ALMOST EQUIVALENCE OF ALGEBRAIC ANOSOV FLOWS

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Abstract. Two flows on two compact manifolds are almost equivalent if there is a homeomorphism from the complement of a finite number of periodic orbits of the first flow to the complement of the same number of periodic orbits of the second flow that sends orbits onto orbits. We prove that every geodesic flow on the unit tangent bundle of a negatively curved 2-dimensional orbifold is almost equivalent to the suspension of some automorphism of the torus. Together with a result of Minikawa, this implies that all algebraic Anosov flows are pairwise almost equivalent. We initiate the study of the Ghys graph —an analogue of the Gordian graph in this context— by giving explicit upper bounds on the distances between these flows.

Introduction

This paper deals with a classification problem which lies at the interplay between topology and dynamical systems.

In dimension 3, Anosov flows are prototypes of flows having chaotic behaviour while being structurally stable. There are two basic constructions of such flows, namely suspensions of automorphisms of the 2-dimensional torus and geodesic flows on negatively curved surfaces —or more generally on 2-dimensional orbifolds. These two classes are called **algebraic Anosov flows**. There also exists surgery and gluing constructions that take one or several Anosov flows and construct a new one, thus yielding non-algebraic examples [FrW80, HaT80, BBY17].

One says that two flows are **topologically equivalent** if there is a homeomorphism of the underlying manifold that sends orbits of the first flow, seen as oriented 1-manifolds, onto orbits of the second. Note that the time-parameter needs not be preserved. The question of whether two flows are topologically equivalent can be answered for algebraic Anosov flows: two geodesic flows are equivalent if and only if the underlying 2-orbifolds are of the same type [Gro76], two suspensions are equivalent if and only if the underlying matrices are conjugated in $\mathrm{GL}_2(\mathbb{Z})$, and a geodesic flow is never equivalent to a suspension.

Following Goodman and Fried and elaborating on the notion of Dehn surgery, a more flexible notion was proposed by Ghys in several talks: two flows are **almost equivalent** if there is a homeomorphism from the complement of a finite number of periodic orbits of the first flow to the complement of the same number of periodic orbits of the second flow that sends orbits onto orbits. Almost equivalence is an equivalence relation on the larger class of pseudo-Anosov flows. A seminal construction of Birkhoff and Fried [Bir17, Fri83] shows that the geodesic flow on a negatively curved surface is almost equivalent to the suspension flow of some explicit automorphism of the torus [Ghy87, Has92]. Some other constructions followed, exhibiting examples of almost-equivalence of some geodesic flows with some suspension flows [Bru94, Deh15]. Ghys asked whether any two transitive Anosov flows with orientable invariant foliations are almost equivalent. Here we give a positive answer for suspension flows and geodesic flows of Anosov type :

Theorem A. *Every algebraic Anosov flow whose invariant foliations are orientable is almost equivalent to the suspension of the map* $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

A weak version of this statement was already proven [Deh13] where almost equivalence was replaced by *almost commensurability* : finite coverings were allowed.

Since orientability of the stable and unstable foliations cannot be broken by removing isolated periodic orbits, one cannot get rid of the orientability assumption.

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The proof goes in two steps. The first one was done by Minakawa. For $A \in \mathrm{SL}_2(\mathbb{Z})$, we denote by $\mathbb{T}^3_A$ the 3-manifold $\mathbb{T}^2 \times [0,1]/_{(x,1)\sim(Ax,0)}$ and by ϕ_{sus} the flow on $\mathbb{T}^3_A$ that is tangent to the $[0,1]$-coordintate, it is called the *suspension flow*.

Theorem B0. [Min13] If $A \in \mathrm{SL}_2(\mathbb{Z})$ has a trace larger than 3, then there exists $B \in \mathrm{SL}_2(\mathbb{Z})$ with $3 \leqslant \mathrm{tr}B < \mathrm{tr}A$ such that the suspension flow on $\mathbb{T}^3_A$ is almost equivalent to the suspension flow on $\mathbb{T}^3_B$.

The second one is new, although several partial results already exist [Deh15, HaM13] :

Theorem C1. If $\mathcal{O}$ is 2-dimensional orientable orbifold with a hyperbolic metric, then there exists a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that the geodesic flow on $\mathrm{T}^1\mathcal{O}$ is almost equivalent to the suspension flow on $\mathbb{T}^3_A$.

We will replicate Minakawa's proof here for two reasons. First the reference [Min13] is a video of a talk given in Tokyo in 2013 where Minakawa announced the theorem. The proof is not in the video, it was outlined in the abstract of the talk and cannot be found online anymore[1]. Second we push Minakawa's result a bit further, as we explain now.

The Gordian graph is the graph whose vertices are isotopy classes of knots in the 3-space and whose edges connect knots which differ by one crossing change in some projection. Let us define the **Ghys graph** G_{Ghys} as the graph whose vertices are pairs of the form (3-manifold, Anosov flow), up to topological equivalence, and whose edges connect two pairs if one can remove one periodic orbit of each pair and obtain two flows that are topologically equivalent. Note that the orientability of the invariant foliations of the flow is an invariant of the connected components of G_{Ghys}. Turning this graph into a metric space (where edges have length 1), the **Ghys distance** d_{Ghys} between two Anosov flows is then the minimal number of periodic orbits one has to remove on both flows in order to obtain the same flow on the same 3-manifold. We denote it by d_{Ghys}.

Remark that, if two pairs (M, X) and (M', X') of Anosov flows with orientable foliations are at distance 1 with respect to the Ghys distance, the manifolds M, M' are connected with a integral Dehn surgery. One could define another (longer) distance where the length of an edge is the absolute value of the coefficient of the surgery. The estimates we give below actually work with this other distance.

Theorem A can be rephrased in terms of the Ghys graph: all algebraic Anosov flows with orientable foliations lie in the same connected component of G_{Ghys}. Now we estimate the Ghys distance. Write X for the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and Y for the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. It is a folklore result that every hyperbolic matrix is conjugated to a positive word in X and Y containing both letters, and this word is unique up to cyclic permutation of the letters [Deh11, Prop. 4.3]. Now we enforce Minakawa's result:

Theorem B1. Let W be a positive word on the alphabet $\{X, Y\}$ containing both letters. Then we have

$$d_{\mathrm{Ghys}}((\mathbb{T}^3_W, \phi_{\mathrm{sus}}), (\mathbb{T}^3_{XW}, \phi_{\mathrm{sus}})) \leqslant 3.$$

Theorems C1 and B1 can both be rephrased in terms of *Birkhoff sections* and we will use this notion to actually prove them:

Definition 1. Given a flow Φ on a compact 3-manifold M, an oriented surface $i : S \to M$ with boundary is a **Birkhoff surface** if

(i) the interior of $i(S)$ is embedded in M and positively transverse to Φ,
(ii) the boundary $i(\partial S)$ is immersed in M and tangent to Φ.

The surface S is a **Birkhoff section** if, moreover, it satisfies the additional condition

(iii) S intersects all orbits within bounded time, *i.e.*, $\exists T > 0$ such that $\Phi^{[0,T]}(i(S)) = M$.

In general, we forget the immersion i and see directly S in M. We use this heavier notation for underlining the behaviour on the boundary: it follows directly from the definition that the oriented boundary $i(\partial S)$ of a Birkhoff section is the union $\cup_{i=1}^{c}\gamma_i$ of finitely many periodic orbits. Each such orbit γ_i is oriented by Φ. Since S is oriented, its boundary is also canonically oriented. Therefore, there exist multiplicities $n_i \in$

[1]Minakawa actually wrote in an email that the result was already announced in 2004 at the 51st Topology Symposium at Yamagata, but we could not find no abstract nor proceedings.

$\mathbb{Z}$ such that $i(\partial S) = \sum_{i=1}^{c} n_i\gamma_i$. Remark that a Birkhoff section where all boundary components have multiplicities ± 1 is the page of an open-book decomposition of the underlying 3-manifold, where the other pages are obtained by pushing the section by the flow.

Given a flow Φ and a Birkhoff section S, there is an induced **first-return map** $f_S : \mathrm{int}(S) \to \mathrm{int}(S)$. Removing the periodic orbits of Φ that form ∂S, we get an almost-equivalence of Φ with the suspension flow of f_S. Theorems C1 and B1 are respectively equivalent to :

Theorem C2. If $\mathcal{O}$ is 2-dimensional orientable orbifold with a hyperbolic metric, then the geodesic flow on $\mathrm{T}^1\mathcal{O}$ admits a genus-one Birkhoff section.

