Mathematics of ray propagation and applications to time reversal and passive imaging

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I started working on the new methods used by Michel Campillo and collaborators in seismology using the time correlation of noisy fields. Providing semi-classical models for these could be interesting for 2 reasons:

- Giving rather quantitative results it may helps in studying the source of the noise
- As in quantum mechanics, it provides intuition on these methods based on more geometry.

I would like to emphasize first several messages:

- In these dynamical problems, it is easier, contrary to what people often do, to use directly the approximation of dynamics, without using mode decompositions

- Polarizations can be really described giving more informations

The aim of these lectures is to show how the modern theory of linear partial differential equations *(pseudo-differential operators)* builded on the geometry of phase space *(symplectic geometry, Hamiltonian formalism)* can be used in order to study PI and TRM in the high frequency regime and under smoothness assumptions.

I will first review the basic formulae:

- Calculus of the field correlation from the correlation of the source

- Calculus of the time reversed wave

I will then give a very brief introduction to ΨDO calculus and applications to asymptotics of previous formulae.

1. Passive imaging

2. Time reversal mirrors

3. Semi-classics: (a) Pseudo-differential operators (ΨDO's); (b) Ray dynamics; (c) Green's function/propapagator; (d) Egorov Theorem.

4. Application to PI and TRM

Important remarks:

I will mostly discuss the case of Schrödinger equation for several reasons:

- Technically simpler than wave equations: first order w.r. to t, dispersive waves
- Closer to my own knowledge, it includes the very geometric case of the Laplace operator on Riemannian manifold
- Starting from wave equation (acoustical waves, seismic waves) we get non trivial dispersion relations when looking at surface waves with a stratified medium.

I will try to avoid writing done too much explicit formulae and concentrating more on the ideas:

“All waves behave in a similar way” (Brillouin, 1960)

“In high frequency regime, many things can be calculated using the classical (ray) dynamics”

1. Passive imaging **Goal:** assuming some source of noise being propagated by a linear wave equation, there is a relation between

- The correlation $C_{A,B}(\tau)$ of noisy waves between 2 points A and B

- The Green function (or the propagator) for the (deterministic, smooth) wave equation without source.

Following Brillouin, it is more or less independent of the kind of waves. We will show an **exact relation in case of a white noise and an asymptotic relation in case of high frequency propagation.**

Here is the starting point:

$$\frac{d\mathbf{u}}{dt} + \widehat{H}\mathbf{u} = \mathbf{f} \tag{1}$$

- $\mathbf{u}(x,t),\ x \in X^d$ the field (scalar or vector)

- $\widehat{H}$ the deterministic smooth (matrix) Hamiltonian, acting on $L^2(X, \mathbb{C}^N)$ includes the attenuation:

$$\exists k > 0,\ \mathrm{Re} < \widehat{H}\mathbf{u}|\mathbf{u} >\geq k\|\mathbf{u}\|^2$$

- $\mathbf{f}(x,t)$ the noisy source field

- A model case will be the Schrödinger operator:

$$-ihu_t - \frac{h^2}{2}\Delta u + V(x)u - ihku = -ihf, \;\; k > 0 \;\; .$$

- A more complicated case will be any kind of wave equation:

$$\mathbf{u}(x,t) := \begin{pmatrix} u \\ u_t \end{pmatrix}$$

and

$$u_{tt} + au_t - \Delta u = f, \;\; a \geq 0$$

which corresponds to

$$\widehat{H} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta & a \end{pmatrix}$$

and

$$\mathbf{f} := \begin{pmatrix} 0 \\ f \end{pmatrix}$$

The *causal* solution of Equation (1) is given by:

$$\mathbf{u}(x,t) = \int_{-\infty}^{0} ds \int_X P(-s,x,y)\mathbf{f}(t+s,y)|dy| \tag{2}$$

where P (called the *propagator*) (closely related to the Green function) is defined as follows:

P is the integral kernel of $\Omega(t) = \exp(-t\widehat{H})$

$$(\Omega(t)\mathbf{v})(x) = \int_X P(t,x,y)\mathbf{v}(y)|dy| \ .$$

In what follows, we will denote $[A](x,y)$ *the integral kernel of the operator* A*.*

$\Omega(t+s) = \Omega(t) \circ \Omega(s)$ rewrites

$$\int_X P(t,x,y)P(s,y,z)|dy| = P(t+s,x,z)$$

Correlation of the field

We define the correlation matrix

$$C_{A,B}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{u}(A,t)\otimes\mathbf{u}^\star(B,t-\tau)dt$$

or

$$C_{A,B}^{\alpha\beta}(\tau) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{u}_\alpha(A,t)\overline{\mathbf{u}_\beta(B,t-\tau)}dt$$

Putting $\mathbf{u}(A,t)$, $\mathbf{u}(B,t-\tau)$ as given by Equation (2), we get:

$$C_{A,B}(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \Phi(T_t f)dt \qquad (3)$$

with

$$\Phi(f) = \textstyle\int_{-\infty}^0 ds \int_{-\infty}^0 ds' \int_{X\times X} |dxdy| \cdots$$
$$\cdots P(-s,A,x)\mathbf{f}(x,s)\otimes\left(P(-s',B,y)\mathbf{f}(y,s'-\tau)\right)^\star$$

Assuming

$$\mathbb{E}(\mathbf{f}(x,s) \otimes \mathbf{f}^\star(y,s')) = \delta(s-s')K(x,y)$$

and ergodicity, we get, for $\tau > 0$:

$$C_{A,B}(\tau) = \int_{-\infty}^{0} ds \int_{X^2} |dx||dy| P(\tau - s, A, x) K(x,y) (P(-s,B,y))^\star \tag{4}$$

and $C^{\alpha\beta}_{A,B}(-\tau) = \overline{C^{\beta\alpha}_{B,A}(\tau)}$.

We can rewrite Equation (4) in an operator form:

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A, B) \tag{5}$$

with

$$\Pi := \int_0^\infty \Omega(s)\mathcal{K}\Omega^\star(s)ds \tag{6}$$

where $\mathcal{K}$ is the integral operator whose kernel is $K(x, y)$. This is the completely general relation between the correlation and the propagator. We will need to compute Π in some asymptotic regime.

We are lead to the following problem: find the high frequency behaviour of $\Omega(\mathrm{s})\mathcal{K}\Omega^\star(\mathrm{s})$ (and $\int\psi(\mathrm{s})\Omega(\mathrm{s})\mathcal{K}\Omega^\star(\mathrm{s})\mathrm{ds}$) under some appropriate assumptions on $\mathcal{K}$ (and ψ).

If f is a white noise, i.e. $\mathcal{K} = \mathrm{Id}$, we have

$$C_{A,B}(\tau) = [\Omega(\tau) \int_0^\infty \Omega(s)\Omega^\star(s)ds](A, B)$$

If we assume $\hat{H} = \widehat{H_0} + k\mathrm{Id}$ with $\widehat{H_0}$ self-adjoint, we get

$$C_{A,B}(\tau) = \frac{e^{-k|\tau|}}{2k} P_0(\tau, A, B) \qquad (7)$$

In general, i.e. for non homogeneous noises, Equation (7) is only valid approximatively ; that is the purpose of what follows.

Case of the wave equation:

$$u_{tt} + 2ku_t - \Delta u = f \tag{8}$$

Let $Q = \sqrt{-\Delta - k^2}$ and $G(t,x,y)$ the integral kernel of $\frac{\sin tQ}{Q}$.

