INTERSECTION NORMS ON SURFACES AND BIRKHOFF CROSS SECTIONS

MARCOS COSSARINI AND PIERRE DEHORNOY

Abstract. A divide is a finite collection of unoriented closed curves in generic position on a real orientable surface. Turaev associated to every divide a (semi-)norm on the first homology group of the surface. The unit ball of the dual norm is the convex hull of finitely many integer points. We give an interpretation of these points in terms of certain coorientations of the divide. Moreover a divide can be lifted to a link in the unit tangent bundle to the surface and, when the divide is formed of geodesic curves, its lift is made of periodic orbits of the geodesic flow. Our main result is a classification statement: when the surface is hyperbolic and the divide is made of geodesics, integer points in the interior of the unit ball of the dual norm classify isotopy classes of Birkhoff sections for the geodesic flow (on the unit tangent bundle to the surface) whose boundary is the symmetric lift of the divide. These Birkhoff sections also yield open-book decompositions of the unit tangent bundle. All results remain true when one replaces the hyperbolic surface by a 2-dimensional orientable hyperbolic orbifold.

Contents

Date: first version: April 21, 2016, last version: Sept 8, 2018.

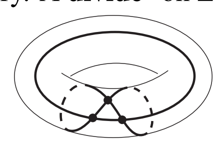
Introduction

This article has two goals. Firstly we study an elementary family of norms, that we call *intersection norms*, on the first homology group of a real orientable surface and we compute their unit balls. These norms we first defined on the torus by Alexander Schrijver [Sch93] and on higher-genus surfaces by Vladimir Turaev as a side remark [Tur02, Section 1.9]. They can be seen as elementary cousins of both the Thurston norm on the second homology of a 3-manifold [Thu86] and the Turaev norm on the first homology of a 2-complex [Tur02].

Secondly, we use these intersection norms to classify certain 2-dimensional objects in some 3-manifolds, namely, we classify up to isotopy Birkhoff cross sections with prescribed boundary for the geodesic flow in the unit tangent bundle to a negatively curved surface.

In this introduction, we give a rather detailed description of all results and of several ideas that underlie the paper. Our hope is that the experts get a good picture by only reading the introduction. The next sections contain the proofs, which are rather elementary (although not necessarily obvious).

Intersection norms. Let Σ be a real compact surface with empty boundary. A **divide**[1] on Σ is a finite collection of unoriented compact circles immersed in Σ and in general position (that is, all multiple points are double points where the two arcs are transverse). A divide can be seen as a graph embedded in Σ, with vertices of degree 4.



The geometric intersection number of two multi-curves is usually defined as the minimum of the number of intersection points of two representants of the free homotopy classes of the two multi-curves that have disjoint double points. We use here a variation of this notion where we fix one multi-curve (the divide), but minimize over the homology class of the second.

So, let γ denote a fixed divide on Σ. For α a path on Σ, the geometric intersection $i_\gamma(\alpha)$ is the minimal number of intersection points with γ of a multi-curve homotopic to α and in general position with respect to γ. Beware that this definition is not symmetric since the divide γ is fixed and not allowed to change in its homotopy class. Given a homology class a in $\mathrm{H}_1(\Sigma;\mathbb{Z})$, we then minimize the intersection number over all closed multi-curves in a in general position with respect to γ. This defines a function $x_\gamma : \mathrm{H}_1(\Sigma;\mathbb{Z}) \to \mathbb{N}$ by

$$x_\gamma(a) := \min_{[\alpha]=a} i_\gamma(\alpha) = \min_{\substack{[\alpha]=a \\ \alpha\pitchfork\gamma}} \Big|\{\alpha \cap \gamma\}\Big|.$$

Theorem A. [Sch93, Tur02] *Let Σ be a compact oriented surface and γ a divide on Σ. The function x_γ extends uniquely to a continuous function $x_\gamma : \mathrm{H}_1(\Sigma;\mathbb{R}) \to \mathbb{R}_+$, which is convex, and linear on rays from the origin. If, furthermore, the divide γ fills Σ in the sense that the complement $\Sigma \setminus \gamma$ is the union of topological discs, then x_γ is a norm (that is, it is different from 0 for non-zero vectors).*

[1]The notions of *divide curves* and *divide knots* were introduced by Nobert A'Campo [A'C98] who, along with Sabir M. Gusein-Zade, studied divides on the disc in the context of singularities [A'C75, GuZ74, GuZ77]. They were later generalized to arbitrary surfaces by Masaharu Ishikawa [Ish04]. This terminology is maybe not so common in the world of surface topologists, but since we will then lift our divides to the unit tangent bundle and consider fibrations of their complement over the circle, as did A'Campo, Gusein-Zade, and Ishikawa, it seems fair to use this terminology here.

The function x_γ is called the **intersection norm** associated to γ. Theorem A is an exact transposition of Bill Thurston's result defining a norm on the second homology group of a 3-manifold [Thu86, Thm 1][2]. Intersection norms are also related to Turaev's norms on the first cohomology of finite 2-complexes [Tur02] in the sense that, for some particular 2-complexes built out of a divide, the Turaev norm reduces to the intersection norm associated to the corresponding divide.

Like the Thurston and Turaev norms, the (semi-)norm x_γ has the property of taking integer values on integer classes. This implies (see [Thu86, Thm2]) that the unit ball, denoted here by B_{x_γ}, is very peculiar : it is a polyhedron with finitely many faces, which are all given by linear equations with integer coefficients. Equivalently, the closed unit ball of the dual norm on $\mathrm{H}^1(\Sigma; \mathbb{R})$, denoted here by $B^*_{x_\gamma}$, is the convex hull of finitely many integer points.[3]

In the case of the Thurston norm, some of these extremal points of the dual ball could be interpreted using Euler classes of fibrations on the circle [Thu86, Thm 3] or more generally of Reebless foliations, see [CC03, chap. 10]. An interpretation of all extremal points was then given in terms of Euler class of taut foliations by David Gabai (unpublished, see [Yaz16, Thm 3.3]), and in terms of flows by Danny Calegari [Cal06].

The analog statement for intersection norms is simpler. A **coorientation of an arc** α in the surface Σ is an orientation on the normal bundle of α in Σ, in other words, a continuous function η from the set of vectors transverse to α (tangent to Σ at points of α, but not tangent to α) to the discrete space $\{+1, -1\}$, such that $\eta(-v) = -\eta(v)$ for each v. Considering a divide γ as a graph whose vertices are the double-points and whose edges are the simple arcs connecting the double points, a **coorientation** of γ is the choice of a coorientation for every edge of γ. A given divide has only finitely many coorientations (the exact number is $2^{|E(\gamma)|}$). An example is depicted with light blue arrows on Figure 1 left. Given a coorientation η of γ, every double point p of γ has 4 adjacent edges. Travelling around p, each of these four edges is crossed either positively with respect to η, or negatively. A coorientation is **Eulerian** if, around every double point, there are two positively and two negatively cooriented edges. For example the coorientation depicted on Figure 1 is Eulerian. A coorientation η can be paired with an oriented curve α in general position using signed intersection. If η is Eulerian, it turns out that the pairing $\eta(\alpha)$ depends only on the homology class of α, so that an Eulerian coorientation η induces an integral cohomology class $[\eta] \in \mathrm{H}^1(\Sigma; \mathbb{Z})$.[4] One can wonder which classes are represented by such Eulerian coorientations.

A first remark is that, representing a class a by a curve α which minimises the geometric intersection with γ, one sees that $|\eta(a)|$ is not larger than $x_\gamma(a)$. A second remark is that the parity of $\eta(a)$ is fixed by γ : indeed, since all intersection points are counted with a coefficient ± 1, the parity of $\eta(\alpha)$ is determined by the parity of $i_\gamma(\alpha)$ and does not depend on η; since γ is a graph of even degree, the parity of $i_\gamma(\alpha)$ does not change if we replace α by a homologous curve. Our first result states that these restrictions are the only ones: the classes of the Eulerian coorientations are

[2]It is likely that Thurston actually thought about intersection norms on the 2-torus when proving [Thu86, Thm 6].

[3]Thurston's proof of this fact is not as natural as one might expect. In particular most available proofs rely on an induction on the dimension, which may seem superfluous. Mikael De La Salle recently gave a more direct proof [Sal16].

[4]According to the universal coefficients theorem for cohomology, for any abelian group G, the cohomology group $\mathrm{H}^1(\Sigma, G)$ is naturally isomorphic to the group $\mathrm{Hom}(\mathrm{H}_1(\Sigma; \mathbb{Z}); G)$ of group homomorphisms from $\mathrm{H}_1(\Sigma; \mathbb{Z})$ to G. However, in this article we can take this identification as the definition of $\mathrm{H}^1(\Sigma; G)$, since the usual definition of cohomology groups (as homology groups of the singular cochain complex) will not be used.

exactly the integer points in $B^*_{x_\gamma}$ that are **congruent to** $[\gamma]_2$ **mod 2** in the previous sense. More interestingly, the extremal points of $B^*_{x_\gamma}$ correspond to some Eulerian coorientations.

Theorem B. *Let Σ be a compact oriented surface and γ a divide on Σ. The dual unit ball $B^*_{x_\gamma}$ in $\mathrm{H}^1(\Sigma;\mathbb{R})$ is the convex hull of the points in $\mathrm{H}^1(\Sigma;\mathbb{Z})$ given by all Eulerian coorientations of γ. Equivalently, for every a in $\mathrm{H}_1(\Sigma;\mathbb{Z})$, we have*

$$x_\gamma(a) = \min_{[\alpha]=a} i_\gamma(\alpha) = \max_{\substack{\eta \text{ Eulerian} \\ \text{coor. of } \gamma}} \eta(a).$$

Moreover every point in $B^*_{x_\gamma} \cap \mathrm{H}^1(\Sigma;\mathbb{Z})$ that is congruent to $[\gamma]_2$ mod 2 is the class of some Eulerian coorientation (see Figure 1).

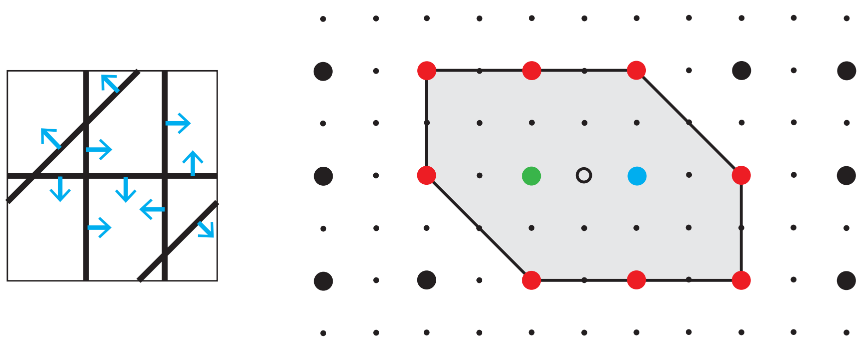


FIGURE 1. Illustration of Theorems B and D. On the left, a divide γ consisting of four geodesics on the torus T^2, and an Eulerian coorientation (blue arrows). Seen as a graph is has 5 vertices and 10 edges. On the right, the dual unit ball $B^*_{x_\gamma} \subset \mathrm{H}^1(\mathrm{T}^2,\mathbb{R})$ of the associated intersection norm. The empty circle denotes the origin. The big dots denote those classes in $\mathrm{H}^1(\mathrm{T}^2,\mathbb{Z})$ congruent to $[\gamma]_2$ mod 2. Among these classes, 10 (in blue, green an red) are in the dual unit ball $B^*_{x_\gamma}$ and correspond to all cohomology classes of Eulerian coorientations of γ (Theorem B). For example, the class corresponding to the blue coorientation is the blue point. The blue and green points lie in the interior of $B^*_{x_\gamma}$, hence describe the two isotopy classes of Birkhoff cross sections for φ_{geod} bounded by $-\overleftrightarrow{\gamma}$, while the 8 red points are on the boundary of $B^*_{x_\gamma}$ and describe isotopy classes of surfaces transverse to φ_{geod}, but not intersecting every orbit, and bounded by $-\overleftrightarrow{\gamma}$ (Theorem D).

Theorem B was actually proven in the case of the torus by Schrijver with different methods (see [Sch92, Thm 9]).

Not only does this result provide an interpretation of the integer points inside the unit ball of the dual norm, it also gives an effective way for computing the norm x_γ, since it reduces the minimisation over an infinite number of curves into a maximisation over a finite number of coorientations.

One can wonder what happens when the divide γ changes, in particular when one is given a Riemannian metric on the surface and one considers a sequence γ_n of divides whose lengths tend to infinity. It is not hard to see (Remark 9 below) that if (γ_n) is a sequence of closed geodesics that tend in the weak topology to the Liouville measure associated to the metric, then the norms x_{γ_n}, appropriately rescaled, tend to the stable norm associated to the metric.

