CLUSTER ALGEBRAS AND CLUSTER CATEGORIES ASSOCIATED WITH TRIANGULATED SURFACES: AN INTRODUCTION

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Abstract. These are the notes of a three hours minicourse given at the school Winterbraids VIII, CIRM Luminy in March 2018.

Introduction

Cluster algebras have been introduced almost 20 years ago by Fomin and Zelevinsky. They have immediately been linked with various fields of mathematics among which representation theory of finite dimensional algebras. This link has been very fruitful from both sides, permitting on one hand a better understanding of cluster algebras, and on the other hand to develop new ideas in representation theory.

The aim of this short note is to give a brief idea of this link. Cluster algebras and cluster categories are defined in a very general setting, but in this course, we focus on the construction coming from triangulated surfaces. The definition of cluster algebra in this context is very natural, and this setup already reflects all questions and difficulties that arise in general. Lots of short courses have been already given on a similar subject, and we refer to the following notes for more complete material [FWZ16], [FWZ17], [Kel11b], [Kel11a], or [Kel12], [Pla].

1. Cluster algebras from triangulated surfaces

Throughout the paper, Σ will be an oriented Riemann surface with non empty boundary, and $\mathcal{M}$ be a finite set of points (called *marked points*) on the boundary of Σ such that there is at least one marked point on each boundary component of Σ. We assume moreover that $(\Sigma, \mathcal{M})$ is not a disc with 1 or 2 marked points.

The aim of this section is to give the definition of the cluster algebra associated to the marked surface $(\Sigma, \mathcal{M})$. The material of this section comes entirely from [FZ02] and [FST08].

1.1. **Triangulations.** In this section, we recall some basic facts on arcs and triangulations of surfaces.

Definition 1.1. A *boundary segment* is a curve on the boundary $\partial\Sigma$ so that it intersects $\mathcal{M}$ only in its endpoints.

By an *arc*, we mean a continuous map $\gamma : [0,1] \to \Sigma$ such that $\gamma_{|]0,1[}$ is injective and such that $\gamma(0)$ and $\gamma(1)$ belong to $\mathcal{M}$. We consider the set of arcs up to isoptopy (fixing endpoints). We moreover assume that an arc is not isotopic to a marked point or to a boundary segment. We denote by $\mathbb{A}(\Sigma, \mathcal{M})$ the set of isotopy classes of arcs.

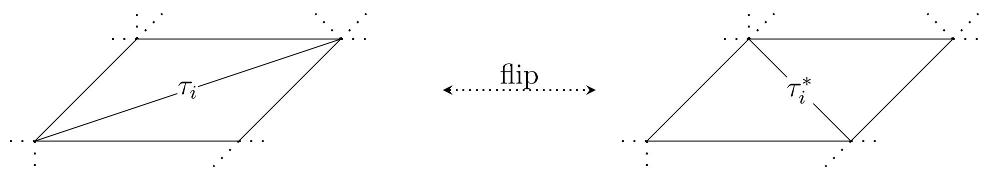
Two arcs are called *compatible* if they have representants in their isotopy class that do not intersect, except possibly at their endpoints.

A *triangulation of* $(\Sigma, \mathcal{M})$ is a maximal collection of pairewise compatible arcs.

Since there is at least one marked point on each boundary component of the boundary, a triangulation (together with the boundary segments) cuts out the surface Σ into triangles.

The following facts are classical (see [FST08, Section 2] for references).

Theorem 1.2.
- *Any arc can be completed into a triangulation.*
- *If $\tau = \{\tau_1, \ldots, \tau_n\}$ is a triangulation of $(\Sigma, \mathcal{M})$, then we have the equality $n = 6g - 6 + 3b + c$ where g is the genus of Σ, b the number of boundary components and c the number of marked points.*
- *Let $\tau = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of $(\Sigma, \mathcal{M})$. For any $i = 1, \ldots, n$ there exists a unique arc (up to isotopy) τ_i^* not isotopic to τ_i such that $\mathfrak{f}_{\tau_i}(\tau) := \{\tau_1, \ldots, \tau_i^*, \ldots, \tau_n\}$ is a triangulation of $(\Sigma, \mathcal{M})$. Such new triangulation is called the* flip *of τ at the arc τ_i.*
- *Any two triangulations τ and τ' can be related by a (non unique) sequence of flips.*



1.2. **Definition of cluster algebra.** The keystone of the definition of cluster algebra associated to a triangulated surface is given by the following result.

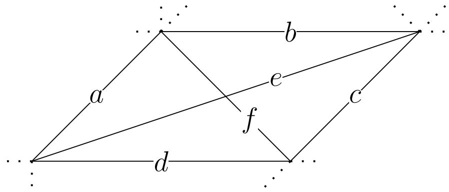
Theorem 1.3. *Let $(\Sigma, \mathcal{M}, \tau)$ be a triangulated surface. Denote by n the number of arcs in τ. Then there exists a map*

$$\mathbf{x}^\tau : \mathbb{A}(\Sigma, \mathcal{M}) \longrightarrow \mathbb{Z}(x_1, \ldots, x_n),$$

such that

- *for any $i = 1, \ldots, n$ $\mathbf{x}^\tau(\tau_i) = x_i$;*

- *for any arcs a, b, c, d e and f in the following local configuration*



we have

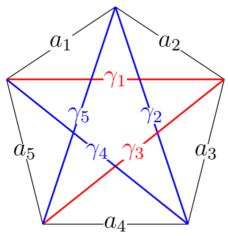
$$\mathbf{x}^\tau(e)\mathbf{x}^\tau(f) = \mathbf{x}^\tau(a)\mathbf{x}^\tau(c) + \mathbf{x}^\tau(b)\mathbf{x}^\tau(d), \tag{1.1}$$

where $\mathbf{x}^\tau(\gamma) = 1$ if γ is a boundary segment.

Note that the relation (1.1) is the Ptolemey relation for the length of sizes of a quadrilateron inscribed in a circle.

Combining this theorem with Proposition 1.2 one sees that the map $\mathbf{x}^\tau$ is uniquely defined. The point here is to see that the map $\mathbf{x}^\tau$ is well-defined.

