COMPLEX VS CONVEX MORSE FUNCTIONS AND GEODESIC OPEN BOOKS

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ABSTRACT. Suppose that Σ is a closed and oriented surface equipped with a Riemannian metric. In the literature, there are four seemingly distinct constructions of open books on the unit (co)tangent bundle of S, having complex, symplectic, contact, and dynamical flavors, respectively. Each one of these constructions is based on either an admissible divide or an ordered Morse function on Σ. We show that the resulting open books are pairwise isomorphic provided that the ordered Morse function is adapted to the admissible divide on Σ. Moreover, we observe that if Σ has positive genus, then none of these open books are planar and furthermore, we determine the only cases when they have genus one pages.

1. INTRODUCTION

Let Σ be a closed and oriented surface equipped with a Riemannian metric. The bundle of cooriented lines tangent to Σ (aka the space of cooriented contact elements), which we denote by $V(\Sigma)$ in this paper, is defined as the set of pairs (q, L) where $q \in \Sigma$ and L is a cooriented line in $T_q\Sigma$. One can identify $V(\Sigma)$ with the unit tangent bundle $ST\Sigma$ as well as with the unit cotangent bundle $ST^*\Sigma$. As an oriented 3-manifold, $ST^*\Sigma$ (resp. $ST\Sigma$) is diffeomorphic to the circle bundle over Σ with Euler number $-\chi(\Sigma)$ (resp. $\chi(\Sigma)$). We orient the circle bundle $V(\Sigma)$ so that it is fiber and orientation preserving diffeomorphic to $ST^*\Sigma$.

Let $DT^*\Sigma$ denote the disk cotangent bundle, which is a disk bundle over Σ with Euler number $-\chi(\Sigma)$, whose boundary is $ST^*\Sigma$. Let λ denote the Liouville 1-form on $T^*\Sigma$. Under the identification of $ST^*\Sigma$ with $V(\Sigma)$, the canonical contact structure $\xi = \ker \lambda|_{ST^*\Sigma}$ on $ST^*\Sigma$ coincides with the canonical contact structure on $V(\Sigma)$, which we again denote by ξ throughout the article.

A *divide* $P \subset \Sigma$ is a generic immersion of the disjoint union of finitely many copies of the unit circle, which is said to be *admissible* if it is connected, each component of $\Sigma \setminus P$ is simply connected and $\Sigma \setminus P$ admits a black-and-white coloring so that two sides of an edge in P have opposite colors, see Figure 1. We say that an ordered Morse function $f : \Sigma \to \mathbb{R}$ (meaning that the higher the index of a critical point the greater its value) is *adapted* to a given admissible divide $P \subset \Sigma$ if $P = f^{-1}(0)$, each double point of P corresponds to a critical point of f of index 1, and each black (resp. white) region of $\Sigma \setminus P$ contains exactly one index 2 (resp. 0) critical point of f. Note that for each admissible divide $P \subset \Sigma$, there exists an ordered Morse function $f : \Sigma \to \mathbb{R}$ adapted to P. We also say that an admissible divide $P \subset \Sigma$ is *convex* with respect to the given metric on Σ, if it is a set of geodesics on Σ, every geodesic on Σ meets P in bounded time, and every region of $\Sigma \setminus P$ can be foliated by concentric closed curves with non-vanishing curvature.

Theorem 1.1. *Suppose that P is an admissible divide on a closed and oriented surface Σ equipped with a Riemannian metric. If $f : \Sigma \to \mathbb{R}$ is an ordered Morse function adapted to P, then the open books described in (1) - (4) below, are pairwise isomorphic under the natural identifications of the 3-manifolds $ST\Sigma$, $ST^*\Sigma$, and $V(\Sigma)$.*

(1) (A'Campo [2] & Ishikawa [23]) *The Morse function $f : \Sigma \to \mathbb{R}$ can be extended to a complex Morse function $f_{\mathbb{C}} : T\Sigma \to \mathbb{C}$, which restricts to a smooth achiral*[1] *Lefschetz fibration $DT\Sigma \to D^2$. There is an induced open book on the boundary $ST\Sigma$.*

(2) (Johns [24]) *The Morse function $f : \Sigma \to \mathbb{R}$ can be extended to an exact symplectic Lefschetz fibration $f_\omega : E \to D^2$, where E is conformally exact symplectomorphic to $(DT^*\Sigma, \omega = d\lambda)$. The induced open book on the boundary $ST^*\Sigma$ supports ξ.*

(3) (Giroux [18]) *The Morse function $f : \Sigma \to \mathbb{R}$ can be modified to an ordered*[2] *ξ-convex (or contact) Morse function $f_\xi : V(\Sigma) \to \mathbb{R}$, which induces an open book for $V(\Sigma)$ that supports ξ.*

(4) (Birkhoff [5] & Fried [15]) *If P is convex, then the lift of P to $ST\Sigma$ is the binding of a "geodesic" open book, that is an open book whose pages are negative*[3] *Birkhoff cross sections of the geodesic flow.*

Remark 1.2. The open books in items (1)-(3) are independent of the given metric on the surface Σ, up to isomorphism. Moreover, when Σ is of genus at least two and the metric is hyperbolic, the geodesic flow is independent of the metric, up to homeomorphism, by a theorem of Gromov [20] (see Section 5 for details) and hence the geodesic open books in item (4) are uniquely determined up to isomorphism. More generally, if Σ is of any genus and is equipped with two different metrics, the resulting geodesic open books in item (4) are isomorphic as long as P is *convex* with respect to both metrics.

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[1] There is orientation error in [23, Lemma 2.6]: The Lefschetz fibration $DT\Sigma \to D^2$ Ishikawa constructs must be *achiral* or equivalently, it must be considered on the cotangent (rather than tangent) disk bundle $DT^*\Sigma$.

[2] The fact that f_ξ is ordered, provided that f is ordered, was observed by Massot [28] along with some other details of Giroux's construction [18, Example 4.9] restricted to dimension three.

[3] We use the term *negative* since the natural orientation given by the geodesic flow of the binding of this open book is opposite to that induced from a natural orientation of the page.

Note that the open books in items (2) and (3) support the canonical contact structure ξ on $ST^*\Sigma \cong V(\Sigma)$ by construction, and the "duals" of the geodesic open books in item (4), when considered for the cogeodesic flow on $ST^*\Sigma$, also support ξ, since the cogeodesic flow agrees with the Reeb flow of the Liouville 1-form λ (see, for example, [16, Theorem 1.5.2]). As we explain in Remark 2.1 below, Ishikawa's Lefschetz fibrations should be considered on $DT^*\Sigma$ (rather than $DT\Sigma$) and thus they induce open books for $ST^*\Sigma$. Therefore, the A'Campo-Ishikawa open books in item (1) support ξ as well when they are viewed for $ST^*\Sigma$, as a consequence of Theorem 1.1.

Corollary 1.3. *For each admissible divide P on a closed and oriented surface Σ, the A'Campo-Ishikawa open book (when viewed for $ST^*\Sigma$) supports the canonical contact structure ξ.*

Our method of proof also implies that for each connected divide on D^2, the open book that A'Campo constructs for $\mathbb{S}^3$ supports the standard contact structure, which can also be deduced from the main result of another article of Ishikawa [22]. Notice that A'Campo already proved [1, Section 4] that the binding of his open book is transverse to the standard contact structure, which is a necessary condition.

Provided that the genus of Σ is at least one, none of the open books mentioned in Theorem 1.1 are planar, by construction. This is consistent with the fact that the support genus of $(ST^*\Sigma, \xi)$ is equal to one for a surface Σ of positive genus, which follows by combining an obstruction to planarity due to Etnyre [13], with constructions of Massot [28] when the genus of Σ is at least two and Van Horn-Morris [32] for a surface Σ of genus one (see [30] for further details). The only case for which the open book in Theorem 1.1 is planar is the one where the divide P is a single embedded circle on S^2. The corresponding open book adapted to $(ST^*S^2 = \mathbb{R}P^3, \xi)$ is planar, whose page is an annulus and monodromy the square of the Dehn twist along the core circle. Here, we characterize those open books in Theorem 1.1 with genus one pages.

Theorem 1.4. *Up to homeomorphism, there are exactly three admissible divides on a closed and oriented surface Σ of genus at least one, as depicted in Figure 8, that yield genus one open books obtained by any of the methods listed in Theorem 1.1. These open books have $4g, 4g+2$, and $4g+4$ binding components, respectively.*

As another consequence of Theorem 1.1, one can draw the following conclusion. The monodromies of the open books in item (3), which are not immediately clear from their construction, can be computed via Theorem 1.1, since the monodromies of the open books in items (1) and (2) are given explicitly as a product of Dehn twists [1, 24] and the monodromies in items (1) and (4) are given as a product of two involutions [3, 12].

Moreover, Theorem 1.1 answers Questions (1) and (3) affirmatively listed at the end of [30], while Theorem 1.4 and the discussion in Section 7 give a negative answer for Question (4).

1.1. **Pairwise identification of the 3-manifolds in Theorem 1.1.** The identification of $V(\Sigma)$ with $ST\Sigma$ is obtained by taking the unit tangent vector positively normal to the given cooriented line. The identification of $V(\Sigma)$ with $ST^*\Sigma$ is obtained by further composing the identification above with the natural bundle map $ST\Sigma \to ST^*\Sigma$ induced by the Riemannian metric on Σ. Equivalently, a more direct identification of $V(\Sigma)$ with $ST^*\Sigma$ is obtained by taking for each cooriented line L in $T_q\Sigma$, the unit covector (a linear map $T_q\Sigma \to \mathbb{R}$) in $T_q^*\Sigma$, whose kernel is L with its coorientation. In other words, the bundle of cooriented lines tangent to Σ has a natural identification with the bundle of rays in $T^*\Sigma$, which therefore can be denoted by $ST^*\Sigma$ (see, for example, [29]), but for the purposes of this paper, we opted to denote the bundle of cooriented lines by $V(\Sigma)$, to distinguish it from the unit cotangent bundle, although they are orientation-preserving diffeomorphic to each other.