Classification of Birkhoff cross sections for geodesic flows. We now turn to our main result. Let M be a compact, orientable smooth n-manifold without boundary, and let X be a non-singular vector field on M. In order to understand the dynamics of X it is desirable to find a **global cross section** for (M, X), namely a compact, orientable hypersurface S without boundary such that

- S is embedded in M,
- S is transverse to X,
- every orbit of X intersects S after a bounded time: we have $\phi^{[0,T]}(S) = M$ for some $T > 0$.

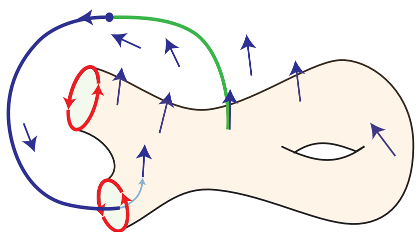
When such a section exists, there is a well-defined first-return map on S and the first-return time is bounded from above by definition and from below by compactness. In this case the manifold M fibers over the circle with fiber S. The pair (M, X) is homeomorphic to $(S\times[0,1]/_{(p,1)\sim(f(p),0)}, \tau_p\frac{d}{dz})$, where τ_p is the first-return time on S and $\frac{d}{dz}$ denotes the vector field tangent to the $[0,1]-$coordinate. The dynamics of X is then, up to the time-reparametrisation function τ, the dynamics of the first-return map f on S.

A standard topological argument shows that two global sections are isotopic if and only if they are homologous. Indeed the flow realizes the isotopy between such homologous sections (see for example the discussion at the beginning of [Thu86, Section 3]). Therefore questions of existence and classification of global sections are of algebraic nature. Indeed, a necessary and sufficient condition for a given homology class σ in $\mathrm{H}_2(M;\mathbb{Z})$ to contain a global section has been described by Sol Schwartzman [Sch57]. The quickier way to express it requires to consider measures as currents and to consider their homology classes: given a X-invariant probability measure μ, the associated 1-current c_μ is the linear functional on the space $\Omega^1(M)$ of 1-forms defined by $c_\mu(\phi) = \int_M \phi(X(p))d\mu(p)$. Since μ is invariant, c_μ is closed as a current, hence it induces a cohomology class $[c_\mu]$ in $\mathrm{H}^1(M)$. The latter is called the **Schwartzman asymptotic cycle** associated to μ. The condition of Schwartzman (restated later by Fuller, Sullivan, and Fried [Ful65, Sul76, Fri82]) is that X admits a global section if and only if the set of all Schwartzman asymptotic cycles lies in the half-space $\{\langle\sigma,\cdot\rangle > 0\}$, where $\langle\cdot,\cdot\rangle$ denotes the algebraic intersection pairing $\mathrm{H}_2(M;\mathbb{R})\times \mathrm{H}_1(M;\mathbb{R}) \to \mathbb{R}$. This implies for example that vector fields on $\mathbb{S}^3$ never admit global sections. Further results of Bill Thurston [Thu86] and David Fried [Fri82] imply that in the case of a pseudo-Anosov flow, the set of homology classes of global sections is an open cone with finitely many extremal rays.

For Σ a Riemann surface, the **unit tangent bundle** $\mathrm{T}^1\Sigma$ is the subset of $\mathrm{T}\Sigma$ of norm 1-vectors. It is a 3-manifold whose points are of the form (p, v) for p a point of Σ and v a tangent vector at p of norm 1. The **geodesic flow** φ_{geod} on $\mathrm{T}^1\Sigma$ is the flow whose orbits are lifts of geodesics: for g an arbitrary geodesic of Σ travelled at speed 1, the orbit of φ_{geod} going through $(g(0), \dot g(0))$ is given by $\varphi^t_{\mathrm{geod}}(g(0), \dot g(0)) = (g(t), \dot g(t))$. The geodesic flow on a negatively curved surface has been studied since Jacques Hadamard who remarked its sensibility to initial condition [Had1898]. It even became the paradigm of 3-dimensional chaotic systems when Dmitri Anosov showed its hyperbolic character [Ano67]. In general the geodesic flow depends heavily on the metric given on the surface. However Mikhail Gromov remarked [Gro76] that the geodesic flows corresponding to any two negatively curved metrics on a surface are actually *topologically conjugated*, meaning that there is a homeomorphism of the tangent bundle sending the oriented orbits the first on the oriented orbits of the second. This is a consequence of the structural stability of Anosov flows. Therefore, as long as we are only interested in the topological properties of the orbits, one can speak of *the* geodesic flow on a negatively curved surface.

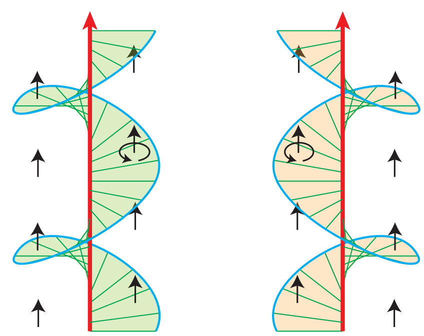
Since the antipodal map $(p, v) \mapsto (p, -v)$ preserves the geodesic flow, its set of asymptotic cycles is symmetric with respect to the origin in $\mathrm{H}_1(\mathrm{T}^1\Sigma; \mathbb{R})$, so that geodesic flows never admit global sections.

In order to make it useful, a relaxation of the notion of global section is desirable. A good solution was proposed by Henri Poincaré and George Birkhoff [Bir17]. For M a real compact, oriented 3-manifold and X a non-singular vector field on M, a **Birkhoff cross section** for (M, X) is compact orientable surface S *with* boundary such that



- the interior $int(S)$ is embedded in M,
- $int(S)$ is transverse to X,
- the boundary ∂S is tangent to X,
- every orbit of X intersects S after a bounded time: we have $\phi^{[0,T]}(S) = M$ for some $T > 0$.

The third condition implies that the boundary of S is the union of finitely many periodic orbits of X, possibly with multiplicities. The second and third conditions may look hard to realize at the same time, but actually it is not the case: in a flow box oriented so that the vector field is vertical, the general picture of a Birkhoff cross section near its boundary is that of one or several coaxial helicoidal staircases. Since the interior of a Birkhoff cross section S is transverse to X, it is cooriented by X. Since M is oriented, this induces an orientation on S, and in turn an orientation of ∂S. On the other hand, ∂S is a collection of periodic orbits of X, so it is oriented by X. For every component γ of ∂S, we can then define the multiplicity of γ as the algebraic number of times one sees γ in ∂S. In the whole text we restrict our attention to Birkhoff cross sections such that **every boundary component has multiplicity** -1. It would be more natural to consider Birkhoff sections such that every boundary component has multiplicity $+1$, but it turns out that there are none. On the other hand classifying mixed sections where the multiplicities are ± 1, or even general sections with arbitrary mulitplicities is an ongoing project.



negative *positive*

If the fourth condition in the definition is not satisfied, namely some orbits do not intersect the considered surface, we simply speak of a **transverse surface**.

It turns out that Birkhoff cross sections exist much more often than global sections. In particular Poincaré noticed that the geodesic flow on a sphere often admits an annulus as Birkhoff cross section. This remark was generalized by Birkhoff who gave a family of Birkhoff cross sections for the geodesic flow [Bir17] (popularized in [Fri83]). Birkhoff's example was then given another presentation by Marco Brunella [Bru94, Description 2]. Our second result is a generalization of Birkhoff's and Brunella's examples.

For γ an unoriented collection of geodesics on a surface Σ, we denote by $\overset{\leftrightarrow}{\gamma}$ the **antithetic lift** of γ in $\mathrm{T}^1\Sigma$, that is, the set of unit tangent vectors based on γ and tangent to γ. The set $\overset{\leftrightarrow}{\gamma}$ forms a link that is invariant by the involution $(p, v) \mapsto (p, -v)$. It is the union of $2|\gamma|$ periodic orbits of φ_{geod}, each component being oriented by the flow. The antithetic lift of a collection of geodesics was actually already used by Birkhoff [Bir17] who (in modern terms) described a fibration of the complement over the circle. It was later called a *divide link* by A'Campo who gave another description of the same fibration (and so did later Giroux [Gir91, Example I.4.8]). Another fibration was sketched by Brunella [Bru94, Description 2]. Here we use Eulerian coorientations in order to describe *many* fibrations :

Theorem C. *Let Σ be a compact oriented Riemann surface and γ a finite collection of closed geodesics on Σ. There is canonical a map S^{BB} (for Birkhoff-Brunella) that associates to every Eulerian coorientation η of γ an oriented surface $S^{BB}(\eta)$ in $\mathrm{T}^1\Sigma$ which is positively transverse to the geodesic flow and whose oriented boundary is $-\overleftrightarrow{\gamma}$. For every η, the Euler characteristic of $S^{BB}(\eta)$ is minus twice the number of double points of γ.*

If two Eulerian coorientations η_1, η_2 of γ are cohomologous and if the associated surfaces $S^{BB}(\eta_1)$ and $S^{BB}(\eta_2)$ are Birkhoff sections for the geodesic flow, then they are isotopic (fixing their common boundary).

It turns out that this new construction gives a description of *all* isotopy classes of Birkhoff cross sections with oriented boundary $-\overleftrightarrow{\gamma}$, instead of one with the previously known constructions. Recall that a collection of curves is **filling** if its complement in the surface is a union of topological discs.

Theorem D. *Let Σ be a surface of genus at least 2 with a negatively curved metric. Let γ be a finite collection of closed geodesics on Σ.*

If γ is filling, then the map $[\eta] \mapsto \{S^{BB}(\eta)\}$ is a one-to-one correspondance between integer points in the open unit ball $\mathrm{int}(B^*_{x_\gamma})$ congruent to $[\gamma]_2 \bmod 2$ and isotopy classes of Birkhoff cross sections for φ_{geod} in $\mathrm{T}^1\Sigma$ with boundary $-\overleftrightarrow{\gamma}$.

If γ is not filling, then there is no surface bounded by $-\overleftrightarrow{\gamma}$ and transverse to the geodesic flow.

Theorem D implies that the collection $\overleftrightarrow{\gamma}$ bounds a Birkhoff cross section with all mutiplicities equal to -1 if and only if the polyhedron $B^*_{x_\gamma}$ contains an integer point congruent to $[\gamma]_2 \bmod 2$ in its interior. This is the case for most choices of γ, but not for all. For example, if there is a closed curve that intersects γ only once, then $\overleftrightarrow{\gamma}$ does not bound a Birkhoff cross section for φ_{geod}, although the interior of $B^*_{x_\gamma}$ may not be empty.

Remark 1. The case of the torus with a flat metric is not covered by Theorem D. In this case, the fact that the unit tangent bundle $\mathrm{T}^1\mathbb{T}^2$ is trivial allows to cut-and-glue horizontal tori to Birkhoff cross sections, so that there are in general infinitely many isotopy classes with a given boudary. However, up to this additional operation, there are still only finitely many classes. These have been classified in a previous work by the second author [Deh15a, Thm 3.12]. The statement is similar, namely equivalence classes of Birkhoff sections are classified by point in the interior of a certain polygon with integral vertices. The statement if even more general since, in this restricted case of the torus, there is no assumption that the boundary of the section is antithetic. One could recover this earlier result in the antithetic case by a proof very similar to that of Theorem D.

Remark 2. A Birkhoff surface for the geodesic flow in $\mathrm{T}^1\Sigma$ bounded by $\overleftrightarrow{\gamma}$ is a global section for the restriction of the flow to $\mathrm{T}^1\Sigma \setminus \overleftrightarrow{\gamma}$. The assumption that the oriented boundary is $-\overleftrightarrow{\gamma}$ can be seen as a restriction on the homology class of the section : it has to lie in a certain affine subspace of $\mathrm{H}_2(\mathrm{T}^1\Sigma, \overleftrightarrow{\gamma}; \mathbb{Z})$ (see Section 3.c). On the other hand, as explained before, the geodesic flow on $\mathrm{T}^1\Sigma$ for Σ a hyperbolic surface is of Anosov type. Its restriction to $\mathrm{T}^1\Sigma \setminus \overleftrightarrow{\gamma}$ is then of pseudo-Anosov type, with singularities along the removed orbits. Thurston fibered faces Theory [Thu86, Section 3] then says that the homology classes of global sections to such a flow (and therefore also the isotopy classes) is a cone in $\mathrm{H}_2(\mathrm{T}^1\Sigma, \overleftrightarrow{\gamma}; \mathbb{R})$ whose extremal rays are directed by integral vectors. David Fried [Fri82] gives an algorithm to explicitly compute these vectors, starting from a Markov partition of the flow. So one directly deduces that the set of Birkhoff cross sections with prescribed boundary is given by the intersection of a cone with an affine plane: it is a polyhedron. However, the determination of this polyhedron using Fried's approach requires an explicit Markov partition for the geodesic flow on $\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}$, which does not exist yet. Even if it did, it is not clear that this would yield an explicit and elementary description of the polyhedron for every collection γ. Theorem D can be rephrased by saying the Thurston's fibered face corresponding to the geodesic flow on $\mathrm{T}^1\Sigma \setminus \overleftrightarrow{\gamma}$ is a multiple of $B^*_{x_\gamma}$. From this perspective, the interest of our paper lies in the elementary and explicit characters of all constructions.