Example 1.4. Let $(\Sigma, \mathcal{M})$ be a disc with 5 marked points. The set $\mathbb{A}(\Sigma, \mathcal{M})$ contains exactly 5 elements $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ as in the following picture:



Fix the triangulation $\tau = \{\gamma_1, \gamma_3\}$. Then we have $\mathbf{x}^\tau(\gamma_1) = x_1$, $\mathbf{x}^\tau(\gamma_3) = x_2$. Since γ_1 and γ_5 are diagonals of the quadrilateron $a_5a_1a_2\gamma_3$, we have the relation $\mathbf{x}^\tau(\gamma_1)\mathbf{x}^\tau(\gamma_5) = 1 + \mathbf{x}^\tau(\gamma_3)$ hence $\mathbf{x}^\tau(\gamma_5) = \frac{1+x_2}{x_1}$. For the same reason, we have $\mathbf{x}^\tau(\gamma_4) = \frac{1+x_1}{x_2}$. Now γ_2 and γ_1 are diagonals of $a_1a_2a_3\gamma_4$, so get

$$\mathbf{x}^\tau(\gamma_2) = \frac{1 + \frac{1+x_1}{x_2}}{x_1} = \frac{1 + x_1 + x_2}{x_1x_2}.$$

We also see that γ_2 and γ_3 are diagonals of $a_2a_3a_4\gamma_5$, and we have

$$\mathbf{x}^\tau(\gamma_2) = \frac{1 + \frac{1+x_2}{x_1}}{x_2} = \frac{1 + x_1 + x_2}{x_1x_2}.$$

We can check similarly that the map $\mathbf{x}^\tau$ is here well defined.

We are now ready to state the main definition of the section.

Definition 1.5. The *cluster algebra* $\mathcal{A}_{(\Sigma,\tau)}$ is the subalgebra of $\mathbb{Z}(x_1, \ldots, x_n)$ generated by the image of the map $\mathbf{x}^\tau$.

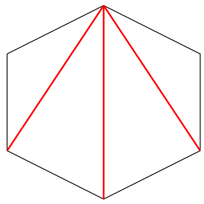
The elements $\mathbf{x}^\tau(\gamma)$, where $\gamma \in \mathbb{A}(\Sigma, \mathcal{M})$, are called *cluster variables*.

The set $\{\mathbf{x}^\tau(\gamma_1), \ldots, \mathbf{x}^\tau(\gamma_n)\}$ where $\{\gamma_1, \ldots, \gamma_n\}$ is a triangulation is called a *cluster*.

Example 1.6. (1) Let $(\Sigma, \mathcal{M})$ be a disc with 4 marked points, and τ a triangulation (given by one arc). Then there are two cluster variables x_1 and $\frac{2}{x_1}$ and the cluster algebra is $\mathcal{A}_{(\Sigma,\tau)} = \mathbb{Z}[x_1, 2x_1^{-1}]$.

(2) In the example above, one has exactly 5 cluster variables which are $\{x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\}$ and five clusters. The set $\{x_1, \frac{1+x_1}{x_2}\}$ is a cluster while the set $\{x_1, \frac{1+x_2}{x_1}\}$ is not since $\{\gamma_1, \gamma_5\}$ is not a triangulation.

(3) Let $(\Sigma, \mathcal{M})$ be the disc with 6 marked points and τ be the following triangulation:

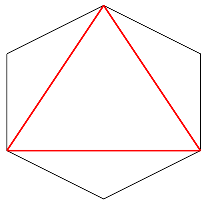


There are 9 cluster variables which are

$$x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{1+x_2}{x_3}, \frac{x_1+x_3}{x_2}, \frac{x_1+x_3+x_2x_3}{x_1x_2},$$

$$\frac{x_1+x_3+x_1x_2}{x_2x_3}, \text{ and } \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3}.$$

Let τ' be the following triangulation:



The 9 cluster variables are now: $x'_1, x'_2, x'_3, \frac{x'_1+x'_2}{x'_3}, \frac{x'_2+x'_3}{x'_1}, \frac{x'_1+x'_3}{x'_2}, \frac{x'_1+x'_2+x'_3}{x'_1x'_2}, \frac{x'_1+x'_2+x'_3}{x'_1x'_3}$ and $\frac{x'_1+x'_2+x'_3}{x'_2x'_3}$.

Here are some first basic facts on the map $\mathbf{x}^\tau$.

Proposition 1.7. (1) *The map $\mathbf{x}^\tau$ is injective, hence the cluster algebra $\mathcal{A}_{(\Sigma,\tau)}$ is finitely generated if and only if the set $\mathbb{A}(\Sigma, \mathcal{M})$ is finite, that is if and only if the surface Σ is a disc.*

(2) *For any triangulation $\tau' = \{\tau'_1, \ldots, \tau'_n\}$ the cluster $\{\mathbf{x}^\tau(\tau'_1), \ldots, \mathbf{x}^\tau(\tau'_n)\}$ is a transcendental basis of $\mathbb{Z}(x_1, \ldots, x_x)$ and the field isomorphism $\mathbb{Z}(x'_1, \ldots x'_n) \to \mathbb{Z}(x_1, \ldots, x_n)$ sending x'_i to $\mathbf{x}^\tau(\tau'_i)$ restricts to an isomorphism $\mathcal{A}_{(\Sigma,\tau')} \to \mathcal{A}_{(\Sigma,\tau)}$.*

So as a subalgebra of $\mathbb{Z}(x_1, \ldots, x_n)$ the cluster algebra $\mathcal{A}_{(\Sigma,\tau)}$ heavily depends on τ, but as an abstract algebra (even as a cluster algebra), it only depends on the surface.

One first remarkable property that can be noted in the first example is the following.

Theorem 1.8. [FZ02]*(Laurent phenomenon) Let* (Σ, τ) *be a triangulated surface. For any* $\gamma \in \mathbb{A}(\Sigma, \mathcal{M})$*,* $\mathbf{x}^\tau(\gamma) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$*, hence the cluster algebra is contained in* $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$*.*

In fact we have more. Combining this result with Proposition 1.7 (2), we obtain the following.

Corollary 1.9. *For any arc* γ *and for any cluster* $\underline{y} = \{y_1, \ldots, y_n\}$*,* $\mathbf{x}^\tau(\gamma)$ *is in* $\mathbb{Z}[\underline{y}^{\pm 1}]$*. In particular we have the inclusion*

$$\mathcal{A}_{(\Sigma,\tau)} \subset \bigcap_{\underline{y} \text{ cluster}} \mathbb{Z}[\underline{y}^{\pm 1}].$$

The algebra $\bigcap_{\underline{y} \text{ cluster}} \mathbb{Z}[\underline{y}^{\pm 1}]$ is called the *upper cluster algebra.* In general it is strictly bigger than the cluster algebra, but it is equal in some cases [GLS06].