We get

$$u(x,t) = \int_0^\infty e^{-ks} ds \int_X G(s,x,y) f(y,t-s)|dy|$$

And the correlation, for $\tau > 0$,

$$C_{A,B}(\tau) = [\cos \tau Q \ \Pi_+ + \sin \tau Q \ \Pi_-](A,B)$$

with

$$\Pi_\pm = \int_{-\infty}^0 ds e^{k(2s-\tau)} \frac{\sin sQ}{Q} \mathcal{K} \begin{cases} \frac{\sin sQ}{Q} \\ \frac{\cos sQ}{Q} \end{cases}$$

In case of a white noise, we get

$$C_{A,B}(\tau) = \frac{e^{-k|\tau|}}{4(Q^2 + k^2)} \left[\frac{\cos \tau Q}{k} + \frac{\sin \tau Q}{Q} \right] (A, B)$$

whose high frequency limit is:

$$C_{A,B}(\tau) \approx \frac{e^{-k|\tau|}}{4\Delta} \left[\frac{\cos \tau \sqrt{-\Delta}}{k} \right] (A, B)$$

The τ derivative of $C_{A,B}(\tau)$ is

$$\approx -\frac{e^{-k|\tau}}{4k} G(\tau, A, B)$$

2. Time reversal mirrors

Goal: we record a wave generated from of source at $t = 0$ during the time intervall $[0, T]$. We want to reemit the recorded wave after reversing time and amplification. We look at that new wave at time $2T$ and want this $u(2T)$ to be quite close of $u(0)$ or at least located at the same point with some amplification.

Here we consider a wave equation inside a Riemannian manifold (X, g) without boundary (for simplicity)

$$u_{tt} - \Delta_g u = f \tag{9}$$

Puting $A := \sqrt{-\Delta_g}$, we get:

$$u(t) = \int_0^\infty \frac{\sin sA}{A} f(t-s)ds$$

The scheme of TRM

We start with $f(x,t) = \delta(t=0)u_0(x)$ and, for $0 \leq t \leq T$, the propagated pulse

$$u(t) = \frac{\sin tA}{A}u_0$$

and we record data from $u(t)$, for $0 \leq t \leq T$, with an operator $\Omega : L^2(X) \to \mathcal{H}$ with $\mathcal{H}$ an auxiliary (finite dimensional) vector space. We call it the *recording operator*.

We then use, for $T \leq t \leq 2T$,

$$f(t) = \psi(2T - t)K\left(\Omega u(2T - t)\right)$$

with $\psi \in C_o^\infty(]O, T[)$ as the r.h.s. of Equation (9). $K : \mathcal{H} \to L^2(X)$ is called the *amplification operator.*

We get easily:

$$u(2T) = \int_0^T \psi(s) \frac{\sin sA}{A} L \frac{\sin sA}{A} u_0 ds$$

with $L := K\Omega$. The problem is to study the operator $R : u_0 \to u(2T)$ and, for example, see how close it is from Id. Using

$$\sin a = \frac{e^{ia} - e^{-ia}}{2i}$$

we get integrals like

$$\int \psi(s) e^{\pm isA} \tilde{L} e^{\pm isA} ds$$

which are similar to the integrals to be studied in PI.

3. Semi-classics

We want a nice class of operators for which we can study the high frequency limits of $U(s)BU^\star(s)$. They are called the pseudo-differential operators (ΨDO's) and were introduced in the sixties by Calderon, Zygmund, Nirenberg, Hörmander as a tool in the study of linear partial differential equations with non constant co-efficients. In some sense, they give the geometrical extension of Hamiltonian formalism of classical mechanics to wave mechanics. In applications to physics, it is often called the **ray** method. The same tools apply to the study of the semi-classical limit of quantum mechanics and to the high frequency limit of wave equations (acoustic, electromagnetic or seismic waves). There is a small paramater $\varepsilon > 0$ in the theory which can be $\hbar$ or ω^{-1}.

- (a) ΨDO's
- (b) Ray dynamics
- (c) Green's function/propapagator
- (d) Egorov Theorem

(a) ΨDO's

ε will be a **small parameter:** in what follows

- ε can be the inverse of the frequency
- ε can be the typical correlation distance of the noisy field, i.e. $K(x,y) = k(x,y,\frac{x-y}{\varepsilon})$

A pseudo-differential operator (ΨDO)on $\mathbb{R}^d$

$$A_\varepsilon := \mathsf{Op}_\varepsilon(a)$$

is defined using a function $a(x,\xi) : \mathbb{R}^d \oplus \mathbb{R}^d \to \mathbb{C}$ on the phase space. a is assumed to be