Remark 3. One may wonder how general Theorem D is, namely whether one can hope for an analog theorem for any (transitive) Anosov flow. The previous remark extends to this context : the set of Birkhoff sections up to isotopy fixing the boundary is described by the integral points inside a certain polyhedron. However we do not know how to describe this polyhedron in general. It seems to be related to linking numbers of periodic orbits of the flow. However linking numbers are only defined for null-homologous links. Ghys [Ghy09] proved that Gauss linking forms describe all linking numbers between periodic orbits (and even invariant measures) for a vector field in a homology sphere. Moreover he showed how to use these Gauss forms to decide whether all finite collections of periodic orbits bound a Birkhoff sections (the so-called left- or right-handed flows). Probably one should then first extend the concept of Gauss linking forms to manifolds that are not rational homology spheres, and see how this helps defining linking of periodic orbits and more generally of invariant measures. Then one could hope that these generalized linking describe exactly the homological information needed to apply Schwartzman's criterion, as we will do in Section 3.

Extension to 2-dimensional orbifolds. The results of this paper can be generalized in the following sense. Instead of considering only orientable surfaces, one can consider orientable 2-dimensional orbifolds, as introduced by Thurston [Thu80]. Such a 2-orbifold O is described by an orientable topological surface Σ_O and charts that are local homeomorphisms $\mathbb{R}^2/(\mathbb{Z}/k\mathbb{Z}) \to \Sigma_O$, where $\mathbb{Z}/k\mathbb{Z}$ acts by rotation on $\mathbb{R}^2$.

There are several possible definitions for the homology of an orbifold that yield different spaces. The one that is useful here is the most elementary: we define $\mathrm{H}_i(O; \mathbb{R})$ to be the space $\mathrm{H}_i(\Sigma_O; \mathbb{R})$. In this context the definition of intersection norms extends trivially. Theorems A and B still hold.

Now the unit tangent bundle $\mathrm{T}^1 O$ is 3-manifold that is a Seifert fibered space over Σ_O. The geodesic flow is well-defined on $\mathrm{T}^1 O$ and it is still of Anosov type. Theorem C extends directly in this context. Concerning Theorem D, it has to be modified for taking into account orbifolds that are homology spheres (a case that does not occur with hyperbolic surfaces).

Theorem E. *Let O be a hyperbolic orientable 2-dimensional orbifold. Let γ be a finite collection of closed geodesics on Σ.*

If Σ_O is a sphere, then T^1O is a rational homology sphere. In this case the lift $\overleftrightarrow{\gamma}$ bounds a Birkhoff section for the geodesic flow in T^1O if and only if γ is filling in Σ_O. If it exists, the Birkhoff section is unique up to isotopy fixing the boundary.

If Σ_O is not a sphere and if γ is filling, then the map $[\eta] \mapsto \{S^{BB}(\eta)\}$ is a one-to-one correspondance between integer points in the open unit ball $\mathrm{int}(B^*_{x_\gamma})$ *congruent to* $[\gamma]_2 \bmod 2$ *and isotopy classes of Birkhoff cross sections for φ_{geod} in $\mathrm{T}^1\Sigma$ with boundary $-\overleftrightarrow{\gamma}$.*

If Σ_O is not a sphere and γ is not filling, then there is no surface bounded by $-\overleftrightarrow{\gamma}$ and transverse to the geodesic flow.

A particular case is when O is a triangular orbifold (that is, a sphere with three conic points). In this case every collection γ of closed geodesic is filling, hence its lift $\overleftrightarrow{\gamma}$ bounds a Birkhoff section (see [Deh17] for another proof of this fact).

Acknowledgments. Pierre D. thanks Étienne Ghys and Adrien Boulanger for related discussions, and Elena Kudryavtseva who initiated this article by asking several questions about Birkhoff cross sections. The authors thank and anonymous referee for useful suggestions, in particular the extension to orbifolds.

1. Intersection norms

In this section we define intersection norms and prove Theorem A[5]. All statements are elementary transcriptions of results of Thurston [Thu86] to the 2-dimensional context of a surface with a divide on it.

For the whole section we fix a compact surface Σ of genus g without boundary, and a divide γ on Σ. Given a closed multi-curve α transverse to γ and such that the multiple points of α and γ are disjoint, there is a finite number of intersection points between α and γ. What we do here is to minimize this finite number over the homology class of α:

Definition 4. (see Figure 2) A divide γ being fixed on Σ, the function $x_\gamma : \mathrm{H}_1(\Sigma;\mathbb{Z}) \to \mathbb{N}$ is defined by

$$x_\gamma(a) := \min_{[\alpha]=a} i_\gamma(\alpha) = \min_{\substack{[\alpha]=a \\ \alpha \pitchfork \gamma}} |\,\alpha \cap \gamma\,|.$$

Since the number of intersection points is an integer, the lower bound is always realized and x_γ takes integral values. A multi-curve that realizes the minimum is declared **x_γ-minimizing**.

The function x_γ has two properties that will turn it into a semi-norm, namely it is linear on rays and convex. To prove the first point we need an elementary remark. Let us recall that a multi-curve is **simple** if it has no double points, that is, if it is an embedding.

Lemma 5 (simplification)**.** *For every divide γ in Σ and for every class a in* $\mathrm{H}_1(\Sigma;\mathbb{Z})$*, there exists a x_γ-minimizing multi-curve in a that is simple.*

Proof. Starting from an arbitrary α_0 in a that is minimizing, we can smooth the double points of α_0 away from γ

[5] Although Theorem A is not new, Turaev only sketched a proof [Tur02]. We include a detailed version, as it is short and elementary.

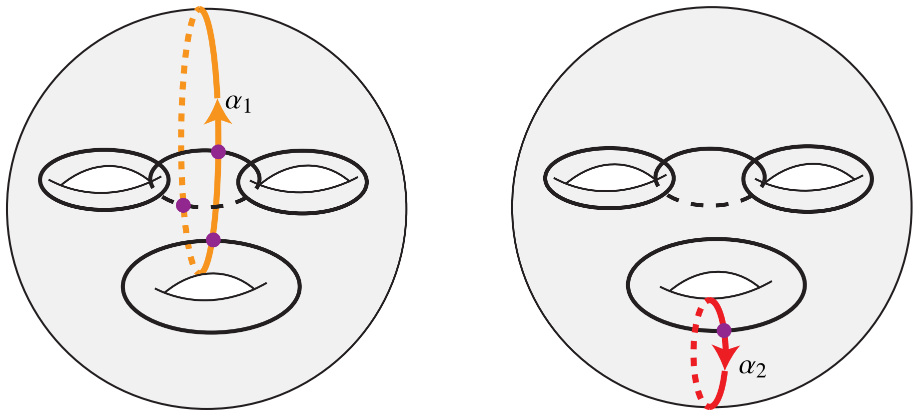
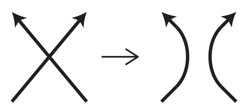


FIGURE 2. A genus 3 surface with a divide γ made of four closed curves (black). On the left the curve α_1 (orange and bold) is transverse to γ and intersects it three times. On the right α_2 (red) is homologous to α_1 since their difference bounds a subsurface, namely the right hemisurface. The curve α_2 intersects γ only once. This number cannot be reduced to 0 in the same homology class, hence α_2 is x_γ-minimizing and we have $x_\gamma([\alpha_1]) = x_\gamma([\alpha_2]) = i_\gamma(\alpha_2) = 1$.



thus turning α_0 into a new multi-curve α which is simple. The two multi-curves are in general not homotopic, but they are homologous, hence the result. □

Lemma 6 (linearity on rays)**.** *For every a in $\mathrm{H}_1(\Sigma;\mathbb{Z})$ and for all $n \in \mathbb{Z}$ one has*

$$x_\gamma(n \cdot a) = |n|\, x_\gamma(a).$$

Proof. Since one does not change the number of intersection points by reversing the orientation of a curve, one has $x_\gamma(-a) = x_\gamma(a)$.

We then assume $n > 0$. Given $a \in \mathrm{H}_1(\Sigma;\mathbb{Z})$, consider a minimizing multi-curve α in a. Since n parallel copies of α intersect γ in $n\, x_\gamma(a)$ points, we have $x_\gamma(n \cdot a) \leq n\, x_\gamma(a)$.

For the other inequality, consider a multi-curve $\alpha^{(n)}$ that minimizes $x_\gamma(n \cdot a)$. By the simplification Lemma 5, we can suppose $\alpha^{(n)}$ simple. Since $\alpha^{(n)}$ is homologous to n copies of α, its number of crossings (counted with signs) with any generic loop is a multiple of n. So, starting from an arbitrary region in the complement $\Sigma \setminus \alpha^{(n)}$ that we label with the number $0 \in \mathbb{Z}/n\mathbb{Z}$, we can label the other regions with the numbers $0, 1, \ldots, n-1 \in \mathbb{Z}/n\mathbb{Z}$ in such a way that the color increases by 1 mod n when one crosses an arc of $\alpha^{(n)}$ positively (from right to left). Therefore $\alpha^{(n)}$ is the union of the n simple multi-curves α_i, each such α_i consisting on the components of $\alpha^{(n)}$ that run leaving the regions labelled i on their right, and the regions labeled $i+1$ on their left. Since they pairwise bound a subsurface of Σ, all of these n multi-curves are homologous. These implies that $\alpha^{(n)}$ is homologous to n copies of any α_i. Since it is also homologous to n copies of α, and $\mathrm{H}_1(\Sigma;\mathbb{Z})$ has no torsion, we conclude that each α_i is homologous to α. Then it has at least $x_\gamma(a)$

intersections with γ, which implies that $\alpha^{(n)}$ has at least $nx_\gamma(a)$ intersections with γ, thus proving the inequality $x_\gamma(n\cdot a) \geq n\, x_\gamma(a)$. □

Lemma 7 (convexity). *For every a, b in $\mathrm{H}_1(\Sigma;\mathbb{Z})$ one has*

$$x_\gamma(a+b) \leq x_\gamma(a) + x_\gamma(b).$$

Proof. The union of two multi-curves that realize $x_\gamma(a)$ and $x_\gamma(b)$ crosses γ in $x_\gamma(a)+x_\gamma(b)$ points, giving $x_\gamma(a+b) \leq x_\gamma(a) + x_\gamma(b)$. □

Proof of Theorem A. Every class in $\mathrm{H}_1(\Sigma;\mathbb{Q})$ is of the form $\frac{1}{q}\,a$ with $a \in \mathrm{H}_1(\Sigma;\mathbb{Z})$ and $q \in \mathbb{N}^*$. We then define $x_\gamma(\frac{1}{q}\,a)$ as $\frac{1}{q}x_\gamma(a)$, and the linearity on rays (Lemma 6) ensures that this definition does not depend on the choice of q and a and that it yields a well-defined function (also denoted by x_γ) from $\mathrm{H}_1(\Sigma;\mathbb{Q})$ to $\mathbb{Q}_+$ that is linear on rays. Now convexity (Lemma 7) implies that this function extends uniquely to a convex function from $\mathrm{H}_1(\Sigma;\mathbb{R})$ to $\mathbb{R}_+$. Indeed the extension can be defined by taking the convex hull of the epigraph (what lies above the graph), or, more precisely, the supremum of the linear functions that are smaller than x_γ. The extension (still denoted by x_γ) is also convex and linear on rays, hence it is a semi-norm on $\mathrm{H}_1(\Sigma;\mathbb{R})$.

If the collection γ decomposes Σ into simply-connected regions, then γ intersects every curve that is not null-homotopic at least once. This implies that x_γ is at least 1 on non-zero integral homology classes, hence x_γ is positive on $\mathrm{H}_1(\Sigma;\mathbb{R}) \setminus \{0\}$. Therefore x_γ is a norm. □

Remark 8. One can easily extend the notion of intersection norms to surfaces with boundary, by allowing divides to contain arcs with endpoints on the boundary of the surface (as did A'Campo originally). One then obtains two norms on $\mathrm{H}_1(\Sigma;\mathbb{R})$ and $\mathrm{H}_1(\Sigma,\partial\Sigma;\mathbb{R})$, depending whether one considers absolute or relative homology classes. Theorem A also holds in the second context.