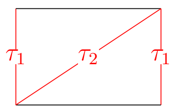
The following was conjectured by Fomin and Zelevinsky in their first paper [FZ02]. It was proved only ten years later.

Theorem 1.10. *(Positivity Conjecture/theorem)*[MSW11] *Let* (Σ, τ) *be a triangulated surface. For any arc* γ*, we have*

$$\mathbf{x}^\tau(\gamma) \in \mathbb{N}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

One very important problem in cluster algebra theory is the construction of basis in general, and basis with positive structure constant in particular. This has been achieved in the surface setting in [MSW13].

Example 1.11. Let $(\Sigma, \mathcal{M})$ be an annulus with two marked points and τ be a triangulation.



We obtain here infinitely many cluster variables X_k, $k \in \mathbb{Z}$ that can be computed using the following induction relation:

$$X_0 = x_1, X_1 = x_2 \text{ and } X_{k+1}X_{k-1} = X_k^2 + 1.$$

Fixing $X_0 = X_1 = 1$, a consequence of Laurent phenomenon and of the positivity conjecture is that one obtains only positive integers for the sequence. This sequence is related with the 4-Somos sequence (see [Mus02]).

1.3. **More general cluster algebras.** As we already mentioned in the introduction of this note, the theory of cluster algebra is much more general than the one explained here. Let us mention quickly different generalizations.

The most natural generalisation would be to consider the case with marked points not only in the boundary but also in the interior of the surface. One problem in this setting is that it is not always possible to flip triangulations: one may obtain "self-folded triangles" (that are not really triangles from a topological point of view), that cannot be fliped. To handle this problem, Fomin Shapiro and Thurston introduced the notion of tagged arcs and tagged triangulations, and get a theorem very similar to Theorem 1.3. We refer to [FST08] for details.

In fact, as we will see in the next section, one can associate a quiver (= a finite oriented graph) to any triangulation, in which every vertex corresponds to an arc. The flip of the triangulation can be translated in a combinatorial way in term of an operation on the corresponding quiver. This operation is called the *mutation*. Fomin and Zelevinsky associated (already in their first paper [FZ02]) a cluster algebra to any quiver without loops and oriented cycle of length 2. We refer here to [Kel] for a very useful java applet that permits (among other things) to mutate any quiver, and compute the cluster variables. The positivity conjecture and the construction of basis have been proved in this general setting [LS15], [GHKK18].

The data of a quiver Q without loops and oriented cycles of length 2 is equivalent to the data of a skewsymmetric matrix B with integral entries. Cluster algebras can also be defined starting with a skewsymmetrizable matrix, that is an integral matrix such that there exists a diagonal matrix D so that DB is skewsymmetric.

One can also defined a cluster algebra starting with a $r \times n$ ($r < n$) matrix with upper part being skew symmetrizable. Such cluster algebras are called *with coefficients*.

Several well studied rings have the structure of a cluster algebra. Let us list some of them:

(1) The homogenous coordinate ring of the Grassmannian $\mathrm{Gr}(k, n)$ has a structure of cluster algebra (with coefficients) the Plücker coordinates being cluster variables. [Sco06]
(2) The ring $\mathbb{C}[\mathrm{SL}_n(\mathbb{C})]$ (or more generally $\mathbb{C}[G/N]$, $\mathbb{C}[N]$, where G is a Lie group of type A, D, E, and N a unipotent subgroup) has a structure of cluster agebra [BFZ05]. For these rings a beautiful categorification quite different from the one presented here have been achieved by Geiss, Leclerc and Schröer, see for instance [GLS06]. We refer to [GLS13] for a survey concerning this point of view.
(3) The ring of functions of the decorated Teichmüller space of a surface with Penner coordinates [FG06].

2. Categorification

Now that the basic definition of a cluster algebra associated to a surface is stated, we can start with the categorification. Usually, in topology, the categorification of a ring is tha data of a category in which the ring appear as an invariant of the category (for example the Grothendieck ring of a monoidal category). Such categorification can be done [HL10]. But this is not the kind of categorification we are interested in here. In fact the categorification we are dealing with is more the categorification of flips (or mutation) than the categorification of the cluster algebra itself. As the reader will see, the cluster algebra does not really appear as an invariant ring of the category. The idea is more to find a category in which the operations of flip/mutation appears naturally.

2.1. **The category** $\mathrm{Rep}(Q_\tau)$**.** We start by stating very basic facts on the representation theory of quivers. We only state here what is really necessary for our exposition. All the material (and much more) can be found for example in [ASS06].

To a triangulation $\tau = \{\tau_1, \dots, \tau_n\}$ of $(\Sigma, \mathcal{M})$, we associate a quiver (= an oriented graph) Q_τ as follows: the vertices $\{1, \dots, n\}$ of Q_τ are in bijection with the (internal) arcs of τ. We put an arrow $i \to j$ if (τ_i, τ_j) form a positive angle (recall that the surface is oriented) in the triangulation τ.

Let us now describe the *category* $\mathrm{Rep}(Q_\tau)$. An *object* V in $\mathrm{Rep}(Q_\tau)$ is given by finite dimensional $\mathbb{C}$-vector spaces V_i for each vertex i, and linear maps $V_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$, so that $V_\beta \circ V_\alpha$ vanishes if α and β are consecutive angles in a triangle of τ. The *dimension vector* of V is the n-uple $(\dim_{\mathbb{C}} V_i, i = 1, \dots, n)$.

A *morphism* $\varphi : V \to W$ between two objects in $\mathrm{Rep}(Q_\tau)$ is given by $\mathbb{C}$-linear maps $\varphi_i : V_i \to W_i$ for each vertex i so that for each arrow $\alpha : i \to j$ the following square is commutative:

$$\begin{array}{ccc} V_i & \xrightarrow{V_\alpha} & V_j \\ \Big\downarrow{\scriptstyle \varphi_i} & & \Big\downarrow{\scriptstyle \varphi_j} \\ W_i & \xrightarrow{W_\alpha} & W_j \end{array} .$$

A morphism φ is an isomorphism if every φ_i is an isomorphism.

