize U: define $U_\delta := R_\delta U = \frac{1}{\delta^{d+1}}\rho\left(\frac{\cdot}{\delta}\right) *_{t,x,z} U$, for a given approximation of identity $\left(\frac{1}{\delta^{d+1}}\rho\left(\frac{\cdot}{\delta}\right)\right)_{\delta>0}$. We take $Supp(\rho) \subset \{x \le 0\}$, so that convolution only involves values of U for $x>0$. Hence, on D,

$$(2.12)\qquad 2T_{\varphi_i}(\partial)U_\delta - P(\varphi_i)U_\delta = iR_\delta F_i + 2[T_{\varphi_i}(\partial), R_\delta]U - [P(\varphi_i), R_\delta]U,$$

and the last two terms go to zero (with δ) in $L^{p+1}(\mathbb{R}^{1+d})$, thanks to Friedrichs'lemma, since φ_i is smooth.

We want to show that the sequence $\left(U_{\delta|_{t=\underline{t}}}\right)_{\delta>0}$ uniformly converges (w.r.t. $\underline{t}$) in $L^2(\mathbb{R}^d)$ as δ goes to zero. So, consider a domain $\mathcal{D}$ included in $\overline{\mathcal{T}} \cap \{t \leq \underline{t}\}$ and delimited by incident rays. Set

$$\mathcal{D}_{\underline{t}} := \mathcal{D} \cap \{t = \underline{t}\},$$

$$\mathcal{D}_{\underline{t}}^{\mu} := \mathcal{D}_{\underline{t}} \cap \{x \geq \mu\} \cap \{\text{ distance to } S \cap \{t = \underline{t}\} \geq \mu\},$$

$$\mathcal{D}^{\mu} := \text{ all points transported from } \mathcal{D}_{\underline{t}}^{\mu} \text{ by the incident flow in } \mathcal{D},$$

for a small parameter $\mu > 0$. In particular, $\overline{\mathcal{D}^{\mu}} \cap \{x = 0\} = \emptyset$.

Now, take a smooth cut-off function χ with values in $[0,1]$, 1 on a neighbourhood of $\overline{\mathcal{D}} \cap \{t = \underline{t}\}$, and zero on $\{t \leq -T\}$:

$$\begin{aligned} &2T_{\varphi_i}(\partial)(\chi U_\delta) - P(\varphi_i)\chi U_\delta = \\ &\qquad i\chi R_\delta F_i + 2\chi[T_{\varphi_i}(\partial), R_\delta]U - \chi[P(\varphi_i), R_\delta]U - 2(T_{\varphi_i}(\partial)\chi)U_\delta \\ &\qquad := G_\delta, \end{aligned} \tag{2.13}$$

and integration of $\mathrm{Re}\left([(2.13)_\delta - (2.13)_{\delta'}] \times \chi(\overline{U_\delta} - \overline{U_{\delta'}})\right)$ on $\mathcal{D}^{\mu}$ writes out:

$$\begin{aligned} &\int_{\mathcal{D}_{\underline{t}}^{\mu}} (1+x)(\partial_t \varphi_i)|U_\delta - U_{\delta'}|^2 = \\ &\qquad = \int_{\mathcal{D}^{\mu}} P(\varphi_i)|\chi(U_\delta - U_{\delta'})|^2 + \mathrm{Re}\left((G_\delta - G_{\delta'})\chi(\overline{U_\delta} - U_{\delta'})\right) \\ &\qquad \leq \int_{\mathcal{T}} |P(\varphi_i)||\chi(U_\delta - U_{\delta'})|^2 + |(G_\delta - G_{\delta'})\chi(\overline{U_\delta} - U_{\delta'})|. \end{aligned} \tag{2.14}$$

Let μ go to zero, so that the left-hand side goes to

$$\int_{\mathcal{D}_{\underline{t}}} (1+x)(\partial_t \varphi_i)|U_\delta - U_{\delta'}|^2 \geq c\|U_\delta - U_{\delta'}\|_{L^2(\mathcal{D}_{\underline{t}})}$$

with $c > 0$. Finally, the right-hand side in (2.14) goes to zero as $\delta, \delta' \to 0$, giving the result.