1.2. **Open book constructions in chronological order.** Our goal here is to bring together seemingly unrelated work in the literature over a span of more than one hundred years, going all the way back to Birkhoff's article [5] on dynamical systems, where he constructs a "surface of section" for the geodesic flow on the unit tangent bundle $ST\Sigma$. The existence of such a surface reduces the study of some dynamical properties of the flow to understanding a self-homeomorphism of this surface. Birkhoff's approach was later popularized by Fried [15], who utilized it to study arbitrary transitive Anosov flows on closed 3-manifolds. With the terminology used in this paper, the surface of section S is a page of an open book for $ST\Sigma$, so that the interior of S is transverse to the geodesic flow, while the binding ∂S is a collection of periodic orbits.

The notion of a divide on D^2, and the link of a divide in $\mathbb{S}^3$, was introduced by A'Campo [1, 2], who was interested in computing geometric monodromies of isolated plane curve singularities. In particular, A'Campo proved that the link of any isolated plane curve singularity appears as the link of a divide, the link of a connected divide is fibered, and that this fibration is a model for the Milnor fibration of the singularity. Note that, for each connected divide, A'Campo's construction gives nothing but an open book for $\mathbb{S}^3$ whose binding is the link of the given divide. The work of A'Campo was later generalized by Ishikawa [23], who showed that the link of an admissible divide[4] on an arbitrary surface Σ (not just D^2) is fibered in $ST\Sigma$ and moreover $DT\Sigma$ admits an achiral Lefschetz fibration over D^2, which induces the same open book on the boundary.

In an other direction, at the turn of this century, a major breakthrough in contact topology was achieved by Giroux [19], who proved that every closed contact manifold is convex, which is equivalent to the fact that every contact manifold admits an adapted open book with Weinstein pages. By combining with the earlier work of Giroux [18], it follows in particular that $V(\Sigma) \cong ST^*\Sigma$ admits an open book [28] that supports its canonical contact structure ξ.

Finally, building on the fundamental work of Seidel [31], Johns [24] constructed an exact symplectic Lefschetz fibration on the symplectic disk cotangent bundle $(DT^*\Sigma, \omega)$, which in turn, restricts to an open book for $ST^*\Sigma$ supporting ξ. The difference from Ishikawa's Lefschetz fibration is that Johns' fibration is not just a smooth fibration but adapted to the canonical symplectic structure $\omega = d\lambda$, meaning that the regular fibers are symplectic submanifolds.

In this article, we relate the four different constructions of open books described in the last four paragraphs, where the first one involves the dynamics of the geodesic flow, the second one is motivated by the study of isolated plane curve singularities, the third one illustrates the power of convexity in contact topology and the last one has a distinctive symplectic flavor.

Remark 1.5. It is not true that every open book adapted to $(ST^*\Sigma, \xi)$ arises from an admissible divide or an ordered Morse function on Σ, using any one of the (eventually equivalent) constructions in Theorem 1.1. This is simply because each open book appearing in Theorem 1.1 has an even number of binding components and there is no reason for this to hold for an arbitrary open book adapted to $(ST^*\Sigma, \xi)$. For example, a genus one open book adapted to $(ST^*T^2 = T^3, \xi)$ with *three* binding components, was constructed by Van Horn-Morris [32] (see also [11, Section 3] for similar examples obtained by the first author). Note that one can always get an adapted open book with an odd number of binding components, by positively stabilizing an adapted open book with an even number of binding components, but this is not the case for the aforementioned example of Van Horn-Morris.

Outline of the paper: First, we review in some details the four distinct constructions above due to A'Campo & Ishikawa, Johns, Giroux, and Birkhoff & Fried in Sections 2 to 5, respectively. Then we describe a proof of Theorem 1.1 in Section 6, while we give a proof of Theorem 1.4 in Section 7.

[4]Birkhoff used the term *primary set of curves* for what Ishikawa (following A'Campo) called an *admissible divide.*

2. Complexification of a Morse function on a surface

In this section, we briefly review Ishikawa's construction [23] of Lefschetz fibrations on the disk (co)tangent bundle of a closed and oriented surface Σ, which is a generalization of the work of A'Campo [1, 2]. These Lefschetz fibrations induce natural open books for the unit (co)tangent bundle by restricting to the boundary.

For any admissible divide $P \subset \Sigma$, there is an ordered Morse function $f : \Sigma \to \mathbb{R}$ adapted to P, satisfying the following conditions

- $P = f^{-1}(0)$,
- each double point of P corresponds to a critical point of f of index 1, and
- each black (resp. white) region of $\Sigma \setminus P$ contains exactly one index 2 (resp. 0) critical point of f.

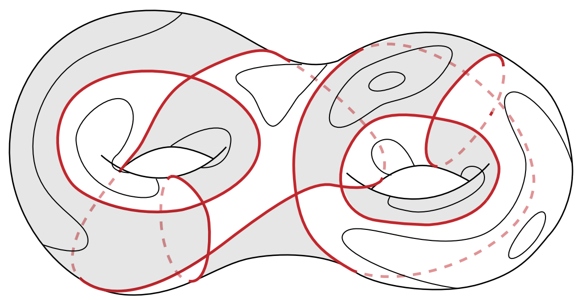

Figure 1. An admissible divide P (in red) on a genus 2 surface, a black-and-white coloring of the complement $\Sigma \setminus P$, and some level sets of an ordered Morse function adapted to P. The divide P consists of six curves ($c(P) = 6$) and includes seven double points ($v(P) = 7$).

The Morse function $f : \Sigma \to \mathbb{R}$ can be extended to an *almost complexified* Morse function $f_{\mathbb{C}} : T\Sigma \to \mathbb{C}$ defined as

$$f_{\mathbb{C}}(x, u) = f(x) + i\eta df_x(u) - \frac{1}{2}\eta^2\chi(x)d^2 f_x(u, u) \tag{2.1}$$

for $x \in \Sigma$ and $u \in T_x\Sigma$. Here the linear map $df_x : T_x\Sigma \to \mathbb{R}$ denotes the differential, and the bilinear map $d^2 f_x : T_x\Sigma \times T_x\Sigma \to \mathbb{R}$ denotes the Hessian of f at the point x. The smooth map $\chi : \Sigma \to \mathbb{R}$ is a suitable cut-off function which vanishes outside of a sufficiently small neighborhood of the critical points, and η is a positive real number, whose role will be clarified below.

The map $f_{\mathbb{C}}$ is almost complexified in the sense that the Morse singularities of f becomes complex Morse singularities of $f_{\mathbb{C}}$. More precisely, the Morse singularities of f of the form $-x_1^2 - x_2^2, -x_1^2 + x_2^2, x_1^2 + x_2^2$ in local coordinates on Σ becomes singularities of $f_{\mathbb{C}}$ of the form $-z_1^2 - z_2^2, -z_1^2 + z_2^2, z_1^2 + z_2^2$, respectively, in local complex coordinates on $T^*\Sigma$, which are all equivalent to the complex Morse singularity $z_1^2 + z_2^2$, under a complex change of coordinates. Moreover, by choosing η sufficiently small, one can guarantee that there are no other critical points of $f_{\mathbb{C}}$. It follows that a point in $T\Sigma$ is a critical point of $f_{\mathbb{C}}$ if and only if it belongs to Σ and it is a critical point of f. The map $f_{\mathbb{C}} : T\Sigma \to \mathbb{C}$ descents to an *achiral* (see Remark 2.1) Lefschetz fibration $DT\Sigma \to D^2$. The singular fiber over $0 \in D^2$ contains a singularity for each double point of P. Corresponding to each index 2 or index 0 critical point of f, there is a singular fiber containing a unique singularity. Moreover, the regular fiber and the vanishing cycles of the Lefschetz fibration $DT\Sigma \to D^2$, are described explicitly in [23] by a method due to A'Campo [2].

Remark 2.1. There is an orientation issue in [23, Lemma 2.6]. Following the notation in [23], suppose that x_1, x_2 are local coordinates on the surface Σ and u_1, u_2 are the corresponding coordinates in the *tangent* fibers, so that $\{x_1, x_2, u_1, u_2\}$ is an oriented chart for $T\Sigma$. But the orientation of this chart is opposite to the complex orientation that is required in the definition of a Lefschetz fibration, since Ishikawa uses the complex chart

$$(z_1, z_2) = (x_1 + iu_1, x_2 + iu_2)$$

for $T\Sigma$ in the proof of [23, Lemma 2.6]. Except the orientation issue, however, everything else works in Lemma 2.6 and the subsequent discussion leading to Proposition 3.1 and Lemma 3.2 in [23]. In other words, the Lefschetz fibration Ishikawa constructs is *achiral* on $DT\Sigma$ and the corresponding Lefschetz fibration $DT^*\Sigma \to D^2$, obtained by reversing the orientation, induces an open book on the boundary $ST^*\Sigma$. It follows, by Theorem 1.1, that this open book supports the contact canonical structure on $ST^*\Sigma$, which is not clear from Ishikawa's work [23].

Remark 2.2. When restricted to $ST\Sigma$, the complexification map $f_{\mathbb{C}}$ in (2.1) vanishes on the link $L(P)$ of the divide P, which is the set of unit vectors tangent to P (cf. [2, 23]). Moreover, provided that η is sufficiently small, this map restricts to a fibration $ST\Sigma \setminus L(P) \to S^1$, obtained by dividing the image by its norm, and therefore inducing an open book for $ST\Sigma$ with binding $L(P)$, hereafter called *A'Campo-Ishikawa* open book. If the unit circle S^1 is parametrized by θ, then we denote the closure of the inverse image of θ under the fibration map as S_θ, the θ-page.

