## Algebras

**Exercise 1.** Let $k$ be a commutative ring. Show that

$$A := \{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & e \end{pmatrix},\ a,b,c,d,e \in k\}$$

is a $k$-algebra.

**Exercise 2.** Let $k$ be a field and $A$ be a finite dimensional $k$-algebra. We consider the map $\phi : A \to \mathrm{End}_k(A)$ sending $a \in A$ to $\varphi_a : b \mapsto ab$.

1. Show that $\phi$ is an injective algebra morphism.
2. Deduce that any finite dimensional $k$-algebra is the subalgebra of a matrix algebra.
3. Construct the corresponding morphism for $\mathbb{C}\mathbb{Z}_2$.
4. Show that $\mathbb{C}\mathbb{Z}_2$ is isomorphic to $\mathbb{C}^2$ and conclude.

**Exercise 3.** A *quiver* (=oriented graph) $Q$ is the data of two finite sets $Q_0$ (the vertices) and $Q_1$ (the arrows) and two maps $s, t : Q_1 \to Q_0$ (source and target). A *path* on the quiver $Q$ is either

- a concatenation of arrows $p = a_n \dots a_1$ with $s(a_{i+1}) = t(a_i)$. Then we define $s(p) := s(a_1)$ and $t(p) := t(a_n)$.
- a element $e_i$ for each $i \in Q_0$ with $s(e_i) = t(e_i) = i$ called a trivial path.

Let $k$ be a unital commutative ring. The path algebra $kQ$ on $Q$ is defined to be the free $k$-module generated by the paths, and whose multiplication is defined by concatenation of paths extended by linearity.

1. Describe $Q_0$, $Q_1$, $s$ and $t$ for
$$Q := 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$
2. Show that $kQ$ is isomorphic to $\mathcal{T}_3(k)$ the subalgebra of upper triangular $3 \times 3$ matrices.
3. Let $Q$ be the quiver with $|Q_0| = |Q_1| = 1$. Show the $kQ$ is isomorphic to $k[X]$.
4. Find a quiver $Q$ such that the algebra $A$ defined in Exercise 1 is isomorphic to $kQ$.

**Exercise 4.** Let $A$ be a $k$-algebra and $M$ be a $A$-$A$-bimodule. We consider the $k$-module $A \oplus M$ and define a multiplication

$$(a, m).(a', m') := (aa', am' + ma').$$

1. Show that this makes $A \oplus M$ a $k$-algebra.

2. Show that the canonical maps $\iota : A \to A \oplus M$ and $\pi : A \oplus M \to A$ are algebra morphisms.

3. Show that $\{0_A\} \oplus M$ is a two-sided ideal of $A \oplus M$.

4. What is $A \oplus M$ for $A = k$ and $M = k$ ?

**Exercise 5.** Let $k$ be an algebraically closed field. Let $A$ be a $k$-algebra of dimension 2.

1. Let $x$ be an element of $A$ which is not in $\mathrm{vect}(1_A)$. Show that there exists a unique algebra morphism $k[X] \to A$ sending $X$ to $x$.

2. Considering its kernel show that any 2-dimensional algebra is isomorphic to $k[X]/(X - \alpha)(X - \beta)$ for some $\alpha, \beta$ in $k$.

3. Deduce that $A$ is isomorphic to $k[X]/(X^2)$ or to $k[X]/(X^2 - 1)$. Are these two algebras isomorphic ?

4. To which of these is $k^2$ isomorphic to ?

5. Same question for $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

## Modules

**Exercise 6.**

1. For $N$, $N'$ submodules of $M$. Show that there is a canonical isomorphism

$$N/(N \cap N') \simeq (N + N')/N'.$$

2. For $M'' \subset M' \subset M$ modules, show that there is a canonical isomorphism

$$(M/M'')/(M'/M'') \simeq M/M'.$$

3. If $f : M \to N$ is a morphism and $N'$ is a submodule of $N$, show that $f^{-1}(N')$ is a submodule of $M$ and that $f$ induces an isomorphism

$$M/f^{-1}(N') \simeq \mathrm{Imf}/\mathrm{N'} \cap \mathrm{Imf}.$$

**Exercise 7.** Let $p$ be a prime number and $n \in \mathbb{Z}$. Define $f : \mathbb{Z}_{p^2} \to \mathbb{Z}_{p^2}$ to be the multiplication by $n$. Compute the kernel and the cokernel of $f$ depending on the value of $n$.