The trace at $\{x = 0\}$ is obtained in a similar way, by integration on a tube of rays with "base" in $\{x = 0\}$. $\square$

Continuity with respect to time allows us to interpret (2.10), when $Supp\ a \cap \{t = -T\} \neq \emptyset$, and to deduce (2.6*(iii)*).

We get Equation (2.6*(ii)*) for U_r in the same way by testing (0.1) against $\overline{a} e^{-i\varphi_r/\varepsilon}$, with $a \in \mathcal{C}_0^\infty(\mathcal{T}_r)$.

This time, loss of regularity (at the boundary) of coefficients in the equation prevents one to deduce directly lemma 2.1 for U_r. Nevertheless, this result is still valid away from the singular ray: In particular, if $\omega \subset\subset Light_0$ (Notation 1.1), U_r admits a trace (in $L^2_{t,z}(\omega)$). Testing (0.1) against $\overline{a}\left(e^{-i\varphi_i/\varepsilon} - e^{-i\varphi_r/\varepsilon}\right)$, with $a \in \mathcal{C}_0^\infty\left(\mathcal{T} \cup Light_0\right)$, and integrating by parts, we deal with only one term:

(2.15)
$$0 = \int_{Light_0} \left[\left((\partial_x\varphi_i)U_i - (\partial_x\varphi_r)U_r\right)\overline{a}\right]_{|x=0} dtdz = \int_{Light_0} \left[(\partial_x\varphi_i)\left(U_i + U_r\right)\overline{a}\right]_{|x=0} dtdz,$$

and since $\partial_x\varphi_{i_{|x=0}}(t)$ does not vanish on $Light_0$, this implies the boundary condition (2.6*(iv)*). $\square$

3 Nonlinear operations on the profiles

Starting from the formal (WKB) equivalence

$$\partial_t u^\varepsilon \sim i(\partial_t\varphi)\mathcal{U}\,(t,x,\varphi/\varepsilon) = i\left((\partial_t\varphi_i)U_i e^{i\varphi_i/\varepsilon} + (\partial_t\varphi_r)U_r e^{i\varphi_r/\varepsilon}\right),$$

we define

Definition 3.1. *When $U_k \in L^{p+1}(\mathcal{T}_k) \hookrightarrow L^{p+1}(\mathcal{T})$, $k = i, r$,*

$$E_k(U_i,U_r) := \oint_{\mathbb{T}^2} F\left(i(\partial_t\varphi)\mathcal{U}(t,x,\theta)\right) e^{-i\theta_k} d\theta \tag{3.1}$$

(where the circle on the integration symbol means the measure $d\theta$ has a total mass normalized to one),

$$\mathcal{E}(U_i,U_r) := E_i(U_i,U_r)e^{i\theta_i} + E_r(U_i,U_r)e^{i\theta_r}. \tag{3.2}$$

Hence, E_k is a continuous operator from $L^{p+1}(\mathcal{T}) \times L^{p+1}(\mathcal{T})$ to $L^{1+1/p}(\mathcal{T})$, and $\mathcal{E}$, from $L^{p+1}(\mathcal{T}) \times L^{p+1}(\mathcal{T})$ to $L^{1+1/p}(\mathcal{T} \times \mathbb{T}^2)$.

The challenge is to understand the link between $\mathcal{F}$ and $\mathcal{E}(U_i,U_r)$, in particular, whether oscillations in the difference $\mathcal{F} - \mathcal{E}$ are propagated by P^ε or not.