Given two representations, one can form the direct sum of them, which is also a representation. An object V in $\mathrm{Rep}(Q_\tau)$ is then said to be *indecomposable* if it is not isomorphic to the sum of two non zero representations.

Proposition 2.1. *The category* $\mathrm{Rep}(Q_\tau)$ *is the category of a finite dimensional modules over a finite dimensional algebra. In particular it* *is abelian, it satisfies the Krull-Schmidt property, i.e. every object is isomorphic to a unique direct sum of indecomposable objects.*

Example 2.2. Let $(\Sigma, \mathcal{M})$ be the triangulated pentagon as in example 1.4. Then the quiver Q_τ is $1 \to 2$. Denote by

$$S_1 = (\mathbb{C} \xrightarrow{0} 0), \quad S_2 = (0 \xrightarrow{0} \mathbb{C})$$

the simple representations (that are the representations such that $\dim_{\mathbb{C}} \bigoplus_i V_i = 1$). They are clearly indecomposable. For $\lambda \in \mathbb{C}$ denote by

$$V_\lambda = (\mathbb{C} \xrightarrow{\lambda} \mathbb{C}).$$

The representation V_0 is clearly decomposable since $V_0 = S_1 \oplus S_2$, whereas if $\lambda \neq 0$, then V_λ is indecomposable isomorphic to V_1.

Now let W be a general object in $\operatorname{Rep}(Q_\tau)$. It is given as follows:

$$W = (\mathbb{C}^m \xrightarrow{M} \mathbb{C}^n)$$

where $M \in \mathcal{M}_{n\times m}(\mathbb{C})$. There exist $P \in \operatorname{GL}(m, \mathbb{C})$ and $Q \in \operatorname{GL}(n, \mathbb{C})$ such that

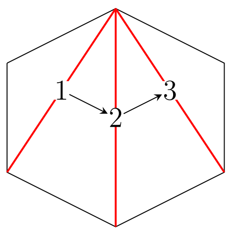
$$QMP^{-1} = \left(\begin{array}{c|c} \mathrm{I}_r & 0 \\ \hline 0 & 0 \end{array}\right),$$

so in other words, there exists an isomorphism

$$W \simeq (S_1)^{m-r} \oplus (S_2)^{n-r} \oplus (V_1)^r.$$

Thus, up to isomorphism, the category $\operatorname{Rep}(Q_\tau)$ has finitely many indecomposables.

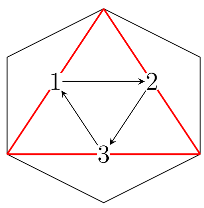
Example 2.3. Let $(\Sigma, \mathcal{M}, \tau)$ be as in Example 1.6(3).



One can check similarly that there are 6 indecomposable objects whose dimension vectors are

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, 1, 0),\ (0, 1, 1),\ (1, 1, 1).$$

Let τ' be the following triangulation:



The category $\mathrm{Rep}(Q_{\tau'})$ has also 6 indecomposable objects whose dimension vectors are

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (0,1,1),\ (1,0,1).$$

Note that here there is no indecomposable with dimension vector $(1,1,1)$ since the composition of any two arrows is zero.

Remark that the dimension vectors are exactly the denominators of the cluster variables. This is of course not a coincidence.

Example 2.4. Take the triangulation of Example 1.11. The corresponding quiver is

$$1 \rightrightarrows 2\ .$$

There are infinitely many representations in this case, which are given as follows:

$$\mathbb{C}^n \underset{\left[\begin{smallmatrix}0\\ \mathrm{I}_n\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}\mathrm{I}_n\\ 0\end{smallmatrix}\right]}{\rightrightarrows}} \mathbb{C}^{n+1}; \qquad \mathbb{C}^{n+1} \underset{\left[\begin{smallmatrix}0 & \mathrm{I}_n\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}\mathrm{I}_n & 0\end{smallmatrix}\right]}{\rightrightarrows}} \mathbb{C}^n, \qquad n\in\mathbb{N}$$

$$\mathbb{C}^n \underset{\left[\mathrm{I}_n\right]}{\overset{\left[\mathrm{J}_n(\lambda)\right]}{\rightrightarrows}} \mathbb{C}^n, \qquad \mathbb{C}^n \underset{\left[\mathrm{J}_n(0)\right]}{\overset{\left[\mathrm{I}_n\right]}{\rightrightarrows}} \mathbb{C}^n,\ n\in\mathbb{N}, \lambda\in\mathbb{C}$$

where $J_n(\lambda)$ is the Jordan bloc of eigenvalue λ.

The category $\mathrm{Rep}(Q_\tau)$ is equivalent to the category of modules over a certain algebra, which has the property to be *gentle*. These algebras has been strongly studied in representation theory. In particular a consequence of the description of the indecomposable modules due Butler and Ringel [BR87] is the following.

Proposition 2.5. *The category* $\mathrm{Rep}(Q_\tau)$ *has finitely many indecomposables if and only if* Σ *is a disc.*

Remark 2.6. If the surface is a disc, or an annulus, one can find triangulations containing no internal triangles (see for instance the examples 1.6 and 1.11). In this case, the quiver Q_τ does not have any oriented cycles and there is no relations involving the arrows. The category $\mathrm{Rep}(Q_\tau)$ is the usual category of representations of the quiver Q_τ. In the case of a disc, the corresponding quiver has underlying graph of Dynkin type $\mathbb{A}_{n-3}$ (where n is the number of marked points on the boundary). In the case of an annulus, the corresponding quiver has underlying graph of Euclidean type $\widetilde{\mathbb{A}}_{n-1}$. In these special cases, the result above is also a consequence of Gabriel's theorem [Gab72].

The quivers without oriented cycle of Dynkin type $\mathbb{D}_n$ can also be understood via triangulations of a once punctured disc. But the quivers of type $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ do not have any geometric descriptions.

Now let us give a more precise description of the indecomposable objects of the category $\operatorname{Rep}(Q_\tau)$ following [BR87] and [ABCJP10]. For $(\Sigma, \mathcal{M})$ a marked surface, define the set $\mathbb{A}^{\text{gen}}(\Sigma, \mathcal{M})$ of homotopy classes of (non oriented) curves in Σ with endpoints in $\mathcal{M}$, that are not contractible, and not homotopic to a boundary segment. Let $\pi_1^{\text{free}}(\Sigma)$ be the set of non contractible loops in Σ up to free homotopy.

