Semi-classical Analysis of Integrable Systems

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Motivation:

Spectra of Laplace operators on compact Riemannian manifolds (RM). Kac's problem 'Can one hear the shape of a drum?'. Using semi-classics leads to nice results.

- *Quantum mechanics* on a RM given by Laplace operator

- *Classical mechanics* given by geodesics

Link between spectrum and length of periodic geodesics [70'-75']: general systems (no integrability conditions).

Under some integrability assumptions, one expects to be able to describe the asymptotic expansion of (part of) the eigenvalues. They are described in terms of “quantum numbers” via the *Bohr-Sommerfeld quantization rules.*

Applications to systems close to integrability

- KAM, including systems classically integrable, but NOT quantum integrable

- Birkhoff (semi-classical) normal forms near equilibria or closed orbits

General remarks:

- *Integrable* / *computable:* in the semi-classical setting, it means that one can "compute" the quantum spectra mod $O(\hbar^\infty)$. It is the case in 1D.

- *Semi-classics* is one of the best ways to get some intuition on quantum mechanics. We need to extend the Hamiltonian formalism. We will expand everything in (formal)power series in $\hbar$. *We will not consider the difficult and important problem of the "summations" of these series*.

  – First term(s) are often computable from the classical mechanics

- New terms (Maslov indices, full expansion beyond Weyl formula)

- New objects: *spectra* are often discrete (quantization rules).

- Tools of *Hamiltonian systems* can be extended: *pseudo-differential operators* (ΨDO*'s)* (Normal forms, KAM theory).

Background A: symplectic geometry

- The 2d-phase space will be a cotangent space $Z = T^\star X_d$
- The Liouville 1-form $\Lambda = \sum_{j=1}^d \xi_j dx_j$ and the symplectic 2-form $\omega = d\Lambda = \sum_{j=1}^d d\xi_j \wedge dx_j$. The Poisson bracket $\{f, g\} = \sum_{j=1}^d \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial \xi_j}$
- Jacobi identity $\{f, \{g, h\}\} +$ cycl. perm. $= 0$
- Hamiltonian dynamics: $H : Z \to \mathbb{R}$ and $\omega(X_H, .) = -dH$:

$$X_H = \sum_j \frac{\partial H}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial \xi_j}$$

- Lagrangian manifolds: $L \subset T^\star X_d$ with $\dim L = d$ and $\omega_{|L} = 0$.

Background B: Liouville integrability:

$F = (F_1, \cdots F_d) : Z \to \mathbb{R}^d$ (the moment map) s.t.

- $\{F_i, F_j\} = 0$ ($\Rightarrow [X_{F_i}, X_{F_j}] = 0$)
- F is a submersion almost everywhere
- $H = \Phi(F_1, \cdots, F_d)$
- F is proper.

The smooth fibers of F are finite union of **Lagrangian tori** invariant by the Hamiltonian flow of X_H. This flow is **quasi-periodic.**

Singular fibers

have been recently the subject of many works:

- classical: San's lectures

- semi-classical: my lectures, mainly the 1D case.

Background C: spectral theory

- Typical examples: Δ_g on a connected RM (X_d, g), Schrödinger operators
$$\hat{H} = -\hbar^2 \Delta_g + V(x)$$
with $V : X_d \to \mathbb{R}$ smooth and confining $\lim_{x\to\infty} V(x) = +\infty$

- Spectrum:
$$\lambda_1(\hbar) < \lambda_2(\hbar) \leq \cdots \leq \lambda_n(\hbar) \leq$$
Basic problem: asymptotic of eigenvalues as $\hbar \to 0$ (large eigenvalues of Δ_g)

- Functional calculus: $F(\hat{H})\varphi_n = F(\lambda_n)\varphi_n$.

Quasi-modes:

If $\|(\widehat{H} - E)u\|_{L^2} \leq \varepsilon \|u\|_{L^2}$ and $\widehat{H}$ self-adjoint, then

$$\text{Spectrum}(\widehat{H}) \cap [E - \varepsilon, E + \varepsilon] \neq \emptyset$$

In applications E and u will depend on $\hbar$ and ε will be a power of $\hbar$.

BUT it is not always true that u is close to some eigenfunction

Example: symmetric double well 1D potential. Quasi-modes localized in each well while eigenfunctions are odd or even.

Background D: Fourier transform

$$\widehat{u}(\xi) = \mathcal{F}_\hbar u(\xi) = \frac{1}{(2\pi\hbar)^{d/2}} \int e^{-i\langle x|\xi\rangle/\hbar} u(x)dx$$

$$u(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int e^{i\langle x|\xi\rangle/\hbar} \widehat{u}(\xi)d\xi$$

Background E: Poisson summation formula (PSF):

$f \in \mathcal{S}(\mathbb{R}^d)$, $\Gamma^\star$ dual lattice of Γ ($\langle \Gamma | \Gamma^\star \rangle \subset 2\pi\mathbb{Z}$):

$$\sum_{\gamma \in \Gamma} f(a+\gamma) = \frac{(2\pi)^{d/2}}{|\Gamma|} \sum_{\gamma^\star \in \Gamma^\star} \widehat{f}(\gamma^\star) e^{i\langle a | \gamma^\star \rangle}$$

with $\widehat{f} = \mathcal{F}_1 f$.

$$d = 1, \ \sum_{l \in \mathbb{Z}} f(a + 2\pi l \hbar) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} f(t) dt + O(\hbar^\infty)$$

Background F: Philosophy of trace formulas

A way to access to spectra is computing traces of $F(\widehat{H})$ in 2 ways

- $Z = \sum F(\lambda_n)$
- Direct calculation of the Schwartz kernel $K_F(x, y)$ of $F(\widehat{H})$ and $Z = \int_X K_F(x, x) dx$.

Examples of $F(E)$'s:

- exp$(-tE)$: heat equation
- exp$(-itE/\hbar)$: Schrödinger equation
- $1/E^s$: zeta function

Exact calculation: Poisson summation formula (PSF), Selberg trace formula (for closed RM with $K \equiv -1$). Does not imply integrability!

The simplest problem: 1D Schrödinger

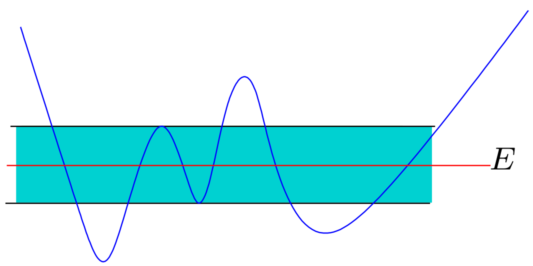
$$\widehat{H}_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x) \ .$$

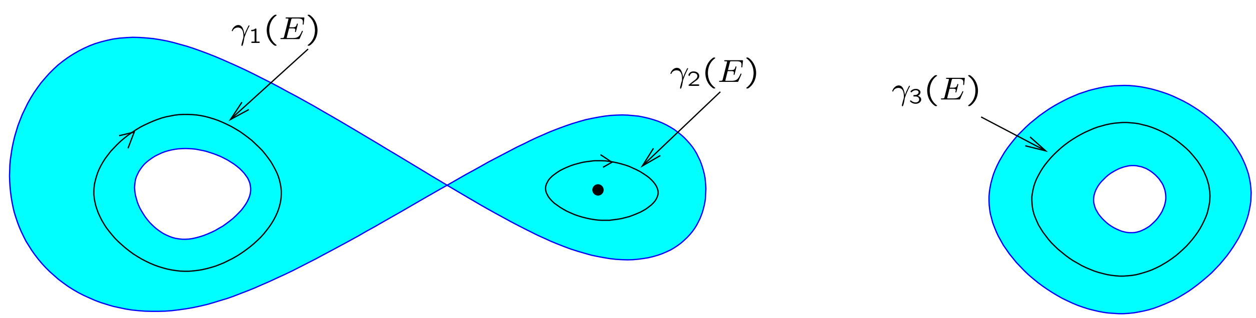
- $-\infty \leq a < b \leq +\infty$ and $V : I =]a,b[\to \mathbb{R}$ smooth
- $-\infty < \inf V = E_0 < E_\infty := \liminf_{x \to \partial I} V(x)$
- Self-adjoint boundary conditions

The spectrum of $\widehat{H}_\hbar$ is discrete in $]-\infty, E_\infty[$:

$$(E_0 <) \lambda_1(\hbar) < \lambda_2(\hbar) < \cdots < \lambda_n(\hbar) < \cdots (< E_\infty) \ .$$

We want to describe the asymptotic behavior of the eigenvalues in term of the classical mechanics. We will denote by $H = \xi^2 + V(x)$ the classical Hamiltonian: $\widehat{H} = H_{\mathsf{W}}$.





Topics:

1. Algebras of pseudo-differential operators: micro-localization and star-products

2. The 1D case: spectrum, regular Bohr-Sommerfeld rules and trace formulas

3. Inverse semi-classical problem

4. A short introduction to FIO's: Egorov's Theorem

5. Semi-classical normal forms

6. Hyperbolic singular points and singular Bohr-Sommerfeld rules

7. $D > 1$: classical, semi-classical and quantum integrability

8. $D > 1$: Bohr-Sommerfeld rules

1. Algebras of pseudo-differential operators, micro-localization and star-products

A *pseudo-differential operator* (ΨDO) on $\mathbb{R}^d$ is given by the formula (Weyl quantization):

$$\mathrm{Op}_{\mathsf{Weyl}}(a)(u)(x) = \frac{1}{(2\pi\hbar)^d} \int e^{i\langle x-y|\xi\rangle/\hbar} a\left(\frac{x+y}{2}, \xi\right) u(y)|dyd\xi| \ ,$$

where a, named the *Weyl symbol* of $A = \mathrm{Op}_{\mathsf{Weyl}}(a)$ is a suitable smooth function. The easiest case is $a \in \mathcal{S}(\mathbb{R}^d \oplus \mathbb{R}^d)$.