Definition 3.2. *A sequence h^ε bounded in $L^{1+1/p}$ has no propagated oscillations if, for every sequence w^ε such that $\partial_t w^\varepsilon$ is bounded in L^{p+1}*

and $P^\varepsilon w^\varepsilon$ is bounded in $L^{1+1/p}$,

$$\int h^\varepsilon \partial_t \overline{w}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.3}$$

We have (*cf.* [5]):

Proposition 3.1. *Let* $\partial_t v^\varepsilon \sim i(\partial_t\varphi)\mathcal{U}(t,x,\varphi/\varepsilon)$ *in* $L^{p+1}(\mathcal{T})$*, and* $\mathcal{U}$ *as mentioned above. Then*

i) $F(\partial_t v^\varepsilon)$ *is bounded in* $L^{1+1/p}(\mathcal{T})$ *and admits unique profiles* $E_k(\mathcal{U})$*;*

ii) if $h^\varepsilon \sim F(\partial_t v^\varepsilon) - \mathcal{E}(\varphi/\varepsilon)$ *in* $L^{1+1/p}(\mathcal{T})$*,* h^ε *has no propagated oscillations.*

Proof.

Thanks to continuity of F from L^{p+1} to $L^{1+1/p}$, and to the fact that h^ε is only defined up to a sequence going to 0 in $L^{1+1/p}$, we can replace $\partial_t v^\varepsilon$ and h^ε with approximations in L^{p+1} and $L^{1+1/p}$, respectively, and assume that

$$\partial_t v^\varepsilon = i(\partial_t\varphi)\mathcal{U}(t,x,\varphi/\varepsilon),\ h^\varepsilon = F(\partial_t v^\varepsilon) - \mathcal{E}(\varphi/\varepsilon), \tag{3.4}$$

with profiles U_k $\mathcal{C}^\infty$ with compact support (on which phases are regular).

Since F is continuous on $\mathbb{R}^2$, the function

$$\mathbf{F}(t,x,z,\theta) := F(i(\partial_t\varphi)\mathcal{U}) \tag{3.5}$$

admits an absolutely convergent Fourier series (with respect to θ),

$$\mathbf{F}(t,x,\theta) = \sum_{\alpha\in\mathbb{Z}^2} c_\alpha(t,x,z)e^{i\alpha.\theta}, \tag{3.6}$$

where

$$c_{(1,0)} = E_i,\ \text{and}\ c_{(0,1)} = E_r. \tag{3.7}$$

Then we extract profiles from $F(\partial_t v^\varepsilon)$ ($k \in \{i, r\}$, $a \in \mathcal{C}_0^\infty$):

$$
\begin{aligned}
\int F(\partial_t v^\varepsilon)\overline{a} e^{-i\varphi_k/\varepsilon} &= \sum_\alpha \int c_\alpha \overline{a} e^{i(\alpha.\varphi - \varphi_k)/\varepsilon} \\
&\underset{\varepsilon \to 0}{\longrightarrow} \int E_k \overline{a},
\end{aligned}
\tag{3.8}
$$

thanks to the dominated convergence theorem, since each integral, α being fixed with $\alpha.\varphi \neq \varphi_k$, has a nonstationary phase (*cf.* Lemma 1.2*i)*) and thus tends to 0.

So as to prove there is no propagated oscillation, using absolute convergence in (3.6), we can replace h^ε with $c_\alpha(t, x, z)e^{i\alpha.\varphi/\varepsilon}$, with $\alpha \notin \{(0,1),(1,0)\}$; we set $h^\varepsilon = ce^{i\psi/\varepsilon}$.

The idea is then to construct a function g^ε tending to 0 in L^{p+1} such that $P^\varepsilon g^\varepsilon \sim \partial_t h^\varepsilon$ in $L^{1+1/p}$: With w^ε just as in Definition 3.2, we get, by integrations by parts (at least formally):

$$
\int h^\varepsilon \partial_t \overline{w}^\varepsilon \simeq -\int P^\varepsilon g^\varepsilon \overline{w}^\varepsilon = -\int g^\varepsilon \overline{P^\varepsilon w^\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 0.
\tag{3.9}
$$

Look for g^ε under the form $g^\varepsilon = \varepsilon d e^{i\psi/\varepsilon}$.