- smooth
- homogeneous near infinity in ξ

$$A_\varepsilon(f)(x) = \frac{1}{(2\pi)^d} \int e^{i(x-y|\xi)} a(x,\varepsilon\xi) f(y) dy d\xi$$

Simple examples:

- $\mathsf{Op}_\varepsilon(\xi_j) = \frac{h}{i}\frac{\partial}{\partial x_j}$

- $\mathsf{Op}_\varepsilon(x_j)$ is the multiplication by x_j

- $\mathsf{Op}_\varepsilon(\chi(\xi))$ is a frequency cut-off

Pseudo-differential operators act nicely on WKB functions:

$$\mathsf{Op}_\varepsilon(a)(A(x)e^{iS(x)/\varepsilon}) \approx a(x, S'(x))A(x)e^{iS(x)/\varepsilon}$$

The integral kernel of $\mathsf{Op}_\varepsilon(a)$ is

$$\varepsilon^{-d}\widehat{a}(x, \frac{x-y}{\varepsilon})$$

where $\widehat{a}(x, X)$ is the partial Fourier transform w.r. to ξ of $a(x, \xi)$.

The main properties are the following ones which hold as $\varepsilon \to 0$:

- Composition:

$$\mathsf{Op}_\varepsilon(a) \circ \mathsf{Op}_\varepsilon(b) \approx \mathsf{Op}_\varepsilon(ab)$$

- Brackets:

$$[\mathsf{Op}_\varepsilon(a), \mathsf{Op}_\varepsilon(b)] \approx \frac{\varepsilon}{i}\mathsf{Op}_\varepsilon\{a, b\}$$

  where

$$\{a, b\} = \sum_{j=1}^{d} \left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} \right)$$

  is the Poisson bracket

Localization properties and Wigner measures:

ΨDO's are almost local; more precisely, if Support$(u) = D \subset X$, we have, on $X \setminus D$, $Pu = O(\varepsilon^\infty)$

More precise quantitative informations are given by Wigner measures.

Wigner measures define the localisation of energy in phase space. The Wigner measure W_u of u is the measure on the phase space defined by

$$\int a dW_u = < u | \mathrm{Op}_\varepsilon(a) u > \quad .$$

The projection of W_u on X is $\approx |u|^2 dx$.

If $P = \mathrm{Op}_\varepsilon(p)$, we have:

$$W_{Pu} \approx |p|^2 W_u$$

Main examples:

1. Hamiltonians:

- Riemannian Laplacian:

$$\mathsf{Op}_\varepsilon(g^{ij}(x)\xi_i\xi_j) = -\varepsilon^2\Delta_g$$

- Schrödinger operators:

$$\mathsf{Op}_\varepsilon(\|\xi\|^2 + V(x)) = -\varepsilon^2\Delta + V(x)$$

2. Noisy fields:

Let us denote by w the white noise on X:

$$\mathbb{E}(w(x)\overline{w(y)}) = \delta(x-y)$$

If $d=1$, w is the "speed" of the Brownian motion. Of course w is a very bad function (a distribution). But, if P is an Hilbert-Schmidt operator, Pw is a random L^2 field.

Examples

Ex 1: $d=1$,

$$f = \mathsf{Op}_1(p)w$$

were $p=p(\omega)$. $|p|^2(\omega)$ is called the power spectrum.

Ex 2: $f = \chi w$ were $\chi : X \to \mathbb{R}$ and $D = \text{support}(\chi)$. We have a random field localized in D.

Ex 3: putting together both examples, it is natural to take $f = \mathrm{Op}_\varepsilon(p)(w)$ with p small at infinity. We get the correlations

$$C_f(x,y) := \mathbb{E}(f(x)\overline{f(y)}) \approx \varepsilon^{-d}|\widehat{p}|^2(x, \frac{x-y}{\varepsilon})$$

We see that ε is the order of magnitude of the correlation distance! The averaged density of energy $\mathbb{E}(W_f)$ is the measure $|p|^2|dxd\xi|$. It is the phase space *power spectrum*.