Theorem B2. Let W be a positive word on the alphabet $\{X, Y\}$ containing both letters at least once. Then $(\mathbb{T}^3_{XW}, \phi_{\mathrm{sus}})$ admits a genus-one Birkhoff section with at most 3 boundary components, and whose induced first-return map is given by the matrix W.

The paper is organized as follows. In Section 1 we present an operation on Birkhoff surfaces that we call the *Fried union*, and we explain two ways to compute the Euler characteristics of a Birkhoff surface that are useful later. In Section 2 we prove Theorem C2, and in Section 3 we prove Theorem B2.

1. Fried sum and Euler characteristics

1.a. **Fried sum.** Here we present an operation introduced by Fried that takes two Birkhoff surfaces and gives a new one [Fri83].

Assume that M is a closed 3-manifold, that X is a vector field on M with induced flow ϕ^t_X, and that $S^{(1)}, S^{(2)}$ are two Birkhoff surfaces. Their boundaries are (not necessarily disjoint) links $\Gamma^{(1)}, \Gamma^{(2)}$ formed of periodic orbits of X, with multiplicities. We write Γ for the link $\Gamma^{(1)} \cup \Gamma^{(2)}$. At the expense of perturbing them transversality to X on can assume that $S^{(1)}$ and $S^{(2)}$ are in transverse position. Then their intersection is a 1-manifold, that is, a union of circles and arcs whose ends lie in Γ.

Definition 1.1. Given two Birkhoff surfaces $S^{(1)}, S^{(2)}$ as above, their **Fried sum**, denoted by $S^{(1)} \overset{F}{\cup} S^{(1)}$, is the surface obtained from $S^{(1)} \cup S^{(2)}$ by desingularizting all circles and arcs of $S^{(1)} \cap S^{(2)}$ transversally to the vector field X (see Figure 1).

One may wonder whether this operation is well-defined, especially along Γ, when the surfaces have boundary components in common. In order to picture this, one can normally blow-up Γ: for every point p in Γ, one replaces p by the normal sphere bundle $S((TM)_p/\mathbb{R}X(p))$, which is topologically a circle. We denote by M_Γ the resultating 3-manifold, it is a compactification of $M \setminus \Gamma$ whose boundary ∂M_Γ consits of tori. The vector field X extends to a vector field X_Γ tangent to ∂M_Γ.

The surface $S^{(1)}$ and $S^{(2)}$ extend to embedded surfaces $(S^{(1)}, \partial S^{(1)})$, $(S^{(2)}, \partial S^{(2)})$ in $(M_\Gamma, \partial M_\Gamma)$. There, the surfaces are still transverse to X and to one another, and, up to perturbing them, one can assume their boundaries are transverse to X_Γ and to one another.

The Fried sum $(S^{(1)}, \partial S^{(1)}) \overset{F}{\cup} (S^{(2)}, \partial S^{(2)})$ is then obtained by desingularizing the arcs and circles of intersection. In particular, the boundary of the resulting surface is obtained by resolving the intersection points of the curves $\partial S^{(1)} \cap \partial S^{(2)}$ in $\partial M_\Gamma = \Gamma \times \mathbb{S}^1$, transversally to X_Γ.

1.b. **Euler characteristics.** In order to check that we obtain tori in the proofs of Theorems C2 and B2, one has to estimate the Euler characteristics of some surfaces. This can be done in several ways.

Firstly, all the vector fields we consider in this article are of Anosov type. In particular there is a 2-dimensionl transverse foliation $\mathcal{F}^s$ which is X-invariant and uniformly contracted by the flow. For every Birkhoff surface S, there is an induced foliation $\mathcal{F}^s \cap S$ on S, whose singularities all lie on ∂S. Also these singularities are all of index $-1/2$. Therefore one can compute the Euler characteristics of S by counting these singularities. They appear exactly at those points of ∂S where the tangent plane TS is tangent to the foliation $\mathcal{F}^s$.

Secondly, one can notice that the Euler characteristics is linear under Fried sum, when computed in the manifold M_Γ. Indeed, if we triangulate both surfaces so that the intersection circles and arcs are in the

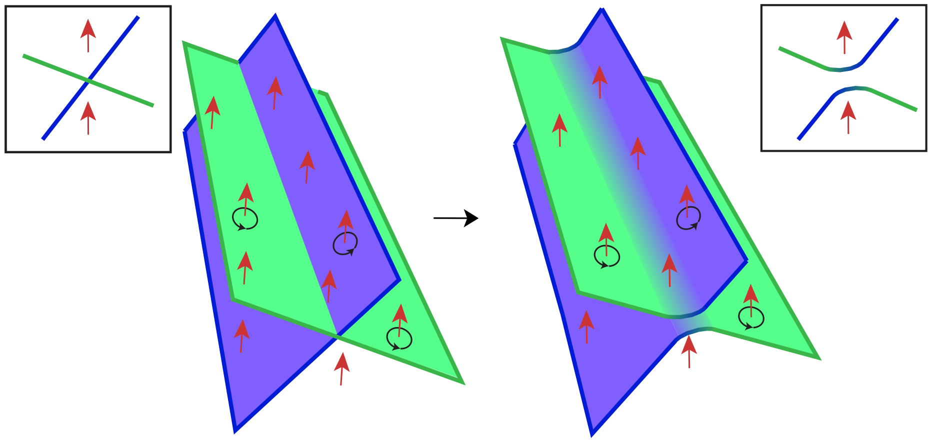


FIGURE 1. Given two surfaces $S^{(1)}, S^{(2)}$ transverse to a vector field X, one consider the link $\Gamma = \partial S^{(1)} \cup \partial S^{(2)}$ and blows up each of its component. In the resulting manifold M_Γ, the two surfaces $(S^{(1)}, \partial S^{(1)})$ and $(S^{(2)}, \partial S^{(2)})$ are surfaces with boundary whose intersection consists of circles and arcs ending on ∂M_Γ. The Fried sum $(S^{(1)}, \partial S^{(1)}) \overset{F}{\cup} (S^{(2)}, \partial S^{(2)})$ in then obtained by desingularizing these circles and arcs. In particular in the boundary $\partial M_\Gamma \simeq \Gamma \times \mathbb{S}^1$, the boundaries $\partial S^{(1)}$ and $\partial S^{(2)}$ are circles transverse to X_Γ and to one another, the desingularization is the unique reconnection of these circles that preserves transversality to X_Γ.

1-skeleton, one checks that the Fried sum can be triangulated with exactly the same number of simplices of each type. Beware that one first needs to remove one disc every time one surface intersects the boundary of the other (so that the resulting surfaces live in the same manifold M_Γ).

Another less elementary argument is that Birkhoff surfaces minimize the genus in their homology classes. Thurston and Fried proved that the Euler characteristics is a linear form on fibered faces. Indeed it is computed by pairing the class of the surfaces with the Euler class of the normal bundle to X. Since the homology class of the Fried sum is the sum of the homology classes of the two surfaces, the result follows. The point here is that we have to take care that the surfaces have to lie in the same fibered face, that is, to be surfaces transverse to the same flow in the same manifold.

2. Genus one-Birkhoff sections for geodesic flows

In this Section we prove Theorem C2.

Gromov remarked that given two hyperbolic surfaces of the same genus, the associated geodesic flows on the unit tangent bundles are equivalent. We note that the statement extends to 2-orbifolds, with the same proof.