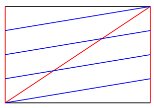
Theorem 2.7. [ABCJP10] *Let $(\Sigma, \mathcal{M}, \tau)$ be a triangulated marked surface. The indecomposable objects of the category* $\operatorname{Rep}(Q_\tau)$ *are in bijection with the following sets:*

- $\mathbb{A}^{\text{gen}}(\Sigma, \mathcal{M}) \backslash \{\tau_1, \ldots, \tau_n\}$;
- $\pi_1^{\text{free}}(\Sigma) \times \mathbb{C}^* / \sim$.

where the equivalence relation $\sim$ is given by $([\gamma], \lambda) \sim ([\gamma^{-1}], \lambda^{-1})$.

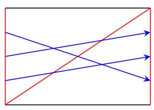
This bijection is in fact very constructive. In particular for $\gamma \in \mathbb{A}^{\text{gen}}(\Sigma, \mathcal{M}) \backslash \{\tau_1, \ldots, \tau_n\}$, if we denote by $M^\tau(\gamma)$ the corresponding representation, then the dimension $\dim_{\mathbb{C}}(M^\tau(\gamma)_i)$ is equal to the number of intersections of γ with the arc τ_i. And the same hold for the objects associated to $([\gamma], \lambda)$ in $\pi_1^{\text{free}}(\Sigma) \times \mathbb{C}^*$.

Example 2.8. Let $(\Sigma, \mathcal{M})$ be as in example 1.11. The representations associated to the arc in blue are as follows:

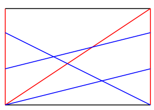


$$\mathbb{C}^3 \underset{\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]}{\rightrightarrows}} \mathbb{C}^2,$$

For $\lambda \neq 0$



$$\mathbb{C}^3 \underset{\mathrm{I}_3}{\overset{\left[\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right]}{\rightrightarrows}} \mathbb{C}^3,$$



$$\mathbb{C}^2 \underset{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}{\overset{\mathrm{I}_2}{\rightrightarrows}} \mathbb{C}^2,$$

2.2. **Cluster-tilting theory in triangulated categories.** The aim of this section is to explain very roughly the basic ideas of cluster-tilting theory. We refer to [Hap88] for basic properties of triangulated categories, and to [Kel12] for more information on cluster-tilting theory.

The framework of cluster-tilting theory takes place in triangulated $\mathbb{C}$-categories with finite-dimensional Hom-spaces and with the Krull-Schmidt property. In a such category $\mathcal{C}$ we denote by [1] the shift functor. Moreover such a category is assumed to be 2-Calabi-Yau, that is there exists a bifunctorial isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(X, Y[1]) \simeq \operatorname{Hom}_{\mathbb{C}}(\operatorname{Hom}_{\mathcal{C}}(Y, X[1]), \mathbb{C}).$$

Definition 2.9. An object X in $\mathcal{C}$ is said to be *rigid* if the space $\mathrm{Hom}_{\mathcal{C}}(X, X[1])$ vanishes.

An object $T = \oplus_{i=1}^{n} T_i$ in $\mathcal{C}$ is said to be *cluster-tilting* if it is basic (i.e. for any $i \neq j$, the indecomposables T_i and T_j are not isomorphic), rigid, and maximal for this property that is if $\mathrm{Hom}_{\mathcal{C}}(X, T[1]) = 0$ then X is isomorphic to a sum of the T_i's.

The main feature of cluster-tilting objects is that they can be mutated.

Theorem 2.10 (Iyama-Yoshino)**.** [IY08] *Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category, and $T = \oplus_{i=1}^{n} T_i$ be a cluster-tilting object in $\mathcal{C}$. Then for any $i = 1, \ldots, n$ there exists a unique indecomposable object T_i^* non isomorphic to T_i such that the object $T' = \mu_{T_i}(T) := T/T_i \oplus T_i^*$ is cluster-tilting.*

Moreover, one can associate a quiver Q_T to any object $T = \sum_{i=1}^{n} T_i$ in $\mathcal{C}$ whose vertices are in bijection with the direct summands T_i's. (This quiver Q_T is the Gabriel quiver of the finite dimensional algebra $\mathrm{End}_{\mathcal{C}}(T)$.) If i is the vertex of Q_T corresponding to the indecomposable summand T_i and if Q_T does not have loops and 2-cycle at i, then the quiver $Q_{T'}$ is the mutation of Q_T at vertex i [BIRS09].

2.3. **Cluster categories of a marked surface.** We now come back to our original setup: let $(\Sigma, \mathcal{M}, \tau)$ be a triangulated surface.

Here is a theorem collecting differents facts concerning the categorification of triangulated surfaces.

Theorem 2.11. *There exists a triangulated 2-Calabi-Yau category $\mathcal{C}_{(\Sigma,\tau)}$ with the following properties:*

(1) *there exists a bijection*

$$T : \mathbb{A}(\Sigma, \mathcal{M}) \longrightarrow \{\textit{rigid indecomposables objects in } \mathcal{C}\}/\mathrm{iso}.$$

(2) *for any two arcs γ and δ, the dimension of $\mathrm{Hom}_{\mathcal{C}}(T(\gamma), T(\delta)[1])$ is the number of intersections between γ and δ in the interior of the surface.*

(3) *for $\tau = \{\tau_1, \ldots, \tau_n\}$ a triangulation, we define $T(\tau) := \bigoplus_{i=1}^{n} T(\tau_i)$. Then the map T induces a bijection*

$$T : \{\tau \textit{ triangulation of } (\Sigma, \mathcal{M})\} \longrightarrow \{\textit{cluster-tilting objects in } \mathcal{C}\}/\mathrm{iso}.$$

Moreover the flip of an arc δ on the left hand side corresponds to the cluster-tilting mutation at the summand $T(\delta)$ on the right hand side.

(4) *for any triangulation τ' there is an equivalence*

$$\mathcal{C}_{\Sigma,\tau}/\langle T(\tau')\rangle \simeq \mathrm{Rep}(Q_{\tau'}),$$

where $\mathcal{C}_{\Sigma,\tau}/\langle T(\tau')\rangle$ is the quotient of the category $\mathcal{C}_{\Sigma,\tau}$ by the ideal of morphisms factoring through a sum of summands of $T(\tau')$.