$$
\begin{aligned}
P^\varepsilon g^\varepsilon &= \left[-\frac{p(d\psi)}{\varepsilon} d + 2iT_\psi(\partial) d + \varepsilon P(\partial) d \right] e^{i\psi/\varepsilon}, \text{ and} \\
\partial_t h^\varepsilon &= \left(i\frac{\partial_t \psi}{\varepsilon} c + \partial_t c \right) e^{i\psi/\varepsilon}.
\end{aligned}
\tag{3.10}
$$

On the support of c, $p(d\psi)$ is smooth and does not vanish (because of nonresonance, Lemma 1.2*i)*). We choose

$$
d = -i\frac{\partial_t \psi}{p(d\psi)} c + \frac{\varepsilon}{p(d\psi)} \left(2T_\psi(\partial) \left[\frac{\partial_t \psi}{p(d\psi)} c \right] - \partial_t c \right),
\tag{3.11}
$$

which implies $g^\varepsilon = \mathcal{O}_{L^{p+1}}(\varepsilon)$, $P^\varepsilon g^\varepsilon - \partial_t h^\varepsilon = \mathcal{O}_{L^{1+1/p}}(\varepsilon)$, and there is really no boundary term in the integrations by parts in (3.9). $\square$

4 Construction of the approximate solution

In this paragraph, we construct a function v^ε satisfying the assumptions of Proposition 3.1 and giving the asymptotics of u^ε (see Paragraph 5). It is obtained via cut-off (near the singular ray S) and regularization of profiles.

Proposition 4.1. *Let u^ε be the solution of (0.1), (0.2), $\mathcal{U}$ an associated profile, and $\mathcal{F}$ a profile for $F(\partial_t u^\varepsilon)$. Then, there is a function $v^\varepsilon \in \mathcal{C}([-T,T], L^2(\mathbb{R}^d))$ $\cap H^1(\mathcal{T})$, supported in $\mathcal{T}$, such that*

$$
\begin{aligned}
&v^\varepsilon \sim \varepsilon\mathcal{U}\left(t,x,z,\varphi/\varepsilon\right) \text{ in } H^1(\mathcal{T}),\\
&\partial_t v^\varepsilon \sim i(\partial_t\varphi)\mathcal{U}\left(t,x,z,\varphi/\varepsilon\right) \text{ in } L^{p+1}(\mathcal{T}),\\
&P^\varepsilon v^\varepsilon \sim -\mathcal{F}\left(t,x,z,\varphi/\varepsilon\right) \text{ in } L^{1+1/p}(\mathcal{T}),\\
&\left(\frac{v^\varepsilon}{\varepsilon}, \nabla v^\varepsilon\right)_{|t=-T} \sim \left(\frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right)_{|t=-T} \text{ in } L^2(\mathcal{T}\cap\{t=-T\}),\\
&\partial_t v^\varepsilon_{|x=0} = 0.
\end{aligned}
\tag{4.1}
$$

Proof.

Our strategy is to use cut-off and regularization on each profile, in order to extend it beyond the singular ray and to apply P on it. This will provide a function

$$v^{\varepsilon,\mu,\rho} := \varepsilon\left(U_i^{\mu,\rho}e^{i\varphi_i/\varepsilon} + U_r^{\mu,\rho}e^{i\varphi_r/\varepsilon}\right), \tag{4.2}$$

where μ and ρ are parameters for cut-off and regularization.

Then we apply the following diagonal process ($h = v, \partial_t v, Pv, \dots$):

Lemma 4.1. *For sequences $\mathcal{H}^\varepsilon$ and $h^{\varepsilon,\alpha}$ such that* $\underline{\lim}_{\alpha\to 0}\overline{\lim}_{\varepsilon\to 0}(h^{\varepsilon,\alpha} - \mathcal{H}^\varepsilon) = 0$*,* i.e.

$$\forall\delta > 0,\ \exists\ \varepsilon(\delta), \alpha(\delta),\ |\alpha(\delta)| \leq \delta, \tag{4.3}$$
$$\forall\varepsilon \leq \varepsilon(\delta),\ \|h^{\varepsilon,\alpha(\delta)} - \mathcal{H}^\varepsilon\| \leq \delta,$$

there is $h^\varepsilon := h^{\varepsilon,\alpha(\varepsilon)}$ such that $\|h^\varepsilon - \mathcal{H}^\varepsilon\| \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Proof.