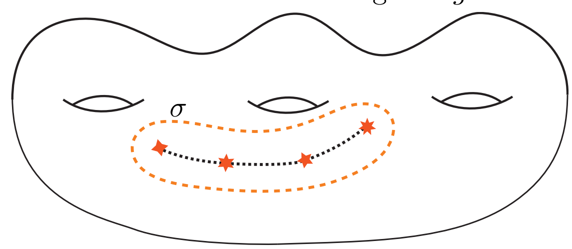
Proposition 2.1 ([Gro76])**.** *Given two compact hyperbolic orientable 2-orbifolds $\mathcal{O}_1, \mathcal{O}_2$ of the same type, then there exists a homeomorphism $\mathrm{T}^1\mathcal{O}_1 \to \mathrm{T}^1\mathcal{O}_2$ sending orbits of the geodesic flow onto orbits of the geodesic flow.*

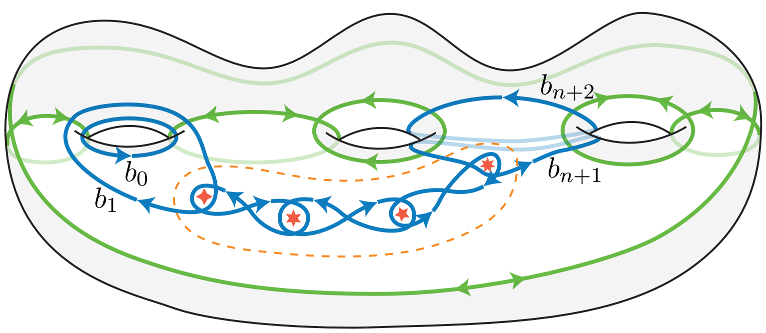
Proof. Since $\mathcal{O}_1, \mathcal{O}_2$ are hyperbolic, their universal cover is $\mathbb{H}^2$ and they are isometric to $\mathbb{H}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$ respectively. Since they are of the same type, there is an isomorphism $f : \Gamma_1 \to \Gamma_2$. Identifying $\partial\mathbb{H}^2$ with $\partial\Gamma_1$ and $\partial\Gamma_2$, f extends to a (Γ_1, Γ_2)-equivariant homeomorphism $\partial\mathbb{H}^2 \to \partial\mathbb{H}^2$.

Now a geodesic on $\mathbb{H}^2$ is represented by a pair of distinct points on $\partial\mathbb{H}^2$ and a unit tangent vector by a positively oriented triple of distinct points (the third point defines a unique canonical projection on the geodesics represented by the first two points). Therefore f extends to a (Γ_1, Γ_2)-equivariant homeomorphism $C_3(\partial\mathbb{H}^2) \to C_3(\partial\mathbb{H}^2)$ where $C_3(\partial\mathbb{H}^2)$ denotes the set of triple of distincts points in $\partial\mathbb{H}^2$, that is, a homeomorphism $\mathrm{T}^1\mathbb{H}^2 \to \mathrm{T}^1\mathbb{H}^2$. Projecting on the first two coordinates one sees that it sends geodesics on geodesics. (Note that since the third coordinate is not the time-parameter, the speed is not at all preserved.) Projecting back to $\mathrm{T}^1\mathbb{H}^2/\Gamma_1 = \mathrm{T}^1\mathcal{O}_1$ and $\mathrm{T}^1\mathbb{H}^2/\Gamma_2 = \mathrm{T}^1\mathcal{O}_2$, we obtain the desired topological equivalence. □

Thanks to Proposition 2.1, the metric is not relevant concerning the existence and the topology of Birkhoff sections for the geodesic flow, as long as it is hyperbolic. However, chossing a suitable hyperbolic metric will help in describing and picturing the construction.

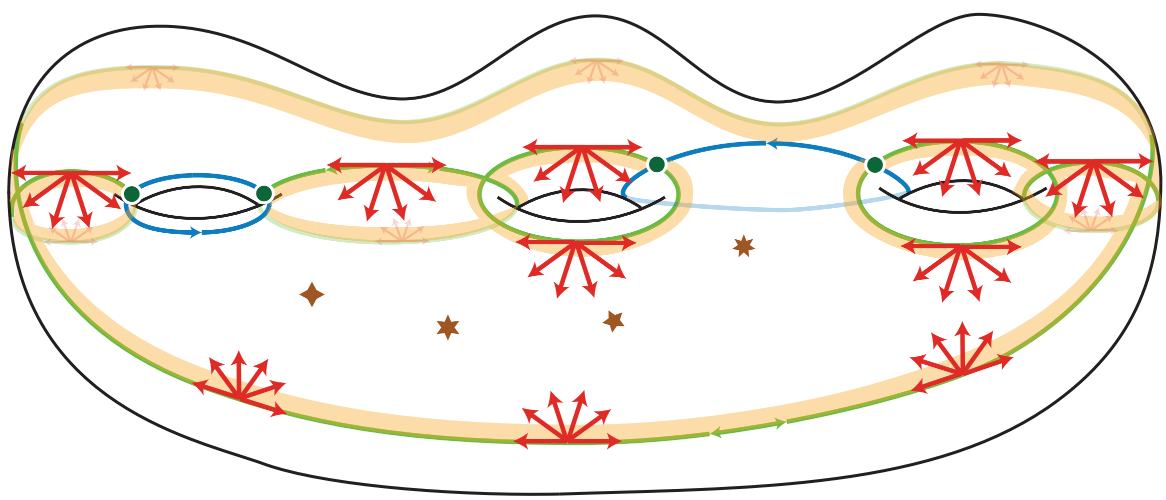
2.a. **Choice of the orbifold metric.** Let $\mathcal{O}_{g;k_1,\dots,k_n}$ denote an orientable orbifold of genus g with cone points of orders $k_1,\dots,k_n$. We choose a hyperbolic metric on $\mathcal{O}_{g;k_1,\dots,k_n}$ in such a way that the cone points are aligned on a short segment. In this way there is a simple closed geodesic that separates the cone points from the handles of $\mathcal{O}_{g;k_1,\dots,k_n}$ which also appears short. We denote by σ such a separating geodesic.



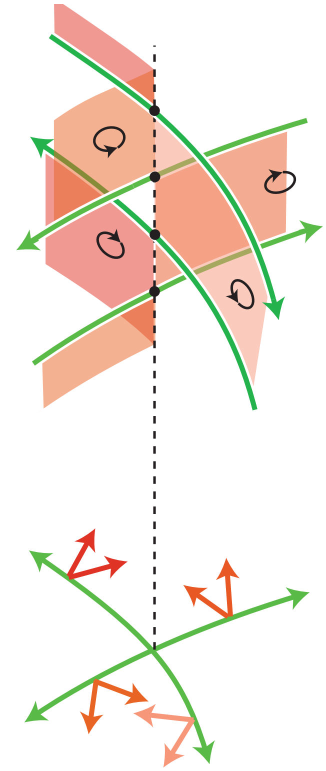


2.b. **Choice of the boundary components.** Suppose first that on $\mathcal{O}_{g;k_1,\dots,k_n}$ no order k_i is equal to 2. We then consider the collection $\Gamma_{g,n}$ of $4g+n+3$ oriented geodesics depicted below. The green lines correspond to $2g$ pairs of geodesics for which both orientations are chosen. The blue lines correspond to $n{+}3$ geodesics for which only one orientation is chosen. All the green geodesics stay in the part of the orbifold containing the handles. Two blue geodesics b_1, b_{n+1} intersect σ, two others b_0, b_{n+2} remain on the handles-side, and the remaining $n{-}1$ on the cone points-side. The geodesics staying in the handles-side (all greens and two blues), considered as unoriented curves, separate the topological surface into four $2g{+}2$-gons. These polygons can be black-and-white colored and we choose to color in black the polygons not containing the cone points and in white the two other faces, one of which contains all cone points.

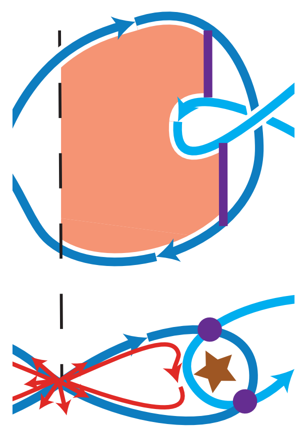
2.c. **Choice of the surface.** We now describe a surface $S_{g;k_1,\dots,k_n}$ in $\mathrm{T}^1\mathcal{O}_{g;k_1,\dots,k_n}$ with boundary $\Gamma_{g,n}$ and we will prove later that it is the desired Birkhoff section. The surface consists of two main parts S_h and S_c connected by a piece S_σ. As suggested by the names S_h lies in the handles-part of $\mathrm{T}^1\mathcal{O}_{g;k_1,\dots,k_n}$, while S_c lies in the cone points-part.

