Mathematical models for passive imaging II: Effective Hamiltonians associated to surface waves.

Yves Colin de Verdière *

September 28, 2006

Abstract

In the present paper which follows our previous paper "Mathematical models for passive imaging I: general background", we discuss the case of surface waves in a medium which is stratified near its boundary at some scale comparable to the wave length. We discuss how the propagation of such waves is governed by effective Hamiltonians on the boundary. The results are certainly not new, but we have been unable to find a precise reference. They are very close to results in adiabatic theory.

Introduction

This paper is strongly related to the first part [3]. We will be more specific and discuss the case of surface waves which are used in seismology in order to image the earth crust.

We consider a medium X with boundary ∂X and assume that the medium is stratified near ∂X. We will discuss how the linear propagation of waves located near the ∂X is determined by an effective Hamiltonian on ∂X. It is interesting enough to remark that this Hamiltonian is no more a differential operator, but only a "pseudo-differential operator" (a ΨDO) with a non-trivial dispersion relation (principal symbol). This situation is well known in physics as giving birth to some kind of wave guides.

A typical motivating situation is that of seismic surface waves propagating along the earth crust: it is well known that, at "macroscopic scales", the earth crust is horizontally stratified such giving birth to the so-called surface seismic waves which admit some more technical names like "Rayleigh" or "Love" waves

*Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); http://www-fourier.ujf-grenoble.fr/˜ ycolver/

(see [1]). They play a crucial role in the passive imaging of the earth crust. Moreover, they are the most dangerous in case of an earthquake.

From the general methods described in our paper [3], the field-field correlation allows to recover the dispersion relation of these surface waves from which we want to recover the transverse (vertical) structure of the crust. After describing a general result which is close in spirit to the adiabatic Theorem in Quantum Mechanics, we will discuss briefly what kind of inverse spectral problem we need to solve at the end and we solve it under a physically realistic monotonicity assumption.

1 The general setting

We will work locally in $X = \{(\mathbf{x}, z) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid z \leq 0\}$. We will consider the very simple case of an acoustic wave equation near the origine of X:

$$\begin{cases} u_{tt} - \operatorname{div}(n \ \operatorname{grad} u) = 0 \\ u(\mathbf{x}, 0) = 0 \end{cases} \tag{1}$$

with

$$n(\mathbf{x}, z) = N(\mathbf{x}, z, \frac{z}{\varepsilon})$$

and $N(\mathbf{x}, z, Z) : \mathbb{R}^{d-1} \times \mathbb{R}_- \times \mathbb{R}_- \to \mathbb{R}_+$ a non negative function which is independent of Z for $Z \leq Z_0 < 0$. There are 2 possible assumptions on the regularity of N:

- **Assumption 1** *N is smooth*
- **Assumption 2** *N is smooth as a function of $(\mathbf{x}, z)$ with values into $L^\infty(\mathbb{R}_-)$ and $N(\mathbf{x}, z, Z) = N(\mathbf{x}, 0, Z) + O(z^\infty)$*

In other words, the medium X admits a quite irregular behaviour in the vertical direction in a very small horizontal band $B_\varepsilon \subset X$ of width εZ_0. We plan to see that Equation (1) admits, as $\varepsilon \to 0$, asymptotic solutions of frequency of order ε^{-1} located in B_ε. Moreover these solutions are determined by solving an effective pseudo-differential equation on the boundary $\partial X = \mathbb{R}^{d-1} \times \{0\}$.

Assumption 3 *We will assume that*

$$0 < \inf_{Z \leq 0} N(\mathbf{x}, O, Z) < N(\mathbf{x}, 0, -\infty) \ .$$

Physically, it means that the propagation speed at some points very close to the boundary is smaller than inside the medium. This kind of assumption is usually satisfied in seismology where the speed of elastic waves into surface sediments layers is smaller than the speed inside the rocks below the sediments.

2 The main result

2.1 A Sturm-Liouville operator

Let us consider, for each $(\mathbf{x},\xi)\in T^\star\partial X$, the self-adjoint differential operator $L_{\mathbf{x},\xi}$ on the half line $Z\leq 0$, with Dirichlet boundary condition at $Z=0$, defined by:

$$L_{\mathbf{x},\xi}v := -\frac{d}{dZ}(N(\mathbf{x},0,Z)\frac{dv}{dZ}) + N(\mathbf{x},0,Z)|\xi|^2 v \tag{2}$$

In case of Assumption 2, $L_{\mathbf{x},\xi}$ is defined in terms of quadratic forms.