Remark 9. One can wonder how the intersection norms compare with other known norms on the first homology of a surface. For example the *stable norm* x_g is defined in terms of a metric g by $x_g(a) = \liminf_{n\to\infty} \min_{\alpha^{(n)}\in na} g(\alpha^{(n)})/n$. On a surface the stabilisation is not necessary, so that $x_g(a) = \min_{\alpha\in a} g(\alpha)$. One can check that if $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of filling geodesics for g, meaning that the sequence of invariant measures on $\mathrm{T}^1\Sigma$ that are concentrated on the lift $\vec{\gamma}_k$ tends in the weak sense to the Liouville measure defined by g on $\mathrm{T}^1\Sigma$, then the rescaled norms $\frac{1}{g(\gamma_k)}x_{\gamma_n}$ tend to the stable norm of g. Equivalently, the rescaled unit balls $g(\gamma_k)B_{x_{\gamma_k}}$ tend to the unit ball of the stable norm.

2. Unit balls and coorientations

For the whole section we fix a surface Σ of genus at least 1 and a divide γ on it. The norm x_γ defined in the previous section has a very peculiar property: it takes integral values on integral classes. This property is shared for example by the ℓ_1- and ℓ_∞-norms on $\mathbb{R}^d$, and their unit balls are polyhedral. Moreover all faces of these unit balls are of the form $\{(x_1,\dots,x_d) \mid \sum x_i y_i = 1\}$ for some $(y_1,\dots y_d) \in \mathbb{Z}^d$. This is not a coincidence.

Theorem 10 (Thm 2 of [Thu86]). *If N is a seminorm on $\mathbb{R}^d$ taking integral values on $\mathbb{Z}^d$, then there is a finite subset F of $\mathbb{Z}^d$ such that $N(x) = \max_{y\in F}\langle x, y\rangle$ for all x in $\mathbb{R}^d$.*

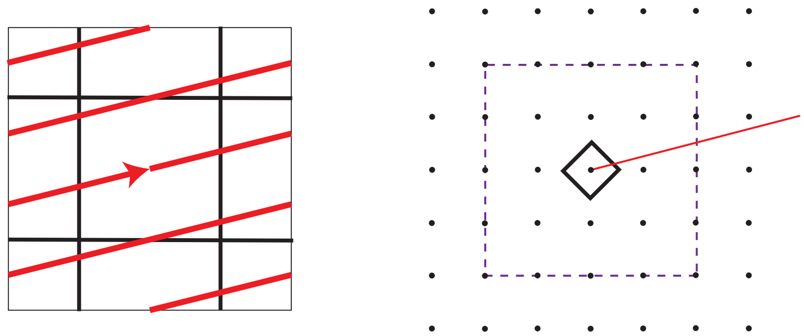


FIGURE 3. A torus with a collection γ (black) made of four curves, two vertical and two horizontal. The curve α (red and bold) intersects γ in 10 points. It is the best for a curve whose homology class is $(4, 1)$. The norm x_γ is actually given by $x_\gamma((p, q)) = 2|p| + 2|q|$ in the canonical coordinates. The unit balls B_{x_γ} (bold) and $B^*_{x_\gamma}$ (dotted) are shown on the right. The faces of B_{x_γ} are defined by integral equations while the vertices of $B^*_{x_\gamma}$ belong to $\mathbb{Z}^2$, as predicted by Thurston's result.

Let us recall that a norm N on a vector space induces a dual norm N^* on the dual by $N^*(y) = \max_{x\in B}\langle x, y\rangle$ where B denotes the unit ball of N. Thurston's result can be restated by saying that the unit ball of the dual norm is the convex hull of finitely many integral points.

In our context, denote by x^*_γ the norm on $\mathrm{H}_1(\Sigma;\mathbb{R})^* \simeq \mathrm{H}^1(\Sigma;\mathbb{R})$ dual to x_γ, by B_{x_γ} the unit ball of x_γ, and by $B^*_{x_\gamma}$ the unit ball of x^*_γ. A direct consequence of Theorem 10 is

Corollary 11 (see Figure 3)**.** *For Σ a compact surface and γ a divide on it, the unit ball $B^*_{x_\gamma}$ is the convex hull in $\mathrm{H}^1(\Sigma;\mathbb{R})$ of finitely many points that belong to $\mathrm{H}^1(\Sigma;\mathbb{Z})$.*

A natural question is whether the vertices of $B^*_{x_\gamma}$ (or equivalently the faces of B_{x_γ}) have a nice interpretation. For example in the context of the Thurston norm, the vertices correspond to the Euler classes of certain taut foliations (Gabai, see [Yaz16, Thm 3.3]), or of certain vector fields [Fri79, Mos92, Cal06]. Here we also have such an interpretation in terms of Eulerian coorientations (Theorem B), and it is the goal of this section to prove it.

2.a. **Coorientations and signed intersections.** Recall that the divide γ is assumed to be self-transversely immersed with only double points. We denote by $V(\gamma)$ the set of double points, that we call **vertices** of γ. Consequently we denote by $E(\gamma)$ the set of connected components of $\gamma\setminus V(\gamma)$, that we call **edges** of γ. This turns γ into a graph of degree 4 embedded in Σ.

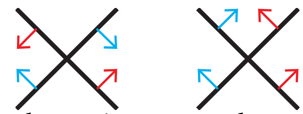
Definition 12. For e an edge of γ, a **coorientation on** e is the choice of one of the two possible ways of crossing e: from left to right, or from right to left. A **coorientation on** γ is a choice of a coorientation for every edge in $E(\gamma)$.

There are $2^{|E(\gamma)|}$ coorientations of γ. A coorientation η may be evaluated on an oriented immersed curve α transverse to γ: one counts +1 for every intersection point of α with γ if the orientation of α coincides with the coorientation of the edge, and -1 if the orientations disagree. Denoting by $\eta(\alpha)$ this intersection pairing, one sees that $\eta(\alpha)$ is an integer satisfying $|\eta(\alpha)| \leq i_\gamma(\alpha)$.

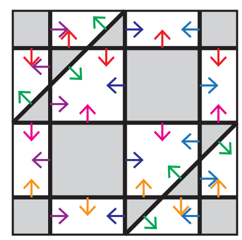
2.b. **Eulerian coorientations.** The question now is whether the above inequality may be turned into an equality for x_γ-minimizing curves, and whether $\eta(\alpha)$ may depend only on the homology class of α so that one can compute η on a single representative. Both questions admit a positive answer if we restrict to some special coorientations, called Eulerian.

Definition 13. A coorientation on γ is **Eulerian** (or **closed**) if it vanishes on boundaries, that is, if for every region $D \subset \Sigma$ whose boundary is transverse to γ one has $\eta(\partial D) = 0$. The set of all global Eulerian coorientations is denoted by $\mathcal{E}ulCo(\gamma)$.

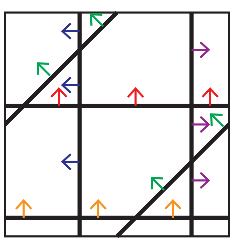
The set $\mathcal{E}ulCo(\gamma)$ is an affine subspace of $\{-,+\}^{E(\gamma)}$. Actually the closing condition is local: for η to be Eulerian it is enough that around every vertex of γ there are as many positively cooriented edges than negatively cooriented. Hence, up to rotation, there are only two types (locally, at each vertex) of Eulerian coorientations:



When one travels straight along γ and encounters a vertex of the first type the coorientation changes, hence the name. For the second type on the other hand, it is as if the coorientation does not see the vertex.



Example 14. If $[\gamma]_2 \in \mathrm{H}^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is zero (meaning that every closed curve intersects γ an even number of times), then the regions of $\Sigma \setminus \gamma$ can be colored in black and white in such a way that adjacent regions have different colors. In this case we can coorient all edges toward the white regions. The obtained global coorientation is Eulerian, all double points being alternating.



Example 15. There always exist global Eulerian coorientations, even when $[\gamma]_2 \in \mathrm{H}^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is not zero. Indeed if the divide γ is the immersion of c curves, it admits at least the 2^c Eulerian coorientations obtained by choosing a coorientation for every component and having only non-alternating vertices.

Lemma 16. *If η is an Eulerian coorientation of γ, then for every multi-curve α, the pairing $\eta(\alpha)$ depends only of the homology class $[\alpha] \in \mathrm{H}_1(\Sigma; \mathbb{Z})$.*

Proof. If two multi-curves α, α' are homologous, then their difference bounds a singular subsurface in Σ. Definition 13 implies that the pairing of the boundary of the image of a surface with an Eulerian coorientation is zero. Hence $\eta(\alpha - \alpha') = 0$, so $\eta(\alpha) = \eta(\alpha')$. □

Lemma 16 implies that every Eulerian coorientation η induces a well-defined cohomology class $[\eta]$ in $\mathrm{H}^1(\Sigma; \mathbb{Z})$. We denote by $[\mathcal{E}ulCo(\gamma)]$ the subset of $\mathrm{H}^1(\Sigma; \mathbb{Z})$ formed by the classes of global Eulerian coorientations on γ. Note that the class of an Eulerian coorientation is easily computed since it is enough to evaluate its pairing with $2g$ curves that generate the homology of Σ. Moreover, Eulerian coorientations give lower bounds on x_γ:

Lemma 17 (Eulerian orientations are in the dual ball)**.** *For every Eulerian coorientation η of γ and for every a in $\mathrm{H}_1(\Sigma; \mathbb{Z})$, we have $\eta(a) \leq x_\gamma(a)$.*

Proof. Let α be a curve in a that realizes $x_\gamma(a)$. Then $\eta(\alpha)$ counts every intersection point of α and γ with a coefficient ± 1, while $x_\gamma(a)$ counts all these intersection points with a coefficient $+1$, hence the inequality. □

2.c. **Eikonal functions.** An Eulerian coorientation is analog to a certain 1-form on the graph dual to γ in Σ. As such it can be seen as the differential of a certain multi-valued function or, equivalently, as the projection to Σ of the differential of a function defined on the universal cover of Σ. This approach is useful for constructing Eulerian coorientations with a prescribed cohomology class, as is needed for proving Theorem B.

We consider for every path α in Σ, the number $\mathrm{Len}_\gamma(\alpha)$ of intersections with γ; this notion of length determines a (not positive definite) distance d_γ on Σ. Two points x, y are **neighbors** if $d_\gamma(x, y) = 1$.

Choose a basepoint $p_0 \in \Sigma$ and construct the universal cover $\tilde\Sigma$ of Σ as the set of homotopy classes $x = \{\alpha\}$ of curves α that begin at p_0 and end at any point $p =: \pi\{\alpha\}$; in particular, let x_0 be the class of the constant curve.

Lift γ to a divide $\tilde\gamma$ in $\tilde\Sigma$. Any closed curve β in Σ based at p_0 determines a deck transformation $T_{\{\beta\}} : x = \{\alpha\} \in \tilde\Sigma \mapsto x' = \{\beta \cdot \alpha\}$, where $\beta \cdot \alpha$ is the concatenation of α after β. The deck transformation $T_{\{\beta\}}$ preserves the distance $d_{\tilde\gamma}$.

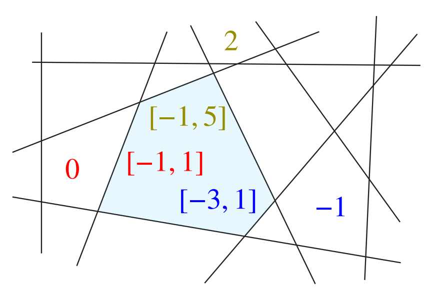
An Eulerian coorientation η on M gives rise to a function $f_\eta : \tilde\Sigma \setminus \tilde\gamma \to \mathbb{Z}$ by the formula $f_\eta\{\alpha\} = \int_\alpha \eta$ consisting in counting with signs the intersection points of α with γ. This function is **eikonal**, meaning that $|f_\eta(x) - f_\eta(x')| = 1$ whenever $d_\gamma(x, x') = 1$ (and also $f_\eta(x) = f_\eta(x')$ whenever $d_\gamma(x, x') = 0$). It is also $[\eta]$-equivariant, meaning that $x' = T_{\{\beta\}}x$ implies $f_\eta(x') - f_\eta(x) = [\eta]([\beta])$.

Definition 18. A function f defined on a subset D of $\tilde\Sigma \setminus \tilde\gamma$ is said **pre-eikonal** if it satisfies $|f(y') - f(y)| \leq d_{\tilde\gamma}(y', y)$ and $f(y') - f(y) \equiv d_{\tilde\gamma}(y', y) \bmod 2$ for every $y, y' \in \tilde\Sigma$.

Observe that a function defined on all of $\tilde\Sigma \setminus \tilde\gamma$ is eikonal if and only if it is pre-eikonal.