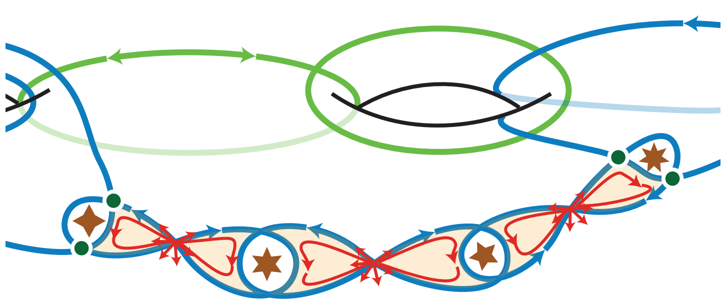


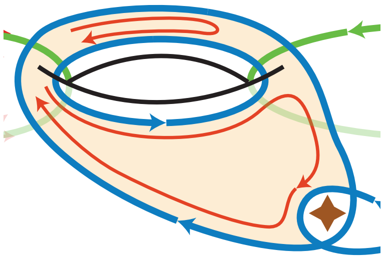
The surface S_h is similar to the one constructed by Birkhoff and Fried [Bir17, Fri83]. It is made of those tangent vectors based on the green geodesics of $\Gamma_{g,n}$ (those that are taken with both orientations) and pointing into the white faces. Therefore for every arc α of a green geodesic bounded by two intersection points, there is an associated rectangle in S_h. The horizontal boundary of this rectangle consists of the two oriented lifts of α, while the vertical boundary consists of some pieces of the fibers of the extremities of α. We depicted on the left what happens in a neighbourhood of the fiber of an intersection point of the green geodesics. There the surface S_h consists of four rectangles. One checks that they glue nicely: for each of the four quadrants, there are exactly two rectangles arriving in this quandrant of tangent vectors, and their orientations agree. Below we depicted where this surface S_h projects on $\mathcal{O}_{g;k_1,\dots,k_n}$. All in all the boundary of S_h consists of the lifts of all green geodesics, plus the fibers of the four points where green and blue geodesics intersect. Also each rectangle contributes by -1 to the Euler characteristics of S_h (1 face, 2 horizontal sides and 4 vertical sides each shared by 2 rectangles, 6 vertices each shared by 3 rectangles). Since there are $4g$ rectangles, the total contribution is $-4g$.

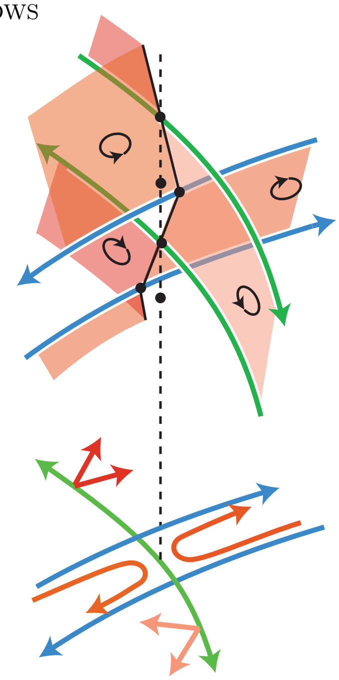


The surface S_c is inspired by the surfaces constructed in [Deh15]. The blue curves form $2n-2$ triangular regions, that we foliate by a vector field (red, below) that look like $n-1$ butterflies. At the (self-)intersection points of the blue curves, a whole sub-segment of the fiber is part of the surface S_c. On the right we show the lift of one triangular face: it consists of one hexagon, three of whose sides correspond to arcs of blue geodesics and three other sides correspond to parts of fibers where the hexagon is connected to an adjacent one. Each hexagon contributes by $-1/2$ to the Euler characteristics (1 face, 3 sides and 3 sides each shared by 2 hexagons, 6 vertices each shared by 2 hexagons), hence the contribution of S_h is $1-n$. The boundary of S_h that is not contained in the link $\Gamma_{g,n}$ consists of four arcs in the fibers of the points where b_1 and b_{n+1} intersect b_2 and b_n.









Finally there is the part S_σ which connects S_h and S_c. It is made of those tangent vectors that are based in the two regions between b_0 and b_1, and the two regions between b_{n+1} and b_{n+2}, and tangent to the two Reeb-like vector fields depicted on the left. It has two connected components. The boundary of the left connected component of S_σ consists of the lifts of b_0 and b_1, plus the parts of the fibers of S_c that are adjacent to S_σ, plus some tangent vectors based on the green geodesics that intersect b_0 and b_1. One checks that S_σ and S_c glue nicely. Concerning S_h, the boundaries do not exactly match: they would if the geodesics b_0 and b_1 would intersect the green geodesics at the same point, and the gluing pattern would be exactly the one at the intersection points of the green geodesics. Here one has to make an isotopy of this picture, so that the two blue geodesics are not exactly one above the other. However, this can be easily done, and the resulting modifiaction of S_h glue nicely with S_σ.

Proof of Theorem C2. Consider the surface $S_{g;k_1,\dots,k_n}$ that is the union of S_h, S_c and S_σ described above (see also Figure 2 where all pieces are put together).

Firstly we claim that its boundary is $\Gamma_{g,n}$ (actually $-\Gamma_{g,n}$ if one takes orientations into account). Indeed the boundary of S_h is made of the lifts of the $2g$ green geodesics, plus some tangent vectors around the intersection of the green geodesics and b_0, b_1, b_{n+1} and b_{n+2}. In the same way the boundary of S_c it made of the lifts of the $n-1$ blue geodesics $b_2, \dots, b_n$, plus some tangent vectors at the intersections of b_1 with b_2 and at intersection of b_n and b_{n+1}. Finally the boundary of S_σ is $b_0 \cup b_1 \cup b_{n+1} \cup b_{n+2}$, plus some tangent vectors where these geodesics intersect the other green and blue ones. All in all, these extra contribution cancel (the orientation being opposite), so that $\partial S_{g;k_1,\dots,k_n} = -\Gamma_{g,n}$.

Secondly we claim that $S_{g;k_1,\dots,k_n}$ has genus one. In order to justify this claim, we compute its Euler characteristics. The part S_h is made of $4g$ rectangles of the form $e\times[0,\pi]$ where e is a edge of a green geodesics located between two double points. Each such rectangle contributes to -1 to the Euler characteristics, so we have $\chi(S_h) = -4g$. The part S_c is made of $2n-2$ hexagons, each if them projecting on a triangle on $\mathcal{O}_{g;k_1,\dots,k_n}$. Each such hexagon has a contribution of $-1/2$ to the Euler characteristics, so we have $\chi(S_c) = 1-n$. Finally the part S_σ is made of two rectangles similar to those of S_h who contribute to -1 each, and two octagon which project on hexagons who also contribute to -1, so $\chi(S_\sigma) = -4$. Adding all contributions we have $\chi(S_{g;k_1,\dots,k_n}) = -4g-n-3$, which is the opposite of the number of boundary components. Hence $S_{g;k_1,\dots,k_n}$ is a torus.

Another way to check that $S_{g;k_1,\dots,k_n}$ is a torus is to count how many times $S_{g;k_1,\dots,k_n}$ intersects the stable direction of the geodesic flow along each boundary component. If this number is 2 for every boundary component, then, following the comments of Section 1.b, the surface $S_{g;k_1,\dots,k_n}$ is indeed a torus. For the surface S_h, one sees that it is tangent to the stable direction of the geodesic flow only in the fibers of the intersection points of the green geodesics, and in such fibers it is tangent four times (one per quadrant). This implies that it is indeed tangent to the stable direction twice per boundary component. For the surface S_c, since the red vector field is assumed to be by convex curves, it cannot be tangent to the stable direction, since the latter is given by horocycles, except at the inflection points of the foliations. In the fibers of such points, the surface S_c is twice tangent to the stable direction, hence the result. Finally for S_σ, the argument is similar.