The spectrum of $L_{\mathbf{x},\xi}$ consists of a finite discrete spectrum and a continuous spectrum $[N(\mathbf{x},0,Z_0)|\xi|^2,+\infty[$. Under **Assumption 3**, $L_{\mathbf{x},\xi}$ admits, for ξ large enough, a non empty discrete spectrum of simple eigenvalues

$$\inf_Z N(\mathbf{x},0,Z)|\xi|^2 < \lambda_1(\mathbf{x},\xi) < \cdots < \lambda_j(\mathbf{x},\xi) < \cdots < \lambda_k(\mathbf{x},\xi) < N(\mathbf{x},0,Z_0)|\xi|^2 \ ,$$

which depend smoothly of $(\mathbf{x},\xi)$. In order to see that, we can interpret $|\xi|^{-2}L_{\mathbf{x},\xi}$ as a semi-classical Schrödinger type operator with an effective Planck constant $|\xi|^{-1}$ and a principal symbol

$$p_{\mathbf{x}}(Z,\zeta) = N(\mathbf{x},0,Z)(\zeta^2+1)$$

which admits a well near $(Z,\zeta)=(0,0)$. We should however take care of the fact that the number k depends on $(\mathbf{x},\xi)$ and goes to ∞ as ξ does.

It leads to an interesting bifurcation problem which we will discuss in Section 4.

2.2 WKB solutions of the stationnary wave equation

The goal of this section is to build the effective surface Hamiltonians which describe the surface waves. Let us start with the:

Lemma 1 *Let us consider the operator $\hat{H}$ defined by:*

$$\hat{H}u := -\varepsilon^2 \mathrm{div}(n\ \mathrm{grad}u) \tag{3}$$

acting on functions on X vanishing at $z=0$ (Dirichlet boundary conditions).

Let us choose $\lambda(\mathbf{x},\xi)$ an eigenvalue of $L_{\mathbf{x},\xi}$ depending smoothly of $(\mathbf{x},\xi)\in U$, where U is a bounded open set of $T^\star\partial X$, and $\varphi(\mathbf{x},\xi,.)$ a normalized associated eigenfunction. There exists:

- *An asymptotic expansion*

$$\Phi_\varepsilon = \sum_{m=0}^{\infty} \varphi_m(\mathbf{x},\xi,\frac{z}{\varepsilon})\varepsilon^m$$

with $\varphi_0=\varphi$ and the $\varphi_m(\mathbf{x},\xi,.)$'s are smoothly dependent of $(\mathbf{x},\xi)$ with values in the domain of $L_{\mathbf{x},\xi}$. The φ_m 's $(m\geq 1)$ are unique if they are assumed to be orthogonal to φ.

- *A symbol*
$$a_\varepsilon(\mathbf{x},\xi)=\sum_{m=0}^{\infty}a_m(\mathbf{x},\xi)\varepsilon^m$$
with $a_0=\lambda$

such that we have the following identity of formal power series in ε*:*

$$\hat{H}\left(\Phi_\varepsilon(\mathbf{x},\xi,\frac{z}{\varepsilon})e^{i\langle\mathbf{x}|\xi\rangle/\varepsilon}\right)=a_\varepsilon(\mathbf{x},\xi)\Phi_\varepsilon(\mathbf{x},\xi,\frac{z}{\varepsilon})e^{i\langle\mathbf{x}|\xi\rangle/\varepsilon}\ . \tag{4}$$

Proof.–

Expanding N by Taylor formula, we get: $N(\mathbf{x},\varepsilon Z,Z)=\sum_{l=0}^{\infty}N_l(\mathbf{x},Z)\varepsilon^l$ with

$$N_l(\mathbf{x},Z)=\frac{1}{l!}\frac{\partial^l N}{\partial z^l}(\mathbf{x},O,Z)\ .$$

Under Assumption 1, for $l\geq 1$, the N_l's are smooth and compactly supported in Z. Under Assumption 2, the N_l's vanish for $l\geq 1$.