We will denote $a_{\mathsf{W}} := \mathrm{Op}_{\mathsf{Weyl}}(a)$. From Fourier inversion formula, one can check that a is uniquely determined by a_{W}.

Examples:

- $(x\xi)_{\mathsf{W}} = \frac{\hbar}{i}\left(x\frac{d}{dx} + \frac{1}{2}\right)$ $[x \star \xi = x\xi + \frac{\hbar}{2i}\{x, \xi\}]$

- $(\|\xi\|^2 + V(x))_{\mathsf{W}} = -\hbar^2\Delta + V(x)$

- $\left(\sum g^{ij}(x)\xi_i\xi_j\right)_{\mathsf{W}} = -\hbar^2\Delta_g - \frac{\hbar^2}{4}\left(\sum \frac{\partial^2 g^{ij}}{\partial x_i \partial x_j}\right)$ if $|dx|_g = |dx|$.

  [This example can be computed using the Moyal formula (see below): enough to compute $\xi_i \star g^{ij}(x) \star \xi_j$.]

Manifolds:

Can be extended to manifolds using local coordinates: give a locally finite atlas (U_α) and $\phi_\alpha \in C_o^\infty(U_\alpha)$ so that $\sum_\alpha \phi_\alpha^2 = 1$. If $a : T^\star X_d \to \mathbb{C}$,

$$\mathrm{Op}(a) = \sum_\alpha \varphi_\alpha \mathrm{Op}_{\mathrm{Weyl}}(a) \varphi_\alpha \ .$$

Symbols:

Need of large classes of symbols in order to include differential operators: $a = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha$.

- $\Sigma^m := \{a | \forall \alpha, \beta, \exists C_{\alpha,\beta}, |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$ with $\langle \xi \rangle = \sqrt{1+|\xi|^2}$: symbols of degree m (ex: polynomials in ξ of degree $\leq m$)

- $S^m := \{a \equiv \sum_{j=0}^{\infty} \hbar^j a_j\}$ with $a_j \in \Sigma^{m-j}$: semi-classical symbols of degree m

- $\Psi^m = \{a_{\mathrm{W}} | a \in S^m\}$: ΨDO of degree m

Using suitable extensions of Lebesgue integrals (Fresnel oscillatory integrals), one can extend Weyl quantization to symbols in S^m.

Remark: larger classes of symbols were used by people, but I will not enter in this!

The main fact is that $\cup_{m\in\mathbb{Z}}\Psi^m$ is a graded algebra and we have explicit formulas for the symbols: if $a \in S^m$ and $b \in S^{m'}$, we have

$$a_{\mathrm{W}} \circ b_{\mathrm{W}} = (a \star b)_{\mathrm{W}}$$

where $a \star b \in S^{m+m'}$ and $a \star b$ is given by the *Moyal formula:*

$$a \star b \equiv \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\hbar}{2i}\right)^j a \left(\sum_{p=1}^{d} \overleftarrow{\partial}_{\xi_p} \vec{\partial}_{x_p} - \overleftarrow{\partial}_{x_p} \vec{\partial}_{\xi_p}\right)^j b$$

$$a \star b = ab + \frac{\hbar}{2i}\{a, b\} + \cdots \ .$$

Rem 1: Moyal formula comes from the stationary phase expansion

Rem 2: Jacobi identity for Poisson bracket is a consequence of the same (trivial) identity for operators brackets

Another algebra: the semi-classical Weyl algebra

Let us consider the space $\mathcal{W}$ of formal powers series in the variables $(x, \xi, \hbar)$ with the grading

$$\mathcal{W} = \oplus_{j=0}^{\infty} \mathcal{W}_j$$

where

$$\mathcal{W}_j = \text{span}\{x^\alpha \xi^\beta \hbar^\gamma \mid |\alpha| + |\beta| + 2\gamma = j\} \ .$$

We have $\mathcal{W}_j \star \mathcal{W}_k \subset \mathcal{W}_{j+k}$.

This graded algebra (called *(semi-classical) Weyl algebra*) will be important for Birkhoff normal forms. The meaning of this grading is action on micro-functions localized at the origin of phase space. Useful for BNF!

A sub Lie-algebra $\mathcal{W}^+$: If $\mathcal{W}_j^+ = \{a \in \mathcal{W}_j | \ a$ even w.r. to $\hbar\}$, $[\mathcal{W}_j^+, \mathcal{W}_k^+] \subset \mathcal{W}_{j+k}^+$.

Moyal product splits into an even and an odd part:

$$a \star b = a \star_+ b + \frac{\hbar}{i} a \star_- b \ ,$$

where $a \star_\pm b$ contains only even powers of $\hbar$, $a \star_+ b = b \star_+ a$ and $a \star_- b = -b \star_- a$.

Brackets:

The symbol of the operator bracket $[a_W, b_W]$ is given by the Moyal bracket

$$[a,b]_\star \equiv \frac{2\hbar}{i} a \star_- b \stackrel{\text{def}}{=} \frac{\hbar}{i} \sum_{j=0}^{\infty} \hbar^{2j} \{a,b\}_j$$

where $\{a,b\}_0$ is the classical Poisson bracket.

Functional calculus:

if $F \in C_o^\infty(\mathbb{R})$ and $\hat{H} = H_{\mathsf{W}}$ with $H \in \Sigma^m$ real valued self-adjoint on $L^2(\mathbb{R}^d)$, we can define $F(\hat{H})$ and we have $F(\hat{H}) = F^\star(H)_{\mathsf{W}}$ with

$$F^\star(H)(z_0) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \left(F^{(k)}(H(z_0))(H - H(z_0))^{\star k} \right)(z_0) \ ,$$

$$F^\star(H) = F(H) - \frac{\hbar^2}{8}\left(F''(H)\mathrm{det}(H'') + \frac{1}{3}F'''(H)H''(X_H, X_H) \right) + O(\hbar^4) \ ,$$

with $\omega(X_H, .) = -dH$.

$F^\star(H)$ contains only even powers of $\hbar$.

L^2 continuity:

If $a \in S^0$, a_W is (uniformly in $\hbar$) continuous from $L^2(\mathbb{R}_d)$ into $L^2(\mathbb{R}_d)$.

Principal symbol:

if $A = \left(\sum_{j=0}^{\infty} \hbar^j a_j\right)_{\mathrm{W}} \in \Psi^m$ the principal symbol is the function $a_0(x, \xi)$.

The Moyal formula shows that the principal symbol of a composition of 2 ΨDO is the usual product of the principal symbols.

Fact: a symbol is invertible near z_0 if and only if $a_0(z_0) \neq 0$. We say that a_{W} is *elliptic* at that point.

WKB functions:

$$u_\hbar(x) \equiv e^{iS(x)/\hbar}\left(\sum_{j=0}^{\infty} \hbar^j b_j(x)\right) \ ,$$

with $S : \mathbb{R}^d \to \mathbb{R}$ smooth and b_j smooth.

It will be convenient to introduce the Lagrangian manifold $\Lambda_S := \{(x, S'(x))\}$.

ΨDO's act on WKB functions as follows, if $a \equiv \sum_{j=0}^{\infty} \hbar^j a_j \in S^m$,

$$a_W u_\hbar(x) \equiv e^{iS(x)/\hbar} \left(\sum_{j=0}^{\infty} \hbar^j c_j(x) \right) \ ,$$

with

1. $c_0(x) = a_0(x, S'(x)) b_0(x)$ (1)

2. If $a_0(x, S'(x)) \equiv 0$:

$$c_1(x) = a_1(x, S'(x)) b_0(x) - i \left(db_0(X_{a_0}) + \frac{1}{2} (\delta b_0) \right) \ (2),$$

   with $\delta := H''_{x_i,\xi_i} + H''_{\xi_i,\xi_j} S''_{x_i,x_j}$.

–*Equation (1)* implies that the principal symbol is a function on $T^\star X_d$.

–*Equation (2)* can be interpreted geometrically on the Lagrangian manifold $\Lambda_S = \{(x, S'(x))\}$. We define the principal symbol ω of the WKB function as $\omega = \pi^\star(b_0|dx|^{\frac{1}{2}})$ and get that the principal symbol of $a_W u$ if $(a_0)_{|\Lambda} = 0$ is

$$-i\mathcal{L}_{X_{a_0}}\omega + a_1\omega \ .$$

Caustics:

If $L \subset T^\star X_d$ is Lagrangian, the caustic set C_L is the set of points of L at which the projection from L onto X is not of rank d.

- If $l_0 \notin C_L$, L is near l_0 the graph of the differential of a function $S(x)$.

- If $l_0 \in C_L$, there are still generating functions: $\varphi(x, \theta),\ \theta \in \mathbb{R}_N$ so that $L = \{(x, \partial_x \varphi) | \partial_\theta \varphi = 0\}$.

Using these φ, one can build natural families of functions extending near caustic points the WKB functions:

$$u_\hbar(x) = (2\pi\hbar)^{-N/2} \int e^{i\varphi(x,\theta)/\hbar} a_\hbar(x, \theta) d\theta$$

with a a symbol of order 0.