We assume $\varepsilon(\delta)$ is strictly increasing with δ (decreasing $\varepsilon(\delta')$ when $\delta' < \delta$), and goes to zero at zero. So there is a reciprocal function $\delta(\varepsilon)$, going to zero too, and we set $\alpha(\varepsilon) := \alpha(\delta(\varepsilon))$.

Given a challenging $\delta > 0$, since $\varepsilon \leq \varepsilon(\delta(\varepsilon)) = \varepsilon$, we know that

$$\|h^{\varepsilon,\alpha(\varepsilon)} - \mathcal{H}^\varepsilon\| \leq \delta(\varepsilon). \tag{4.4}$$

We conclude using the fact that $\delta(\varepsilon)$ goes to zero. $\square$

The main difficulty comes from the boundary condition $\partial_t v^\varepsilon_{|x=0} = 0$, which can be lost if the regularization is not performed carefully.

1-Begin with the cut-off of U_r along S. We use the local coordinates b from the proof of Lemma 1.1 ($Light_0 = \{b_1 < 0\}$) and the homeomorphism Z_r: Given a smooth function χ_0 on $\mathbb{R}$ with value 1 on $\{r \leq -1\}$, 0 on $\mathbb{R}_+$, we set

$$(4.5)\qquad \begin{aligned} \chi^\mu(s,y) &:= \chi_0\left(\frac{b_1}{\mu}\right), \\ \chi^\mu_r(t,x,z) &:= \chi\left(\frac{\pi(Z_r^{-1}(t,x,z))}{\mu}\right), \end{aligned}$$

with $\pi(s,y,\eta) = (s,y)$. We write $\chi^\mu_r U_r := U^\mu_r$, which equals U_r away from S, and vanishes on a neighbourhood (of size μ) of S.

We do the same for U_i along S thanks to χ^μ_i ($\chi^\mu_i U_i := U^\mu_i$), preserving $(U^\mu_i + U^\mu_r)_{|x=0} = 0$.

The interesting property of these cut-offs is commutation with transport operators:

$$(4.6)\qquad [T_{\varphi_i}(\partial), \chi^\mu_i] = [T_{\varphi_r}(\partial), \chi^\mu_r] = 0.$$

Hence, U^μ_i and U^μ_r satisfy

$$(4.7)\qquad \begin{cases} 2T_{\varphi_k}(\partial)U^\mu_k - P(\varphi_k)U^\mu_k = i\chi^\mu_k F_k := iF^\mu_k \text{ on } \mathcal{T}_k,\ F^\mu_k \underset{\mu\to 0}{\longrightarrow} F_k\ L^{1+1/p}(\mathcal{T}_k) \\ U^\mu_{i\ |t=-T} = g^\mu \underset{\mu\to 0}{\longrightarrow} g\ L^2 \\ (U^\mu_i + U^\mu_r)_{|x=0} = 0 \end{cases}$$

and coefficients in the equations are smooth on the union $\mathcal{T}^\mu$ of supports of the U^μ_k.

2-A tangent regularization (to $\{x=0\}$, in the variables t, z) provides approximations $U^{\mu,\rho_1}_k := R^{\rho_1}U^\mu_k$, whose derivatives with respect to t, z are in $L^{p+1}(D)$, and which still satisfy the boundary condition on $\{x=0\}$.

