Decomposition

Artinian-Noetherian

Exercise 1. Let M be a module and let $f \in \mathrm{End}_A(M)$.

1. Show that if M is Artinian and f is injective, then f is an automorphism.
2. Show that if M is Noetherian and f is surjective, then f is an automorphism.

Exercise 2. Let A be a Noetherian ring. The aim is to show that $A[X]$ is Noetherian (Hilbert's basis theorem). Let I be a left ideal of $A[X]$.

1. For any $n \in \mathbb{N}$, define I_n to be the set of $a \in A$ such that there exists a polynomial in I with aX^n as term of higher degree. Show that the I_n form a ascending chain of finite type ideals of A, and denote by n_0 an integer such that $I_{n_0} = I_m$ for any $m \geq n_0$.
2. For each $i \leq n_0$, denote by $\{a_{i,j}, j = 1 \dots, m_i\}$ a generating set of I_i, and P_{ij} be a polynomial in I such that the non zero term of higher degree is $a_{ij}X^i$. Show that I is generated by the P_{ij} and conclude.

Indecomposable and connected

Exercise 3. Let Q be a quiver. Show that if k has no zero divisor, then kQ is connected if and only if Q is connected.

Exercise 4. Let Q be the quiver $1 \to 2 \to 3$. Prove that the following representations are indecomposable

$$\begin{array}{ccc} k \to 0 \to 0 & 0 \to k \to 0 & 0 \to 0 \to k \\ k = k \to 0 & 0 \to k = k & k = k = k \end{array}$$

Show that there are no other indecomposable representations up to isomorphism.

Exercise 5. Let k be a field. Is A a local algebra in the following cases?

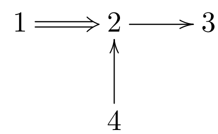
1. $A = \mathcal{T}_n(k)$;
2. $A = \{X \in \mathcal{T}_n(k), X_{ii} = X_{jj} \forall i, j\}$?

Simple and composition series

Exercise 6. 1. Let Q be a quiver without any oriented cycles. Show that the only simple representations are the $(S_i)_{i\in Q_0}$ where $(S_i)_i = k$ and $(S_i)_j = 0$ $i \neq j$. Is it true if Q has oriented cycles ?

2. Let A be a Noetherian and Artinian algebra. Show that there are finitely many isomorphism classes of simples. Is it true if A is assumed to be only Noetherian ?

Exercise 7. What are the composition series of kQe_1 where Q is the following quiver ?



Show that if Q is a quiver without oriented cycles and V is a representation of Q, then $\ell(V) = \dim_k V$ and the simple S_i appears exactly $\dim V_i$ times as a decomposition factor of V.

Exercise 8. Let $A = \mathcal{M}_2(\mathbb{R})$.

1. Show that A has a unique simple module up to isomorphism. What is the length of a composition series for A ?
2. Find an infinite family of composition series for A.

Semi-simple

Exercise 9. Let M be a A-module. Show that the following facts are equivalent.

1. M is semi-simple;
2. for any monomorphism $j : L \to M$ and any morphism $f : L \to L'$, there exists a morphism $f' : M \to L'$ such that $f'j = f$.
3. for any epimorphism $p : M \to N$ and any morphism $g : N' \to N$ there exists a morphism $g' : N' \to M$ such that $pg' = g$.

Exercise 10. Let k be a field, $G = \mathfrak{S}_3$, and $\rho : G \to \mathrm{GL}(k^3)$ be the representation acting by permuting the coordinates.

1. If $\mathrm{char} k \neq 3$, decompose ρ into a sum of irreducible representations.
2. Show that ρ is not semi-simple if $\mathrm{char} k = 3$.

Exercise 11. Determine how many isomorphism classes of 8-dimensional semi-simple algebras over $\mathbb{C}$.

Deduce that there exists non isomorphic groups G and G' with $kG \simeq kG'$.