Lemma 19 (Extension)**.** *Every pre-eikonal function $f : D \to \mathbb{Z}$ extends to an eikonal function $\overline{f} : \tilde\Sigma \setminus \tilde\gamma \to \mathbb{Z}$.*

Proof. (Based on footnote of [Whi34].) To define $\overline{f}(x)$, we first observe that it must lie in the interval $[f(y) - d_{\tilde\gamma}(x, y), f(y) + d_{\tilde\gamma}(x, y)]$ for every $y \in D$.



So we can define $\overline{f}(x)$ as the highest common point $\overline{f}(x) := \min_{y \in D} f(y) + d_{\tilde\gamma}(x, y)$ of these intervals, after checking that they do have a common point, because they intersect pairwise. And indeed they do, for otherwise there would exist two points y, y' in S such that $f(y) + d_{\tilde\gamma}(x, y) < f(y') - d_{\tilde\gamma}(x, y')$, which implies $f(y') - f(y) > d_{\tilde\gamma}(x, y) + d_{\tilde\gamma}(x, z) \geq d_{\tilde\gamma}(y, y')$, a contraction to pre-eikonality.

We claim that the extension $\overline{f}$ is pre-eikonal (and therefore eikonal, since it is defined in all of $\Sigma \setminus \gamma$).

Indeed, to prove that $|f(x') - f(x)| \leq d_{\tilde\gamma}(x, x')$, it is enough to check that

$$\left|(f(y) + d_{\tilde\gamma}(x', y)) - (f(y) + d_{\tilde\gamma}(x, y))\right| \leq d_{\tilde\gamma}(x, x')$$

for each y, which follows from the triangle inequality in the form $|d_{\tilde\gamma}(x', y) - d_{\tilde\gamma}(x, y)| \le d_{\tilde\gamma}(x, x')$. To prove that $f(x') - f(x) \equiv d_{\tilde\gamma}(x, x')$ modulo 2, we write

$$\begin{aligned} f(x') - f(x) &= (f(y') + d_{\tilde\gamma}(x', y')) - (f(y) + d_{\tilde\gamma}(x, y)) \text{ for certain } y, y' \in D \\ &\equiv d_{\tilde\gamma}(y, y') + d_{\tilde\gamma}(x', y') - d_{\tilde\gamma}(x, y) \bmod 2 \text{ since } f \text{ is pre-eikonal} \\ &\equiv d_{\tilde\gamma}(y, y') + d_{\tilde\gamma}(x', y') + d_{\tilde\gamma}(x, y) \text{ since plus and minus coincide mod 2} \\ &\equiv d_{\tilde\gamma}(x, x') \text{ since homotopic paths have congruent length mod 2.} \end{aligned}$$

□

Note that a pre-eikonal function admits in general several eikonal extensions. The one we picked in the proof is the *highest* one. It has the advantage of admitting a simple definition.

2.d. **Proof of Theorem B.** By Lemma 17, for every Eulerian coorientation η, the class $[\eta]$ belongs to $B^*_{x_\gamma}$, so we have $[\mathcal{E}ulCo(\gamma)] \subset B^*_{x_\gamma}$. Conversely, by Thurston's Theorem, extremal points of $B^*_{x_\gamma}$ belong to $\mathrm{H}^1(\Sigma; \mathbb{Z})$. Therefore it is enough to show that for every integer point n in the closed ball $B^*_{x_\gamma} \subset \mathrm{H}^1(\Sigma; \mathbb{R})$ that is congruent to $[\gamma]_2$ mod 2 there exists an Eulerian coorientation whose cohomology class is n. This is the content of Lemma 22 below.

Choose a basepoint p_0 in $\tilde\Sigma \setminus \tilde\gamma$. Denote by D its orbit under the deck action. For every closed curve α based at p_0 and for $y = \{\alpha\} \in D$ we set $f_n(y) := n([\alpha])$.

Lemma 20. *The function $f_n : D \to \mathbb{Z}$ is a n-equivariant pre-eikonal function.*

Proof. Let y, y' be two points in D that we write as $y = \{\alpha\}$ and $y' = T_{\{\beta\}}(y) = \{\beta\cdot\alpha\}$ for some closed curves α, β based at p_0. By definition we have $f_n(y') - f_n(y) = n([\beta\cdot\alpha]) - n([\alpha]) = n([\beta])$, so f_n is n-equivariant. Furthermore, if we choose β in the form $\beta = \alpha\cdot\beta'\cdot\alpha^{-1}$ with β' of minimum length, that is, $\mathrm{Len}_\gamma(\beta') = d_{\tilde\gamma}(y, y')$, we see that $|f_n(y') - f_n(y)| = |n([\beta'])| \le \mathrm{Len}_{\tilde\gamma}(\beta') = d_{\tilde\gamma}(y, y')$. Finally we have $f_n(y) - f_n(y') = n([\beta']) \equiv \mathrm{Len}_\gamma(\beta') = d_{\tilde\gamma}(y', y)$ mod 2. Therefore f_n is pre-eikonal. □

By the Extension Lemma 19, we can extend f_n to an eikonal function $\overline{f_n}$. We chose $\overline{f_n}$ as in the proof, namely by the formula $\overline{f_n}(x) = \min_{y\in D} f_n(y) + d_{\tilde\gamma}(x, y)$.

Lemma 21. *The function $\overline{f_n}$ is n-equivariant.*

Proof. If $x = \{\alpha\}$ and $x' = T_{\{\beta\}}(x) = \{\beta\cdot\alpha\}$, then to prove that $f_n(x') - f_n(x) = n([c])$ we just need to observe that the function $f_n : D \to \mathbb{Z}$, as seen from x', looks the same, but $n([\beta])$ units higher, than as seen from x. More precisely, the contribution $f(y) + d_{\tilde\gamma}(x, y)$ of each $y = \{\alpha\} \in D$ to the formula $\overline{f_n}(x) = \min_{y\in D} f(y) + d_{\tilde\gamma}(x, y)$ is $n([\beta])$ units less than the contribution $f_n(y') + d(x', y')$ of its image $y' = T_{\{\beta\}}(y)$ to the formula $\overline{f_n}(x') = \min_{y'\in D} f_n(y') + d(x', y')$, because $f_n(y') - f_n(y) = n([\beta])$ and $d_{\tilde\gamma}(x', y') = d_{\tilde\gamma}(x, y)$. □

Lemma 22. *There is a unique Eulerian coorientation η on γ whose lift $\tilde\eta$ satisfies $\int_b \tilde\eta = \overline{f_n}(x') - \overline{f_n}(x)$ whenever $x = \{\alpha\}$ and $x' = \{\beta\cdot\alpha\}$. This η satisfies $[\eta] = n$.*

Proof. Since $\overline{f_n}$ is an eikonal function, there exists a unique coorientation $\tilde\eta$ of $\tilde\gamma$ whose integral on each path equals the variation of $\tilde\eta$. To prove that it descends to a coorientation η of γ, we only need to check that it is invariant by deck transformations. Indeed if x and z are neighbors, and x', z' are the respective images via a deck transformation $T_{\{\beta\}}$, then $\tilde\eta(x', z') = \overline{f_n}(z') - \overline{f_n}(x') = (\overline{f_n}(z) + n([\beta])) - (\overline{f_n}(x) + n([\beta])) = \overline{f_n}(z) - \overline{f_n}(x) = \tilde\eta(x, z)$, as required. Finally, to see that $[\eta] = n$, note that if β is a closed loop in Σ based at a point p, and α is a curve from p_0 to p, then both the startpoint x_0 and the endpoint $x = T_{\{\alpha\cdot\beta\cdot\alpha^{-1}\}}(x_0)$ of the loop $\alpha\cdot\beta\cdot\alpha^{-1}$ are in D, and we have

$$\int_\beta \eta = \int_{\alpha\cdot\beta\cdot\alpha^{-1}} \eta = \overline{f_n}(T_{\{\alpha\cdot\beta\cdot\alpha^{-1}\}}x_0) - \overline{f_n}(x_0) = n([\alpha\cdot\beta\cdot\alpha^{-1}]) = n([\beta]). \qquad \square$$

Remark 23. The fact that the unit ball has finitely many faces and that its faces are given by Eulerian coorientations is related to Dylan Thurston's Smoothing Lemma [Thu]. Indeed Eulerian coorientations can be smoothed (in a non-unique way if the coorientation has alternating double points) without changing their algebraic intersection with other transverse curves. In particular extremal coorientations yield smoothings of the original collection.

3. Birkhoff cross sections with antithetic boundary for the geodesic flow

In this part, we make an additional assumption: now Σ denotes a Riemann surface with strictly negative curvature. The divide γ now consists of finitely many periodic geodesics on Σ.

In this setting, the geodesic flow $(\varphi^t_{\mathrm{geod}})_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathrm{T}^1\Sigma$ is the flow whose orbits are lifts of geodesics. Namely for g a geodesic parametrized at speed one, the orbit of φ_{geod} going though the point $(g(0), \dot g(0)) \in \mathrm{T}^1\Sigma$ is $\varphi^t_{\mathrm{geod}}((g(0), \dot g(0)) = (g(t), \dot g(t))$. For every oriented periodic geodesic g on Σ, there is one periodic orbit of φ_{geod} corresponding to the oriented lift of g and denoted by $\vec{g}$. If g now denotes an unoriented geodesic on Σ, there are two associated periodic orbits of φ_{geod}, one for each orientation. We denote by $\overset{\leftrightarrow}{g}$ the union of these two periodic orbits, it is an oriented link in $\mathrm{T}^1\Sigma$ that is invariant under the involution $(p, v) \mapsto (p, -v)$. A link of the form $\overset{\leftrightarrow}{g}_1 \cup \cdots \cup \overset{\leftrightarrow}{g}_k$ is called an **antithetic link**[6].

Let us recall from the introduction that, given a complete flow $(\phi^t)_{t\in\mathbb{R}}$, a compact surface S with boundary is **transverse** to ϕ^t if its interior is transverse to the orbits of the flow and its boundary is the union of finitely many periodic orbits[7]. A **Birkhoff cross section** for ϕ^t is then a transverse surface S that intersects every orbit of ϕ^t. A small analysis and a compactness argument show that around the boundary S necessarily looks like a helix, so that the first-return time on $\mathrm{int}(S)$ is bounded.

In this section, we give a construction that associates to every Eulerian coorientation a surface transverse to the geodesic flow (3.a). Then we recall some facts on the existence of global sections for vector fields (3.b), before making some elementary algebraic topology for describing homology classes of surfaces with boundary (3.c). Finally we put pieces together to prove that the construction actually exhausts all possible surfaces, thus proving Theorems C and D (3.d).

3.a. **Constructions of Birkhoff cross sections with antithetic boundary.** We now explain how to associate to every Eulerian coorientation of γ a surface bounded by $\overset{\leftrightarrow}{\gamma}$ and transverse to φ_{geod}, thus proving the first part of Theorem C.

From now on we fix a global coorientation η (not yet Eulerian) of γ. For every edge e of γ (*i.e.* segment between two double points), we consider the set $R^{e,\eta}$ of those tangent vectors based on e and pairing positively with η. This is a rectangle in $\mathrm{T}^1\Sigma$ of the form $e \times [-\pi, \pi]$ (see Figure 4). Is is bounded by the two lifts of e in $\mathrm{T}^1\Sigma$ (called the **horizontal part** of $\partial R^{e,\eta}$) and two halves of

[6] This term was suggested by Bruce Bartlett.

[7] Often in the literature a transverse surface is only defined locally. The condition we add here on the boundary is not standard. However we keep the name for avoiding a heavier expression.

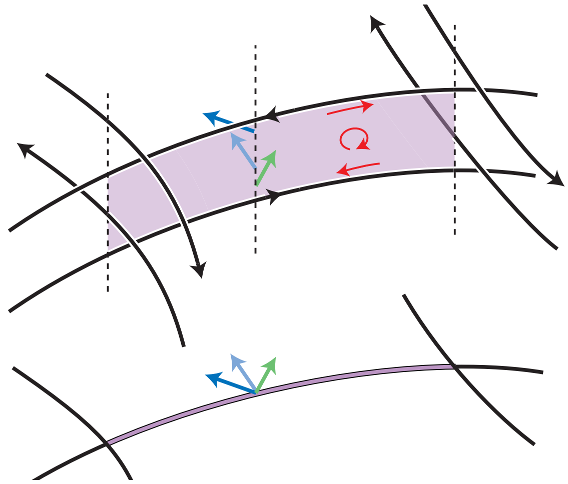


FIGURE 4. Bottom: an edge e of γ and a coorientation η on it. Top: the corresponding rectangle $R^{e,\eta}$ in $\mathrm{T}^1\Sigma$. The dotted lines represent the fibers of some points of Σ, that is, each point on these lines represent a unit tangent vector to Σ. Since the fibers are actually circles, the top and bottom extremities of the dotted lines should be glued. $R^{e,\eta}$ is transverse to φ_{geod} and the induced coorientation is shown in red. The induced orientation of the horizontal boundary of $R^{e,\eta}$ (red) is opposed to the orientation of the flow (black). Thus the surfaces we will construct are Birkhoff cross sections whose boundary components have negative multiplicity.

the fibers of the extremities of e (called the **vertical part** of $\partial R^{e,\eta}$). Note the interior of $R^{e,\eta}$ is transverse to the geodesic flow φ_{geod} while the horizontal part of $\partial R^{e,\eta}$ is tangent to it. We then orient $R^{e,\eta}$ so that φ_{geod} intersects it positively. One checks that the induced orientation on $\partial R^{e,\eta}$ is opposite to the one given by φ_{geod}.