Traces:

If $F \in C_o^\infty(\mathbb{R})$ and if H is proper, we can compute the trace of $F(H_{\mathsf{W}})$ as

- Trace$(F(H_{\mathsf{W}})) = \sum F(\lambda_n(\hbar))$

- Trace$(F(H_{\mathsf{W}})) = \frac{1}{(2\pi\hbar)^d} \int F^\star(H) |dx d\xi|$

Identification of both expressions gives information on the asymptotic of eigenvalues; in particular, the Weyl formula:

$$\#\{\lambda_n(\hbar) \leq E\} \sim \frac{1}{(2\pi\hbar)^d} \int_{H \leq E} |dx d\xi|$$

II. Tuesday, June 17

The space $\mathcal{A}(X)$

Let us consider a family of functions $u_\hbar(x)$ so that the L_2 norm is locally $O(\hbar^{-m})$ for some m. We will denote $\mathcal{A}(X)$ the space of such admissible functions on X.

Basic Example: WKB functions $u_\hbar(x) \equiv e^{iS(x)/\hbar}\left(\sum_{j=0}^{\infty} b_j(x)\hbar^j\right)$. S **real valued**. Not true if S complex and $\Im S < 0$ somewhere!

Micro-support:

The micro-support MS($u_\hbar$) describes the localization of $u_\hbar$ in the phase space:

- $(x_0, \xi_0) \notin \text{MS}(u_\hbar)$ if and only if $\exists \varphi \in C_o^\infty(\mathbb{R}_d)$, $\varphi(x_0) \neq 0$ and $\mathcal{F}_\hbar(\varphi u_\hbar)(\xi) = O(\hbar^\infty)$ in some neighborhood of ξ_0.

- Another way to say that: $(x_0, \xi_0) \notin \text{MS}(u_\hbar)$ if and only if there exists $a \in C_o^\infty(T^\star \mathbb{R}_d)$ with $a(x_0, \xi_0) \neq 0$ and $a_W u_\hbar = O(\hbar^\infty)$.

  Ex. of WKB functions:

$$\text{MS}\left(e^{iS(x)/\hbar}\left(\sum b_j(x)\hbar^j\right)\right) = \{(x, S'(x))\}$$

We have:

- $$\mathrm{MS}(a_{\mathrm{W}}(u_\hbar)) \subset \mathrm{MS}(u_\hbar)$$

  with equality if a is elliptic.

- $$\mathrm{MS}(u_\hbar) \subset \mathrm{MS}(a_{\mathrm{W}}(u_\hbar)) \cup a_0^{-1}(0)$$

  $a_0^{-1}(0)$ (the set of points in phase space where a_{W} is not elliptic) is called the *characteristic set* of the ΨDO.

Example of quasi-modes: $(\widehat{H} - E)u_\hbar = O(\hbar^\infty)$.

$$\mathrm{MS}(u_\hbar) \subset \{H = E\}$$

Let us put $\Sigma_E := \{H = E\}$.

Micro-functions:

We plan to be able to work locally in the phase space, for that, we will define the space of micro-functions $\mathcal{M}(U)$ in an bounded open set U of $T^\star\mathbb{R}_d$ by:

$$\mathcal{M}(U) = \mathcal{A}(\mathbb{R}_d)/\{u_\hbar \mid \mathsf{MS}(u_\hbar) \cap U = \emptyset\}$$

ΨDO's act on $\mathcal{M}(U)$ for every U.

Sheaf of micro-functions:

As defined before, micro-functions are only a *presheaf:* it is not always possible to glue together compatible micro-functions on an open covering of $T^\star\mathbb{R}_d$. One needs to compactify the fibers: this can be done by adding the sphere bundle

$$S^\star\mathbb{R}_d := \{(x, \infty\xi) | (x, \xi) \in T^\star\mathbb{R}_d \setminus 0\}$$

and the *extended micro-support:* $(x_0, \infty\xi_0) \notin \widehat{MS}(u_\hbar)$ iff

$$\mathcal{F}_\hbar(\phi u)(\xi) = O(\hbar^\infty / \langle\xi\rangle^\infty)$$

in a conical neighborhood of $(x_0, \infty\xi_0)$ and for a $\phi \in C_o^\infty(\mathbb{R}_d)$ with $\phi(x_0) \neq 0$.

Micro-solutions:

Let us consider $U \subset T^\star X_d$. We want to solve $(\hat{H} - E)u = O(\hbar^\infty)$ in U. This is not possible with a non-trivial u if $U \cap \Sigma_E = \emptyset$.

Let us assume that dH (or X_H) does not vanishes on $\Sigma_E \cap U$. Then we start with a Lagrangian manifold $L \subset \Sigma_E \cap U$.

Outside the caustic, L is the graph of S' with S a solution of the Hamilton-Jacobi equation $H(x, S'(x)) = 0$. We can find a WKB microsolution whose micro-support is L.

If $d = 1$, S is uniquely defined (up to a constant) and one can check that there exists an unique WKB solution modulo multiplication by a power series in $\hbar$.

Caustics:

If $L \subset T^\star X_d$ is Lagrangian, the caustic set C_L is the set of points of L at which the projection from L onto X is not of rank d.

- If $l_0 \notin C_L$, L is near l_0 the graph of the differential of a function $S(x)$.

- If $l_0 \in C_L$, there are still generating functions: $\varphi(x, \theta)$, $\theta \in \mathbb{R}_N$ so that $L = \{(x, \partial_x \varphi) | \partial_\theta \varphi = 0\}$.

Using these φ, one can build natural families of functions extending near caustic points the WKB functions:

$$u_\hbar(x) = (2\pi\hbar)^{-N/2} \int e^{i\varphi(x,\theta)/\hbar} b_\hbar(x, \theta) d\theta$$

with b a symbol of order 0.

Conclusion:

if $d = 1$, near each regular point of Σ_E, there is an (essentially unique) Lagrangian solution of $(\widehat{H} - E)u_\hbar = O(\hbar^\infty)$.

Question to be discussed later: what about singular points of Σ_E*?*

An important micro function: the spectral density

$D(E,\hbar)=\sum \delta(\lambda_n(\hbar))$ whose $\hbar$-Fourier transform if

$$Z=(2\pi\hbar)^{-\frac{1}{2}}\sum e^{-it\lambda_n(\hbar)/\hbar}\ .$$

The mathematical expression of the *Gutzwiller trace formula* [YCdV, Chazarain, Duistermaat-Guillemin (73'-75')] can be rephrased as saying that this micro-function is equivalent to a sum of contributions of micro-functions associated to the periodic orbits: the micro-support of D is the set of pairs (t,E) so that the Hamiltonian H admits an orbit of period t and energy E. In the integrable case, this formula is a corollary of PSF. Application to Kac's problem.

2. The 1D case: spectrum, regular Bohr-Sommerfeld rules and trace formulae

The goal is to get a complete description of the semi-classical spectrum of the Schrödinger operator in 1D, for a smooth Morse potential.

- We will start by describing the uniform asymptotic expansion of the eigenvalues **far from the critical values.** Today!

- Then we will come to the hard part which is the description of this expansion **around the critical values.**

There are 2 parts in this kind of problems:

1. Building approximate eigenfunctions and eigenvalues (quasi-modes)

2. Showing that there are NO other eigenvalues.

Let us start with our 1D Schrödinger operator: $\hat{H} = H_{\mathsf{W}}$ with $H = \xi^2 + V(x)$.

- The critical values of V: $E_0 = \min H < E_1 < \cdots$

- The wells: if $I_N =]E_{N-1}, E_N[$, the wells of order N are the connected component of $\{V < E_N\}$.

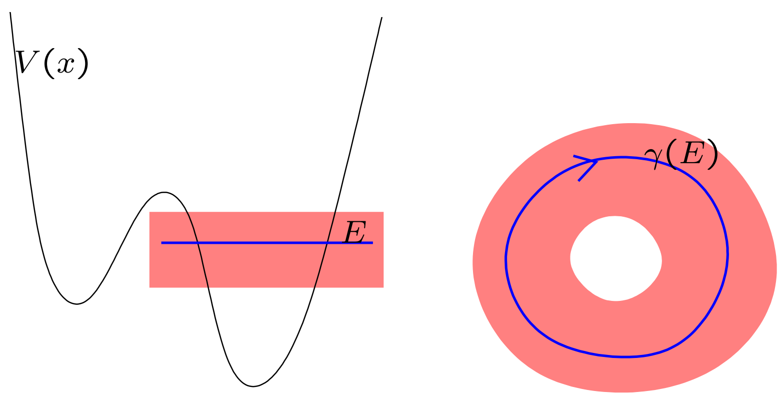
The regular part of the semi-classical spectrum splits according to the wells. We will first construct quasi-modes for each well.

The semi-classical action

For each well and $E \in I_N$:

$$S(E) \equiv \sum_{j=0}^{\infty} \hbar^j S_j(E) \text{ with}$$

- $S_0(E) = \int_{\gamma(E)} \xi dx$ with $\gamma(E)$ a connected component of $H^{-1}(E)$ (a periodic orbit) the classical action
- $S_1(E) = -\pi$ the Maslov correction
- $S_2(E) = -1/24 \int_{\gamma(E)} \det(H'')dt$
- $S_{2j+1}(E) \equiv 0$


$V(x)$
E
$\gamma(E)$

Bohr-Sommerfeld rules

$e^{iS(E)/\hbar}$ is the monodromy of the WKB solutions of

$$(\widehat{H} - E)u = O(\hbar^\infty) \ .$$

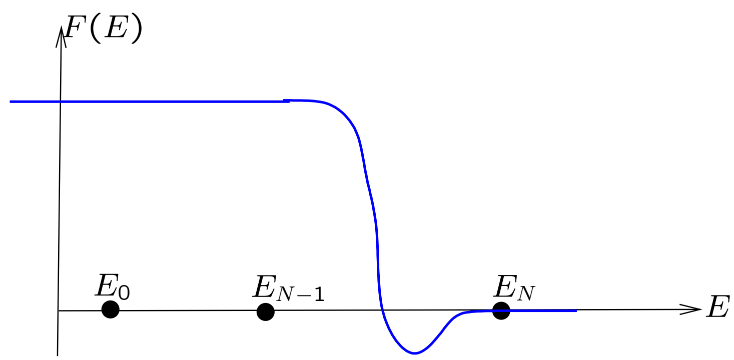
BS rules: $S_\hbar(E) \in 2\pi\hbar\mathbb{Z}$. They describe the spectrum (outside the critical values of V) mod $O(\hbar^\infty)$ as the union of spectra associated to the wells.