(5) *for any triangulation τ' there exists an equivalence of triangulated categories $\mathcal{C}_{(\Sigma,\tau)} \simeq \mathcal{C}_{(\Sigma,\tau')}$.*

The general construction of the category $\mathcal{C}_{(\Sigma,\tau)}$ follows from [Ami09, Kel11c]. It is defined as the quotient of certain derived categories of differential graded algebras (see [Ami11] for details). But several previous constructions have been given in some more specific situations: a geometric construction in the case where Σ is a disc can be found in [CCS06] while a general construction for any acyclic quiver is done in [BMR$^+$06]. In the latter the construction is completely algebraic: the cluster category is defined as the orbit category of the derived category of the path algebra of the quiver under a certain auto-equivalence. An "intermediate" generalization has been given in [Ami09], starting from algebras of global dimension 2 instead of path algebras. All these different constructions enter into the general definition [Ami11].

The facts (1), (2) and (3) are shown in [BZ11], while (4) follows from [BMR07]. Point (5) follows from [KY11].

Note that by the Calabi-Yau property, the dimension of the space $\mathrm{Hom}_{\mathcal{C}}(T(\gamma), T(\delta)[1])$ is the same as the dimension of $\mathrm{Hom}_{\mathcal{C}}(T(\delta), T(\gamma)[1])$. This makes sense with point (2) since the number of intersections between γ and δ is the same as the one of δ with γ.

As a consequence of point (5), we may write $\mathcal{C}_\Sigma$ instead of $\mathcal{C}_{\Sigma,\tau}$. But this notation may be a bit dangerous since the equivalence in (4) is not canonical (see appendix in [CS17]).

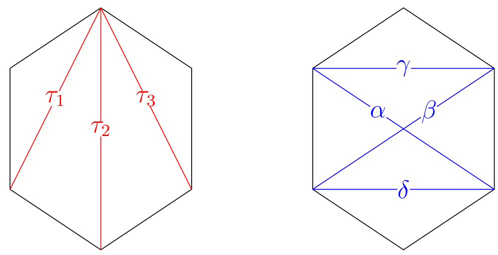
Let us give more details about point (4) above. For any object $X \in \mathcal{C} := \mathcal{C}_{\Sigma,\tau}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(T(\tau')[-1], X)$ has a natural structure of right $\mathrm{End}_{\mathcal{C}}(T(\tau')[-1])$-module. Since $[-1]$ is an auto-equivalence of the triangulated category $\mathcal{C}$, we have $\mathrm{End}_{\mathcal{C}}(T(\tau')[-1]) \simeq \mathrm{End}_{\mathcal{C}}(T(\tau'))$. Moreover, one can show that the algebra $\mathrm{End}(T(\tau'))$ is isomorphic to the quotient of the path algebra of the quiver $Q_{\tau'}$ by the relations $\alpha\beta = 0$ for any arrows corresponding to consecutive angles in a triangle. Therefore we have an equivalence between the category $\mathrm{mod}\,\mathrm{End}(T(\tau')[-1])$ and the category $\mathrm{Rep}(Q_{\tau'})$. Hence $\mathrm{Hom}_{\mathcal{C}}(T(\tau')[-1], -)$ is a functor $\mathcal{C} \to \mathrm{Rep}(Q_{\tau'})$. Since the object $T(\tau')$ is rigid, this functor sends any summand of $T(\tau')$ to the zero representation, and we obtain a functor $\mathcal{C}/\langle T(\tau')\rangle \to \mathrm{Rep}(Q_{\tau'})$. One result in [BMR07] shows that this functor is an equivalence of categories. As a corollary, we obtain that the indecomposable objects of $\mathcal{C}_\Sigma$ are in bijection with

$$\mathbb{A}^{\mathrm{gen}}(\Sigma, \mathcal{M}) \cup (\pi_1^{\mathrm{free}}(\Sigma) \times \mathbb{C}^*)/\sim .$$

Moreover if γ is an arc that do not belong to τ', the object $T(\gamma)$ is sent to $M^\tau(\gamma)$ through the equivalence $\mathcal{C}_\Sigma/\langle T(\tau)\rangle \simeq \mathrm{Rep}(Q_\tau)$.

The autoequivalence [1] of the category $\mathcal{C}_\Sigma$ can be interpreted as an element of the mapping class group of Σ fixing globally the marked points. For each boundary component $\partial_i\Sigma$ denote n_i the number of marked point on $\partial_i\Sigma$. Then denote by σ_i the fractional $\frac{1}{n_i}$ Dehn twist around $\partial_i\Sigma$, that is a homeomophism sending each marked point of $\partial_i\Sigma$ to the next one in the counterclockwise order. Then the homeomorphism of Σ associated with [1] is the product $\sigma = \prod_{i=1}^b \sigma_i$, that is for any generalised arc γ, we have an isomorphism $T(\gamma)[1] \simeq T(\sigma\gamma)$.

Example 2.12. Let $(\Sigma, \mathcal{M}, \tau = \{\tau_1, \tau_2, \tau_3\})$ be the following triangulated surface, and let $\alpha, \beta, \gamma, \delta$ be the other arcs of Σ.



By the description of the shift functor above, there is an isomorphism $T(\alpha)[1] \simeq T(\beta)$.

The object $T(\alpha)$ seen a representation of Q_τ is $M^\tau(\alpha)$, that is the representation: $(\mathbb{C} \to \mathbb{C} \to 0)$

2.4. **Cluster characters and applications.** As we have just seen, the operation of flip appears naturally in the cluster category associated to a surface. In fact, the link between the category and the cluster algebra is much stronger: in particular, one can recover the entire cluster algebra from the cluster category, this can be done using a certain map called a cluster character (or a Caldero-Chapoton map). We refer to [Pla] for more details.