We see that we are reduced to compute

$$\Psi:=\hat{H}(e^{i\langle\mathbf{x}|\xi\rangle/\varepsilon}\varphi(\mathbf{x},\frac{z}{\varepsilon}))\ ,$$

with $n(\mathbf{x},z)=N(\mathbf{x},z/\varepsilon)$. We find

$$\Psi=e^{i\langle\mathbf{x}|\xi\rangle/\varepsilon}\left(L_0\varphi+\varepsilon L_1\varphi+\varepsilon^2L_2\varphi\right)$$

with $L_0=L_{\mathbf{x},\xi}$,

$$L_1\varphi=-2in\langle\xi|\nabla_{\mathbf{x}}\varphi\rangle-i\langle\xi|\nabla_{\mathbf{x}}n\rangle\varphi$$

$$L_2\varphi=-\langle\nabla_{\mathbf{x}}n|\nabla_{\mathbf{x}}\varphi\rangle>-n\Delta_{\mathbf{x}}\varphi\ .$$

We start with $\varphi_0=\varphi$ an eigenfunction of $L_{\mathbf{x},\xi}$ with eigenvalue λ. By induction, we pick all terms in ε^m and get from Equation (4):

$$L_{\mathbf{x},\xi}\varphi_m=\lambda\varphi_m+a_m\varphi_0+R(\varphi_0,\cdots,\varphi_{m-1})\ , \tag{5}$$

with

$$R(\varphi_0,\cdots,\varphi_{m-1})=\sum_{l=1}^{m-1}a_{m-l}\varphi_l-\sum_{l<m,\ k+l+j=m}L_j^k\varphi_l$$

where L_j^k is given as L_j with N replaced by N_k.

We first choose the unique a_m so that

$$a_m\varphi_0+R(\varphi_0,\cdots,\varphi_{m-1})$$

is orthogonal to φ_0, then we can solve Equation (5). In order to show that φ_m lies in the domain of $L_{\mathbf{x},\xi}$, it is enough to show that $R(\varphi_0,\cdots,\varphi_{m-1})$ is in $L^2(Z)$.

□

2.3 A short review on micro-functions

In order to help the reader, we give here a very short review on micro-functions and ΨDO's. Good references for more details could be [7] or [6].

Let us first give the main definitions:

Definition 1 *A family of functions (distributions) $f_\varepsilon : X \to \mathbb{C}$ is said to be* admissible *if for any function $\chi \in C_o^\infty(X)$, there exists some real number s so that the Sobolev norms $\|\chi f_\varepsilon\|_s$ are at most of polynomial growth w.r. to ε^{-1}. We will denote $\mathcal{A}(X)$ the vector space of such families.*

Definition 2 *The* frequency set *or* microsupport*, denoted $WF(f_\varepsilon)$, of an admissible family f_ε is the closed subset of $T^\star X$ which is defined as follows in canonical local coordinates (x,ξ):*

$$(x_0,\xi_0) \notin WF(f_\varepsilon) \ if \ and \ only \ if \ \exists \chi \in C_o^\infty(X), \ \chi(x_0) \neq 0, \\ \mathcal{F}_\varepsilon(\chi f_\varepsilon)(\xi) = O(\varepsilon^\infty) \ for \ \xi \ close \ to \ \xi_0 \ .$$

Here $\mathcal{F}_\varepsilon u$ is the $\varepsilon-$Fourier transform:

$$\mathcal{F}_\varepsilon u(\xi) = (2\pi\varepsilon)^{-d/2} \int e^{-i\langle x|\xi\rangle/\varepsilon} u(x)|dx| \ .$$

Definition 3 *If X is a smooth manifold and U is an open set in $T^\star X$, $\mathcal{M}(U)$ the space of* microfunctions in U, *is the quotient*

$$\mathcal{M}(U) := \mathcal{A}(X)/\{f_\varepsilon | WF(f_\varepsilon) \cap U = \emptyset\} \ .$$

We will denote $f_\varepsilon = O(\varepsilon^\infty)$ in U the property

$$WF(f_\varepsilon) \cap U = \emptyset \ .$$

Definition 4 *If $a(x,\xi,\varepsilon)$ is a suitable function, called a* symbol, *the* pseudo-differential operator *(ΨDO) $A = \mathrm{Op}_\varepsilon(a)$ is defined by:*

$$Au(x) := (2\pi\varepsilon)^{-d} \int e^{i\langle x-y|\xi\rangle/\varepsilon} a(x,\xi,\varepsilon) u(y) |dy d\xi| \ .$$