As depicted in Figure 2, the interior of S_0 (resp. S_π) consists of the unit vectors tangent to the level sets of f in black (resp. white) regions, while the interior of $S_{\pi/2}$ (resp. $S_{3\pi/2}$) consists of the unit vectors along the divide P pointing into black (resp. white) regions of $\Sigma \setminus P$, except around the double points of the divide where

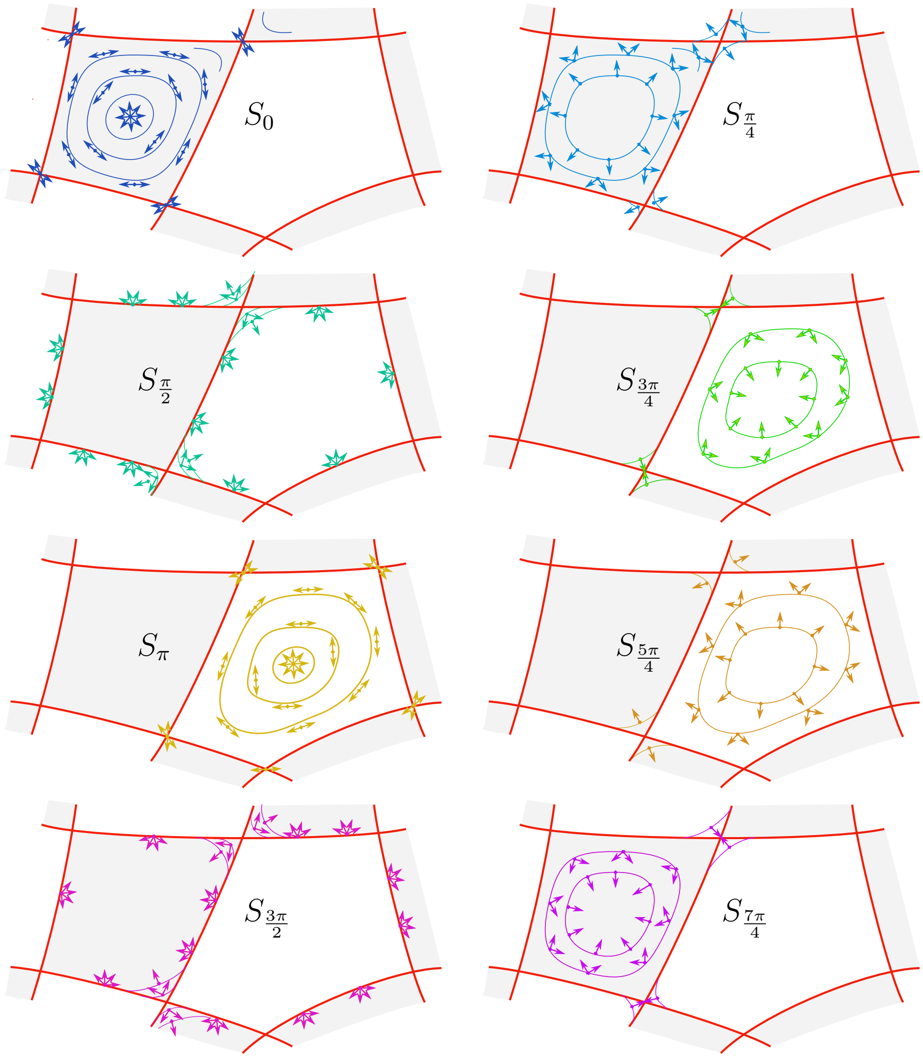

FIGURE 2. Some of the pages S_θ of the A'Campo-Ishikawa open book for $ST\Sigma$, viewed as sets of vectors tangent to Σ.

the term involving the Hessian in Equation (2.1) smooths the surface and separate the surfaces associated to different values of θ. Around the double points of P, where the function χ does not vanish, the surfaces S_θ are depicted in Figure 3.

The topological type of the pages of the A'Campo-Ishikawa open book can be easily determined as follows. Denote by $v(P)$ the number of double points of a divide P and by $c(P)$ the number of circles that form P (see Figure 1, for an example).

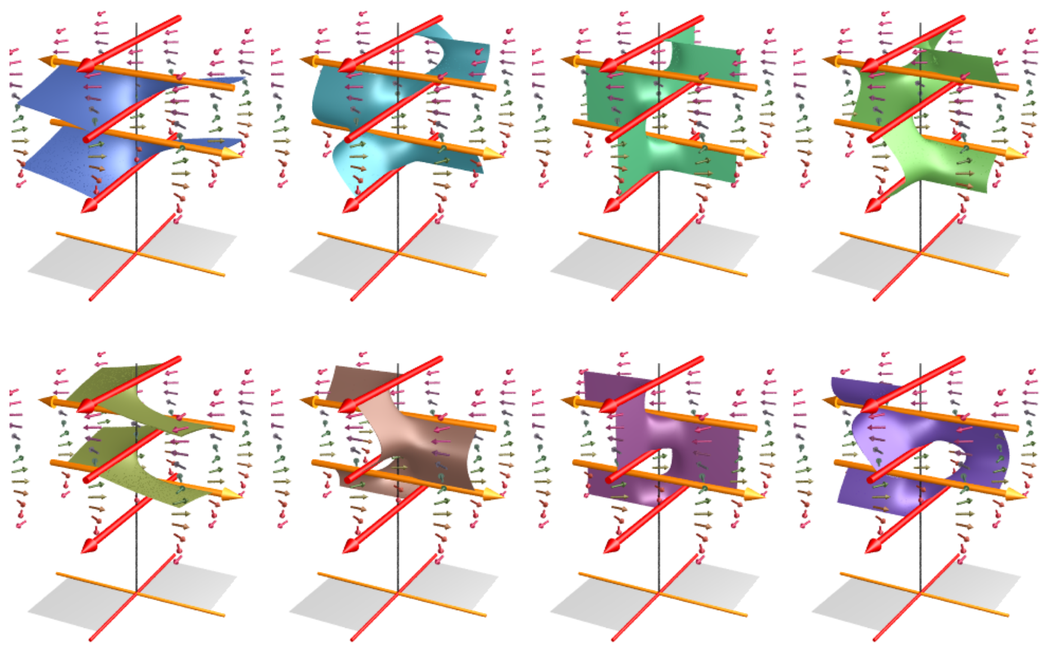

FIGURE 3. The pages S_θ of the A'Campo-Ishikawa open book for $ST\Sigma$ around the fiber of a double point of a divide P, for $\theta = \frac{k\pi}{4}$ with $k = 0, \ldots, 7$.

Proposition 2.3. *For any admissible divide P, the A'Campo-Ishikawa open book has $2c(P)$ binding components and its page genus is given by $1 + v(P) - c(P)$.*

Proof. Since all the pages have the same genus, we focus on one page, say $S_{\frac{\pi}{2}}$, in particular. First notice the boundary of $S_{\frac{\pi}{2}}$ is the link $L(P)$ of the divide, which has two components for every circle in P. Now consider $P \subset \Sigma$ as a graph whose vertices are the double points of P and whose edges are the segments of P connecting these double points. Then the surface $S_{\frac{\pi}{2}} \subset ST\Sigma$ is made of one rectangle in the fiber of every segment of P. The horizontal sides of these rectangles are in the link $L(P)$, hence in the boundary of $S_{\frac{\pi}{2}}$. The vertical sides can be divided into two segments, which are glued with the rectangle associated to the next and previous edge of P respectively, when rotating around the corresponding vertex of P, as depicted in Figure 3. One checks that every rectangle contributes by -1 to the Euler characteristics of $S_{\frac{\pi}{2}}$. Since P has twice more edges than vertices, we deduce that $\chi(S_{\frac{\pi}{2}}) = -2v(P)$. The number of boundary components of $S_{\frac{\pi}{2}}$ is $2c(P)$. Hence the genus of $S_{\frac{\pi}{2}}$ is $1 + v(P) - c(P)$. □

3. An exact symplectic Lefschetz fibration on the cotangent bundle

Let λ denote the Liouville 1-form on the cotangent bundle $T^*\Sigma$. It is well-known that the symplectic disk cotangent bundle $(DT^*\Sigma, \omega = d\lambda)$ is a Weinstein domain and in fact a Stein filling of its contact boundary $(ST^*\Sigma, \xi = \ker \lambda)$, see, for example, [8]. Therefore, according to [4, 25], the Stein filling $DT^*\Sigma$ admits a smooth Lefchetz fibration over D^2.

Here we briefly review Johns' construction [24] of an *exact symplectic* Lefschetz fibration on $(DT^*\Sigma, \omega = d\lambda)$, using methods other than those described in [4, 25]. Apparently inspired by the work A'Campo on the complexification of smooth Morse functions (but not referring to Ishikawa's work [23]), Johns starts with an *educated guess* for the fiber and the vanishing cycles of a smooth Lefschetz fibration $E \to D^2$. By a standard method (see, for example, [31]) the total space E can be equipped with an exact symplectic form so that the regular fibers are symplectic submanifolds of E and the vanishing cycles are Lagrangian submanifolds of a regular fiber. The crux of the matter is that the construction of the fiber depends on the choice of a smooth Morse function $f : \Sigma \to \mathbb{R}$ and for each critical point of f, there is a vanishing cycle of the fibration. More precisely, the fiber is constructed by a smooth plumbing of some annuli, each one of which represents a neighborhood of a vanishing cycle corresponding to a critical point of f of index zero or one. The plumbing is dictated by the way that the 1-handles are attached to the 0-handles in the handle decomposition of Σ given by f. Moreover, the vanishing cycles corresponding to the critical points of f of index two are obtained from the vanishing cycles corresponding to the critical points of f of index zero and one, by a simultaneous "Lagrangian surgery" where they meet on the fiber.

To summarize, based on the handle decomposition of the surface Σ given by f, Johns describes how to construct (the diffeomorphism type of) the regular fiber of an exact symplectic Lefschetz fibration $f_\omega : E \to D^2$, and the Lagrangian vanishing cycles explicitly on this fiber. The map f_ω can be viewed as, roughly speaking, a symplectic convexification of the smooth Morse function f with respect to ω, as opposed to a contact convexification due to Giroux, which we will discuss in the next section.