I will assume that we know the existence of the semi-classical action and show how we can compute it using only the Moyal product. This is a consequence of a Trace formula.

Trace formula

We will assume that there is only 1 well, i.e. $H^{-1}(-\infty, E_N[)$ is connected.

$F : \mathbb{R} \to \mathbb{R}$ with F constant on $]-\infty, E_{N-1}+\varepsilon]$ and $F(E) \equiv 0$ if $E \geq E_N - \varepsilon$.



Trace $F(H_{\mathsf{W}})$ can be computed using BS rules and PSF or from the ΨDO calculus. Identification of both results gives the values of the S_j's.

- Using PSF + deformation argument:

$$\mathrm{Tr} F(H_{\mathsf{W}}) \equiv \frac{1}{2\pi\hbar}\left(\iint F(H)dL - \int F'(E)\left(\sum_{j=1}^{\infty}\hbar^{2j}S_{2j}(E)\right)dE\right) .$$

- Using $F^\star(H) = \sum_{j=0}^{\infty}\hbar^{2j}F_j(x,\xi)$ (Moyal product!)

$$\mathrm{Tr} F(H_{\mathsf{W}}) \equiv \frac{1}{2\pi\hbar}\sum_{j=0}^{\infty}\hbar^{2j}\iint F_j(x,\xi)dL$$

rem: we do not see S_1 in the trace formula!

The Weyl law $\#\{\lambda_j \leq E\} \sim \frac{1}{2\pi\hbar}\text{area}(\{H \leq E\})$ is a consequence of

$$\text{Trace}F(H_{\mathsf{W}}) \sim \frac{1}{2\pi\hbar}\int F(H)dL$$

Proof of trace formula: Let $J =]E_{N-1}, E_N[$, $A = H^{-1}(J)$ and $D = H^{-1}(]-\infty, E_{N-1}])$. Let $\tilde{H}$ with

- $\tilde{H} = H$ in A
- $\tilde{H}$ has no critical point in $A \cup D \setminus z_0$
- $\tilde{H} = \frac{1}{2}\left(x^2 + \xi^2\right)$ near z_0.

It is enough to prove the formula for $\tilde{H}$ because $F_\star(H)$ and $F_\star(\tilde{H})$ coïncide in $A \cup D$.

- Formula OK for $F = F_1 \in C_o^\infty(]0, E_N[)$ from PSF assuming no other eigenvalues:

$$\sum F(S^{-1}(2\pi\hbar n) = \frac{1}{2\pi\hbar}\int F(E)S'(E)dE + O(\hbar^\infty)$$

- Formula OK for $F = F_2$ with $\mathrm{Supp}(F) \subset]-\infty, \varepsilon[$: explicit calculation for harmonic oscillator: here comes the Maslov index!

$$\sum_{n=0}^{\infty} F((n+\frac{1}{2})\hbar) = \frac{1}{2}\sum_{n=-\infty}^{\infty} F_1((n+\frac{1}{2})\hbar)$$

where F_1 is the even extension of F.

- Every $F = F_1 + F_2$.

Link with heat expansions

An important tool in the study of Laplace operators on RM is the “heat expansion”: the heat equation $u_t = \Delta_g u$ with $u(0) = f$ is solved as $u(t) = \exp(t\Delta_g)f$. Taking the trace, we get: $Z(t) = \text{Trace}\,(\exp(t\Delta_g)) = \sum_{n=1}^{\infty} e^{t\lambda_n}$. One shows that $Z(t)$ admits the following expansion as $t \to 0^+$:

$$Z(t) \equiv \frac{1}{(4\pi t)^{d/2}} \left(\sum_{j=0}^{\infty} a_j t^j \right)$$

with $a_0 = \text{vol}(X_d)$, $a_1 = (1/6) \int \tau |dx|_g$ (τ= scalar curvature).

Putting $t = \hbar^2$, we can rewrite $Z(t) = \text{Trace}\,(F(\hat{H}))$ with $\hat{H} = -\hbar^2 \Delta_g$ and $F(E) = e^{-E}$. This fits well with our previous approach modulo the fact that e^{-E} is not compactly supported.

Calculation of S_j's

Using trace formula, we get the S_j's for $j \geq 2$ (from Moyal formula).

For S_0 and S_1, enough to look at $\tilde{H}$. Trace formula with $F \in C_o^\infty(]0, E_N[)$ gives $S_0' = T(E)$, $S_1' = 0$:

$$\iint F(H)dL = \int F(E)T(E)dE \ .$$

The integration constants are checked from harmonic oscillator where $S_0(E) = \int_{\gamma(E)} \xi dx$ and $S_1 = \pi$.

No other eigenvalues:

Let $F \in C_o^\infty(J, \mathbb{R})$ and let us compare

- $\tilde{Z}_F = \sum F(\tilde{\lambda}_n(\hbar))$ where $\tilde{\lambda}_n$ are the eigenvalues given by $S_0(\tilde{\lambda_n}) = 2\pi\hbar(n + \frac{1}{2})$

- $Z_F = \text{Trace}(F(H_{\mathrm{W}}))$

We have

- $\tilde{Z}_F = \sum_j S_0^{-1}(2\pi\hbar(n + \frac{1}{2})) = \frac{1}{2\pi\hbar}\left(\int F(S_0^{-1}(u))du\right) + O(\hbar)$

- $Z_F = \frac{1}{2\pi\hbar}\int F(H)dL + O(\hbar)$

Both expressions agree to $O(\hbar)$ which do not allows missing eigenvalues.

Another proof is done by using local uniqueness of micro-local solutions: it proves a priory that the solutions are WKB mod $0(\hbar^{\infty})$.

Gutzwiller trace formula

Let D be the spectral density distribution in the interval J. Then from PSF, we get formally:

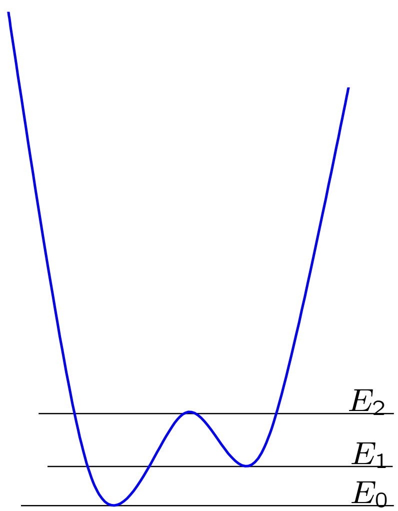
$$D \equiv \sum_{\alpha=1}^{N(J)} \sum_{m \in \mathbb{Z}} D_{\alpha,m}(E)$$

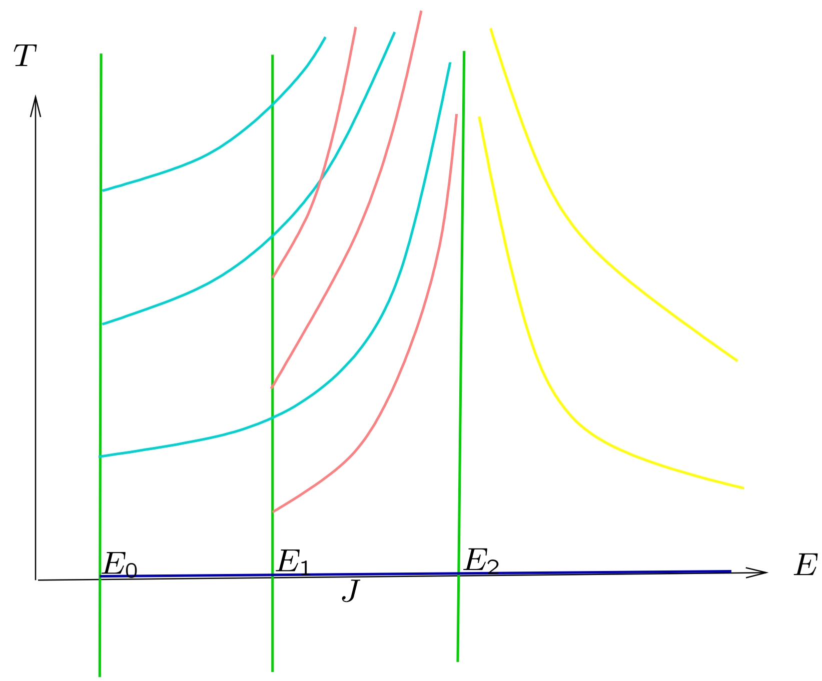
where $\gamma_\alpha,\ \alpha = 1, \cdots, N(J)$, are the periodic orbits associated to the wells in the interval J and

$$D_{\alpha,m}(E) \equiv \frac{1}{2\pi\hbar} S'_\alpha(E) e^{imS_\alpha(E)/\hbar}$$

$$= \frac{(-1)^m T_\alpha(E)}{2\pi\hbar} e^{im\int_{\gamma_\alpha(E)}/\hbar} \Big(1 + im\hbar S_{\alpha,2}(E) + O(\hbar^2)\Big)$$