Thirdly we check that $S_{g;k_1,\dots,k_n}$ is transverse to the geodesic flow. Concerning S_h, it is obvious since an orbit of the geodesic flow not transverse to S_h would be tangent to a green geodesic, hence it is actually the lift of a green geodesic. Concerning S_c and S_σ, we have to check that the vector fields that define these surfaces have non-zero curvature everywhere. For S_c, it is the union of foliations of $2n-2$ triangles whose

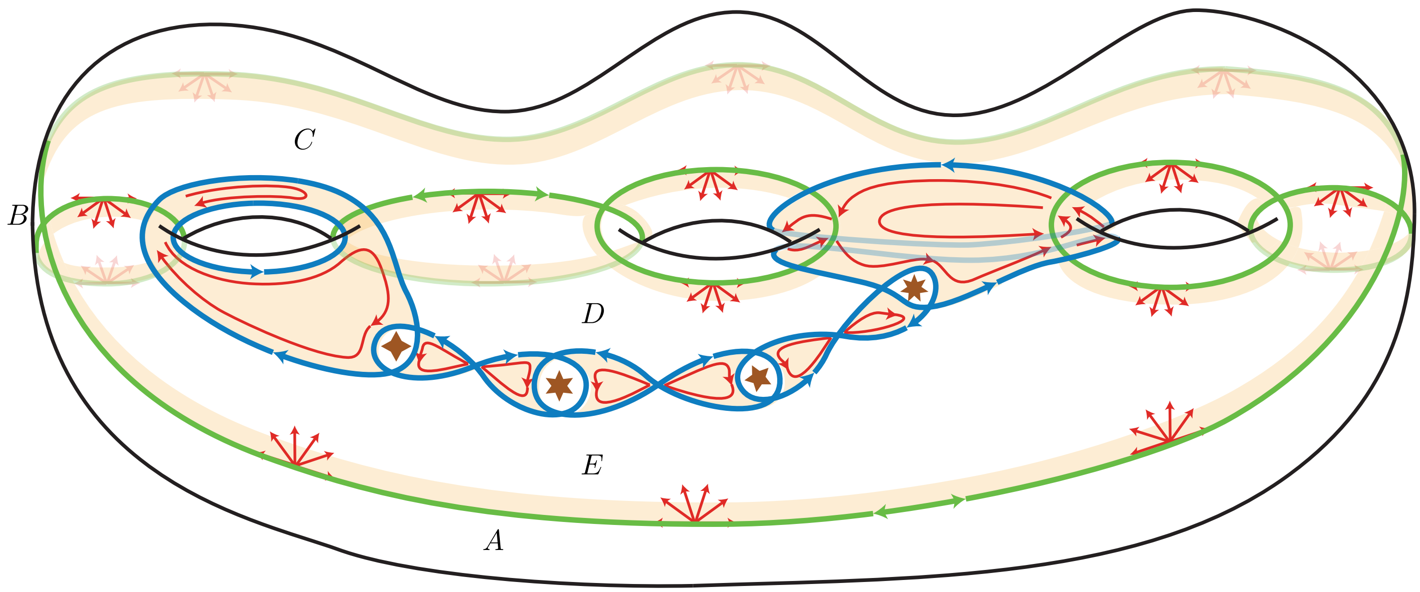


FIGURE 2. A genus one Birkhoff section for the geodesic flow on an orbifold of genus 3 with 4 cone points (indicated by brown stars). The boundary is depicted with bold green and blue lines. The Birkhoff section itselft is depicted with the red arrows and lines: a part of it lies in the fibers of the points on the green geodesics: at those points the section consists of a segment of tangent vectors pointing in one of the two adjacent sides, another part of the section lies in some regions determined by blue geodesics: in those regions the section consists of thoses vectors tangent to the oriented foliation sketched by the red oriented lines. Every green segment contributes -1 to the Euler characteristic of the surface, every foliated blue n-gon contributes $\frac{2-n}{2}$, and the 4 mixed blue/green n-gons contributes $\frac{4-n}{2}$. Hence the total Euler characteristics is -19. Since there are 19 boundary components, the genus is 1.

boundary are geodesics. As in [Deh15], one can indeed achieve such a foliation by convex curves. For S_h, it is the folication of convex 4- or 6-gons with a Reeb-like vector field, which can also be done with convex curves.

Finally we have to check that $S_{g;k_1,\dots,k_n}$ intersects every orbit in bounded time. Since all regions of $S_{g;k_1,\dots,k_n}$ delimited by the green and blue geodesics have no topology, any geodesic on $S_{g;k_1,\dots,k_n}$ must intersect a green or a green blue within a bounded time. Denoting by A, B, C, D et E the large regions of $S_{g;k_1,\dots,k_n}$ as on Figure 2 (forgetting only those regions on which S_c and S_σ project), one checks that everytime a geodesics goes from A to B or E it intersects $S_{g;k_1,\dots,k_n}$, also from C to B or D, and from D to E. Therefore, in order not to cross $S_{g;k_1,\dots,k_n}$, an orbit of the geodesic flow should never visit A or C since exiting these regions force an intersection with SSS. Since going directly from D or E to B can only be made via the fiber of a double point of a green geodesic, this also forces an intersection, so that an orbit not intersecting $S_{g;k_1,\dots,k_n}$ should stay in the D- and E-regions. Once again this is impossible since going from D to E forces an intersection. This concludes the proof of Theorem C2. ☐

3. Removing fixed points on the torus

Theorem B0 is due to Minakawa. However its proof is only given in the abstract of a talk. We write it here, with some extra information on the first-return maps (Theorem B2).

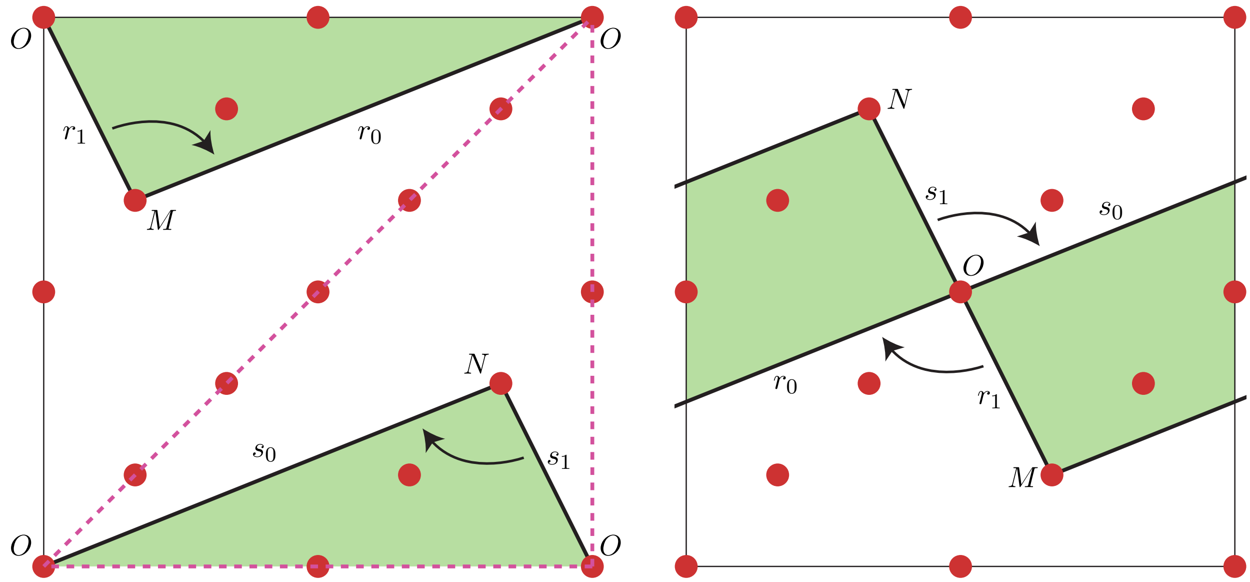


FIGURE 3. The parallelogram P_{XW} (green) in the torus $\mathbb{T}^2$, here seen as the square $[0,1]^2$ on the left, and as the square $[-1/2, 1/2]^2$ on the right. The red dots denote the fixed points of W. The segment r_1 is sent by W on r_0, and s_1 is sent on s_0. If W is not of the form X^mY^n or Y^nX^m, then the point N lies in the interior of the dotted triangle, otherwise it lies on the vertical or diagonal border of this triangle.

Assume we are given a word W on the alphabet $\{X, Y\}$ that contains both letters. Consider the manifold $\mathbb{T}^3_{XW}$ with the suspension flow ϕ_{sus}. It has natural global sections given by the horizontal tori $\mathbb{T}^2_* := \mathbb{T}^2 \times \{*\}$. The goal is to find a genus-one Birkhoff section whose first-return map is given by the matrix W.