The hardest part of Johns' article is his proof that the exact symplectic 4-manifold E with convex boundary is conformally exact symplectomorphic to $(DT^*\Sigma, \omega)$, after smoothing the corners of E, which he accomplishes by following the steps (i)-(iii) below.

(i) There is an exact Lagrangian embedding $\Sigma \subset E$.

(ii) Critical points of f_ω are in Σ, $f_\omega(\Sigma) = [a, b] \subset \mathbb{R}$, and $f_\omega|_\Sigma = f$ (up to reparametrizing Σ and $\mathbb{R}$ by diffeomorphisms).

(iii) E is conformally exact symplectomorphic to $(DT^*\Sigma, \omega)$.

Recall that a conformal exact symplectomorphism between exact symplectic manifolds $(E_i, \omega_i, \theta_i)$ (where $\omega_i = d\theta_i$ is a symplectic form on E_i), for $i = 1, 2$, is a map

$$\psi : (E_1, \omega_1, \theta_1) \to (E_2, \omega_2, \theta_2)$$

such that $\psi^*\theta_2 = K\theta_1 + dh$, for some $K > 0$ and $h : E_1 \to \mathbb{R}$. In particular, $\psi^*\omega_2 = K\omega_1$, which indeed implies that ψ is a conformal symplectomorphism.

The main step in the proof is (i), where Johns uses a *Milnor style* handle decomposition of Σ given by the Morse function f, to embed Σ into E "handle by handle" as an exact Lagrangian submanifold. The statement (ii) follows by the construction of the embedding $\Sigma \subset E$. The last step (iii) is achieved by describing a retraction of E, using a Liouville type flow, onto a small Weinstein neighborhood of Σ in E, symplectomorphic to $DT^*\Sigma$.

4. Convex Morse functions on the bundle of hyperplanes

Our goal in this section is to review Giroux's construction [18] of an open book adapted to the contact 3-manifold $(V(\Sigma), \xi)$, where ξ is the canonical contact structure on the bundle $V(\Sigma)$ of cooriented lines tangent to Σ. We begin with some general facts about convex contact manifolds, and then give some details of Giroux's proof of the convexity of the canonical contact structure on the bundle of cooriented hyperplanes $V(M)$ for an arbitrary smooth manifold M^{n+1} in Section 4.1, before we turn our attention to the case $n = 1$ (i.e, $M = \Sigma$, a surface) in Section 4.2.

Definition 4.1. [18] A contact structure ζ is called *convex* if it is invariant under the flow of a vector field X which is gradient-like for a Morse function. Such a function is called ζ-convex (or *contact*) Morse, and X is called a *contact* vector field. The *characteristic hypersurface* C_X is defined as the set of points where X is tangent to ζ.

In his groundbreaking work, Giroux proved that every contact structure is convex, which implies that every contact manifold admits an adapted open book by Theorem 4.2 below.

Theorem 4.2 (Giroux). *Let (V, ζ) be a closed contact manifold of dimension $2n + 1$. Suppose that X is a contact vector field on V, which is gradient-like for an ordered Morse function $F : V \to \mathbb{R}$. Let L be a regular level set of F above the critical values of index n and below the critical values of index $n + 1$. Then there is an open book adapted to (V, ζ) whose binding K is the transverse intersection of L and the characteristic hypersurface C_X. Moreover, the link $K = L \cap C_X$ cuts $L \cup C_X$ into four pages of this open book.*

Conversely, any open book for V supporting ξ comes from this construction.

4.1. **Convexity of the canonical contact structure on the bundle of hyperplanes.** Let $V(M)$ denote the bundle of cooriented hyperplanes tangent to a closed manifold M^{n+1} and ξ denote the canonical contact structure on $V(M)$. Note that $V(M)$ can be identified with the unit cotangent bundle ST^*M and ξ is given by the kernel of the Liouville 1-form λ, under this identification (see, for example, [16, page 32]). In the following, we review Giroux's construction [18, Example 4.8] of a ξ-convex Morse function on $V(M)$.

Any diffeomorphism $\psi : M \to M$ lifts to a contactomorphism $\widetilde{\psi}$ of $(V(M), \xi)$ defined by

$$\widetilde{\psi}(x, H) = (\psi(x), d\psi_x(H)).$$

It follows that any vector field X on M lifts to a contact vector field $\widehat{X}$ on $(V(M), \xi)$ by lifting the flow of X.

Let $f : M \to \mathbb{R}$ be a Morse function and X be a gradient-like vector field for f such that at each critical point of p of f, the eigenvalues of D_pX are *real* and *simple*. Then there is a contact vector field $\widehat{X}$ for $(V(M), \xi)$ obtained as the lift of X as explained above. To show that $(V(M), \xi)$ is convex, it suffices to construct a Morse function $f_\xi : V(M) \to \mathbb{R}$ such that $\widehat{X}$ is gradient-like for f_ξ. For any critical point p of f, let $V_p(M) := \pi^{-1}(p)$ denote the sphere fiber, where $\pi : V(M) \to M$ is the bundle projection. Since $\widehat{X}$ projects to X, it is vertical above the critical points of f, i.e., it is tangent to V_p. The restriction of $\widehat{X}$ to $V_p(M)$ is the gradient of some Morse function $g_p : V_p(M) \to \mathbb{R}$, having exactly $2n + 2$ critical points, corresponding to the $n + 1$ hyperplanes generated by n eigendirections of D_pX.

We set

$$f_\xi := f + \sum_{p \in Crit(f)} \chi_p g_p, \tag{4.2}$$

where χ_p is a suitably defined cut-off function which vanishes outside of a sufficiently small neighborhood of the critical points.

4.2. **Canonical contact structure on the bundle of cooriented lines.** We restrict our attention to the case $n = 1$, namely we take M to be a closed and oriented surface Σ. Suppose that $f : \Sigma \to \mathbb{R}$ is an ordered Morse function and X is a gradient-like vector field for f on Σ (see Figure 4 for an example). The vector field X lifts to a vector field $\widehat{X}$ on $V(\Sigma)$, which is gradient-like for the modified lift $f_\xi : V(\Sigma) \to \mathbb{R}$. As observed by Massot [28], the ξ-convex Morse function $f_\xi : V(\Sigma) \to \mathbb{R}$ defined by Equation (4.2) is *ordered* provided that $f : \Sigma \to \mathbb{R}$ is

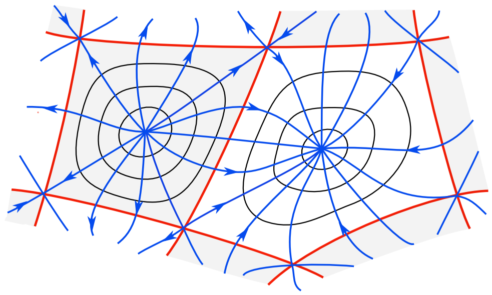

FIGURE 4. A gradient-like vector field for an ordered Morse function associated to a divide on a surface.

an *ordered* Morse function[5]. In particular the singularities of f lift to quadruples of singularities of f_ξ in $V(\Sigma)$, as for example, we depicted on the left in Figure 5 for an index 1 singularity that lifts to two index 1 and two index 2 singularities.

Now one can apply Theorem 4.2 to find an open book, which we call a *Giroux* open book, that supports the canonical contact structure ξ on $V(\Sigma)$ using the ξ-convex ordered Morse function $f_\xi : V(\Sigma) \to \mathbb{R}$. On the right in Figure 5, we depicted the characteristic surface $C_{\widehat{X}}$ (yellow) and the regular level $f_\xi^{-1}(0)$ (green) around the fiber of a double point of the divide. They intersect along the link consisting of the cooriented lines based at P and tangent to ξ.

Suppose that P is an admissible divide on Σ and let $f : \Sigma \to \mathbb{R}$ be an ordered Morse function adapted to P. Recall that $c(P)$ is the number of circles in P and $v(P)$ is the number of double points of P. By our assumption, $v(P)$ is equal to the number of index 1 critical points of f.

Proposition 4.3. [29] *For any admissible divide P, the Giroux open book has $2c(P)$ binding components and its page genus is given by $1 + v(P) - c(P)$.*

Proof. The indices of the critical points of $f_\xi : V(\Sigma) \to \mathbb{R}$ can be computed from the indices of the critical points of $f : \Sigma \to \mathbb{R}$, which in turn, determines the topology of the characteristic surface $C_{\widehat{X}}$. This is because $f_\xi|_{C_{\widehat{X}}}$ is a Morse function whose critical points are exactly the critical points of f_ξ and moreover an index i critical point of f_ξ gives a critical point of $f_\xi|_{C_{\widehat{X}}}$, whose index is i if $i \leq 1$, and $i-1$, otherwise. Since $C_{\widehat{X}}$ is the union of the two pages of an open book supporting ξ, the topology of the page is determined by the number of connected components of the binding, which is equal to twice the number of circles in $P = f^{-1}(0)$. Since

[5]Massot [28] uses a self-indexed Morse function f, but the conclusion holds true if self-indexed is replaced by ordered. Here we assume that f is ordered and $f^{-1}(0)$ includes all index 1 critical points.