Micro-support(D): the energy-period picture

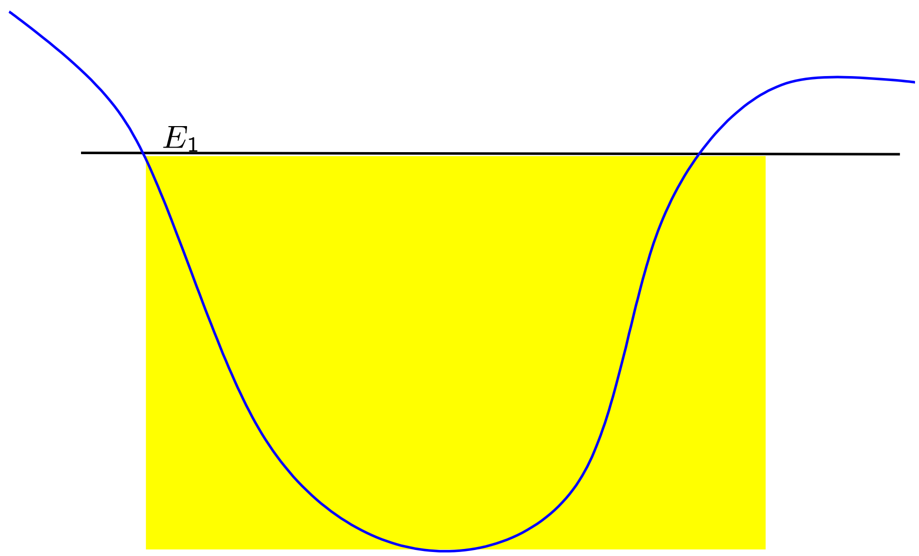




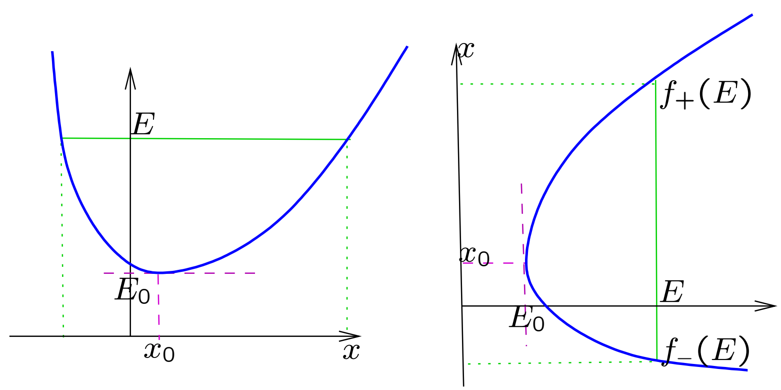
3. Inverse semi-classical problem

Kac's problem revisited: *can we get the potential $V(x)$ from the semi-classical asymptotic of the eigenvalues of the Schrödinger operator ?* YES.

Theorem 1 (YCdV 2007) *Let $V(x)$ a smooth Morse one-well potential: then V is determined below E_1 from the semi-classical spectrum below E_1 modulo $o(\hbar^2)$.*



From the trace formula, we know that we can recover $S_0(E)$ and $S_2(E)$. Moreover from Weyl formula, we get E_0 and $V''(x_0)$ ($V(x_0) = E_0 = \min V$). This implies that we can recover the functions $T(E)$ (the period) and $U(E) = \int_{\gamma(E)} V'' dt$ for $E \leq E_1$. We can rewrite both integrals using 2 functions $f_+(E)$ and $f_-(E)$ ($E_0 \leq E \leq E_1$).

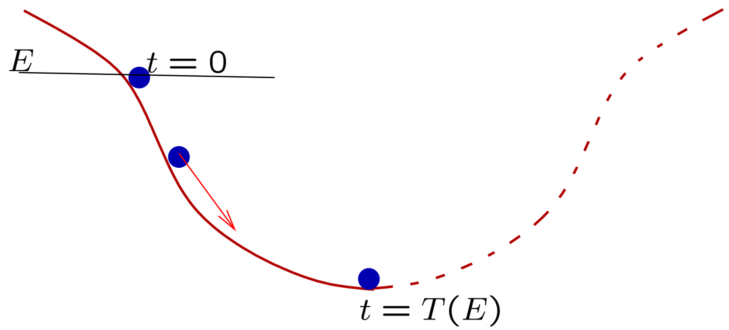


Elementary calculus gives:

$$T(E) = \int_{E_0}^{E} \frac{f'_+(y) - f'_-(y)}{\sqrt{E-y}} dy$$

$$U(E) = \int_{E_0}^{E} \frac{d}{dy} \left(\frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \right) \frac{dy}{\sqrt{E-y}}$$

Abel's toboggan problem (1826): recovering the shape of a toboggan from the arrival times



Tool: consider $\mathcal{A}(f)(E) = \int_{E_0}^{E} \frac{f(y)}{\sqrt{E-y)}} dy$, then $\mathcal{A} \circ \mathcal{A}(f)(E) = \pi \int_{E_0}^{E} f(y) dy$, hence one can recover f from $\mathcal{A}(f)$. Apply this to $T(E)$ and $U(E)$!

Remark: this implies that if V is even, V can be recovered from the period function.

III. Thursday, June 19

FIO's and normal forms

4. A short introduction to FIO's (local theory)

For any bounded open set in $T^\star X$, we have defined the set $\mathcal{M}(U)$ of *micro-functions* in U (admissible functions mod functions which are $O(\hbar^\infty)$ in U). Similarly we can define the algebra $\Psi(U)$ of ΨDO's in U (isomorphic to the algebra of symbols in U with the Moyal product; a quotient algebra) acting on $\mathcal{M}(U)$.

Theorem 2 (Egorov, Duistermaat-Singer) *Let Φ be an graded isomorphism of $\Psi(U)$ onto $\Psi(V)$. Then:*

- *There exists a canonical diffeo χ of U onto V so that*

$$\sigma_{\mathsf{ppal}}(\Phi(a_W)) = \sigma_{\mathsf{ppal}}(a_W) \circ \chi^{-1}$$

- $\exists \widehat{\chi} : \mathcal{M}(U) \to \mathcal{M}(V)$ *so that* $\Phi(a_W) = \widehat{\chi} a_W \widehat{\chi}^{-1}$*;* $\widehat{\chi}$ *is called a FIO or quantized canonical transformation*

- *If* $\chi = \mathrm{Id}$*, there exists an elliptic* ΨDO*,* a_W*, so that* $\Phi(b_W) = a_W \circ b_W \circ a_W^{-1}$ *[Φ is inner]*

- *If U is topologically simple enough, the map $\Phi \to \chi$ is surjective onto the symplectic diffeos of U onto V [existence of FIO's]*

An exact sequence of groups

$$0 \to \mathsf{Inn}(\Psi(U)) \to \mathsf{Aut}(\Psi(U)) \to \mathsf{Sympl}(U) \to 0$$

How to use that: if χ is chosen, choose any Φ (associated to an operator $\widehat{\chi}$) whose associated canonical transformation is χ. Then everything works with ΨDO'S!

Ex: quantization of *twist maps*

Definition 1 $\chi : U_{y,\eta} \to V_{x,\xi}$ *is a* twist map *if and only if the map* $p : (y, \eta) \to (x, y)$ *so that* $\chi(y, \eta) = (x, \xi)$ *is a diffeomorphism from* U *onto an open set* $W \subset X \times X$*.*

χ *is exact if and only if* $\beta = \chi^\star(\alpha_V) - \alpha_U$ *is exact. This imply the existence of a function* $S : W \to \mathbb{R}$*, called* generating function *so that* $dS \circ p = \beta$ *and* $\chi(y, -\partial S/\partial y) = (x, \partial S/\partial x)$

Not all canonical transformations are twist maps, but if U is simple enough, all canonical transformations are compositions of twist maps.

If $\chi : U \to V$ is a twist map of generating function S, we define $\widehat{\chi}$ by

$$\widehat{\chi} u(x) = (2\pi\hbar)^{-d/2} \int e^{iS(x,y)/\hbar} a(x,y) u(y) dy \ ,$$

with a a symbol in $S^{-\infty}$. Using stationary phase expansion, one sees that, if a does not vanishes (we say that $\widehat{\chi}$ is elliptic) on W, $\widehat{\chi}$ is invertible from $\mathcal{M}(U)$ into $\mathcal{M}(V)$.

The Weyl algebra statement:

There exists an exact sequence of groups

$$0 \to \mathcal{I} \to_1 \mathrm{Aut}(\mathcal{W}) \to_2 \mathrm{Sympl}(\mathbb{C}^{2d}) \to 0$$

where

- $\mathcal{I}$ is the group of automorphisms Φ_S of $\mathcal{W}$ given by $\Phi_S w = e^{iS/\hbar} \star w \star e^{-iS/\hbar}$ with $S \in \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \cdots$

- $\mathrm{Aut}(\mathcal{W})$ is the group of all automorphisms of the graded algebra $\mathcal{W}$

- The arrow $\to_2$ is given as the restriction of Φ to $\mathcal{W}_1 = (\mathbb{R}^{2d})' \otimes \mathbb{C}$.