The main idea is to add to $\mathbb{T}^2_{2/3}$ an embedded pair of pants $\mathbb{P}$ whose interior is transverse to the flow and whose boundary is made of 3 periodic orbits. The union $\mathbb{T}^2_{2/3} \cup \mathbb{P}$ will not be a surface, but the Fried sum $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ will be a Birkhoff surface. There are two points to check: firstly that the Fried sum still has genus 1 (this is where the choice of $\mathbb{P}$ is subtle, since most choices would lead to higher genus sections), secondly that the first return map is given by the matrix W in an adapted basis.

3.a. **Finding a nice pair of pants.** For W an arbitrary product of the matrices X and Y containing both letters, we are interested in the matrix XW, that we denote by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Also we set $t := \mathrm{tr}(XW) = a + d$. One can explicitely write some fixed points for XW, namely the points of the form $\frac{k}{t-2}\left(\begin{smallmatrix} d-1 \\ -c \end{smallmatrix}\right)$ for $k \in \mathbb{Z}$.[2] Indeed one has

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (d-1)/(t-2) \\ -c/(t-2) \end{pmatrix} = \begin{pmatrix} (1-a)/(t-2) \\ -c/(t-2) \end{pmatrix} = \begin{pmatrix} (d-1)/(t-2) \\ -c/(t-2) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Denote by O, M, N the respective projections on $\mathbb{T}^2$ of the points $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} (d-1)/(t-2) \\ -c/(t-2) \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} (a-1)/(t-2) \\ c/(t-2) \end{smallmatrix}\right)$. By the previous computation, O, M, and N are fixed by XW. But the computation gives more. Denote by r_1 the projection on $\mathbb{T}^2$ of the segment $[\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} (d-1)/(t-2) \\ -c/(t-2) \end{smallmatrix}\right)]$ in $\mathbb{R}^2$ and by r_0 the projection of $[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} (d-1)/(t-2) \\ -c/(t-2) \end{smallmatrix}\right)]$. The segments r_1 and r_0 both connect O to M and they do not intersect on $\mathbb{T}^2$. Then XW sends r_1 on r_0. Similarly define s_1 as the projection of $[\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} (a-1)/(t-2) \\ c/(t-2) \end{smallmatrix}\right)]$ and s_0 as the projection of $[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} (a-1)/(t-2) \\ c/(t-2) \end{smallmatrix}\right)]$. As before, s_1 and s_0 both connect O to N and they do not intersect. A similar computation shows that XW sends s_1 on s_0.

[2] Since W has $t-2$ fixed points on $\mathbb{T}^2$, the points we described may or may not be all of the fixed points, depending on the value of $\gcd(d-1, -c)$.

Define P_{XW} as the parallelogram on $\mathbb{T}^2$ whose edges are r_0, r_1, s_0 and s_1 in this order (see Figure 3 where $XW = (\begin{smallmatrix} 3 & 8 \\ 4 & 11 \end{smallmatrix})$). Its vertices are O, M, O, N is this order. In order to use P_{XW}, we have to know exactly when it embeds in $\mathbb{T}^2$.

Lemma 3.1. *(see Figure 3) With the previous notations, the interior of P_{XW} is embedded in $\mathbb{T}^2$, as well as the interiors of its sides r_0, r_1, s_0, s_1. If W is of the form XY^n or Y^nX for some $n \geqslant 1$, then the vertices M and N correspond to the same point of $\mathbb{T}^2$. Otherwise the three vertices O, M, and N correspond to different points on $\mathbb{T}^2$.*

Proof. We claim that the point N lies in the closed triangle bounded by the points $(0,0), (1,0), (1,1)$. Indeed its coordinates are $(a-1, c)/(t-2)$. Since we have $a, d \geqslant 1$, we have $0 \leqslant a-1 \leqslant a+d-2$, hence the first coordinates lies in $[0,1]$. Then, writing $W = (\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix})$ with $a', b', c', d' > 0$, one has $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = XW = (\begin{smallmatrix} a'+c' & b'+d' \\ c' & d' \end{smallmatrix})$, and so $c \leqslant a-1$. This proves that N is indeed under the first diagonal. By symmetry the point M lies in the triangle whose vertices are $(0,0), (1,1), (0,1)$, and so the triangles OMO and ONO have disjoint interiors.

Now we have to check when the point N lies on the boundary of the triangle $(0,0), (1,0), (1,1)$. Since c is positive, it cannot lie on the horizontal side.

The point N lies on the diagonal if one has $a-1 = c$, which means $a' = 1$. This implies that W is of the form $(\begin{smallmatrix} 1 & m \\ n & mn+1 \end{smallmatrix}) = Y^nX^m$ for some $m, n \geqslant 1$, and XW is then equal to $(\begin{smallmatrix} n+1 & mn+m+1 \\ n & mn+1 \end{smallmatrix})$. In particular one has $t = mn+n+2$, so that N has coordinates $(n,n)/(mn+n) = (\frac{1}{m+1}, \frac{1}{m+1})$. Therefore P_{XW} is not embedded at N only in the case $m = 1$, in which case one has $M = N$.

Finally N lies on the vertical side $[(1,0),(1,1)]$ if one has $a-1 = t-2$, which means $d = 1$. That means that XW is of the form $(\begin{smallmatrix} mn+1 & m \\ n & 1 \end{smallmatrix}) = X^mY^n$ for some $m \geqslant 2$ and $n \geqslant 1$. In this case, N has coordinates $(mn, n)/mn = (1, 1/m)$. So P_{XW} is not embedded at N only in the case $m = 2$, in which case one also has $M = N$.

Summarizing, P_{XW} fails to embed only at O in general, except when W is of the form XY^n or Y^nX for some $n \geqslant 1$, in which case P_{XW} fails to embded at O and $M = N$ only. ☐

Now we define our nice pair of pants. Recall that $\mathbb{T}^3_{XW}$ is the 3-manifold $\mathbb{T}^2 \times [0,1]/_{(p,1)\sim(XW(p),0)}$.

Definition 3.2. For W a matrix which is positive product of X and Y that contains both letters, define the surface $\mathbb{P}^\perp$ in $\mathbb{T}^3_{XW}$ as the union of the parallellogram $P_{XW} \times \{1/3\}$ in $\mathbb{T}^2_{1/3}$ with the rectangles $r_1 \times [1/3, 1]$, $r_0 \times [0, 1/3]$, $s_1 \times [1/3, 1]$, and $s_0 \times [0, 1/3]$. Define $\mathbb{P}$ as the surface obtained from $\mathbb{P}^\perp$ by smoothing it and making it transversal to the suspension flow (as explained in [Fri83]).

Since $\mathbb{P}^\perp$ is made from one parallellogram $P_{XW} \times \{1/3\}$ which is positively transverse to ϕ_{sus} and four rectangles (actually two in the manifold $\mathbb{T}^3_{XW}$ since $s_1 \times \{1\}$ is identified with $s_0 \times \{0\}$) tangent to it, one can indeed smooth it to make it transverse to ϕ_{sus}.

Denote by γ_O the orbit $O \times [0,1]$ of ϕ_{sus}, and simillarly introduce $\gamma_M := M \times [0,1]$ and $\gamma_N := N \times [0,1]$. Write Γ for the link $\gamma_M \cup \gamma_N \cup \gamma_O$.

Lemma 3.3. *(see Figure 4.) In the previous context, the surface $\mathbb{P}$ is a Birkhhoff surface which is topologically a pair of pants. Moreover one has $i(\partial\mathbb{P}) = -\gamma_M - \gamma_n + 2\gamma_O$ if W is not of the form XY^n or Y^nX, and $i(\partial\mathbb{P}) = -2\gamma_M + 2\gamma_O$ otherwise.*

Proof. First we assume that W is not of the form X^mY^n or Y^nX^m.

Topologically, the surface $\mathbb{P}$ is made of one parallellogram P_{XW} and two vertical rectangles. Counting the contributions, we see that it has Euler characteristics-1. Alternatively, one can count how much the tangent space to the boundary $\partial\mathbb{P}$ intersects the stable direction of the suspension flow. These intersection points appear when, at a vertex of P_{XW}, the interior of P_{XW} intersects the stable direction of XW. This happens only twice (since the angles at the vertices O, M and N add up to a complete turn), hence the Euler characteristics is -1.