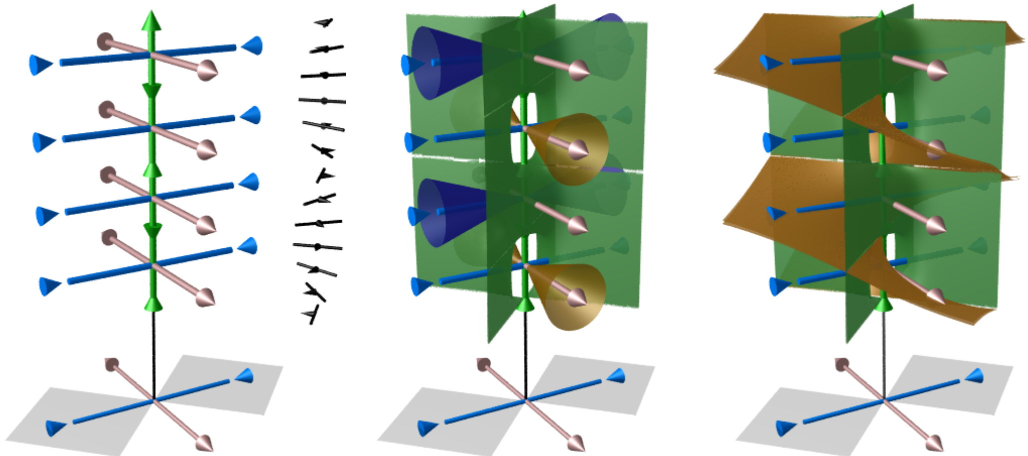

FIGURE 5. Given a vector field X on a surface Σ and a hyperbolic fixed point p, the differential of the flow shrinks tangent lines toward the expanding eigendirection. The lift $\widehat{X}$ in the bundle $V(\Sigma)$ has four critical points in the fiber above p, which correspond to the cooriented lines tangent to the eigendirections as depicted on the left. When X is a (pseudo-) gradient of an ordered Morse function f on Σ, one can lift f to $V(\Sigma)$ and modify it using Equation (4.2) into an ordered Morse function f_ξ, so that the critical points of f_ξ are the critical points of $\widehat{X}$ and the levels of f_ξ away from the fibers of the critical points of f are the lifts of the levels of f. In the center we depicted two critical level sets (blue and yellow) and the intermediate regular level set (green) $f_\xi^{-1}(0)$. On the right, we depicted the level set $f_\xi^{-1}(0)$ (green) and the characteristic surface $C_{\widehat{X}}$ (yellow).

$v(P)$ is equal to the number of critical points of index one of the Morse function f, by our assumption, the Euler characteristic of the page is given by $-2v(P)$, and thus the genus of the page is given by $1 + v(P) - c(P)$. □

It follows that the topology of the pages of the Giroux open book agrees with that of the A'Campo-Ishikawa open book (see Proposition 2.3).

5. GEODESIC FLOW AND BIRKHOFF SECTIONS

In this section we recall Birkhoff's classical construction [5] of a negative Birkhoff cross section for the geodesic flow from a convex divide (which he called a primary set of curves) on a surface. The geodesic flow Φ is the flow on $ST\Sigma$, whose orbits are the lifts of the geodesics on Σ. More precisely, if g is a geodesic with unit speed on the Riemann surface Σ, then the orbit of Φ going through $(g(0), \dot{g}(0))$ is given by

$$\Phi^t(g(0), \dot{g}(0)) = (g(t), \dot{g}(t)).$$

Although the geodesic flow depends on the choice of a Riemannian metric on Σ, Gromov [20] showed that the geodesic flows corresponding to two negatively curved metrics on a surface are topologically conjugated, which means that there is a homeomorphism between the unit tangent bundles of Σ with respect to these metrics, taking the oriented orbits of one onto the other. We conclude that as far as the topological (rather than dynamical) properties are concerned, there is essentially a unique geodesic flow on a negatively curved surface.

Definition 5.1. Suppose that X is a non-singular vector field on a closed and oriented 3-manifold M. A *Birkhoff cross section*[6] for (M, X) is a compact orientable surface S with boundary such that S is embedded in M, X is transverse to the interior of S, the boundary ∂S is tangent to X, and every orbit of X intersects S after a bounded time (see Figure 6 left).

It follows that ∂S is the union of finitely many periodic orbits of X, and near its boundary, a Birkhoff cross section S looks like a *helicoidal staircase* (see Figure 6 right). Notice that S is cooriented by X since the interior of S is transverse to X. Therefore, there is an induced orientation on S (and on its boundary ∂S), since M is oriented. On the other hand, ∂S has a natural orientation as a collection of periodic orbits of the vector field X. A Birkhoff cross section S is said to be a *positive* (resp. *negative*) if for each component of ∂S, the natural orientation given by X coincides with (resp. is opposite of) its orientation inherited as the boundary of S.

Notice that a positive Birkhoff cross section S for (M, X) is a page of an open book for M whose oriented binding is ∂S. On the other hand, if S is a negative Birkhoff cross section for (M, X), then $\overline{S}$ is a positive Birkhoff cross section for $(\overline{M}, X)$ and hence $\overline{S}$ a is a page of an open book for $\overline{M}$ whose oriented binding is $\overline{\partial S}$.

Definition 5.2. We say that an admissible divide P on a surface Σ equipped with a Riemannian metric is *convex* if every curve in P is a closed geodesic, every geodesic on Σ intersects P in bounded time, and every region of $\Sigma \setminus P$ can be foliated by concentric curves with non-vanishing curvature. In this context, a foliation of the regions of $\Sigma \setminus P$ by concentric curves is called a *convex* foliation.

[6] also called Poincaré-Birkhoff section, or global cross section

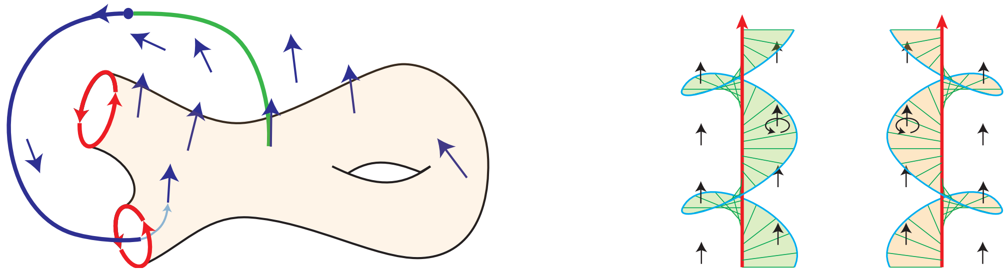

FIGURE 6. Birkhoff cross sections: on the left the general picture, and on the right how it looks around boundary components, assuming the flow is locally vertical, for a negative and a positive Birkhoff cross section, respectively.

Note that every admissible divide on a surface with constant curvature is convex. An admissible divide may fail to be convex if, for example, there is a closed geodesic staying in one region of $\Sigma \setminus P$, as depicted in Figure 7.

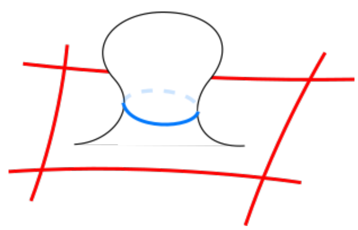

FIGURE 7. A divide (red) that is not convex: the geodesic (blue) does not intersect the divide.

Theorem 5.3 ([5, 15]). *Assume that P is a convex divide and $\mathcal{F}$ is convex foliation of $\Sigma \setminus P$. The set of pairs (p, v), where p belongs to some black region of $\Sigma \setminus P$ and v is a unit vector tangent to $\mathcal{F}$ is a Birkhoff cross section for the geodesic flow on $ST\Sigma$.*

The proof of Theorem 5.3 is straightforward. Denoting by $S^\bullet$ the set of tangent vectors mentioned in Theorem 5.3, it is easily seen that $S^\bullet$ is a surface whose boundary is the link $L(P)$ of the divide P. Hence the boundary $\partial S^\bullet$ is indeed tangent to the geodesic flow. Since the foliation $\mathcal{F}$ is convex, the contact between leaves of this foliation and geodesic is only of order 1 (because of the assumption on the curvature), hence of order 0 in the tangent bundle. This means that the interior of $S^\bullet$ is transverse to the geodesic flow. Finally since every geodesic on Σ intersects P in bounded time, every orbit of the geodesic flow intersects $S^\bullet$ in bounded time.

Remark 5.4. Theorem 5.3 provides one Birkhoff cross section canonically associated to a given divide P, which corresponds to the page S_0 depicted in Figure 2. Another natural Birkhoff cross section is the set of pairs (p, v), where p belongs to some white region of $\Sigma \setminus P$ and v is a unit vector tangent to $\mathcal{F}$. It corresponds to the page S_π, also depicted in Figure 2.

Remark 5.5. Isotopy classes of *negative* Birkhoff cross sections whose boundary is symmetric were classified by Cossarini-Dehornoy [9]. Marty proved that a flow cannot admit at the same time positive and negative Birkhoff sections [27], hence there are no *positive* Birkhoff cross sections for geodesic flows.

6. Proof of Theorem 1.1

The proof of Theorem 1.1 follows by combining Proposition 6.2, Proposition 6.3, and Lemma 6.8 below.

Definition 6.1. Suppose that P is an admissible divide on a closed and oriented surface Σ which is equipped with a Riemannian metric and let $f : \Sigma \to \mathbb{R}$ be an ordered Morse function adapted to P. Then

- $\overline{\pi}_1 : DT\Sigma \to D^2$ denotes Ishikawa's *achiral* Lefschetz fibration and $\overline{\mathrm{OB}}_1$ the induced open book for $ST\Sigma$. Similarly, $\pi_1 : DT^*\Sigma \to D^2$ denotes the corresponding Lefschetz fibration, obtained by reversing the orientation of $\overline{\pi}_1$, and OB_1 the induced open book for $ST^*\Sigma$ (see Remark 2.1).
- $\pi_2 : DT^*\Sigma \to D^2$ denotes Johns' Lefschetz fibration and OB_2 the induced open book for $ST^*\Sigma$, that supports the canonical contact structure ξ. (Note that $f_\omega : E \to D^2$ described in Section 3 induces a Lefschetz fibration $DT^*\Sigma \to D^2$, which we denote by π_2 here.)
- OB_3 denotes Giroux's open book for $V(\Sigma)$ that supports the canonical contact structure ξ.
- $\overline{\mathrm{OB}}_4$ denotes the geodesic open book for $ST\Sigma$ induced by the geodesic flow, whose pages are *negative* Birkhoff cross sections, provided that P is *convex* with respect to the metric on Σ, and OB_4 denotes the corresponding open book for $ST^*\Sigma$, whose pages are *positive* Birkhoff cross sections.