(b) Ray dynamics

If $H(x,\xi)$ is the Hamiltonian function, the associated ray dynamics is defined by the vector field X_H defined by:

$$\begin{cases} \frac{dx_j}{dt} & = \frac{\partial H}{\partial \xi_j} \\ \frac{d\xi_j}{dt} & = -\frac{\partial H}{\partial x_j} \end{cases}$$

If $H = \frac{1}{2}\|\xi\|^2 + V$, we get Newton equations. If $H = \frac{1}{2} g^{ij}\xi^i\xi^j$, we get the geodesics.

We will denote by ϕ_t the flow of X_H:

$$\frac{d}{dt}(\phi_t(z)) = X_H(\phi_t(z))$$

If H is an Hermitian matrix, we consider the dispersion relation

$$\mathbb{D}(x,\xi,\omega) := \mathrm{det}(H(x,\xi) - \omega \mathrm{Id}) \ .$$

The local solutions of $\mathbb{D} = 0$, $\omega = h_j(x,\xi)$, give several dynamics with polarizations the eigenspaces

$$\mathrm{ker}(H - h_j \mathrm{Id}) \ .$$

Elastic waves:

$$u_{tt} = (\lambda + \mu)\mathrm{grad\ div} u + \mu \Delta u$$

$$\mathbb{D} = \mathrm{det}\left((\omega^2 - \mu \|\xi\|^2)\delta_{ij} - (\lambda + \mu)\xi_i \xi_j\right)$$

$$\mathbb{D} = (\omega^2 - (\lambda + 2\mu)\|\xi\|^2)(\omega^2 - \mu\|\xi\|^2)^2$$

The first factor corresponds to $P-$waves, while the second corresponds to $S-$waves.

(c) Green's function/propapagator

Let us assume that our wave dynamics, $U(t) = \exp(-t\widehat{H})$, is generated by $\widehat{H} = \frac{i}{\varepsilon}\mathrm{Op}_\varepsilon H$. What is the semi-classical behaviour of P?

For simplicity, we will assume that the dynamics is dispersive and we are outside caustic points.

$P(t,x,y)$ is a sum of contribution from rays going from y to x in time t. If such a ray is generic, the contribution is a WKB function.

Dispersion relation The dispersion relation $\mathbb{D}(\omega, x, \xi) = 0$ is a frequency dependent hypersurface in the phase space defined by the vanishing of the (determinant of) the symbol of the wave equation. Rays starting in $\mathbb{D}_\omega$ stays inside.

- In the case of Schrödinger equation, $\mathbb{D} = \{\omega = \|\xi\|^2 + V(x)\}$.
- In the case of wave equation: $\mathbb{D} = \{\omega^2 = \|\xi\|^2$

Dispersive waves

From the dispersion relation, we get the ray dynamics which is the Hamiltonian dynamics on $\mathbb{D}$. We say that the wave is dispersive if the speed $\|dx/dt\|$ depends on ω.

- Schrödinger equation is dispersive

- Wave equations are not

- Effective Hamiltonian of surface waves with horizontally stratified media are dispersive

Caustic points

In order to give a more effective definition of dispersivity using what is called caustics:

Let us start with an Hamiltonian $H(x, \xi)$. Let us consider a ray $(x_0(t), \xi_0(t))$, $0 \leq t \leq T$. We will say that $x_0(T)$ lies inside the caustic of $x_0(0)$ if the map $\xi \to x(T)$ with $(x(t), \xi(t) = \varphi_t(x_0(0), \xi)$ is not locally bijective.

For example, if $H = \frac{1}{2} g^{ij}(x) \xi_i \xi_j$, we recover the usual defintion of conjugate points.

Van Vleck formula

Non caustic points are usefull in deriving nice approximate formulae (WKB) for the propagator.