5. Semi-classical normal forms

- Classical normal form: elliptic case

- Classical normal form: hyperbolic case

- Semi-Classical normal form: elliptic case

- Application to spectra near a ND minimum of H

- Semi-Classical normal form: hyperbolic case

Classical normal form: elliptic case

If H admits at the point z_0 a non-degenerate minimum E_0, there exists a canonical transformation χ so that:

$$(H \circ \chi) - E = \mathcal{E}\left(\Omega_e - \alpha_0(E)\right) \ .$$

with $\mathcal{E}(0,0,E_0) \neq 0$, $\alpha(E_0) = 0$ and $\Omega_e = \frac{1}{2}(y^2 + \eta^2)$. Moreover $\alpha_0(E)$ is uniquely defined for $E \geq E_0$ (compute the area of $H \leq E$).

$\sim$*Morse Lemma with a volume form*

Classical normal form: hyperbolic case

If H admits at the point z_0 a non-degenerate saddle point E_0, there exists a canonical transformation χ so that:

$$(H \circ \chi) - E = \mathcal{E}\left(\Omega_h - \alpha_0(E)\right) \ .$$

with $\mathcal{E}(0, 0, E_0) \neq 0$, $\alpha_0(E_0) = 0$ and $\Omega_h = y\eta$. Moreover the Taylor expansion of $\alpha_0(E)$ is uniquely defined.

Semi-Classical normal form: elliptic case

$$(\widehat{\chi})_2^{-1}\,(H_{\mathsf{W}} - E)\,(\widehat{\chi})_1 = (\Omega_e)_{\mathsf{W}} - \alpha(E,\hbar)$$

where

- χ is the canonical transformation for the classical NF.
- $\widehat{\chi}_1,\ \widehat{\chi}_2$ are elliptic OIF's associated to χ
- $\alpha(E,\hbar) \equiv \sum_{j=0}^{\infty} \alpha_j(E)\hbar^{2j}$.

Application to spectra near a ND minimum of H:

Using the fact that $(\Omega_e)_{\mathrm{W}}$ is the harmonic oscillator whose spectrum is $\{(n+\frac{1}{2})\hbar | n = 0, \cdots\}$, we get a good quasi-mode and approximate spectrum given by $\{\alpha^{-1}\left((n+\frac{1}{2})\hbar, \hbar\right) | n = 0, \cdots\}$.

Regular BS rules extend smoothly at the local ND minimas: $\alpha(E,\hbar) = (n+\frac{1}{2})\hbar$

Semi-Classical normal form: hyperbolic case

$$\widehat{\chi}_2^{-1}\,(H_{\mathsf{W}} - E)\,\widehat{\chi}_1 = (\Omega_h)_{\mathsf{W}} - \alpha(E,\hbar)$$

where

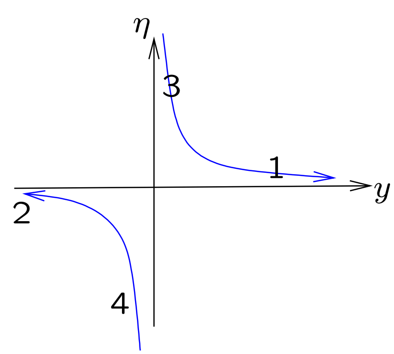
- χ is the canonical transformation for the classical NF.
- $\widehat{\chi}_j$ are elliptic FIO's associated to χ
- $\alpha(E,\hbar) \equiv \sum_{j=0}^{\infty} \alpha_j(E)\hbar^{2j}$.

Application to the local scattering matrix

The equation $((y\eta)_{\mathrm{W}} - \alpha)u = 0$ in $\mathcal{M}(U)$ with $(0,0) \in U$, admits a 2-dimensional free module of solutions over $\mathbb{C}_\hbar$, generated by

- $\varphi_1(y) = [Y(y)|y|^{-\frac{1}{2}+i\alpha/\hbar}]$ and $\varphi_2(y) = [Y(-y)|y|^{-\frac{1}{2}+i\alpha/\hbar}]$

- or by φ_3 and φ_4 defined by their Fourier transform: $\mathcal{F}_\hbar\varphi_3 = Y(\eta)|\eta|^{-\frac{1}{2}-i\alpha/\hbar}$ and $\mathcal{F}_\hbar\varphi_4 = Y(-\eta)|\eta|^{-\frac{1}{2}-i\alpha/\hbar}$.



There is an associated change of coordinates $u = x_1\varphi_1 + x_2\varphi_2 = x_3\varphi_3 + x_4\varphi_4$ and we define the local (unitary) scattering matrix by:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = T(\alpha) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$T(\alpha) = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} + \frac{\alpha}{\hbar}\right) e^{\alpha(\frac{\pi}{2} + i \log \hbar) - i\frac{\pi}{4}} \begin{pmatrix} 1 & ie^{-\alpha\pi/\hbar} \\ ie^{-\alpha\pi/\hbar} & 1 \end{pmatrix}$$

If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define the transmission amplitude as $t = |a|^2 = |d|^2$ and the reflexion amplitude as $r = 1 - t$, we get

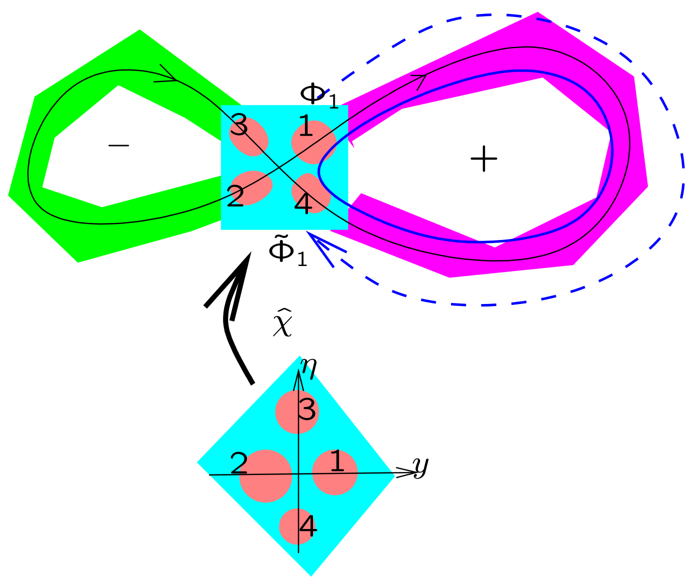
$$t = \frac{1}{1 + e^{-2\alpha\pi/\hbar}}, \ r = \frac{1}{1 + e^{+2\alpha\pi/\hbar}}$$

Where is the local scattering matrix useful?

If $|E - E_0| >> \hbar$, then either t or r is negligible. This implies that we can restrict to —$E - E_0| = O(\hbar^{1-\varepsilon})$. Only the Taylor expansions of the α_j's are relevant.

6. Hyperbolic singular points and singular Bohr-Sommerfeld rules

We want to describe the semi-classical spectrum in an interval K containing the local ND max E_{crit} of V. Let us denote by $K^- = K \setminus]E_{\text{crit}}, +\infty[$.



6.a: Defining the singular actions $S_\pm^{\text{sing}}(E)$

We will define 2 formal series expansions $S_\pm^{\text{sing}}(E) \equiv \sum_{j=0}^\infty S_{j,\pm}^{\text{sing}}(E)\hbar^j$ where the $S_{j,\pm}^{\text{sing}}(E)$ are smooth on K.

Let us denote, for $j = 1, \cdots, 4$: $\Phi_j = \widehat{\chi}(\phi_j)$. For $E \leq E_{\text{crit}}$, by following Φ_1 along γ_+, we get a WKB function $\tilde{\Phi}_1$ which we can compare with Φ_4: we get

$$\tilde{\Phi}_1 = e^{iS_+^{\text{sing}}(E)/\hbar}\Phi_4$$

Similarly:

$$\tilde{\Phi}_2 = e^{iS_-^{\text{sing}}(E)/\hbar}\Phi_3$$

The formal series

$$S_\pm^{\text{sing}} = \sum \hbar^j S_{j,\pm}^{\text{sing}}(E)$$

are called the *singular actions*. They depend on the local NF. They are smooth(!!) w.r.t. to $E \in K$.

6.b Singular actions as a regularization of smooth actions:

Let us compute $S_\pm^{\text{sing}}(E)$ for $E \in K_-$. There are 2 smooth periodic orbits $\gamma_\pm(E)$ whose BS actions are $S_\pm^{\text{smooth}}(E)$ (non smooth at $E = E_{\text{crit}}$. Let us consider the difference; if $E < E_{\text{crit}}$, the coefficients t_{42} and t_{31} of the local scattering matrix $T(E)$ are exponentially small. Hence $t_{41} = \exp(iS_+^{\text{Stirling}}/\hbar) + O(\hbar^\infty)$ and $t_{32} = \exp(iS_-^{\text{Stirling}}/\hbar) + O(\hbar^\infty)$. Let us compute the expansion of $S_\pm^{\text{Stirling}}(E)$. Using Stirling formula, we get:

$$S_+^{\text{Stirling}}(E) \equiv \alpha\left(\log|\alpha| - 1\right) + \hbar\frac{\pi}{4} + \sum_{j=1}^{\infty} \beta_j \left(\frac{\hbar}{\alpha}\right)^{2j}$$

and using the expansion of α:

$$S_+^{\text{Stirling}}(E) \equiv \alpha_0(E)\left(\log|\alpha_0(E)| - 1\right) + \hbar\frac{\pi}{4} + \sum_{j=1}^{\infty} s_{2j}(E)\hbar^{2j}$$

For $E \in K_-$, we can extend Φ_4 following the part of $\gamma_+(E)$ close to the singularity giving $\tilde{\Phi}_4$ and $\tilde{\Phi}_4 = (t_{41})^{-1}\Phi_1$. So we get the relation, for $E < E_{\text{crit}}$:

$$S_\pm^{\text{smooth}} = S_\pm^{\text{sing}} - S_\pm^{\text{Stirling}} + O(\hbar^\infty) \text{ mod } 2\pi\hbar\mathbb{Z}$$

This gives also:

$$S_\pm^{\text{sing}}(E) \equiv S_\pm^{\text{smooth}}(E) + S_\pm^{\text{Stirling}}(E)$$

From that we can compute the expansion of $S_\pm^{\text{sing}}(E)$ for $E \leq E_{\text{crit}}$:

$$S_\pm^{\text{sing}}(E) \equiv S_{0,\pm}^{\text{sing}}(E) + \hbar S_{1,\pm}^{\text{sing}}(E) + \sum_{j=1}^{\infty} S_{2j,\pm}^{\text{sing}}(E)\hbar^{2j}$$

The Taylor expansions at $E = E_{\text{crit}}$ of the $S_{2j,\pm}^{\text{sing}}(E)$ are well defined.