We will show below that the open books OB_1, OB_2 and OB_3 are pairwise isomorphic for any surface Σ and OB_4 is isomorphic to any one of them when Σ is a Riemannian surface such that P is convex (see Definition 5.2), which is the content of Theorem 1.1. As a corollary, it follows that they all support the canonical contact structure ξ on $ST^*\Sigma \cong V(\Sigma)$.

Given a Morse function f which satisfies the hypothesis of Theorem 1.1, for $i \in \{0, 1, 2\}$, denote by m_i the number of critical points of index i.

Proposition 6.2. *The Lefschetz fibrations π_1 and π_2 on $DT^*\Sigma$ are isomorphic.*

Proof. We claim that π_1 and π_2 are isomorphic, meaning that there is a diffeomorphism from the regular fiber of π_1 to that of π_2 taking the ordered vanishing cycles of π_1 to that of π_2. In [30], we proved this result only for one particular Morse function, which gives the minimum genus Lefschetz fibration on $DT^*\Sigma$. Here we prove it for *all* Morse functions satisfying the assumptions of Theorem 1.1. Notice that an isomorphism between the Lefschetz fibrations π_1 and π_2 induces an isomorphism of the open books OB_1 and OB_2.

The regular fiber F_1 and the ordered vanishing cycles

$$\alpha_1, \ldots, \alpha_{m_0}, \beta_1, \ldots, \beta_{m_1}, \gamma_1, \ldots, \gamma_{m_2}$$

of π_1 are described explicitly in [2, 23] based on the divide $P = f^{-1}(0)$. The fiber F_1 is obtained by starting with m_0 disjoint roundabouts (certain embedded annuli), one for each double point of P, and then connecting them with twisted bands, one for each edge in P connecting any two double points. The curves $\beta_1, \ldots, \beta_{m_1}$ are the core circles of the roundabouts, the curves $\alpha_1, \ldots, \alpha_{m_0}$ appear as the boundaries of the white regions, while $\gamma_1, \ldots, \gamma_{m_2}$ appear as the boundaries of the black regions in $\Sigma \setminus P$.

On the other hand, the regular fiber F_2 and the ordered vanishing cycles

$$a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_1}, c_1, \ldots, c_{m_2}$$

of π_2 are described explicitly in [24] based on the handle decomposition of Σ given by the Morse function f. By our assumptions on f, the surface Σ is obtained by gluing m_1 1-handles to the disjoint union of m_0 disks and then gluing m_2 disks to close off the resulting boundary. The fiber F_2 is obtained by starting with m_0 disjoint annuli $A_0, \ldots, A_{m_0}$ (one for each 0-handle) and plumbing m_1 disjoint annuli $B_0, \ldots, B_{m_1}$ (one for each 1-handle), onto this union, according to the handle decomposition of Σ. More precisely, a B-annulus is plumbed for each 1-handle, exactly corresponding to the feet of that 1-handle in the handle decomposition of Σ. Notice that the vanishing cycle a_i is the core circle of A_i, and the vanishing cycle b_j is the core circle of B_j, while the vanishing cycles $c_1, \ldots, c_{m_2}$ are obtained by a (left) Lagrangian surgery on the set of curves $a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_1}$ at the points where they intersect.

To see that F_1 is diffeomorphic to F_2, we view the construction of F_1 slightly differently. Rather than starting with the roundabouts as we described above, we consider the disjoint union of an annulus neighborhood of each of the curves $\alpha_1, \ldots, \alpha_{m_0}$ and view the roundabouts as a plumbing onto this disjoint union. This was already discussed in [30] for twisted 1-handles and a similar proof holds for the untwisted 1-handles as well, which implies that F_1 is obtained in the same way as F_2 from this point of view. Moreover, this also proves the isomorphism of the ordered vanishing cycles, since first of all α_i can be identified with a_i and β_j can be identified with b_j. Recall that $c_1, \ldots, c_{m_2}$ are obtained by a surgery on $a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_1}$ at the points where they intersect. Similarly, one can check that the curves $\gamma_1, \ldots, \gamma_{m_2}$ are obtained from $\alpha_1, \ldots, \alpha_{m_0}, \beta_1, \ldots, \beta_{m_1}$ at the points where they meet, so that the set $\gamma_1, \ldots, \gamma_{m_2}$ can be identified with the set $c_1, \ldots, c_{m_2}$ and hence the ordered vanishing cycles of π_1 are mapped to that of π_2 under the above diffeomorphism of F_1 and F_2, which finishes the proof of the isomorphism of π_1 and π_2. □

Proposition 6.3. *The open book* OB_1 *for* $ST^*\Sigma$ *and the open book* OB_3 *for* $V(\Sigma)$ *are isomorphic.*

Proof. This is essentially contained in the first author's PhD thesis [10, Chapter 1], where tangent bundles were considered rather than cotangent bundles. Here we work with $V(\Sigma) \cong ST^*\Sigma$ which is more natural from the contact geometric point of view. We would like to show that OB_1 for $ST^*\Sigma$ is orientation-preserving isomorphic to OB_3 for $V(\Sigma)$. Instead, we first set up an isomorphism between $\overline{\mathrm{OB}}_1$ and OB_3, and then derive the conclusion by reversing the orientation on the pages of $\overline{\mathrm{OB}}_1$, as explained in Remark 6.9 below.

Lemma 6.4. *Under the identification of* $ST\Sigma$ *with* $V(\Sigma)$*, the binding of* $\overline{\mathrm{OB}}_1$ *corresponds to the binding of* OB_3*.*

Proof. The binding of $\overline{\mathrm{OB}}_1$ (see Remark 2.2 and Figures 2 and 3) is the set of unit vectors tangent to the divide P. Under the identification of $ST\Sigma$ with $V(\Sigma)$, each unit tangent vector represents a cooriented line, by taking its normal. Now we show that the set of these cooriented lines normal to P is the binding of OB_3 for $V(\Sigma)$. Recall that the characteristic surface $C_{\widehat{X}}$ (see Section 4.2) consists of those cooriented lines containing the gradient-like vector field X for f, which can be assumed to be the gradient ∇f away from the critical points of f. Notice that we fixed f from the beginning of the proof, but we are free to choose the gradient-like vector field X, as long as it is in a certain form near the critical points of f as we explained in Section 4.1.

The binding of OB_3 is given by the intersection of $f_\xi^{-1}(0)$ with $C_{\widehat{X}}$. First of all, we observe that $f_\xi^{-1}(0)$ does not intersect the fibers above the Morse charts around index 2 or index 0 critical points. Away from the index 1 critical points of f, the surface $f_\xi^{-1}(0)$ consists of all the circle fibers above the divide $P = f^{-1}(0)$. Since $X = \nabla f$ away from the critical points, there are two points in each such circle fiber that belongs to $C_{\widehat{X}}$, corresponding to the normal directions to P. So, we conclude that, possibly except for a neighborhood of index 1 critical points, the binding of OB_3 consists of cooriented normal lines along P. Nevertheless, one can check that this is also the case on a Morse chart around each index 1 critical point of f either by looking at Figure 5 right, or by the following argument.

Using local coordinates x_0, x_1 for a Morse chart centered at an index 1 critical point, and denoting the corresponding cotangent fiber coordinates by $y_0 = \cos\theta$ and $y_1 = \sin\theta$, we have (cf. [28]):

$$f(x_0, x_1) = -x_0^2 + x_1^2$$

$$X = -x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} \tag{6.3}$$

$$\xi = \cos\theta dx_0 - \sin\theta dx_1$$

$$f_\xi(x, y) = -x_0^2 + x_1^2 + \widetilde{\eta}\chi(x)(y_0^2 - y_1^2) = -x_0^2 + x_1^2 + \widetilde{\eta}\chi(x)\cos 2\theta \tag{6.4}$$

$$\widehat{X} = -x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} - \sin 2\theta\frac{\partial}{\partial\theta}$$

where $\widetilde{\eta} > 0$ is a sufficiently small real number. Notice that

$$\xi(\widehat{X}) = 0 \iff -x_0\cos\theta = x_1\sin\theta.$$

It follows that the binding $f_\xi^{-1}(0) \cap C_{\widehat{X}}$ consists of the following four disjoint line segments

$\theta = \pi/4$ and $-x_0 = x_1$

$\theta = 3\pi/4$ and $x_0 = x_1$

$\theta = 5\pi/4$ and $-x_0 = x_1$

$\theta = 7\pi/4$ and $x_0 = x_1$

in the trivialized circle bundle over this Morse chart. Therefore, since $P = f^{-1}(0)$ is given by the lines $x_0 = \pm x_1$ in this chart, we conclude that the binding consists of cooriented normal lines along P, on this chart as well, which finishes the proof of the lemma. □

Next we show that the four distinct pages of $\overline{\mathrm{OB}}_1$ coincides with that of OB_3, under the identification of $ST\Sigma$ with $V(\Sigma)$. The four pages we have in mind are the pages whose images are purely real or purely imaginary under the fibration map defining $\overline{\mathrm{OB}}_1$ given in Remark 2.2, namely the pages S_0, $S_{\pi/2}$, S_π and $S_{3\pi/2}$ of $\overline{\mathrm{OB}}_1$.

Lemma 6.5. *Under the identification of $ST\Sigma$ with $V(\Sigma)$, the union $S_0 \cup S_\pi$ corresponds to the characteristic surface $C_{\widehat{X}}$, which is the union of two pages in OB_3.*

Proof. The interior of the page S_0 (resp. S_π) of $\overline{\mathrm{OB}}_1$ consists of the unit vectors tangent to the level sets of f in black (resp. white) regions of $\Sigma \setminus P$ (see Figure 2). In particular, the whole circle fiber above a critical point of index 2 (resp. index 0) is included in S_0 (resp. S_π). We claim that the set of corresponding conormal lines of all the level sets in black and white regions belongs to the characteristic surface $C_{\widehat{X}} \subset V(\Sigma)$, up to isotopy. This is clear outside of a neighborhood of the critical points, since the vector field X is assumed to be equal to ∇f, which is indeed normal to the level sets of f. In addition, X is normal to the level sets in a neighborhood of an index 1 critical point by (6.3), see Figure 5.