In quantum mechanics, they are called Van Vleck formulae:

$$P(T, A, B) \sim \sum_{\gamma \in R_{AB}^T} P_\gamma \tag{10}$$

with R_{AB}^T the set of rays from B to A in time T and

$$P_\gamma \sim a_\gamma(\varepsilon) e^{\frac{i}{\varepsilon} S(\gamma)}$$

where S is the Lagrangian action $S(\gamma) = \int_0^T (\xi dx + \omega dt)$.

Let us remark that as a function of A and B, S is a generating function of the flow at time T.

Formally, as it is well known, vV formulae can be derived from Feynman path integral, by applying stationnary phase:

$$(FPI)\ P(T,A,B)=\int_{\Omega^T_{AB}} e^{iS(\gamma)/h}d\gamma$$

where

- Ω^T_{AB} is the set of paths in the configuration space from B to A

- $d\gamma$ is a (mathematically ill defined) measure on Ω^T_{AB}

- $S(\gamma)=\int_0^T \mathcal{L}(\gamma(t),\dot{\gamma}(t))dt$ is the (Lagrangian) action integral.

Non caustic condition is equivalent to non degeneracy of the Hessian of S.

(d) Egorov Theorem

Let us consider first the scalar case with no attenuation ($\widehat{H}$ is self-adjoint, unitary dynamics) $U(t) = \exp(-it\widehat{H}/\varepsilon)$ with $\widehat{H} = \mathsf{Op}_\varepsilon(H)$.

Théorème 1 (Egorov, 70's) *If* $A = \mathsf{Op}_\varepsilon(a)$,

$$A_t := U(-t)AU(t) \approx \mathsf{Op}_\varepsilon(a \circ \phi_t)$$

where ϕ_t *is the Hamiltonian flow of* H.

Proof:

it is enough to look at the derivative, say at $t = 0$*:*

$$\frac{d}{dt}_{|t=0} A_t = \frac{i}{\varepsilon}[\widehat{H}, A]$$

and by the Ψ*DO calculus:*

$$\frac{d}{dt}_{|t=0} A_t \approx \mathsf{Op}_\varepsilon\{H, a\}$$

and remember

$$\{H, a\} = X_H a \ (= \frac{d}{dt}_{|t=0}(a \circ \phi_t)\ )\ .$$

4a. Application to PI

We will assume that the noisy field f is the image $\mathrm{Op}_\varepsilon(p)w$ where w is the white noise on $X \times \mathbb{R}$ and $p = p(x,\xi)$. The source correlation is then of the form $k(x,(x-y)/\varepsilon)\delta(s-t)$ where k is the partial Fourier transform of $|p|^2$ w.r. to ξ. Applying the previous tools, we see, that we are able to compute the leading terms in the behaviour of $C_{A,B}(\tau)$: assuming a scalar dynamics, we get