Furthermore, the initial data for U_i become

(4.8) $$U_i^{\mu,\rho_1}{}_{|t=-T} = g^{\mu,\rho_1} \underset{\rho_1\to 0}{\longrightarrow} g^\mu \ L^2,$$

and left-hand sides of the equations are

(4.9) $$iF_k^{\mu,\rho_1} - 2[R^{\rho_1}, T_{\varphi_k}(\partial)]U_k^\mu + [R^{\rho_1}, P(\varphi_k)]U_k^\mu,$$

converging to iF_k in $L^{1+1/p}(\mathcal{T}^\mu)$, thanks to Friedrichs'lemma for the first commutator, writing

(4.10) $$[R^{\rho_1}, P(\varphi_k)]U = (R^{\rho_1} - Id)\, P(\varphi_k)U + P(\varphi_k)\,(U - U^{\rho_1})$$

and using usual properties of regularizations for the second one.

3-Setting $V_1^{\mu,\rho_1} := U_i^{\mu,\rho_1} - U_r^{\mu,\rho_1}$ and $V_2^{\mu,\rho_1} := U_i^{\mu,\rho_1} + U_r^{\mu,\rho_1}$, we have for these $L^{p+1}(D)$ functions, regular in t, z:

(4.11) $$\begin{cases} \partial_x V^{\mu,\rho_1} = A\partial_{t,z}V^{\mu,\rho_1} + BV^{\mu,\rho_1} + c \in L^{1+1/p}(\mathcal{T}_k) \\ (V_1^{\mu,\rho_1} + V_2^{\mu,\rho_1})_{|t=-T} = 2g^{\mu,\rho_1} \\ V_2^{\mu,\rho_1}{}_{|x=0} = 0 \end{cases}$$

with A and B regular matrices.

We then extend $V_2^{\mu,\rho_1} \in W^{1,1+1/p} \cap L^{p+1}(D)$ by zero on $\{x < 0\}$: This is a function in $W^{1,1+1/p} \cap L^{p+1}(\mathbb{R}^{1+d})$, and by regularization, $V_2^{\mu,\rho_1,\rho_2} \underset{\rho_2\to 0}{\longrightarrow} V_2^{\mu,\rho_1}$ $W^{1,1+1/p} \cap L^{p+1}(D)$. In addition, if we choose a regularization by convolution with $\dfrac{1}{\rho_2}\gamma(x/\rho_2)$, where $Supp\gamma \subset \{x \geq 0\}$, the condition $V_2^{\mu,\rho}{}_{|x=0} = 0$ is still valid.

Concerning V_1^{μ,ρ_1}, we can extend it by symmetry and regularize it, or simply choose a regularization by convolution with $\frac{1}{\rho_2}\gamma(-x/\rho_2)$ (where only values of V_1^{μ,ρ_1} on $\{x \geq 0\}$ play a role), to obtain a smooth approximation on $\mathbb{R}^{1+d}$, $V_1^{\mu,\rho_1,\rho_2} \underset{\rho_2 \to 0}{\longrightarrow} V_1^{\mu,\rho_1}\ W^{1,1+1/p} \cap L^{p+1}(D)$.

4-Back to $U_i^{\mu,\rho}$ and $U_r^{\mu,\rho}$, $\rho = (\rho_1, \rho_2)$, we know they are smooth functions such that

$$
U_i^{\mu,\rho}|_{t=-T} \underset{\rho_2 \to 0}{\longrightarrow} g^{\mu,\rho_1}\ L^2,
$$

$$
(U_i^{\mu,\rho} + U_r^{\mu,\rho})_{|x=0} = 0, \tag{4.12}
$$

$$
U_k^{\mu,\rho} \underset{\rho_2 \to 0}{\longrightarrow} U_k^{\mu,\rho_1}\ W^{1,1+1/p} \cap L^{p+1}(D),
$$

so that, for $k = i, r$,

$$
2T_{\varphi_k}(\partial)U_k^{\mu,\rho} - P(\varphi_k)U_k^{\mu,\rho} \underset{\mu \to 0}{\longrightarrow} iF_k^{\mu,\rho_1}\ L^{1+1/p}(D) \tag{4.13}
$$

just like at Alinea **2**.