On the other hand, over a critical point of index 2 or index 0, the whole fiber in $V(\Sigma)$ is included in $C_{\widehat{X}}$. In addition, by choosing the vector field X to be sufficiently close to ∇f on a Morse chart around an index 2 (resp. index 0) critical point, we can guarantee that the piece of S_0 (resp. S_π) is isotopic to the piece of $C_{\widehat{X}}$ above that chart. In other words, we consider an isotopic fibration map defining the open book $\overline{\mathrm{OB}}_1$, where the isotopy only takes place above the Morse charts around index 2 or index 0 critical points. Therefore, this isotopy does not affect the discussion in the proof of Lemma 6.4, since the binding in each open book does not intersect the fibers above the Morse charts around the critical points of index 2 or index 0. Hence, we finish the proof of the lemma, by adding in the bindings of both open books. □

Lemma 6.6. *Under the identification of $ST\Sigma$ with $V(\Sigma)$, the union $S_{\pi/2} \cup S_{3\pi/2}$ corresponds to the level set $f_\xi^{-1}(0)$, which is the union of two pages in* OB_3.

Proof. The interior of the page $S_{\pi/2}$ (resp. $S_{3\pi/2}$) of $\overline{\mathrm{OB}}_1$ consists of unit vectors along the divide P pointing into black (resp. white) regions of $\Sigma \setminus P$. Notice that $f_\xi^{-1}(0)$ does not intersect the fibers above the Morse charts around index 2 or index 0 critical points. Moreover, away from the index 1 critical points of f, the zero-level set of f_ξ consists of all the circle fibers above the divide P and thus all cooriented lines are in $f_\xi^{-1}(0)$, where the binding of OB_3 separates the two pages. To summarize, away from the index 1 critical points, the union $S_{\pi/2} \cup S_{3\pi/2}$ corresponds to the level set $f_\xi^{-1}(0)$.

Near each index 1 critical point, we understand both $f_\xi^{-1}(0)$ and the union $S_{\pi/2} \cup S_{3\pi/2}$ explicitly in local coordinates (again, see Figure 5 right). According to (6.4), the surface $f_\xi^{-1}(0)$ is given locally by

$$\{(x_0, x_1, y_0, y_1) \mid -x_0^2 + x_1^2 + \widetilde{\eta}\chi(x)(y_0^2 - y_1^2) = 0\}$$

while according to Remark 2.2, the union $S_{\pi/2} \cup S_{3\pi/2}$ is given locally by

$$\{(x_0, x_1, u_0, u_1) \mid -x_0^2 + x_1^2 - \frac{1}{2}\eta^2\chi(x)(u_1^2 - u_0^2) = 0\}.$$

So, we see that these surfaces coincide near each index one critical point, by taking $\widetilde{\eta} = \eta^2/2$ and using identical cut-off functions $\chi : \Sigma \to \mathbb{R}$. □

Lemma 6.7. *Suppose that* OB *(resp.* $\widetilde{\mathrm{OB}}$*) is an open book for a closed and oriented* 3*-manifold* M *(resp.* $\widetilde{M}$*). Let* S *(resp.* $\widetilde{S}$*) be a fixed oriented page of* OB *(resp.* $\widetilde{\mathrm{OB}}$*). If there is an orientation preserving diffeomorphism* $\varphi : M \to \widetilde{M}$ *such that* $\varphi(S) = \widetilde{S}$*, then* OB *is isomorphic to* $\widetilde{\mathrm{OB}}$*.*

Proof. The result follows by observing that $M \setminus S$ is diffeomorphic to a trivial S-bundle over I, while $\widetilde{M} \setminus \widetilde{S}$ is diffeomorphic to a trivial $\widetilde{S}$-bundle over I. □

We conclude the proof of Proposition 6.3 by Lemma 6.5 and Lemma 6.6 that the four pages of $\overline{\mathrm{OB}}_1$ coincides with the four pages of OB_3 under the aforementioned diffeomorphism between $ST\Sigma$ and $V(\Sigma)$. This is more than enough to conclude that OB_1 is isomorphic to OB_3 by Lemma 6.7, after reversing the orientation of the pages of $\overline{\mathrm{OB}}_1$ first as explained in Remark 6.9. □

Lemma 6.8. *The open books* OB_1 *and* OB_4 *for* $ST^*\Sigma$ *are isomorphic, provided that* P *is convex with respect to the given metric on* Σ*.*

Proof. The binding of both open books correspond to the link $L(P)$ consisting of those vectors tangent to P, hence they coincide. The geometric description in Remark 2.2 of the page S_0 of the open book $\overline{\mathrm{OB}}_1$ of A'Campo & Ishikawa coincides with the page of the open book $\overline{\mathrm{OB}}_4$ of Birkhoff & Fried given by Theorem 5.3, based on a set of geodesics satisfying the assumptions of a divide (see the top left of Figure 2). Indeed they both correspond to the set of vectors tangent to the level sets of f in the black regions of $\Sigma \setminus P$. By reversing the orientations, we get the desired isomorphism of OB_1 and OB_4 using Lemma 6.7. □

In fact, one can show that these two open books have *four* "common" pages, namely the pages $S_0, S_{\pi/2}, S_\pi$, and $S_{3\pi/2}$ from OB_1.

Remark 6.9. Let S be an oriented surface with boundary and h an element of the mapping class group $\mathrm{MCG}(S, \partial S)$, which is a product of positive Dehn twists. Note that the sign of the Dehn twist depends on the orientation of S. Let M be the oriented 3-manifold which is the total space of an abstract an open book (S, h) with page S and monodromy h. Let η be the positive Stein fillable contact structure supported by (S, h). The total space of the open book $(\overline{S}, h)$ is $\overline{M}$, where h is now a product of *negative* Dehn twists in $\mathrm{MCG}(\overline{S}, \partial\overline{S})$. Moreover, $\overline{\eta}$ is the positive contact structure on $\overline{M}$ supported by $(\overline{S}, h)$, which is overtwisted. On the other hand, η is a negative contact structure on $\overline{M}$.

Remark 6.10. The manifolds $ST^*\Sigma$ and $DT^*\Sigma$ are orientable, even if Σ is a *nonorientable* surface. While Johns' construction of a symplectic Lefschetz fibration on $DT^*\Sigma$ works for any surface Σ, Ishikawa only deals with oriented surfaces. On the other hand, both constructions work for an oriented surface with nonempty boundary and the proof of Proposition 6.2 extends to cover that case.

7. Lefschetz fibrations and open books of minimal genus

A particular admissible divide on a closed and oriented surface Σ of genus $g > 1$ was given by Fried [15], following Birkhoff [5]. This divide consists of $2g + 2$ simple closed curves, with $2g + 2$ double points, as we depicted at the top right of Figure 8, for $g = 3$. Fried noticed that, when Σ is hyperbolic, the corresponding Birkhoff cross section has genus one, which can also be recovered from Proposition 2.3. The associated Lefschetz fibration $DT^*\Sigma \to D^2$, whose fiber is of genus one (*minimal possible*) with $4g+4$ boundary components, was explicitly constructed by the second author in [30]. Here we simply observe that this construction does not give the minimal number of boundary components of a fiber among all genus one Lefschetz fibrations, answering Question (4) at the end of [30] negatively.

Proposition 7.1. [7] *For any closed and oriented surface Σ of genus $g \geq 1$, there is an explicit genus one Lefschetz fibration $DT^*\Sigma \to D^2$ whose fiber has $4g$ boundary components. As a consequence, there is an explicit genus one open book adapted to $(ST^*\Sigma, \xi)$, whose binding has $4g$ components.*

Proof. The set of $2g$ simple closed curves on Σ depicted at the bottom row of Figure 8 yields an admissible divide on Σ. Since each of these curves intersect exactly two other curves, the regular fiber of the Lefschetz fibration $DT^*\Sigma \to D^2$ obtained by using the methods of either Ishikawa or Johns is of genus one, with $4g$ boundary components. Moreover, the monodromy of this Lefschetz fibration can be explicitly computed as a product of Dehn twists. It follows that, there is an explicit genus one open book adapted to $(ST^*\Sigma, \xi)$, whose binding has $4g$ components. Furthermore, by Theorem 1.1, this open book is isomorphic to the Giroux open book as well as the geodesic open book. □

An open book for a closed 3-manifold induces a Heegaard splitting, where the Heegaard surface is the union of two pages along the binding. If the page of the open book is a surface of genus h with k boundary components, then the Heegaard surface is of genus $2h + k - 1$. According to Boileau-Zieschang [6], for any closed and oriented surface Σ of genus g, the Heegaard genus of $ST^*\Sigma$ is $2g + 1$. Therefore, the binding of any *genus one* open book for $ST^*\Sigma$ must have at least $2g$

[7]This statement was independently proven by C. Bonatti (unpublished, private communication).