Consider now the 2-dimensional complex $S^\times(\eta)$ that is the union of the rectangles $R^{e,\eta}$ for all edges e of γ.

Lemma 24. *The 2-complex $S^\times(\eta)$ described above has boundary $-\overleftrightarrow{\gamma}$ if and only if the coorientation η is Eulerian.*

Proof. Since $S^\times(\eta)$ is the union of one rectangle per edge of γ, the horizontal boundary of $S^\times(\eta)$ is always in $\overleftrightarrow{\gamma}$. Since the orientation is opposite to the geodesic flow, it is actually $-\overleftrightarrow{\gamma}$.

What we have to check is that the vertical boundary is empty if and only if η is Eulerian. At every double point v of γ there are four incident rectangles, corresponding to the four adjacent edges. Now the vertical boundary of a rectangle $R^{e,\eta}$ is oriented upwards (that is, trigonometrically) at the right extremity of e (when cooriented by η) and downwards at the left extremity. Then the vertical boundary in a vertex of γ is empty if only if two adjacent edges are cooriented in a direction, and two others in the opposite direction: this means that η is Eulerian around v. Conversely, if η is Eulerian, then up to rotation there are two local configurations around v (that we called alternating and non-alternating), and one checks that in both cases, the vertical boundary is empty (see the left parts of Figures 5 and 6). ◻

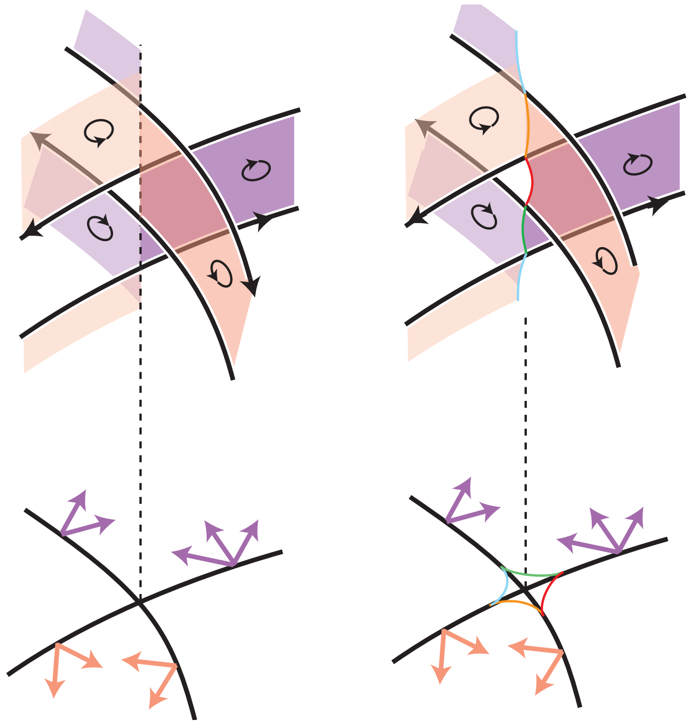


FIGURE 5. On the left, the complex $S^{\times}(\eta)$ around the fiber of an alternating double point of γ. Every point of the fiber of v is adjacent to exactly two rectangles. On the right the surface $S^{BB}(\eta)$ is obtained by smoothing $S^{\times}(\eta)$.

When η is Eulerian, the complex $S^{\times}(\eta)$ is not a topological surface if η has some non-alternating points: as depicted on Figure 6, there are edges adjacent to four faces. But it is the only obstruction and we can desingularize such segments. Also if we want a smooth surface, we have to smooth $S^{\times}(\eta)$ is a neighborhood of the fibers of the double points. In this way, we obtain a smooth surface, transverse to φ_{geod}.

Definition 25. For η an Eulerian coorientation, the associated **BB-surface** is the surface $S^{BB}(\eta)$ obtained from $S^{\times}(\eta)$ by desingularizing and smoothing the fibers of the double points of γ (see the right parts of Figures 5 and 6).

For example, the BB-surface associated to a Birkhoff coorientation (Example 14) is isotopic to the construction suggested by Birkhoff [Bir17] and popularized by Fried [Fri83]. Also the BB-surface associated to a Brunella coorientation (Example 15) has been introduced by Brunella [Bru94, Description 2].

3.b. **Asymptotic cycles and existence of sections.** The question whether a given vector field admits a global section (*i.e.*, with empty boundary) has been given a very satisfactory answer by Schwartzman and Fuller [Sch57, Ful65], then expanded by Fried [Fri82]. A very elegant proof in terms of *foliated currents* was also provided by Sullivan [Sul76].

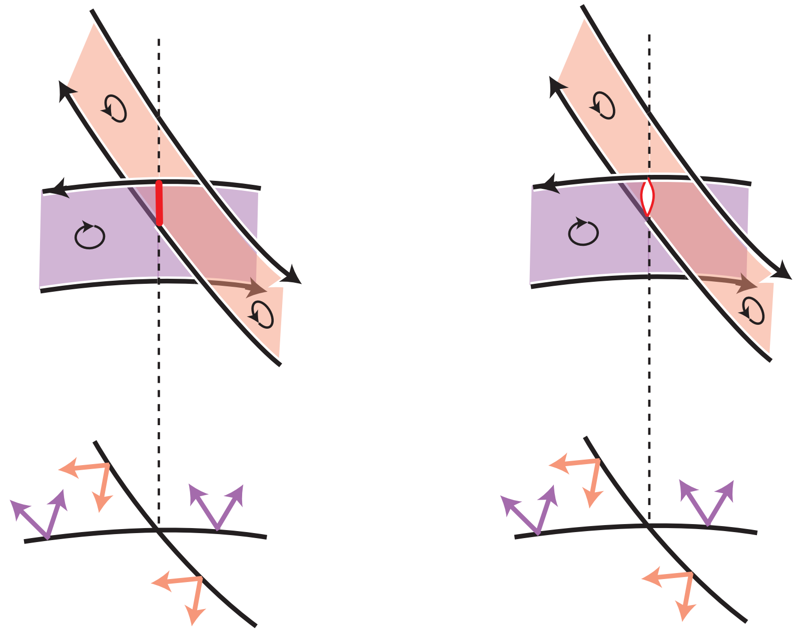


FIGURE 6. On the left, the complex $S^{\times}(\eta)$ around the fiber of a non-alternating double point of γ. Every point of the fiber of v is adjacent to an even number of rectangles. On the right the surface $S^{BB}(\eta)$ is obtained by desingularizing $S^{\times}(\eta)$ on the portion of the fiber where four rectangles meet.

A preliminary remark: if two surfaces S_1 and S_2 in a manifold M are global sections to a flow ϕ and they are homologous, then they are isotopic, and the isotopy is realized by the flow. Indeed[8] one can consider the infinite cyclic covering of $\hat{M} \to M$ associated to the morphism $\pi_1(M) \to \mathbb{Z}$ given by the intersection with $[S_1] = [S_2]$. Then S_1 and S_2 lift into $\mathbb{Z}$ disjoint copies $t^m\hat{S}_1$ and $t^n\hat{S}_2$ in $\hat{M}$, all transverse to the lift of the flow. Now following the flow starting from $\hat{S}_1$, one reaches $\hat{S}_2$, so we have a surjective map $\hat{S}_1 \to \hat{S}_2$ of local degree 1, and since $\hat{S}_1$ is transverse to the flow it is of total degree 1. Similarly we have a surjection $\hat{S}_2 \to \hat{S}_1$ of local degree 1. By composing the two, we get of surjection $\hat{S}_1 \to \hat{S}_1$ of total degree 1, hence a bijection. Therefore the maps $\hat{S}_1 \to \hat{S}_2$ and $\hat{S}_2 \to \hat{S}_1$ are actually bijections, and the flow hence induces an isotopy $\hat{S}_1 \to \hat{S}_2$. Projecting back in M, we obtained the desired isotopy $S_1 \to S_2$.

For X a vector field in a compact manifold M, we denote by $k_X(p,t)$ a closed curve obtained by concatenating the piece of orbit $\phi^{[0,t]}(p)$ starting at p of length t with an arc connecting $\phi^t(p)$ to $\phi^0(p)$ of bounded length. The class $[k_X(p,t)]$ in $\mathrm{H}_1(M;\mathbb{Z})$ then depends on the choice of the closing segment, but only in a bounded way, so that the limit $\lim_{t\to\infty}\frac{1}{t}[k_X(p,t)]$, if it exists, does not depend on this choice. An **asymptotic cycle** of X is then the limit of a sequence of the form $\{\frac{1}{t_n}[k_X(p_n,t_n)] \mid p_n \in M, t_n \to \infty\}$ in $\mathrm{H}_1(M;\mathbb{R})$. The set of asymptotic cycles of X is denoted by $\mathcal{S}chw(X)$. Sullivan [Sul76] reinterpreted it by showing that every X-invariant measure μ induces a *foliated cycle* c_μ that is actually a positive combination of asymptotic cycles.

[8]This mimics the folklore argument in knot theory that the fiber of a fibration minimizes the genus, but it is not so easy to find a reference for this statement.

Theorem 26. [Sch57, Ful65] *A vector field X on a closed M admits a global section whose homology class is $\sigma \in \mathrm{H}_2(M, \partial M; \mathbb{Z})$ if and only σ intersects positively every asymptotic cycle, namely for every $c \in \mathcal{S}chw_(X)$ one has $\langle \sigma, c\rangle > 0$.*

This theorem is beautiful, but unfortunately, for many vector fields X, the point 0 belongs to $\mathrm{Conv}(\mathcal{S}chw_(X))$, so that X admits no global section at all. This is where Birkhoff cross sections come in.

3.c. **Classes of surfaces with given boundary.** Now we work in our restricted setting: Σ is a negatively curved surface, γ is a finite collection of periodic geodesics and $\overset{\leftrightarrow}{\gamma}$ denotes the antithetic lift of γ. In order to apply Theorem 26 for finding Birkhoff cross sections, we need to work in the complement $\mathrm{T}^1\Sigma \setminus \overset{\leftrightarrow}{\gamma}$ and in particular to determine the space $\mathrm{H}_2(\mathrm{T}^1\Sigma, \overset{\leftrightarrow}{\gamma}; \mathbb{Z})$. In this section we show that the homology classes of surfaces bounded by $-\overset{\leftrightarrow}{\gamma}$ form an affine space and we give a canonical origin to this space.

Lemma 27. *The sequence $0 \to \mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z}) \overset{i}{\to} \mathrm{H}_2(\mathrm{T}^1\Sigma, \overset{\leftrightarrow}{\gamma}; \mathbb{Z}) \overset{\partial}{\to} \mathrm{H}_1(\overset{\leftrightarrow}{\gamma}; \mathbb{Z})$, where the first map is the inclusion map and the second is the boundary map, is exact.*[9]

Proof. This is just a part of the long exact sequence associated to the pair $(\mathrm{T}^1\Sigma, \overset{\leftrightarrow}{\gamma})$, see [Hat02, Thm 2.16], plus the remark that $\mathrm{H}_2(\overset{\leftrightarrow}{\gamma}; \mathbb{Z})$ is zero. □

The homology classes of those surfaces whose boundary is $-\overset{\leftrightarrow}{\gamma}$ correspond to the preimages under ∂ of the point $(-1, -1, \dots, -1) \in \mathrm{H}_1(\overset{\leftrightarrow}{\gamma}; \mathbb{Z}) \simeq \mathbb{Z}^{2|\gamma|}$. Hence they form an affine space directed by $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z})$. Indeed, given two surfaces with the same boundary, their difference induces a class in $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z})$. Now using the fact that $\mathrm{T}^1\Sigma$ is a circle bundle with non-zero Euler class, we get $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z}) \simeq \mathrm{H}_1(\Sigma; \mathbb{Z})$: a non-trivial class in $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z})$ can be represented by the set of the fibers over a cycle in $\mathrm{H}_1(\Sigma; \mathbb{Z})$.