Theorem 2.13 (Caldero-Chapoton [CC06], Palu [Pal08])**.** *Let $(\Sigma, \mathcal{M}, \tau = \{\tau_1, \ldots, \tau_n\})$ be a triangulated surface, and let $\mathcal{C}_\Sigma$ be the associated category. There exists a map*

$$X^\tau : \mathrm{Obj}(\mathcal{C}_\Sigma) \longrightarrow \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

such that the following properties hold:

(1) *for any $M, N \in \mathcal{C}_\Sigma$, we have $X^\tau(M \oplus N) = X^\tau(M)X^\tau(N)$;*
(2) *for any $i = 1, \ldots, n$, $X^\tau(T(\tau_i)) = x_i$;*
(3) *if $\dim_k \mathrm{Hom}_{\mathcal{C}}(M, N[1]) = 1$, then we have*

$$X^\tau(M)X^\tau(N) = X^\tau(B) + X^\tau(B')$$

where B and B' appear in the non split triangles

$$M \longrightarrow B \longrightarrow N \longrightarrow M[1] \;\text{ and }\; N \longrightarrow B' \longrightarrow M \longrightarrow N[1]\,.$$

Note that since the space $\mathrm{Hom}_{\mathcal{C}}(M,N[1])$ is one dimensional, then all non zero morphisms $M\to N[1]$ are multiplication of a scalar one of each other, and hence have isomorphic cones. Hence in point (3) above, the objects B and B' are uniquely determined.

The maps $\mathbf{x}^\tau$ and $X^\tau\circ T$ coincide on the arcs of τ. Moreover combining Property (3) above together with Property (2) of Theorem 2.11 implies that $X^\tau\circ T$ satisfies the equation (1.1). Hence for any arc γ of Σ, we have that

$$\mathbf{x}^\tau(\gamma)=X^\tau(T(\gamma)).$$

In fact, the map $\mathbf{x}^\tau$ has been extended to any generalised arc and any closed curve on the surface in [MSW11] using some resolving arc relations called *Skein relations.* Using the bijection (2.3) one can show that the maps $\mathbf{x}^\tau$ and X^τ coincide on generalized arcs [BZ13]. They also coincide on primitive closed curves, but not on non-primitive ones.

Example 2.14. Let us come back to Examples 1.6 (2) and 2.12.

The arcs α and β have exactly one intersection. Hence by Theorem 2.11(2), the spaces $\mathrm{Hom}_{\mathcal{C}}(T(\alpha),T(\beta)[1])$ and $\mathrm{Hom}_{\mathcal{C}}(T(\beta),T(\alpha)[1])$ are both one dimensional. We are in the setup of Theorem 2.13(3). Let us compute the extensions B and B'. As just seen before, any non-zero morphism between $T(\beta)$ and $T(\alpha)[1]$ is an isomorphism. Hence there is a triangle:

$$T(\alpha)\longrightarrow 0\longrightarrow T(\beta)\longrightarrow T(\alpha)[1]\ .$$

Completing one of the non-zero morphism from $T(\alpha)$ to $T(\beta[1])$ we obtain the following triangle in the cluster category

$$(\dagger)\qquad T(\beta)\longrightarrow T(\gamma)\oplus T(\delta)\longrightarrow T(\alpha)\longrightarrow T(\beta)[1]\ .$$

Then one easily checks that

$$\begin{array}{rcl} X^\tau(T(\alpha)).X^\tau(T(\beta)) & = & \mathbf{x}^\tau(\alpha).\mathbf{x}^\tau(\beta) \\ & = & \frac{x_1+x_1x_2+x_3}{x_2x_3}.\frac{x_1+x_3+x_2x_3}{x_1x_2}; \\ & = & \frac{x_1+x_3+x_1x_2+x_2x_3}{x_1x_2x_3}\frac{x_1+x_3}{x_2}+1 \\ & = & \mathbf{x}^\tau(\delta)\mathbf{x}^\tau(\gamma)+1 \\ & = & X^\tau(T(\delta)\oplus T(\gamma))+X^\tau(0). \end{array}$$

Moreover, since $T(\alpha)[-1]\simeq T(\tau_2)$ and $T(\beta)[1]\simeq T(\tau_2)$, applying the functor $\mathrm{Hom}_{\mathcal{C}}(T(\tau)[-1],-)$ to the triangle $(\dagger)$, we obtain a short exact sequence in $\mathrm{Rep}(Q_\tau)$:

$$0\longrightarrow M^\tau(\beta)\longrightarrow M^\tau(\gamma)\oplus M^\tau(\delta)\longrightarrow M^\tau(\alpha)\longrightarrow 0\ ,$$

which is the short exact sequence

$$0\longrightarrow(0\to\mathbb{C}\to\mathbb{C})\longrightarrow\begin{array}{c}(\mathbb{C}\to\mathbb{C}\to\mathbb{C})\\ \oplus\\ (0\to\mathbb{C}\to 0)\end{array}\longrightarrow(\mathbb{C}\to\mathbb{C}\to 0)\longrightarrow 0$$

There exists an explicit general formula (see [Pal08]) for the application X^τ, we give here the formula in the case where Q_τ does not contain oriented cycle [CC06]:

$$X^\tau(M) = \frac{1}{\prod_i x_i^{d_i}} \sum_{\underline{e}\leq \underline{d}} \chi(\mathrm{Gr}_{\underline{e}}(FM)) \prod_i x_i^{m_i(\underline{d},\underline{e})} \tag{2.1}$$

where $FM \in \mathrm{Rep}(Q_\tau)$ is the representation corresponding to M, where $\underline{d} = (d_1, \ldots, d_n)$ is its dimension vector, where $\mathrm{Gr}_{\underline{e}}(FM)$ is the Grassmanian of subrepresentations of FM with dimension vector $\underline{e}$ and where

$$m_i(\underline{e}, \underline{d}) = \sum_{j\to i} e_j - \sum_{i \to j}(d_j - e_j).$$

Example 2.15. Let τ and Q_τ be as in Example 1.11. Take for FM the indecomposable representation with dimension vector $\underline{d} = (1, 2)$. One then checks that for $\underline{e} = (1, 0), (1, 1)$ there is no submodule of FM of dimension $\underline{e}$. For $\underline{e} = (0, 0)$ (resp. $(0, 2)$, resp. $(1, 2)$) the Grassmaniann of subrepresentations of dimension $\underline{e}$ is a point. Following the formula (2.1) the corresponding monomial is x_1^4 (resp. x_2^2, resp. 1). And finally for $\underline{e} = (0, 1)$ the Grassmannian of subrepresentations of FM is isomorphic to $\mathbb{P}^1(\mathbb{C})$ so the coefficient of the monomial x_1^2 is 2. Finally we obtain

$$X^\tau(M) = \frac{x_1^4 + 2x_1^2 + x_2^2 + 1}{x_1 x_2^2}.$$

One easily checks that $X^\tau(M) = X_{-2}$ where the sequence (X_k) is defined in Example 1.11.