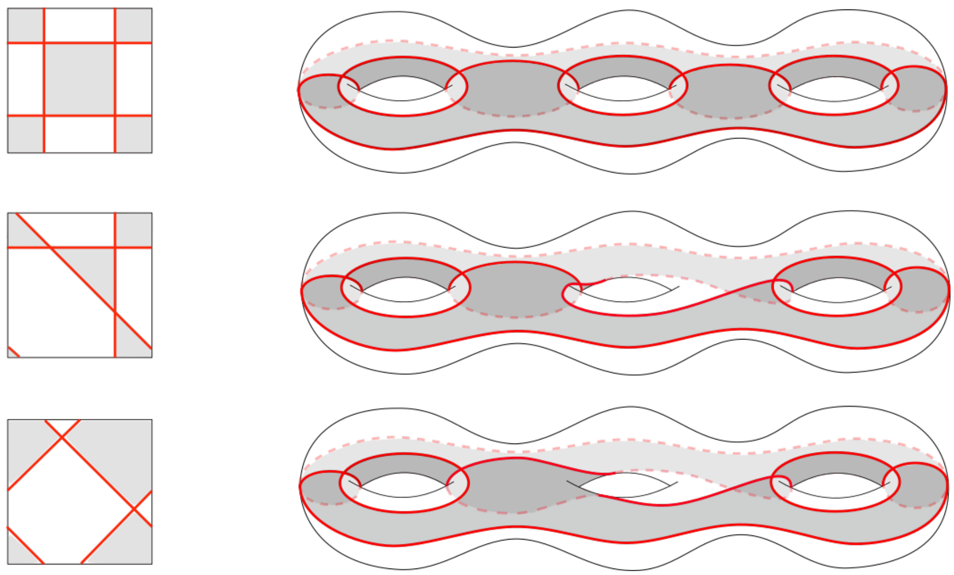

FIGURE 8. Three admissible divides that yield genus one Lefschetz fibrations on $DT^*\Sigma$. The top row corresponds to the Birkhoff-Fried divide which consists of $2g+2$ curves that we depicted on the left for a torus and on a genus 3 surface on the right. The middle row corresponds to Brunella's improvement with $2g+1$ curves. It is obtained by smoothing one crossing, thus connecting the two white regions. The bottom row corresponds to an additional improvement with $2g$ curves only, obtained by smoothing another crossing and connecting the two black regions.

connected components, which implies that the fiber of any *genus one* Lefschetz fibration $DT^*\Sigma \to D^2$ must have at least $2g$ boundary components. In other words, for an oriented surface Σ of genus at least two, the binding number $\mathrm{bn}(ST^*\Sigma, \xi)$ [14] satisfies the following inequality

$$2g \leq \mathrm{bn}(ST^*\Sigma, \xi) \leq 4g.$$

This simple observation indicates that there is some room for improvement for the result stated in Proposition 7.1, using methods other than discussed in this paper. For example, a genus one open book adapted to $(ST^*T^2 = T^3, \xi)$ with *three* binding components, was constructed by Van Horn-Morris [32] (see also [11, Section 3] for similar examples obtained by the first author).

Next we observe that Proposition 7.1 can not be improved utilizing the constructions described in this article. Notice that Theorem 7.2 below implies Theorem 1.4 from the introduction.

Theorem 7.2. *Up to homeomorphism, there are exactly three admissible divides (depicted in Figure 8) on a genus $g \geq 1$ surface Σ that yield genus one Lefschetz fibrations $DT^*\Sigma \to D^2$.*

Proof. Let P be an admissible divide on Σ. Recall that $c(P)$ denotes the number of curves that form P and $v(P)$ the number of double points in P. By Proposition 2.3, the genus of the associated Lefschetz fibration $DT^*\Sigma \to D^2$ is $1 + v(P) - c(P)$. Hence this genus is equal to 1 if and only if $c(P) = v(P)$. Since P is admissible, its complement can be black-and-white colored, which means that every curve in P has an even number of double points (counting self-intersection points twice). Since P is connected, every component of P has at least two double points. By counting all double points twice (one for every curve that traverses the point), one obtains $2v(P) \geq 2c(P)$, with equality if and only if every curve in P has exactly two double points on it. One can check that this is indeed the case for all the divides depicted in Figure 8.

Assume now that P is an admissible divide so that every curve in P has exactly two double points on it. Consider the dual graph $\Gamma(P)$ whose vertices are the curves in P and whose edges are the double points. Note that $\Gamma(P)$ is a graph with loops and multiple edges. Since P is admissible, $\Gamma(P)$ is connected, for otherwise the complement $\Sigma \setminus P$ would have a non-trivial topology. Our assumption on the number of double points implies that the vertices of $\Gamma(P)$ all have degree 2. Hence $\Gamma(P)$ is a cycle, and therefore P is cyclic chain of circles.

If one removes a small open disk in each region of $\Sigma \setminus P$, one obtains a surface with boundary, which deformation-retracts on a neighborhood Σ_P of P. The surface with boundary Σ_P is then a cyclic chain of circular ribbons, say $r_1, \ldots, r_k$, where $k = c(P)$. One can recover Σ_P by gluing one by one all the ribbons, starting with r_1, then gluing r_2, then r_3, and so on, inductively. Finally one adds r_k and glues back r_k to r_1. Since Σ_P is orientable, every ribbon is a standard annulus, and there is no freedom in the homeomorphism type of the gluing, except at the end when gluing r_k back to r_1. Indeed, as shown in Figure 9, the open chain $r_1 \cup \cdots \cup r_k$ has four boundary components, which are color-coded. When gluing r_k back to r_1, one has to chose which boundary component is glued with which one.

If k is even, there are exactly two possible cases that lead to two different divides. In the first case, one glues each boundary component with itself and therefore Σ_P has four boundary components. When filling these boundary components with disks, thus recovering Σ, one checks that Σ is made of four k-gons glued along

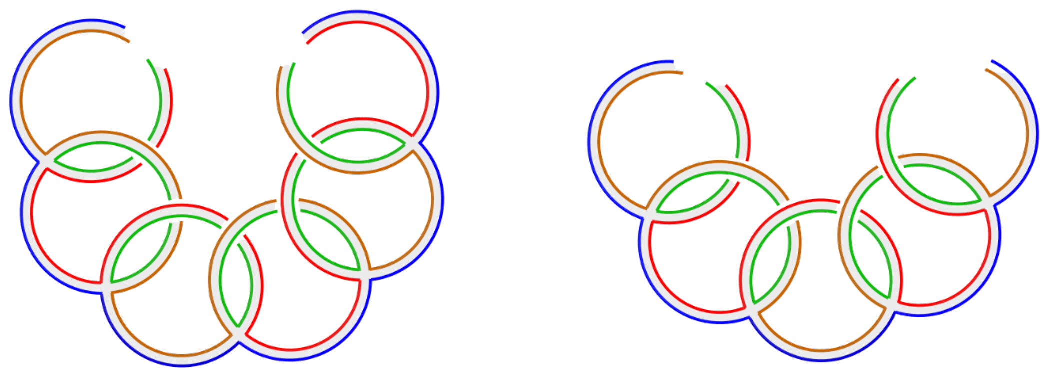

FIGURE 9. Proof of Theorem 7.2 is illustrated for even k on the left and for odd k on the right. When reconstructing a neighborhood Σ_P of P in Σ, one has a chain of ribbons $r_1, \ldots, r_k$. The surface Σ_P with boundary is obtained by connecting the two open bands in both figures. When k is even, there are two possibilities: either one glues each (colored) boundary with itself, or one glues blue with green, green with blue, red with brown, and brown with red. In the first case Σ_P has four boundary components of length k, while in the second it has two boundary components of length $2k$. When k is odd, there are also two possibilities: either one glues blue with blue, brown with brown, red with green, and green with red, or one glues blue with brown, brown with blue, red with red, and green with green. In either case Σ_P has three boundary components, two of length k and one of length $2k$.

their edges. It has genus $\frac{k-2}{2}$, so that $k = 2g + 2$ and the divide P corresponds to the Birkhoff-Fried divide [15] on Σ (as depicted at the top row of Figure 8). In the second case, one glues each boundary component with another one so that Σ_P has two boundary components. When recovering Σ, it is made of two $2k$-gons glued along their edges. It has genus $\frac{k}{2}$, so that $k = 2g$ and the divide P is the one depicted at the bottom row of Figure 8.

Finally when k is odd, there is only one possibility up to homeomorphism: two boundary components have to be glued to themselves, and the two others are glued one with another. Thus Σ is made of two k-gons and one $2k$-gon glued along their edges. It has genus $\frac{k-1}{2}$, so that $k = 2g + 1$ and the divide P corresponds to the Brunella divide depicted at the middle row of Figure 8. $\square$

Now we turn our attention to the monodromies associated to the genus one geodesic open books obtained by restricting to the boundary of the three genus one Lefschetz fibrations in Theorem 7.2. Since the pages have genus one, after contracting their boundary components into points, the monodromies correspond to isotopy classes of orientation-preserving homeomorphisms of the 2-dimensional torus. Such an isotopy class corresponds to a conjugacy class in $\mathrm{SL}_2(\mathbb{Z})$, and every non-elliptic conjugacy class in $\mathrm{SL}_2(\mathbb{Z})$ contains a positive product of the matrices $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which is unique up to cyclic permutation. The monodromy of the Birkhoff-Fried geodesic open book was computed by Ghys [17] and Hashiguchi [21]. Moreover, Brunella already computed the monodromy of the geodesic open book induced by his divide [7]. Furthermore, the monodromy corresponding to the geodesic open book of Proposition 7.1 can be computed using similar techniques, or the method of Dehornoy-Liechti [12]. Here is a summary of these computations:

geodesic open book	number of binding components	monodromy
Birkhoff-Fried	$4g+4$	$L^{g-1}R^2L^{g-1}R^2$
Brunella	$4g+2$	$L^{g-1}R^4L^{g-1}R^2$
Prop 7.1	$4g$	$L^{g-1}R^4L^{g-1}R^4$

Cossarini and Dehornoy [9] generalized the work of Birkhoff [5], Fried [15] and Brunella [7], to construct and classify *negative* Birkhoff cross sections of the geodesic flow associated to arbitrary Eulerian coorientations on a given finite set of geodesics on a hyperbolic surface Σ. Moreover, Marty [26] computed the monodromy of the corresponding open books for $ST\Sigma$ as a product of Dehn twists along curves explicitly described on a page. These open books (when viewed for $ST^*\Sigma$) support the canonical contact structure ξ as well. The following questions naturally arise as a result of this discussion: Are these monodromy factorizations pairwise Hurwitz equivalent? Is the total space of the Lefschetz fibration filling each of these open books diffeomorphic to $DT^*\Sigma$?

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