From the previous discussion we deduce that if we are given an explicit surface S_0 bounded by $-\overset{\leftrightarrow}{\gamma}$, the classes of the other surfaces bounded by $-\overset{\leftrightarrow}{\gamma}$ differ from $[S_0]$ by a class in $\mathrm{H}_1(\Sigma)$. In our context, there is a natural choice of such an origin, for which the computation of the intersection numbers with asymptotic cycles of the geodesic flow will be easy. We denote by $S^{\times}_{\pm}$ the rational chain in $\mathrm{C}_2(\mathrm{T}^1\Sigma, \overset{\leftrightarrow}{\gamma}; \mathbb{Q})$ that is half the sum of all rectangles of the form $R^{e,\eta}$ (see Figure 7) and by $\sigma_\pm$ its homology class in $\mathrm{H}_2(\mathrm{T}^1\Sigma, \overset{\leftrightarrow}{\gamma}; \mathbb{Q})$:

$$S^{\times}_{\pm} := \frac{1}{2} \sum_{e \in \gamma, \eta_e = \pm} R^{e,\eta_e}, \qquad \sigma_\pm := [S^{\times}_{\pm}].$$

In other words, we consider the set of all tangent vectors base at points of γ. Remember that every rectangle is cooriented by the geodesic flow, hence oriented. Therefore, $S^{\times}_{\pm}$ is also oriented. Its boundary is then exactly $-\overset{\leftrightarrow}{\gamma}$ (thanks to the $\frac{1}{2}$ factor). The chain $S^{\times}_{\pm}$ is not a surface since the fibers of the double points of γ are singular. As it is rational the class $\sigma_\pm$ might not be realized by a surface, but $2\sigma_\pm$ is always an integer class.[10]

[9] An erroneous version of this statement is in [Fri82, Lemma 6], where it is claimed that the boundary map is surjective and admits a section. It is not true in general, unless the manifold is a homology sphere.

[10] Actually, $\sigma_\pm$ is realized by a surface if and only if $[\gamma]_2$, the class of γ with $\mathbb{Z}/2\mathbb{Z}$-coefficients, is 0. In this case, the homology class of Birkhoff's coorientation η_B (Example 14) is 0, and $S^{BB}(\eta_B)$ lies in the class $\sigma_\pm$. Also the class $\sigma_\pm$ is equal to $\frac{1}{2}[S^{BB}(\eta) + S^{BB}(-\eta)]$ for every Eulerian η. Hence it is always realized as the average of two surfaces without any assumption on $[\gamma]_2$.

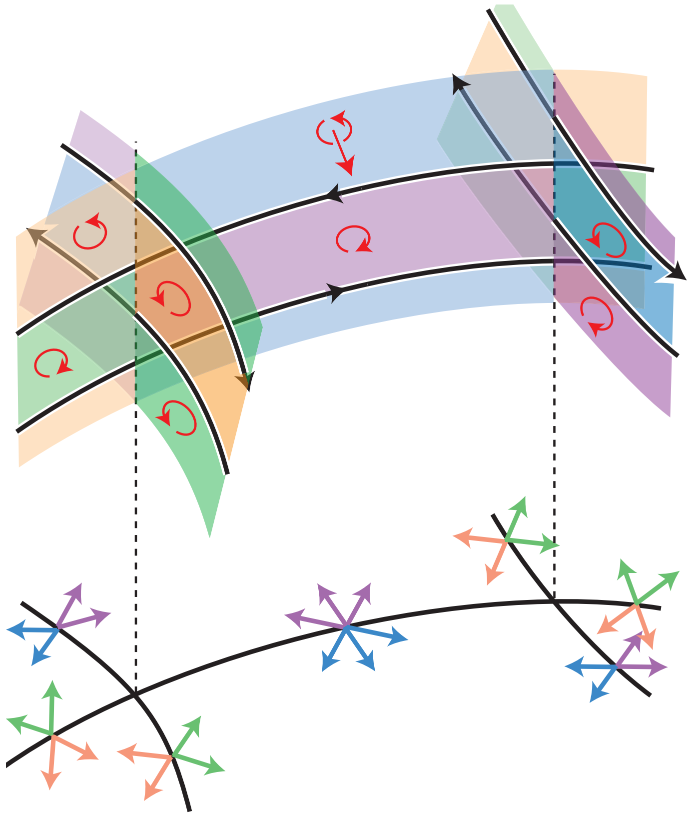


FIGURE 7. The 2-chain $S^{\times}_{\pm}$ is half of the sum of all rectangles R^{e,η_e}. It is cooriented by the geodesic flow, hence oriented (in red). Its boundary, taking orientations into account, is then $-\overset{\leftrightarrow}{\gamma}$.

Lemma 28. *For $\underset{\rightarrow}{\alpha}$ a collection of oriented periodic geodesics on Σ, none of which is a component of γ, the algebraic intersection $\langle\sigma_\pm, \vec{\alpha}\rangle$ is equal to $+\frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}|$.*

This lemma appears in a different form in [DIT17] where it is used to prove that the linking number of two collections $\overset{\leftrightarrow}{\gamma}_1, \overset{\leftrightarrow}{\gamma}_2$ in $\mathrm{T}^1\Sigma$ is actually equal to $|\{\gamma_1 \cap \gamma_2\}|$.

Proof. Since $S^{\times}_{\pm}$ is positively transverse to the geodesic flow, all intersection points of $\vec{\alpha}$ with $S^{\times}_{\pm}$ count positively. Since every rectangle has coefficient $\frac{1}{2}$ in $S^{\times}_{\pm}$, every intersection point contributes for $+\frac{1}{2}$ to the algebraic intersection. Finally $\vec{\alpha}$ intersects $S^{\times}_{\pm}$ exactly in the fiber of the intersection points of $\underset{\rightarrow}{\alpha}$ and γ. □

The connection with intersection norms is now straightforward:

Corollary 29. *For $\underset{\rightarrow}{\alpha}$ a collection of oriented periodic geodesics on Σ, none of which is a component of γ, the intersection $\langle\sigma_\pm, \vec{\alpha}\rangle$ is at least equal to $x_\gamma([\underset{\rightarrow}{\alpha}])$, with equality if and only if $\underset{\rightarrow}{\alpha}$ is a x_γ-minimizing collection of geodesics.*

3.d. **Proofs of Theorems C and D.** Denote by $\mathcal{S}chw_{\overleftrightarrow{\gamma}} \subset \mathrm{H}_1(\mathrm{T}^1\Sigma \setminus \overleftrightarrow{\gamma}; \mathbb{R})$ the set of all asymptotic cycles of the geodesic flow φ_{geod} restricted to $\mathrm{T}^1\Sigma \setminus \overleftrightarrow{\gamma}$. Also denote by π_* the canonical projection from $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{R})$ to $\mathrm{H}_1(\Sigma; \mathbb{R})$. The next statement is the key statement connecting Birkhoff sections and intersection norms.

Lemma 30. *A class $\sigma \in \mathrm{H}_2(\mathrm{T}^1\Sigma, \overleftrightarrow{\gamma}; \mathbb{R})$ intersects positively every element of $\mathcal{S}chw_{\overleftrightarrow{\gamma}}$ if and only if the class $\pi_*(\sigma - \sigma_\pm) \in \mathrm{H}_1(\Sigma; \mathbb{R})$ lies in the interior of $\frac{1}{2}B^*_{x_\gamma}$.*

Proof. By the shadowing property for pseudo-Anosov flows, the projectivization of $\mathcal{S}chw_{\overleftrightarrow{\gamma}}$ is the convex hull of the cycles given by periodic orbits. Hence it is enough to estimate the intersection of σ with all periodic orbits of φ_{geod}.

We use the bracket to denote the intersection, and the index reminds the space where the objects live. For every periodic orbit $\vec{\alpha}$ of φ_{geod}, by Lemma 28, we have

$$\begin{aligned}\langle \sigma, \vec{\alpha} \rangle_{\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}} &= \langle \sigma - \sigma_\pm, \vec{\alpha} \rangle_{\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}} + \langle \sigma_\pm, \vec{\alpha} \rangle_{\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}} \\ &= \langle \sigma - \sigma_\pm, \vec{\alpha} \rangle_{\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}} + \frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}| \\ &= \langle \pi_*(\sigma - \sigma_\pm), \underset{\rightarrow}{\alpha} \rangle_\Sigma + \frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}|.\end{aligned}$$

Hence $\langle \sigma, \vec{\alpha} \rangle_{\mathrm{T}^1\Sigma\setminus\overleftrightarrow{\gamma}}$ is positive if and only if $-\langle \pi_*(\sigma - \sigma_\pm), \underset{\rightarrow}{\alpha} \rangle_\Sigma$ is smaller than $\frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}|$.

Now the term $-\langle \pi(\sigma - \sigma_\pm), \underset{\rightarrow}{\alpha} \rangle_\Sigma$ depends only on the class $[\underset{\rightarrow}{\alpha}] \in \mathrm{H}_1(\Sigma; \mathbb{Z})$, while the other term $\frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}|$ is larger that $\frac{1}{2}x_\gamma([\underset{\rightarrow}{\alpha}])$, with equality if $\underset{\rightarrow}{\alpha}$ is x_γ-minimizing (Corollary 29).

We now treat separately the cases $[\underset{\rightarrow}{\alpha}] \neq 0$ and $[\underset{\rightarrow}{\alpha}] = 0$ in $\mathrm{H}_1(\Sigma; \mathbb{Z})$.

Since there is a x_γ-minimizing geodesic in every non-zero homology class, the inequality $-\langle \pi_*(\sigma - \sigma_\pm), \underset{\rightarrow}{\alpha} \rangle_\Sigma < \frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}|$ is true for all non null-homologous geodesics $\underset{\rightarrow}{\alpha}$ if and only if the inequality $-\langle \pi_*(\sigma - \sigma_\pm), a \rangle_\Sigma < \frac{1}{2}x_\gamma(a)$ is true for every non-zero homology class.

If $\underset{\rightarrow}{\alpha}$ is null-homologous and γ is filling (meaning that the complement $\Sigma \setminus \gamma$ is a union of topological discs), we have $\frac{1}{2}|\{\underset{\rightarrow}{\alpha} \cap \gamma\}| > 0$ and $-\langle \pi_*(\sigma - \sigma_\pm), \underset{\rightarrow}{\alpha} \rangle_\Sigma = 0$. If $\underset{\rightarrow}{\alpha}$ is null-homologous and γ is not filling, all terms equal 0, so that the orbit $\vec{\alpha}$ does not intersect σ. However, in this case, the interior of $B^*_{x_\gamma}$ is empty, so the statement is true.

Summarizing the two previous paragraphs, we find that, if γ is filling, the class σ intersects positively every element of $\mathcal{S}chw_{\overleftrightarrow{\gamma}}$ if and only if for every class $a \in \mathrm{H}_1(\Sigma; \mathbb{Z})$ we have the inequality $-\langle \pi_*(\sigma - \sigma_\pm), a \rangle_\Sigma < \frac{1}{2}x_\gamma(a)$, which means exactly that the point $-\pi_*(\sigma - \sigma_\pm)$ belongs to $\frac{1}{2}B^*_{x_\gamma}$. Since the latter is symmetric about the origin, this amounts to $\pi_*(\sigma - \sigma_\pm)$ belonging to $\frac{1}{2}B^*_{x_\gamma}$. □

As a byproduct of the proof, we obtain that a class $\sigma \in \mathrm{H}_2(\mathrm{T}^1\Sigma, \overleftrightarrow{\gamma}; \mathbb{R})$ intersects non-negatively every asymptotic cycle if and only if $\pi_*(\sigma - \sigma_\pm) \in \mathrm{H}_1(\Sigma; \mathbb{R})$ lies in the closed unit ball $\frac{1}{2}B^*_{x_\gamma}$.

Proof of Theorem C. For η an Eulerian coorientation, we consider the surface $S^{BB}(\eta)$ (Definition 25). By construction it is transverse to the geodesic flow. One easily checks that every rectangle of the form $R^{e,\eta}$ contributes to -1 to the Euler characteristics, hence $\chi(S^{BB}(\eta))$ is $-|E(\gamma)|$. Since γ is a graph of degree 4, one has $|E(\gamma)| = 2|V(\gamma)|$, so that $\chi(S^{BB}(\eta)) = -2|V(\gamma)|$.

Now if η_1 and η_2 are cohomologous, the class $[S^{BB}(\eta_1) - S^{BB}(\eta_2)] \in \mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z})$ projects by π onto $[\eta_1 - \eta_2] = 0$. Since π_* is actually an isomorphism we have $[S^{BB}(\eta_1) - S^{BB}(\eta_2)] = 0$, which in turn implies $[S^{BB}(\eta_1)] = [S^{BB}(\eta_2)]$ in $\mathrm{H}_2(\mathrm{T}^1\Sigma, \overleftrightarrow{\gamma}; \mathbb{Z})$. Now since $S^{BB}(\eta_1)$ and $S^{BB}(\eta_2)$ are both transverse to φ_{geod} and homologous, the flow actually realizes an isotopy between them. □

Proof of Theorem D. Lemma 27 and the paragraph after imply that real homology classes of surfaces bounded by $-\overleftrightarrow{\gamma}$ form an affine space directed by $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{R})$. The class $\sigma_\pm$ defined in 3.c gives a canonical origin to this space. It is a half-integer class, and its double $2\sigma_\pm$ is congruent to $[\gamma]_2$ mod 2. Therefore the doubles of all integer classes correspond to the sublattice of $\mathrm{H}_2(\mathrm{T}^1\Sigma; \mathbb{Z})$ of those points congruent to $[\gamma]_2$ mod 2.