Calculations for $S_0^{\mathsf{sing}}(E)$

For $E < E_{\mathsf{crit}}$:

$$S_{0,\pm}^{\mathsf{sing}}(E) = \int_{\gamma_\pm(E)} \xi dx + \alpha_0(E)\left(\log|\alpha_0(E)| - 1\right)$$

and

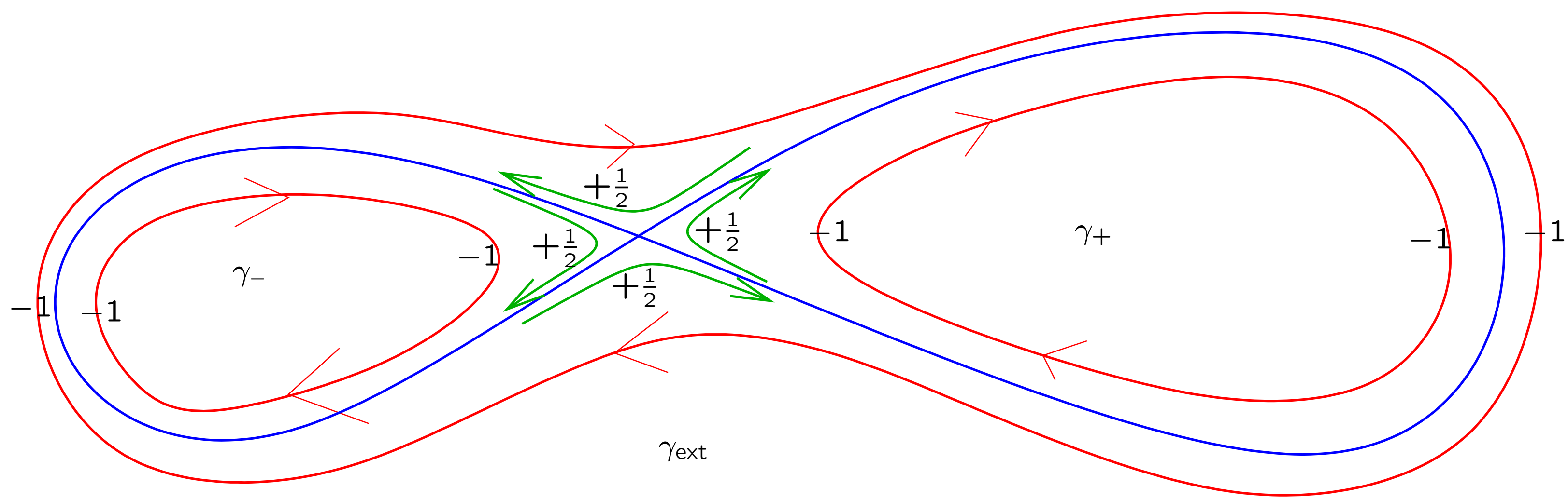
$$\frac{d}{dE} S_{0,\pm}^{\mathsf{sing}}(E) = T_\pm(E) + \alpha_0'(E)\log|\alpha_0(E)|$$

Regularization of the period:

$$T_\pm^{\mathsf{sing}}(E_{\mathsf{crit}}) = \lim_{E \to E_{\mathsf{crit}}^-} \Big(T_\pm(E) + \alpha_0'(E)\log|\alpha_0(E)|\Big)$$

Calculation of $S_1^{\text{sing}}(E)$

$S_1^{\text{sing}}(E) = S_1^{\text{smooth}}(E) + \frac{\pi}{4}$: singular Maslov index.



$m_{\text{sing}}(\gamma_+) = m_{\text{sing}}(\gamma_-) = -3/2$; $m_{\text{sing}}(\gamma_{\text{ext}}) = -3$

6.c Singular BS rules:

Let us look at a solution of $(\hat{H} - E)u = O(\hbar^\infty)$ for E close to E_{crit}. We can assume $u \equiv x_j \Phi_j$ near the singular point. We have the following relations:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = T(\alpha(E)) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$x_4 = e^{iS_+^{\text{sing}}(E)/\hbar} x_1, \ x_3 = e^{iS_-^{\text{sing}}(E)/\hbar} x_2 \ .$$

So that the singular BS quantization rules are:

$$\det \left(\begin{pmatrix} 0 & e^{iS_-^{\text{sing}}(E)/\hbar} \\ e^{iS_+^{\text{sing}}(E)/\hbar} & 0 \end{pmatrix} - T(\alpha(E)) \right) = 0$$

6.d Application to the symmetric double well:

The singular BS rules gives the transition between 2 qualitatively very different spectra:

- For $E > E_{\text{crit}}$, we have a regular spacing of size $\sim \hbar/T(E)$.

- For $E < E_{\text{crit}}$, we have the so called parity doublets:

$$\lambda_{2j} - \lambda_{2j-1} = O(\hbar^{\infty}) \text{ and } \lambda_{2j+1} - \lambda_{2j} \sim \hbar/T(E) \ .$$

We introduce the parameter

$$p(E) = \Big(\lambda_{2j} - \lambda_{2j-1}\Big) \,/\, \Big(\lambda_{2j+1} - \lambda_{2j}\Big)$$

Then $p(E)$ is increasing from 0 to $\frac{1}{2}$ and we can check $p(E_{\text{crit}}) = 1/4$.

V. Saturday, June 21

7. $d > 1$: classical and semi-classical integrability

Let H be an integrable Hamiltonian: $\exists\ F = (F_1, \cdots F_d) : T^\star X \to \mathbb{R}^d$ (the moment map) s.t.

- $\{F_i, F_j\} = 0$
- F' is a submersion almost everywhere
- $H = \Phi(F_1, \cdots, F_d)$
- F is proper.

Semi-classical integrability: exists $\hat{F}_j$ self-adjoint ΨDO's of principal symbols F_j so that

- $[\hat{F}_i, \hat{F}_j] = 0$
- $H_{\mathrm{W}} = \tilde{\Phi}(\hat{F}_1, \cdots, \hat{F}_d)$

Question: Given H a Liouville integrable system, is it true that H_{W} is a semi-classical integrable system?

Examples:

- *Surfaces of revolution:* $\hat{F}_1 = \hbar^2 \Delta_g$, $\hat{F}_2 = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$

- *Liouville surfaces:* $ds^2 = (A(x)+B(y))(dx^2+dy^2)$ with $(x,y) \in (\mathbb{R}/T_1\mathbb{Z}) \times (\mathbb{R}/T_2\mathbb{Z})$, $A, B > 0$.

$$\hat{F}_1 = \hbar^2 \Delta_g = \hbar^2 \frac{\Delta_{\text{Eucl}}}{A(x)+B(y)}$$

$$\hat{F}_1 = \hbar^2 \frac{A(x)\partial_{yy} - B(y)\partial_{xx}}{A(x)+B(y)}$$

- *Laplacians on ellipsoïds*

- *Resonant QBNF in 2D: an example with a* $2:1$ *resonance*

$$\widehat{F}_1 = \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + x^2\right) + \left(-\hbar^2\frac{\partial^2}{\partial y^2} + y^2\right),$$

$$\widehat{F}_2 = y\left(\hbar^2\frac{\partial^2}{\partial x^2} + x^2\right) - \hbar^2\frac{\partial}{\partial y}\left(2x\frac{\partial}{\partial x} + 1\right) \quad .$$

- *High energy levels for Schrödinger on S^2:* Let us consider $\hat{H} = -\hbar^2\Delta + \hbar^2 V$ where Δ is the Laplace operator on S^2 with the canonical metric and V is smooth real valued. Then have $\hat{H} = A + \hbar^2 B$ where

  - A and B are ΨDO's of order 0,

  - $\sigma(A) = \sigma(\hbar^2\Delta) = \{\hbar^2 k(k+1) | k = 0, 1, \}$ with multiplicities $2k+1$,

  - the principal symbol of B is the average $\bar{V}$ of V on the closed geodesics of S^2

  - A **and** B **commute**

Recipe: consider $C = \hbar\sqrt{-\Delta + V}$ and $U(t) = e^{-itC/\hbar}$. $U(2\pi) = -\mathrm{Id} + \hbar R$ where the R is a ΨDO whose principal symbol can be calculated. Take the log of $\mathrm{Id} - \hbar R$ in order to build A.

It is a quantum averaging method: the associated classical integrable system is $(\frac{1}{2}\|\xi\|^2, \bar{V})$.

The previous result is not true for manifolds which are not Zoll (periodic geodesic flow). For example, the torus: the semi-classical spectrum splits into the *stable eigenvalues* (KAM) and the *unstable eigenvalues*.