5-Finally, we set $v^{\varepsilon,\mu,\rho} := \varepsilon\left(U_i^{\mu,\rho}e^{i\varphi_i/\varepsilon} + U_r^{\mu,\rho}e^{i\varphi_r/\varepsilon}\right)$, and compute

(4.14)

$$
P^\varepsilon v^{\varepsilon,\mu,\rho} = i\sum_{k=i,r}\left(2T_{\varphi_k}(\partial)U_k^{\mu,\rho} - (P\varphi_k)U_k^{\mu,\rho}\right)e^{i\varphi_k/\varepsilon} + \varepsilon\sum_{k=i,r}(PU_k^{\mu,\rho})e^{i\varphi_k/\varepsilon}.
$$

Thanks to convergence properties of the approximations, each equivalent in (4.1) is improved using Lemma 4.1 for the quantities $v^{\varepsilon,\mu,\rho}$, $\partial_t v^{\varepsilon,\mu,\rho}$, $P^\varepsilon v^{\varepsilon,\mu,\rho}$, ... (fixing successively μ, ρ_1 and ρ_2, and then letting ε go to zero). □

5 Asymptotics for u^ε

We now prove the difference between the approximate solution v^ε from Paragraph 4 and the exact solution u^ε goes (strongly) to zero:

Proposition 5.1. *Let u^ε be the solution of (0.1), (0.2) –or precisely, the subsequence defining profiles–, and v^ε an approximate solution provided by Proposition 4.1. Then,*

$$\sup_{t\in[-T,T]}\left(\|\nabla(u^\varepsilon - v^\varepsilon)\|_{L^2_{x,z}} + \frac{1}{\varepsilon^2}\|u^\varepsilon - v^\varepsilon\|_{L^2_{x,z}}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{5.1}$$

$$\|\partial_t u^\varepsilon - \partial_t v^\varepsilon\|_{L^{p+1}(\Omega_T)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{5.2}$$

where $\Omega_T :=]-T,T[\times]0,\underline{x}[\times B(0,\underline{r})$.

Proof.

We establish an energy estimate, testing $P^\varepsilon(u^\varepsilon - v^\varepsilon) \in L^{1+1/p}(\Omega_t)$ against $\partial_t(u^\varepsilon - v^\varepsilon) \in L^{p+1}(\Omega_t)$, where $\Omega_t :=]-T,t[\times]0,\underline{x}[\times B(0,\underline{r})$.

Integration by parts gives

$$
\begin{aligned}
2\mathrm{Re}\int_{\Omega_t} P^\varepsilon(u^\varepsilon - v^\varepsilon) =& \partial_t(\overline{u^\varepsilon - v^\varepsilon}) \\
=& \left\| \sqrt{1+x}\partial_t(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(t) \\
&+ \left\|\partial_{x,z}(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(t) + \frac{1}{\varepsilon^2}\left\|u^\varepsilon - v^\varepsilon\right\|^2_{L^2_{x,z}}(t) \\
&- \left\| \sqrt{1+x}\partial_t(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(-T) \\
&- \left\|\partial_{x,z}(u^\varepsilon - v^\varepsilon)\right\|^2_{L^2_{x,z}}(-T) \\
&- \frac{1}{\varepsilon^2}\left\|u^\varepsilon - v^\varepsilon\right\|^2_{L^2_{x,z}}(-T).
\end{aligned} \tag{5.3}
$$

We already know, from construction of v^ε, that the norms of the traces at $t = -T$ go to zero with epsilon. Showing the integral above goes to zero suffices to prove (5.1).