Now we have to determine which of these integer classes yield Birkhoff cross sections. By Schwartzman-Fuller Theorem 26, a class σ contains a Birkhoff cross section if and only if it intersects positively every asymptotic cycles. By Lemma 30 this means that the difference $\pi_*(\sigma - \sigma_\pm)$ lies inside $\frac{1}{2}B^*_{x_\gamma}$, or equivalently that $2\pi_*(\sigma - \sigma_\pm)$ lies inside $B^*_{x_\gamma}$.

Finally surfaces that are transverse to φ_{geod} correspond to homology classes that intersects non-negatively every asymptotic cycle, allowing certain intersection nuumbers to be zero. This means that the boundary of $B^*_{x_\gamma}$ is now authorized. □

4. Extension to orientable 2-orbifolds

We explain here how the results extend to 2-dimensional orbifolds. Actually the only point that is not straightforward is Theorem D which requires a new argument.

Definition 31 (chap.13 of [Thu80])**.** A **Riemannian orientable 2-dimensional orbifold** O is given by an orientable topological surface Σ_O together with an atlas $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ of charts of the form $\phi_\alpha : U_\alpha \to D_\alpha/(\mathbb{Z}/k_\alpha\mathbb{Z})$, with D_α a 2-dimensional Riemannian disc on which $\mathbb{Z}/k_\alpha\mathbb{Z}$ acts by rotations, and such that the chart changes $\phi_\alpha \circ \phi_\beta^{-1}$ are isometries.

Actually the orbifolds to which our theorems extend are the hyperbolic ones. Such a 2-orbifold is always *good* in the sense of Thurston, namely it is a quotient of a hyperbolic surface by a finite automorphism group.

For our purpose we define the **first homology group** $\mathrm{H}_1(O; \mathbb{R})$ to be simply $\mathrm{H}_1(\Sigma_O; \mathbb{R})$. The definition of intersection norms then extend directly and Theorems A and B hold.

We now turn to Theorems C and D. First we have to define unit tangent bundles to orbifolds and geodesic flows. If D is a Riemannian disc on which $\mathbb{Z}/k\mathbb{Z}$ acts by rotation (with a fixed point), then $\mathbb{Z}/k\mathbb{Z}$ also acts on the unit tanget bundle T^1D. The action on T^1D is free, since the vectors tangent to the fixed point are rotated. Hence the quotient $\mathrm{T}^1D/(\mathbb{Z}/k\mathbb{Z})$ is a 3-manifold (actually it is a solid torus).

Definition 32. Given a Riemannian orientable 2-orbifold $O = (\Sigma_O, (U_\alpha, \phi_\alpha)_{\alpha\in A})$, its **unit tangent bundle** is the 3-manifold T^1O defined by the atlas $(\hat{U}_\alpha, \hat{\phi}_\alpha)_{\alpha\in A}$, where $\hat{U}_\alpha = \mathrm{T}^1U_\alpha$ and $\hat{\phi}_\alpha(x, v) = (\phi_\alpha(x), d(\phi_\alpha)_x(v))$. It is equipped with a canonical projection $\pi : \mathrm{T}^1O \to O$.

If O is of the form Σ/Γ for some hyperbolic surface Σ, then T^1O is simply the quotient $(\mathrm{T}^1\Sigma)/\Gamma$.

The **geodesic flow** on T^1O is defined as in the non-singular case by $\varphi^t_{\mathrm{geod}}(\gamma(0), \dot\gamma(0)) = (\gamma(t), \dot\gamma(t))$, where γ is any geodesic travelled at speed 1.

With these definitions, the constructions of Section 3.a (the BB-surface $S^{BB}(\eta)$ associated to an Eulerian coorientation) can be transposed and Lemmas 24, 27, and 28 remain true.

Now, for O a hyperbolic 2-orbifold, the unit tangent bundle T^1O is a 3-manifold, and we have $\mathrm{H}_2(\mathrm{T}^1O; \mathbb{R}) \simeq \mathrm{H}_1(O; \mathbb{R})$. Indeed closed curves in Σ_O lift by π^{-1} to closed surfaces in T^1O. The fact that the unit tangent to a conic disc $D/(\mathbb{Z}/k\mathbb{Z})$ is a torus whose core is the singular fiber implies that cohomologous curves lift to cohomologous surfaces, so that π^{-1} induces a well-defined map $\pi_*^{-1} : \mathrm{H}_1(O;\mathbb{R}) \to \mathrm{H}_2(\mathrm{T}^1O;\mathbb{R})$. The orbifold Euler characteristics of O is negative by hyperbolicity, so that the Euler number of T^1O (as a Seifert fibered space) is also negative, hence the map π_*^{-1} is an isomorphism.

Now Corollary 29 holds, but Lemma 30 needs to be adapted. Firstly remark that if Σ_O is a homology sphere, x_γ is the zero-function, so there is no possible interesting version of Lemma 30 in this case. Secondly, if Σ_O is not a homology sphere, Lemma 30 holds, but one argument needs to be developed, namely:

Lemma 33. *For O a Riemannian orientable 2-orbifold and γ a geodesic divide on O, for every non-zero homology class a in $\mathrm{H}_2(O;\mathbb{R})$, there is a x_γ-minimizing geodesic in a.*

Proof. Let β we a x_γ-minimizing curve such that $[\beta] = a$. As in the case of a standard surface we want to strenghten β to make it geodesic without changing the geometric intersection with γ. Far from the cone points, one can perform isotopies that shorten β with respect to the hyperbolic metric . Since γ is geodesic, these isotopies cannot increase the number of intersection (that is, no Reidemeister-II move is involved).

Around a cone point, one can work in a local cone chart. This amounts to work on a standard disc where everything in invariant under a rotation. Then one can also perform length-decreasing isotopies in an equivariant way, and this does not increase the number of intersection points with γ. □

Theorem C holds with no modification in the proof, and Theorem D has to be changed into Theorem E in order to treat the case of an orbifold whose underlying surface is a sphere.

Proof of Theorem E. Suppose that Σ_O is a sphere. Then $\mathrm{T}^1\Sigma_O$ is a rational homology sphere (in this case, $\mathrm{H}_1(\mathrm{T}^1\Sigma_0 0;\mathbb{Z})$ is finite, but not reduced to the trivial group, unless Σ_O is a sphere with three cone points of respective orders $2, 3$, and 7). If γ is filling, then the class $\sigma_\pm$ intersects every asymptotic cycle, so it contains a Birkhoff section. Since $\mathrm{H}_2(\mathrm{T}^1\Sigma_0 0;\mathbb{Z})$ is trivial, all Birkhoff sections are homologous, hence isotopic relatively to their boundary.

If γ is not filling, then there exists a geodesic α not intersecting γ on Σ_O. Both its oriented lifts do not intersect $S^\times_\pm$, hence there is an asymptotic cycle whose algebraic intersection with $\sigma_\pm$ is zero. Hence the class $\sigma_\pm$ contains no Birkhoff section. Since it is the unique class with boundary $-\overleftrightarrow{\gamma}$, there is no Birkhoff section bounded by $-\overleftrightarrow{\gamma}$ at all.

Finally if Σ_O is not a sphere and γ is filling, the norm x_γ is non-degenerate, and the proof of Theorem D translates directly. □

5. Questions

On intersection norms. If Σ is a flat torus, then the minimal intersection is always realized by geodesics, which are unique in their homology class. Hence if the divide γ is the union of k geodesics $\gamma_1, \dots, \gamma_k$, then $i_\gamma(\alpha) = \sum_{i=1}^k i_{\gamma_i}(\alpha)$. This implies that the dual ball B^*_γ coincides with the Minkowski sum $B^*_{\gamma_1} + \dots + B^*_{\gamma_k}$. Since the segment $[-1, 1] \times \{0\} \subset \mathbb{R}^2$ is the dual unit ball $B^*_{x_\gamma}$ for γ the vertical circle on the torus, every segment containing 0 in the middle is the dual unit ball of some closed circle on the torus. Therefore every convex polygon in $\mathbb{R}^2$ whose vertices are integral and congruent mod 2 is of the form $B^*_{x_\gamma}$ for some γ. This was already remarked by

Thurston [Thu86] and by Schrijver [Sch93]. In higher dimension the situation is probably more intricate.

Question 34. Which polyhedra of $\mathbb{R}^{2g}$ with integer vertices can be realized as the dual unit ball $B^*_{x_\gamma}$ for some γ in Σ_g?

As we finish the paper, a partial answer is given by Abdoul Karim Sane [San18] who proves that some polyhedra in $\mathbb{R}^4$ cannot be dual unit ball of any intersection norm on a genus 2-surface.

Also, if Σ is a torus and γ is a union of geodesics, then the above remarks imply that the number of self-intersection points of γ is exactly $1/4$ of the area of $B^*_{x_\gamma}$ (check on Figure 1). Is there an analog statement in higher genus?

Question 35. Which information concerning γ can be read on $B^*_{x_\gamma}$? Is the number of self-intersection points of γ a certain function defined on $B^*_{x_\gamma}$?

This information is interesting since the this number is exactly the opposite of the Euler characteristic of every Birkhoff cross section bounded by $\overleftrightarrow{\gamma}$. Note that the number of self-intersection points is homogenous of degree 2, so we should look for degree 2 functions on polyhedra in $\mathbb{R}^{2g}$: does it correspond to some symplectic capacity?

Motivated by our application we only defined the intersection norm for a collection of immersed curves, but one can directly extend it for an arbitrary embedded graph. One can wonder which properties extend to this case and which information on the embedded graphs are encoding in this norm. For example when the graph is Eulerian (*i.e.,* all vertices have even degree) the connection with Eulerian coorientations remains.

On Birkhoff cross sections. Our constructions and our classification result deal only with Birkhoff cross sections bounded by an *antithetic* collection of periodic orbits of the geodesic flow, that is, invariant under the involution $(p, v) \mapsto (p, -v)$. However the only restriction *a priori* for being the boundary of a Birkhoff cross section is to be a boundary, that is, to be null-homologous. Our results here say nothing about the classification, or even the existence, of Birkhoff cross sections with arbitrary null-homologous boundary. In this case, the theory of Schwartzman-Fuller-Thurston-Fried and the remarks of Sections 3.b and 3.c still apply, so that these sections still correspond to the point inside a certain polytope in $\mathrm{H}^1(\Sigma; \mathbb{R})$. However we have no analog for the coorientations and the explicit constructions derived from them.

Question 36. Is there a natural generalization of the polytope $B^*_{x_\gamma}$ to non-antithetic finite collections $\vec{\gamma}$ of closed orbits of the geodesic flow φ_{geod}, so that integer points in this polytope classify surfaces bounded by $\vec{\gamma}$ and transverse to φ_{geod}?

In the case of the flat torus, this question is answered in [Deh15a, Thm 3.12] where a polygon $P_{\vec{\gamma}}$ classifying transverse surfaces bounded by $\vec{\gamma}$ is defined for *every* null-homologous collection $\vec{\gamma}$.

What would probably unlock the situation in the higher genus case would be to have, for every null-homologous collection $\vec{\gamma}$, *one* explicit surface bounded by $\vec{\gamma}$ (not necessarily transverse), that is, an analog of $\sigma_\pm$ when $\vec{\gamma}$ is not antithetic. Such an explicit point allows to compute its intersection with every other periodic orbit $\vec{\alpha}$ of φ_{geod}. These intersection numbers are all we need in order to describe explicitly the asymptotic directions of φ_{geod} in $\mathrm{T}^1\Sigma \setminus \vec{\gamma}$. Generalising the constructions of [Deh15b] is a possibility here.

More generally, one can wonder whether there exists a generalization to all flows of the intersection norm x_γ in the following sense:

Question 37. For every 3-dimensional flow X, is there an object that describes all isotopy classes of Birkhoff cross sections?

A starting point would be to try with an Anosov flow that is not the geodesic flow, and see whether Gauss linking forms [Ghy09] could play this role.

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Marcos Cossarini
EPFL, CH-1015 Lausanne, Suisse
marcos.cossarini@epfl.ch

Pierre Dehornoy
Univ. Grenoble Alpes, CNRS, IF, F-38000 Grenoble, France
pierre.dehornoy@univ-grenoble-alpes.fr
http://www-fourier.ujf-grenoble.fr/~dehornop/