Then one checks that $\mathbb{P}$ has three boundary components: one that is a longitude of γ_M and whose orientation is opposed to ϕ_{sus}, one that is a longitude of γ_N and that is also opposed to ϕ_{sus}, and one that is a curve wrapping twice along γ_O, with the same orientation as ϕ_{sus}. The fact that along γ_O there is only one boundary component and not two can be checked in two ways: first the Euler characteristics is odd,

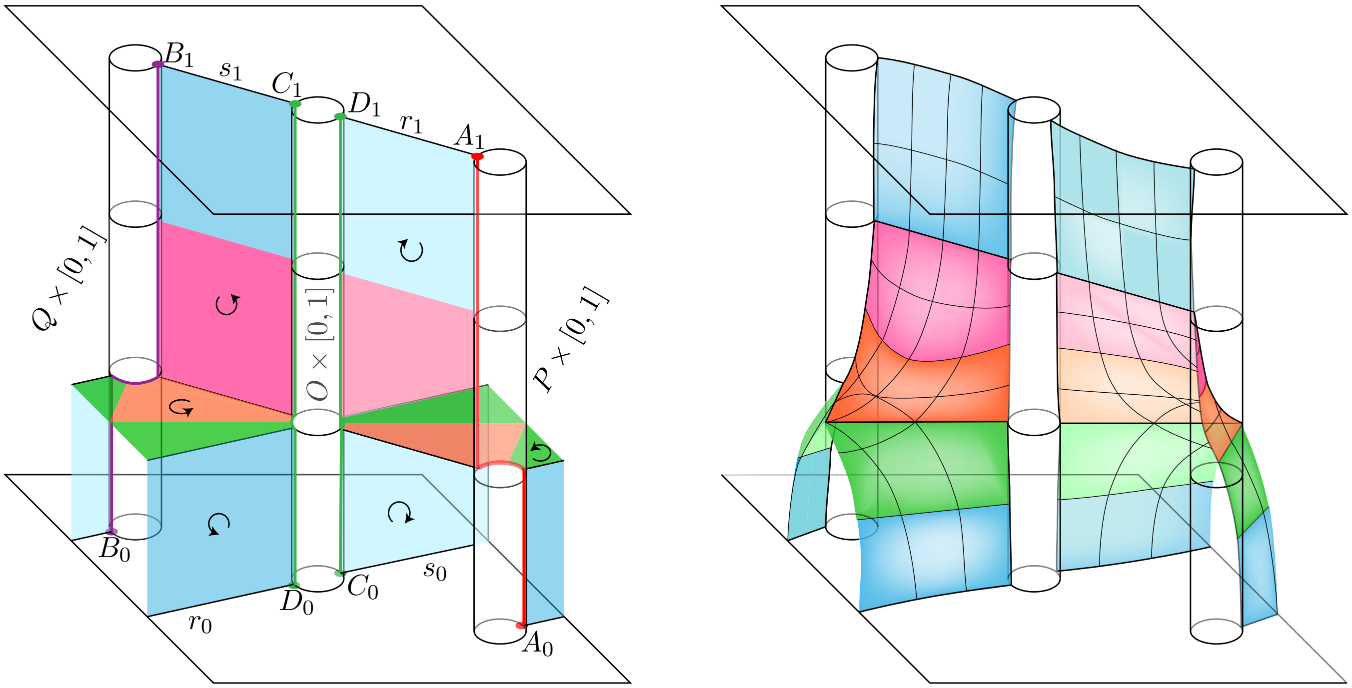


FIGURE 4. The ready-made $\mathbb{P}^\perp$ and the tailored $\mathbb{P}$ in $\mathbb{T}^3_{XW,\Gamma}$, the 3-manifold obtained from $\mathbb{T}^3_{XW}$ by blewing-up the three periodic orbits $O \times [0,1]$, $M \times [0,1]$, and $N \times [0,1]$. The map XW identifies the segments $r_1 \times \{1\}$ with $r_0 \times \{0\}$, $s_1 \times \{1\}$ with $s_0 \times \{0\}$, the points $A_1 \times \{1\}$ with $A_0 \times \{0\}$, etc. One sees that $\mathbb{P}^\perp$ and $\mathbb{P}$ indeed have three boundary components : $O \times [0,1]$ along which $\mathbb{P}^\perp$ wraps twice (in green), $M \times [0,1]$ minus once (in red), and $N \times [0,1]$ minus once (in purple).

so the total number of boundary components has to be odd, or one pays attention at who connects to who when identifying $\mathbb{T} \times \{1\}$ with $\mathbb{T} \times \{0\}$.

If W is of the form XY^n and Y^nX, then the link Γ has only two components γ_M and γ_O. The surface $\mathbb{P}$ is topologically the same, but now the two boundary components that were longitudes of γ_M and γ_N are two parallel longitudes of γ_M, with negative orientation. □

3.b. **The Fried sum $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$.** The surface $\mathbb{T}^2_{2/3}$ and $\mathbb{P}$ are two Birkhoff surfaces for ϕ_{sus}. The first one has empty boundary and cuts all the orbits, while the second one has non-empty boundary and does not cut all orbits. They are transverse one to the other and interset along two arcs, namely $r_1 \times \{2/3\}$ and $s_1 \times \{2/3\}$. Their union is therefore not a surface in $\mathbb{T}^3_{XW}$. However, we can consider their Fried sum $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ (see Section 1). Since $\mathbb{T}^2_{2/3}$ is already a Birkhoff section (*i.e.*, cuts all orbits), so is $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$. We denote by $\mathbb{T}^3_{XW,\Gamma}$ the 3- manifold $\mathbb{T}^3_{X,W}$ where the three orbits γ_M, γ_N and γ_O have been blown up.

Lemma 3.4. *(see Figure 6) If W is not of the form XY^n or Y^nX, the Fried sum $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ has genus 1 and four boundary components. Its boundary is embedded, except along γ_O which has multiplicity 2. Otherwise it has genus 1 and three boundary components. Its boundary is not embedded : γ_M and γ_O have multiplicities -2 and 2 respectively.*

Proof. First we assume that W is not of the form XY^n or Y^nX.

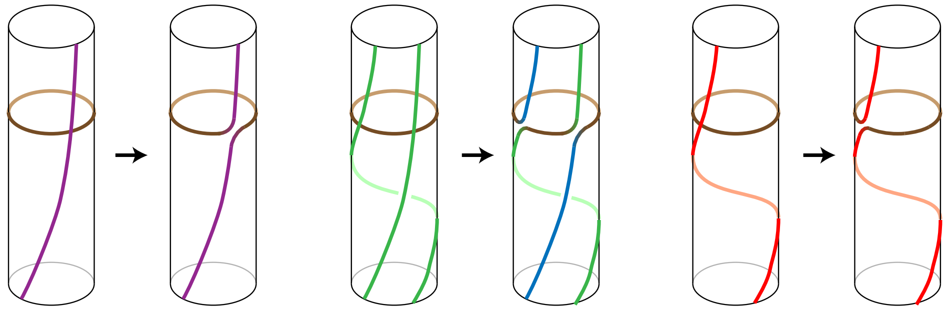


FIGURE 5. The Fried union of the boundaries $\partial\mathbb{T}^2_{2/3}$ and $\partial\mathbb{P}$ on the three components of $\partial\mathbb{T}^3_{XW,\Gamma}$. One sees that $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ has two boundary components along γ_O, and so four boundary components in total.

We compute the Euler characteristics of $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ in the 3-manifold $\mathbb{T}^3_{XW,\Gamma}$. Since $\mathbb{T}^2_{2/3}$ intersects γ_M, γ_N and γ_O in three points, one has $\chi(\mathbb{T}^2_{2/3}\cap\mathbb{T}^3_{XW,\Gamma}) = -3$. Since $\mathbb{P}$ is a pair of pants, and the Euler characteristics is additive under Fried sum (see Section 1.b), one has $\chi(\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}) = -4$.

Now we have to count the boundary components has $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$. In general there are fomulas involving the multiplicities and gcd's, but here one can make the count by hand (see Figure 5): along γ_M and γ_N, there is still one boundary component. Along γ_O, the unique boudary component of $\partial\mathbb{P}$ intersects the meridan disc corresponding to $\partial\mathbb{T}^2_{2/3}$ twice, and then turns into two boundary components (here also one could see that since the Euler characteristics is even, the number of boundary components has to be even, hence there cannot be only one boundary component along γ_O). Therefore $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ has 4 boundary components. Since its Euler characteristics is -4, it is a torus.