**Exercise 8.** Let $A = \mathcal{M}_n(k)$ , and let $M = k^n$ be the $A$-module defined by left multiplication.

1. Show that $M$ has no trivial submodules.

2. Let $B := \mathcal{T}_n^+(k)$ be the subalgebra of $A$ of upper triangular matrices. And let ${}_BM$ the restriction of the module $M$. Determine all the submodules of $M$.

3. For each $X \subset Y \subset M$, determine the structure of the quotient $Y/X$.

**Exercise 9.** Let $k$ be a field. For $M$ be $k[X]$-module denote by $\rho_M : k[X] \to \mathrm{End}_k(M)$ the corresponding algebra map.

1. Show that two a finite dimensional $k[X]$-modules $M$ and $N$ are isomorphic if and only if the endomorphisms $\rho_M(X)$ and $\rho_N(X)$ are conjugate.
2. Deduce that for any $n$, there are infinitely isomorphism classes of $k[X]$-modules of dimension $n$.
3. Describe the isomorphism classes in the case $n = 2$.

## Hom **and** $\otimes$

**Exercise 10.** Let $k$ be a field and $M$ and $N$ be $k$-vector spaces.

1. Show that there exists an isomorphism $\mathrm{Bilin}_k(M \times N, P) \simeq \mathrm{Hom}_k(M \otimes_k N, P)$.
2. Show that if $M$ and $N$ are finite dimensional, we have $\dim_k M \otimes_k N = \dim_k(M) \dim_k(N)$.
3. Denote by $M^* = \mathrm{Hom}_k(M, k)$ the $k$-dual. Show that there exists an isomorphism $\mathrm{Hom}_k(M, N) \simeq M^* \otimes_k N$ of $M$ and $N$ are finite dimensional.

**Exercise 11.** Compute $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$.

**Exercise 12.** Let $G$ be a group $k$ be a field and $(V, \rho_V)$ and $(W, \rho_W)$ be representations. We define a structure of representation on $\mathrm{Hom}_k(V, W)$ by

$$g.f = \rho_W(g) \circ f \circ \rho_V(g)^{-1}.$$

Traduce this structure in terms of $kG$-modules. Can we generalize this construction for any algebra $A$ ?

**Exercise 13.** Let $M$ be a $A$-module.

1. Show that $M \otimes_k M$ has a natural structure of $A \otimes_k A$-module. Deduce its natural $A$-module structure induced by the diagonal embedding $A \to A \otimes_k A$.
2. Show that $\tau : M \otimes_k M \to M \otimes_k M$ defined by $\tau(x \otimes y) = y \otimes x$ is a $A$-module morphism.
3. Defining $SM := \mathrm{Ker}(\tau - \mathrm{Id_M})$ and $\Lambda M := \mathrm{Ker}(\tau + \mathrm{Id_M})$. Show that if $k$ is a field of characteristic different from 2 there is an isomorphism $M = SM \oplus \Lambda M$.

**Exercise 14.** Let $k$ be a commutative ring and $A$ be a $k$-algebra. Show the following isomorphism of $k$-algebras $\mathcal{M}_n(k) \otimes_k A \simeq \mathcal{M}_n(A)$.

**Exercise 15.** Let $M$ be a $A$-module. Assume that $N$ is a submodule of $M$ which is finitely generated and such that $M/N$ is finitely generated. Show that $M$ is finitely generated.