This integral writes out as

$$
\begin{aligned}
&\int_{\Omega_t} \left(P^\varepsilon v^\varepsilon + F(\partial_t u^\varepsilon)\right)\partial_t(\overline{v^\varepsilon - u^\varepsilon}) \\
&= \int_{\Omega_t} \left(P^\varepsilon v^\varepsilon + \mathcal{F}\left(t,x,z,\varphi/\varepsilon\right)\right)\partial_t(\overline{v^\varepsilon - u^\varepsilon}) \\
&+ \int_{\Omega_t} \left(\mathcal{E}\left(t,x,z,\varphi/\varepsilon\right) - \mathcal{F}\left(t,x,z,\varphi/\varepsilon\right)\right)\partial_t(\overline{v^\varepsilon - u^\varepsilon}) \\
&+ \int_{\Omega_t} \left(F(\partial_t v^\varepsilon) - \mathcal{E}\left(t,x,z,\varphi/\varepsilon\right)\right)\partial_t(\overline{v^\varepsilon - u^\varepsilon}) \\
&- \int_{\Omega_t} \left(F(\partial_t v^\varepsilon) - F(\partial_t u^\varepsilon)\right)\partial_t(\overline{v^\varepsilon - u^\varepsilon}).
\end{aligned} \tag{5.4}
$$

The first term on the right-hand side is the product of a function tending strongly to zero in $L^{1+1/p}$ and of a function bounded in L^{p+1}; consequently, it tends to zero.

The second term corresponds to testing $\partial_t(v^\varepsilon - u^\varepsilon) \in L^{p+1}$ (which has null profiles) against a function in $L^{1+1/p}(\Omega_t \times \mathbb{T}^2)$ (smooth with respect to θ) evaluated at $\theta = \varphi/\varepsilon$; again, this term goes to zero.

For the third term, we can apply Proposition 3.1, which states $F(\partial_t v^\varepsilon) - \mathcal{E}(\varphi/\varepsilon)$ has no propagated oscillations; $\partial_t(v^\varepsilon - u^\varepsilon)$ is an admissible test function, so product tends to zero.

Finally, Property (2.3*(ii)*), *i.e.* dissipativity of the nonlinearity, ensures the inequality

$$\text{(5.5)} \qquad \mathrm{Re} \int_{\Omega_t} \left(F(\partial_t v^\varepsilon) - F(\partial_t u^\varepsilon)\right) \partial_t(\overline{v^\varepsilon - u^\varepsilon}) \geq c \, \|\partial_t(v^\varepsilon - u^\varepsilon)\|_{L^{p+1}}^{p+1} ,$$

so that this last quantity, when added to (5.3), is controlled by the three preceding terms, and gives (5.2) together with (5.1). □

Corollary 5.1. *The (whole) sequence u^ε of solutions of (0.1), (0.2) is equivalent in $H^1(\Omega_T) = H^1(]-T,T[\times]0,\underline{x}[\times B(0,\underline{r}))$ to $\varepsilon\mathcal{U}(t,x,z,\varphi/\varepsilon) = U_i(t,x,z)e^{i\varphi_i/\varepsilon}$ $+U_r(t,x,z)e^{i\varphi_r/\varepsilon}$, and the pair of profiles (U_i, U_r) is the unique solution of*

$$\text{(5.6)} \qquad \begin{cases} 2T_{\varphi_i}(\partial)U_i - P(\varphi_i)U_i + \dfrac{1}{i}E_i(U_i,U_r) = 0 \ \ on \ \mathcal{T}_i \\[2ex] 2T_{\varphi_r}(\partial)U_r - P(\varphi_r)U_r + \dfrac{1}{i}E_r(U_i,U_r) = 0 \ \ on \ \mathcal{T}_r \\[2ex] U_{i|_{t=-T}} = g \\[2ex] (U_i + U_r)_{|x=0} = 0. \end{cases}$$

Proof.

Nonlinear terms in the profiles equations are given by Proposition (3.1),*i)* –hence, existence of profiles shows existence of a solution to (5.6). Uniqueness is again a consequence of dissipativity ((2.3),*(ii)*), inherited by the system (5.6) (thanks to the fact that $(U,V) \mapsto \left(\oint_{\mathbb{T}^2} |(\partial_t\varphi_i)Ue^{-i\theta_i} + (\partial_t\varphi_r)Ve^{-i\theta_r}|^{p+1} d\theta\right)^{1/(p+1)}$ is a norm on $\mathbb{C}^2$, for example). □

ACKNOWLEDGMENTS

I would like to thank Guy Métivier for bringing this problem to my attention, and Christophe Cheverry for explanations on his point of view.

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