In the case $W = XY^n$ or Y^nX, the Euler characteristics of $\mathbb{T}^2_{2/3} \cap M_\Gamma$ is only -2, so that $\chi(\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P})$ is -3. A similar argument shows that it has 3 boundary components: one along γ_O and two along γ_M. Hence it is a torus. $\square$

3.c. **Computing the first-return map.** We are left with the computation of the first-return map f of ϕ_{sus} on the surface $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$. Note that since ϕ_{sus} is an Anosov flow, its stable and unstable foliations print on $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ two invariant foliations that are uniformly contracted/expanded by f. Since these foliations are orientable, they have exactly 2 singularities on every boundary component of $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$, hence can be extended into foliations of the surface $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ where all boundary components are contracted into a point. This imply that f is an Anosov map of the torus, hence it is given by a matrix in $\mathrm{SL}_2(\mathbb{Z})$.

Let us first remark that one easily sees that f has less fixed points than XW. Indeed the surface $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ intersects every closed orbit of the suspension flow the same or a larger number of times than $\mathbb{T}^2$. In particular the number of periodic orbits that are interestected only once by $\mathbb{T}^2_{2/3}\overset{F}{\cup}\mathbb{P}$ is smaller or equal than by $\mathbb{T}^2$. It cannot be equal since the orbit through O is now an order 2 point for f (and one sees on Figure 3 that it is likely to be much smaller, since every fixed point for W that sits inside P_{XW} becomes a higher-period periodic point for f). This argument implies than the trace of f is strictly smaller than the trace of XW, thus proving Minakawa's Theorem B0. Here we want to compute precisely the first-return map, in order to gain information on the Ghys graph.

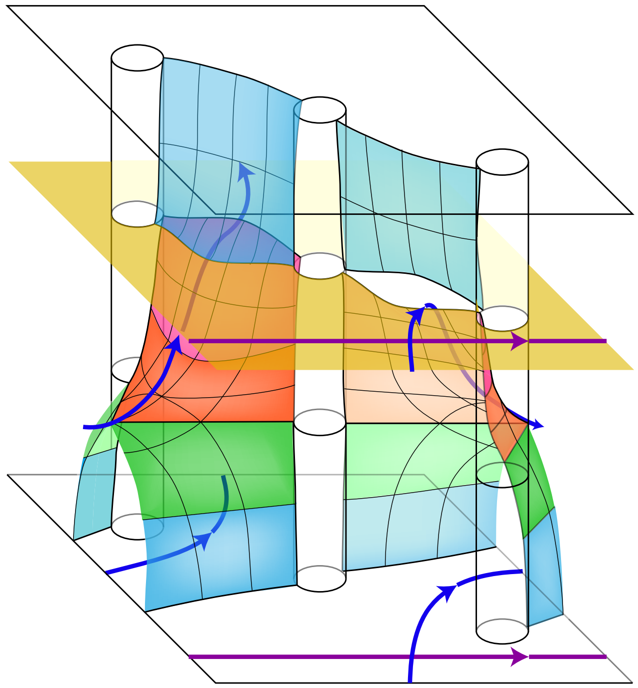


FIGURE 6. The desingularized sum $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ in the 3-manifold $\mathbb{T}^3_{XW,\Gamma}$. Also the curves $\alpha' \times \{0\}$ and $\beta' \times \{0\}$ and their images when pushed toward $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ along ϕ_{sus}.

Since $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ is a torus with boundary, we first exhibit two closed curves α, β that intersects transversaly and exactly once: this ensures that they form a basis for the homology of $\mathbb{T}^2$.

There are natural candidates, namely any pair of vectors that form basis of $\mathbb{T}^2_{2/3}$. In order to make the computation easier we choose for α a curve in $\mathbb{T}^{2/3}_2$ that avoids $(r_1 \cup s_1) \times \{2/3\}$ and whose homology class is $\left(\begin{smallmatrix} d \\ -c \end{smallmatrix}\right)$, and similarly we choose β whose class is $\left(\begin{smallmatrix} d-b \\ a-c \end{smallmatrix}\right)$. This is possible since $r_1 \cup s_1$ is contractible.

Pushing α and β along the flow they meet $\mathbb{T}^2 \times \{1\}$ where they are identified with $\alpha' \times \{0\}$ and $\beta' \times \{0\}$, with α' having homology class $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} d \\ -c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and β' having class $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} d-b \\ a-c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$.

Pushing α' further, it does not meet $\mathbb{P}$, hence meets $\mathbb{T}^2_{2/3}$ directly along $\alpha' \times \{2/3\}$. On the other hand, β' goes once into the "tunnel" formed by $\mathbb{P}$ (see Figure 6), so when pushing it along ϕ_{sus} it is sent on a curve β'' that starts somewhere on $\alpha' \times \{2/3\}$, goes toward $r_1 \times \{2/3\}$, then takes a half-pipe toward $s_1 \times \{2/3\}$ and goes back to its inital point (see Figure 6). In particular in the canonical homological coordinates α is sent on the curve $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and β on $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Hence the first-return map on $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ along ϕ_{sus} is given by $\left(\begin{smallmatrix} d & d-b \\ -c & a-c \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a-c & b-d \\ c & d \end{smallmatrix}\right) = W$.

Proof of Theorem B2. Let W be a word both letters X and Y, and write $W = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Then a', b', c', and d' are all positive. Moreover, $XW = \left(\begin{smallmatrix} a'+c' & b'+d' \\ c' & d' \end{smallmatrix}\right)$ has all coefficients positive.

One then considers the pair of pants $\mathbb{P}$ given by Definition 3.2, and the Fried sum $\mathbb{T}^2_{2/3} \overset{F}{\cup} \mathbb{P}$ of Subsection 3.b, which is a genus-one Birkhoff section for ϕ_{sus}. The computation of Subsection 3.c then shows that the induced first-return map is given in the basis (α, β) by W. $\square$

4. Remarks and perspectives

Suspensions of automorphisms of the torus, up to topological equivalence, correspond to conjugacy classes (in $\mathrm{GL}_2(\mathbb{Z})$) of matrices in $\mathrm{SL}_2(\mathbb{Z})$. As explained before, such conjucacy classes correspond to finite words in X, Y, up to cyclic permutation and exchanging the letters X and Y (thanks to the conjugacy by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$). Theorem B2 and its counterpart when we replace X by Y then says that, when two words differ by adding or removing one letter, the Ghys distance between the suspensions is at most 3.

Denote by $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$ the restriction of the Ghys graph to suspensions of hyperbolic automorphisms of $\mathbb{T}^2$. There are two other natural graphs to compare $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$ with : first the conjugacy graph $G_{\mathrm{SL}_2(\mathbb{Z})}(X, Y)$ which is the quotient of the Cayley graph associated to the generators X, Y of $\mathrm{SL}_2(\mathbb{Z})$ by the conjugacy relation; second the word graph $G_+(X, Y)$ which is the graph whose vertices are positive words in X and Y and to words are connected if they differ by adding or removing one letter. In this way, $G_+(X, Y)$ is naturally a subgraph of $G_{\mathrm{SL}_2(\mathbb{Z})}(X, Y)$, which is naturally a subgraph (up to multiplying the lengths of the edges by at most 3) of $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$.

Question 4.1. What are the geometries of $G_+(X, Y)$, $G_{\mathrm{SL}_2(\mathbb{Z})}(X, Y)$, and $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$? Are they hyperbolic?

Question 4.2. Are the graphs $G_+(X, Y)$, $G_{\mathrm{SL}_2(\mathbb{Z})}(X, Y)$, and $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$ quasi-isometric?

If the answer is negative, it means that there are shortcuts in $G_{\mathrm{Ghys}}(\mathbb{T}^3_*)$ that do not exist in $G_+(X, Y)$, or $G_{\mathrm{SL}_2(\mathbb{Z})}(X, Y)$. Can we find these shortcuts?

On the other hand, in order to prove that there are no shortcuts, one should probably find lower bounds in the Ghys distance. Even forgetting about the flow, this does not seem an easy question.

Question 4.3. Are there explicit lower bounds on the Ghys distance?

Signatures seem a promising place to look at. We know that Christopher-Lloyd Simon is working on this project.

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