A notable application of this cluster character map is the following statement.

Theorem 2.16. [CIKLFP13] *The cluster monomials are linearly independent. So in the surface setup, the set*

$$\{x^\tau(\tau_1')^{m_1} \ldots x^\tau(\tau_n')^{m_n}, \tau' \text{ triangulation of } \Sigma,\ m_i \in \mathbb{N}\}$$

is a linearly independent set.

Using combinatorial methods, Musiker, Schiffler and Williams have shown in [MSW13] that this set can be completed in a basis using multiarcs on the surface. Note that this theorem [CIKLFP13] is stated in the more general setup starting for any quiver. A more general construction of a basis has been achieved in [GHKK18].

As previously said in the introduction, cluster-tilting theory has also been very useful for solving pureley representation theoretic problems. This was not the purpose of this series of lectures.

References

[ABCJP10] Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin, and Pierre-Guy Plamondon. Gentle algebras arising from surface triangulations. *Algebra Number Theory*, 4(2):201–229, 2010.

[Ami09] Claire Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)*, 59(6):2525–2590, 2009.

[Ami11] Claire Amiot. On generalized cluster categories. In *Representation of algebras and related topics*, EMS Ser. Congr. Rep., pages 1–53. Eur. Math. Soc., Zürich, 2011.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. Cluster structures for 2-Calabi-Yau categories and unipotent groups. *Compos. Math.*, 145(4):1035–1079, 2009.

[BMR$^+$06] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[BMR07] Aslak Bakke Buan, Robert J. Marsh, and Idun Reiten. Cluster-tilted algebras. *Trans. Amer. Math. Soc.*, 359(1):323–332, 2007.

[BR87] M. C. R. Butler and Claus Michael Ringel. Auslander-Reiten sequences with few middle terms and applications to string algebras. *Comm. Algebra*, 15(1-2):145–179, 1987.

[BZ11] Thomas Brüstle and Jie Zhang. On the cluster category of a marked surface without punctures. *Algebra Number Theory*, 5(4):529–566, 2011.

[BZ13] Thomas Brüstle and Jie Zhang. A module-theoretic interpretation of Schiffler's expansion formula. *Comm. Algebra*, 41(1):260–283, 2013.

[CC06] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[CCS06] Philippe Caldero, Frédéric Chapoton, and Ralf Schiffler. Quivers with relations and cluster tilted algebras. *Algebr. Represent. Theory*, 9(4):359–376, 2006.

[CIKLFP13] Giovanni Cerulli Irelli, Bernhard Keller, Daniel Labardini-Fragoso, and Pierre-Guy Plamondon. Linear independence of cluster monomials for skew-symmetric cluster algebras. *Compos. Math.*, 149(10):1753–1764, 2013.

[CS17] Ilke Canakci and Sibylle Schroll. Extensions in Jacobian algebras and cluster categories of marked surfaces. *Adv. Math.*, 313:1–49, 2017. With an appendix by Claire Amiot.

[FG06] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[FWZ16] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to cluster algebras. chapters 1-3. Preprint https://arxiv.org/abs/1608.05735, 2016.

[FWZ17] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to cluster algebras. chapters 4-5. Preprint https://arxiv.org/abs/1707.07190, 2017.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[Gab72] Peter Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972.

[GHKK18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608, 2018.

[GLS06] Christof Geiß, Bernard Leclerc, and Jan Schröer. Rigid modules over preprojective algebras. *Invent. Math.*, 165(3):589–632, 2006.

[GLS13] Ch. Geiss, B. Leclerc, and J. Schröer. Cluster algebras in algebraic Lie theory. *Transform. Groups*, 18(1):149–178, 2013.

[Hap88] Dieter Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[HL10] David Hernandez and Bernard Leclerc. Cluster algebras and quantum affine algebras. *Duke Math. J.*, 154(2):265–341, 2010.

[IY08] Osamu Iyama and Yuji Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. *Invent. Math.*, 172(1):117–168, 2008.

[Kel] Bernhard Keller. https://webusers.imj-prg.fr/ bernhard.keller/quivermutation/.

[Kel11a] B. Keller. Cluster algebras and cluster categories. *Bull. Iranian Math. Soc.*, 37(2):187–234, 2011.

[Kel11b] Bernhard Keller. Algèbres amassées et applications (d'après Fomin-Zelevinsky, ...). *Astérisque*, (339):Exp. No. 1014, vii, 63–90, 2011. Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026.

[Kel11c] Bernhard Keller. Deformed Calabi-Yau completions. *J. Reine Angew. Math.*, 654:125–180, 2011. With an appendix by Michel Van den Bergh.

[Kel12] Bernhard Keller. Cluster algebras and derived categories. In *Derived categories in algebraic geometry*, EMS Ser. Congr. Rep., pages 123–183. Eur. Math. Soc., Zürich, 2012.

[KY11] Bernhard Keller and Dong Yang. Derived equivalences from mutations of quivers with potential. *Adv. Math.*, 226(3):2118–2168, 2011.

[LS15] Kyungyong Lee and Ralf Schiffler. Positivity for cluster algebras. *Ann. of Math. (2)*, 182(1):73–125, 2015.

[MSW11] Gregg Musiker, Ralf Schiffler, and Lauren Williams. Positivity for cluster algebras from surfaces. *Adv. Math.*, 227(6):2241–2308, 2011.

[MSW13] Gregg Musiker, Ralf Schiffler, and Lauren Williams. Bases for cluster algebras from surfaces. *Compos. Math.*, 149(2):217–263, 2013.

[Mus02] Gregg Musiker. Cluster algebras, somos sequences and exchange graphs. 2002.

[Pal08] Yann Palu. Cluster characters for 2-Calabi-Yau triangulated categories. *Ann. Inst. Fourier (Grenoble)*, 58(6):2221–2248, 2008.

[Pla] Pierre-Guy Plamondon. Cluster characters. In *Homological methods, representation theory, and cluster algebras*, CRM Short Courses, pages 101–125.

[Sco06] Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)*, 92(2):345–380, 2006.

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