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B)$$

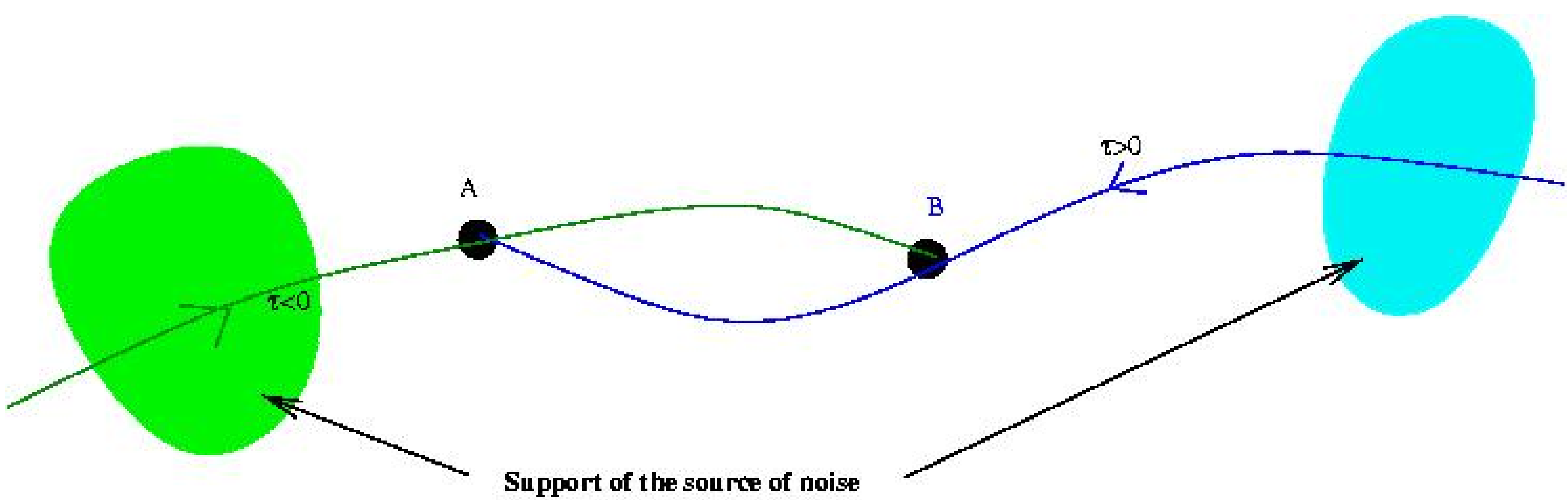
where Π is a ΨDO whose symbol can be explicitely computed as a (convergent) integral over the trajectories from B to A in time τ. More precisely, if such a trajectory γ satisfies $\gamma(0) = B,\ \gamma(\tau) = A$, it is an integral over $t \leq 0$. The non-vanishing corresponds to the fact that this negative part of γ crosses the support of the noise f.

If B and A are not conjugate along γ it gives a WKB formula for $C_{A,B}(\tau)$.

We cannot apply directly Egorov Theorem to $\Omega(s)\mathcal{K}\Omega^\star(s)$. We first split $\Omega(s) = U(s)A(s)$ where $U(s)$ is unitary and $A(s)$ is a contracting ΨDO (the attenuation) and $\Omega^\star(s) = A^\star(s)U(-s)$. We can apply Egorov Theorem to compute

$$\Pi = \int_0^\infty U(s)A(s)\mathcal{K}A^\star(s)U(-s)ds \ .$$

It implies that Π is a ΨDO whose symbol can be commputed.



A
B
τ<0
τ>0
Support of the source of noise

Time reversal symmetry

On the classical level, it corresponds to the fact that the dispersion relation is invariant by $(x, \xi) \to (x, -\xi)$. On the quantum level, it means that the symbol satisfies $H(x, -\xi) =^t H(x, \xi)(= H^\star(x, \xi))$. In that case, the correlation $C_{AB}(\tau)$ and $C_{AB}(-\tau)$ share the same oscillating part, but in general with different amplitudes.

4b. Application to TRM

We will assume that the operator $B = K\Omega$ is a ΨDO whose symbol is localized in phase-space (frequency cut-off + place of recording).

We need to evaluate the following integrals

1.
$$I = \int \psi(s) e^{isA} B e^{isA} ds$$
with B a compactly supported ΨDO

2.
$$II = \int \psi(s) e^{-isA} B e^{isA} ds$$

The first one is small: indeed we can rewrite it as

$$I = \int \psi(s)B(s)e^{2isA}ds$$

and integrating by part we get

$$I = \int \psi(s)B_k(s)A^{-N}e^{2isA}ds$$

which is of order ε^N.

The second one is more interesting, because we can apply Egorov Theorem and we get

$$II = \mathrm{Op}\left(\int \psi(s) b \circ \phi_s ds\right)$$

where ϕ_s is the geodesic flow.

It immplies that R (R is the operator which associate $u(2T)$ to $u(0)$) is a ΨDO whose symbol does not vanish if T is large enough. More precisely $T \geq T_0$ is enough, where T_0 is what is called the diameter of the Riemannian manifold (speed=1 !).

Using ergodicity:if we assume that the geodesic flow is ergodic, the averages

$$\frac{1}{T}\int_0^T b \circ \phi_s ds$$

will converge to the space average and it implies that the symbol of R will be independant of x: R is just a frequency cut-off.