The joint spectrum

Assuming H_{W} to be semi-classically integrable, we can consider the joint spectrum $\subset \mathbb{R}_d$: $\exists$ (φ_α) an ONB of $L^2(X_d)$ which is an eigenbasis for all $\widehat{F}_j$'s.

$$\widehat{F}_j \varphi_\alpha = \lambda_{j,\alpha} \varphi_j$$

The set of points $\lambda_\alpha = (\lambda_{1,\alpha}, \cdots, \lambda_{d,\alpha}) \in \mathbb{R}_d$ is the joint spectrum.

We can try to extend what has been done in 1D case to this case. In particular:

- (Regular) BS rules

- Singular BS rules describing the spectra close to the critical values of the moment map (in the generic case)

The first part is rather well known, while the second has been recently studied by the Grenoble school.

Action-angle coordinates:

- c: regular value of the momentum map F
- T a compact connected component of $F^{-1}(c)$

There exists an **exact** symplectic diffeomorphism ("exact" means that $\chi^\star(\xi dx) = \eta dy + dS$)

$$\chi : \mathcal{U} \to \mathcal{V}$$

with $\mathcal{U} = \{(y, \eta) \in T^\star(\mathbb{R}^d / 2\pi\mathbb{Z}^d) | \eta \in U\}$ and $\mathcal{V}$ a neighborhood of T, so that $F_j \circ \chi(y, \eta) = G_j(\eta)$:

$$H \circ \chi = K(\eta_1, \cdots, \eta_d) \ .$$

Quasi-periodicity:

The fibers of F near c are finite union of tori.

$$\chi^{-1} X_H = \sum \frac{\partial K}{\partial \eta_j} \frac{\partial}{\partial \theta_j} \ .$$

$$\phi_t \left(\chi(\theta_0, \eta_0) \right) = \chi(\theta_0 + t\omega(\eta), \eta_0) \ ,$$

with

$$\omega = \left(\frac{\partial K}{\partial \eta_j} \right).$$

8.a Bohr-Sommerfeld rules: quasi-modes

Let us look at the system:

$$(\star)\ \left(\widehat{F}_j - \varepsilon_j\right) u = O(\hbar^\infty),\ j = 1, \cdots, d\ ,$$

for $\varepsilon = (\varepsilon_j)$ close to c a regular value of the momentum map.

Fact: $(\star)$ admits an unique (micro-function) solution near each z_0 so that $F(z_0) = \varepsilon$ modulo multiplication by an element of $\mathbb{C}_\hbar$.

Can be proved by the normal form method: $(\star)$ *reduces to* $\frac{\hbar}{i}\partial_{y_j} u = O(\hbar^\infty)$ *near* 0.

If T_ε is one of the tori $\subset F^{-1}(\varepsilon)$ (continuously depending on ε), then one can look at a basis $\gamma_1(\varepsilon), \cdots, \gamma_d(\varepsilon)$ of $H_1(T_\varepsilon)$, and at the holonomies $e^{iS_j(\varepsilon)/\hbar}$ of the solutions of $(\star)$.

The Bohr-Sommerfeld rules are $S(\varepsilon) = (S_j(\varepsilon)) \in 2\pi\hbar\mathbb{Z}^d$.

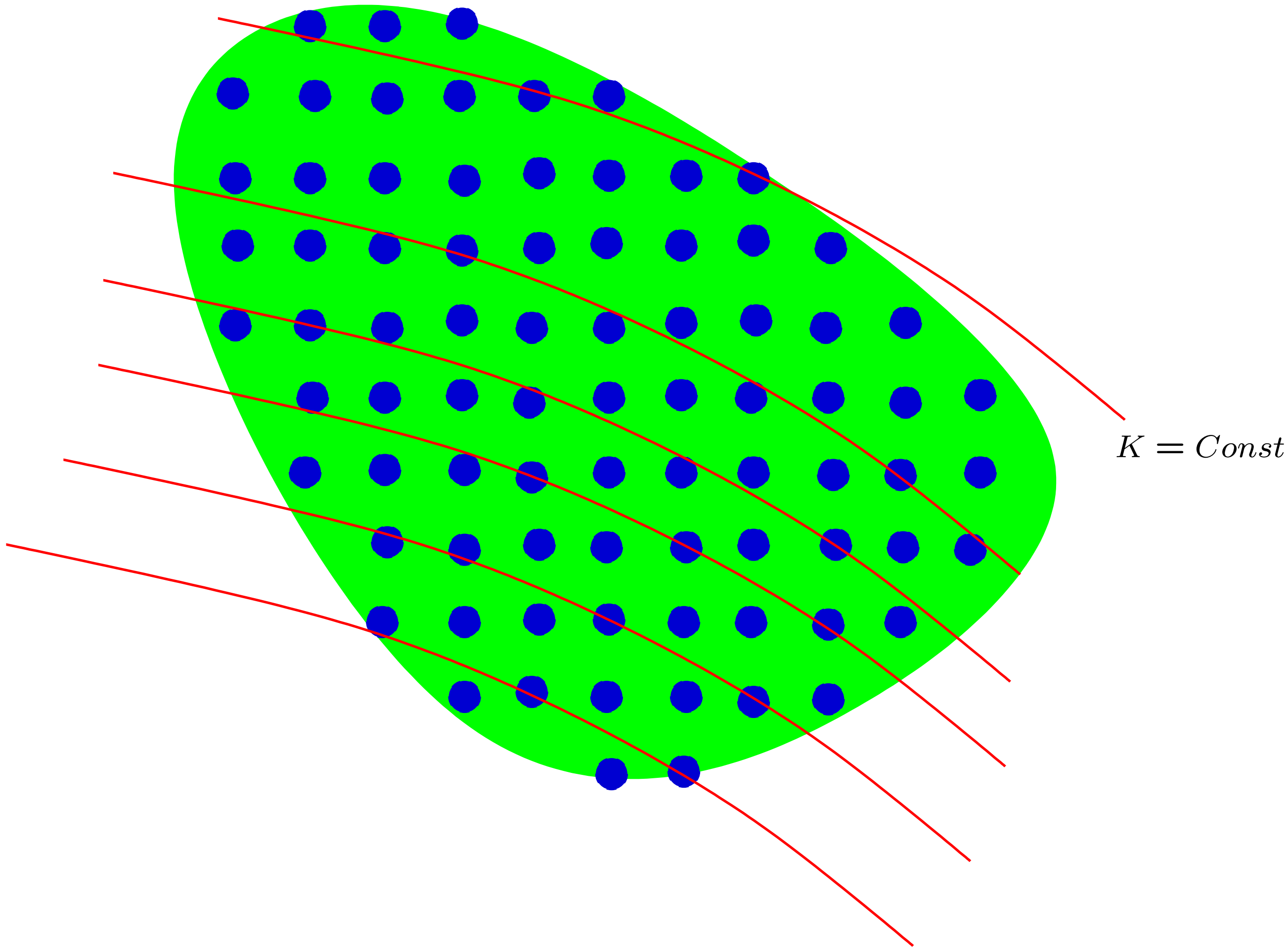
Let us make them more explicit:

$$S_j(\varepsilon) \equiv S_{j,0}(\varepsilon) + \hbar S_{j,1}(\varepsilon) + \sum_{l=2}^{\infty} \hbar^l S_{j,l}(\varepsilon) \ ,$$

with

- $S_{j,0}(\varepsilon) = \int_{\gamma_j(\varepsilon)} \xi dx$
- $S_{j,1}(\varepsilon) = m_j\pi/2$ with $m_j \in \mathbb{Z}$ the Maslov index.
- $S_{j,l}(\varepsilon)$ are not easy to capture (work of Littlejohn and co) except in the case of separable system (Liouville surfaces, ...).

From the classical action-angle Theorem, we know that $S : V_c \to \mathbb{R}^d$ is a local diffeo. Hence the description of the spectrum as a deformed lattice:


$K = Const$

8.b Bohr-Sommerfeld rules: no other eigenvalues near c

Let $\Phi \in C_o^\infty(\mathbb{R}^d)$ supported in a neighborhood of c. Computing Trace $\left(\Phi(\hat{F}_1, \cdots, \hat{F}_d)\right)$

- BS rules and PSF

- Functional calculus

shows that the number of missing eigenvalues is “small”.

In fact, there is no other eigenvalues... because we are able to describe all micro-local solutions!

VIII.c Lattice point problem:

The spectral counting function for an (non singular) integrable system reduces to counting integral points in a bounded smooth domain:

$$N(\hbar) = \#\{\Omega \cap \hbar\mathbb{Z}^2\}$$

Trivially $N(\hbar) \sim \hbar^{-2}|\Omega|$, this is the Weyl formula for integrable systems.

A special case, the *circle problem:* what is the best estimates for $R(\lambda) = \#\{(m,n) \in \mathbb{Z}^2 | m^2 + n^2 \leq r^2\} - \pi r^2$? Conjectured $O(r^{\frac{1}{2}+\varepsilon})$.

One can estimate the remainder from the decay of the Fourier transform of 1_Ω; If $\mathrm{Curv}(\partial\Omega) > 0$, we have

$$\mathcal{F}_1(1_\Omega)(\xi) = 0(|\xi|^{-3/2})$$

This implies

$$N(\hbar) \sim \hbar^{-2}|\Omega| + O\left(\hbar^{-2/3}\right)$$

which is a remainder term smaller than the general remainder term in the Weyl formula ($O(\hbar^{-1})$).

Spectral test for integrability [to be worked out!]

A much better test for integrability would be the Gutzwiller trace formula which gives the regularized density of states. Oscillations of this density are much stronger in the integrable case. The spectral statistics are different.

This subject has been developed by physicists and is related to random spectra and random matrix theory....

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