The main result is that ΨDO's act on microfunctions:

Theorem 1 *If $a(x,\xi,\varepsilon)$ belongs to a suitable class of symbols (for example, all derivatives of a are uniformly bounded independently of ε), then*

$$WF\left(\mathrm{Op}_\varepsilon(a)(f_\varepsilon)\right) \subset WF(f_\varepsilon) \ .$$

Moreover the action of $\mathrm{Op}_\varepsilon(a)$ on $\mathcal{M}(U)$ depends only on the values of a in some neighbourhood of $\overline{U}$.

2.4 Reformulation of Lemma 1 in terms of micro-functions

From Lemma 1, we get the

Theorem 2 *Let us fix* $(\mathbf{x}_0, \xi_0) \in T^\star \partial X$ *and assume that we have a smooth eigenvalue* $\lambda(\mathbf{x}, \xi)$ *of* $L_{\mathbf{x},\xi}$ *defined in some bounded neighbourhood* U' *of* $(\mathbf{x}_0, \xi_0)$. *Let us give* U, *an open neighbourhood of* $(\mathbf{x}_0, \xi_0)$, *so that* $\overline{U} \subset U'$ *and* $\chi \in C_o^\infty(U')$ *which is* $\equiv 1$ *on* $\overline{U}$.

The map:

$$J_\varepsilon \left(e^{i\langle . | \xi \rangle / \varepsilon} \right)(\mathbf{x}, z) := \frac{1}{\sqrt{\varepsilon}} \chi(\mathbf{x}, \xi) e^{i\langle \mathbf{x} | \xi \rangle / \varepsilon} \Phi_\varepsilon \left(\mathbf{x}, \xi, \frac{z}{\varepsilon} \right)$$

defines, by passing to the quotient, a linear map $\overline{J_\varepsilon} : \mathcal{M}(U) \to \mathcal{M}(V)$, *where*

$$V := \{ (\mathbf{x}, z; \xi, \zeta) \in T^\star X \mid (\mathbf{x}; \xi) \in U \} ,$$

which satisfies

- $\overline{J_\varepsilon}$ *is microlocally unitary*
- $\overline{J_\varepsilon}$ *is independent of the choice of* χ
- $WF\left(\overline{J_\varepsilon}(u)\right) \cap V \subset \{z = 0\}$
- $\hat{H}\overline{J_\varepsilon} = \overline{J_\varepsilon}\hat{H}_{\text{eff}}$, *where* $\hat{H}_{\text{eff}}$ *is a* ΨDO *of full left symbol* $a_\varepsilon(\mathbf{x}, \xi)$ *and of principal symbol* $\lambda(\mathbf{x}, \xi)$

Proof.–

The map J_ε is well defined because $\Phi_\varepsilon(\mathbf{x}, \xi, Z)$ is well defined for $(\mathbf{x}, \xi) \in U'$. Let us explain first how $\overline{J_\varepsilon}$ is defined. We first can define A_ε by

$$A_\varepsilon = \mathrm{Op}_\varepsilon \left(\chi(\mathbf{x}, \xi) \Phi_\varepsilon(\mathbf{x}, \xi, .) \right) ,$$

as a ΨDO whose symbol takes values in $L^2(\mathbb{R}^-)$. As such, by Theorem 1, it is well defined from $\mathcal{M}(U)$ to $\mathcal{M}(U; L^2(\mathbb{R}^-))$. We now use the unitary map $E_\varepsilon : \mathcal{M}(U; L^2(\mathbb{R}^-)) \to \mathcal{M}(V)$ which is defined by $E_\varepsilon : f(x, .) \to \varepsilon^{-\frac{1}{2}} f(\mathbf{x}, z/\varepsilon)$ and define $J_\varepsilon = D_\varepsilon \circ A_\varepsilon$.

The rapid decay of $\Phi_\varepsilon(\mathbf{x}, \xi, Z)$ w.r. to Z implies the property $WF\left(\overline{J_\varepsilon}(u)\right) \cap V \subset \{z = 0\}$.

The unitarity comes from the symbolic calculus: $A_\varepsilon^\star \circ D_\varepsilon^\star \circ D_\varepsilon \circ A_\varepsilon$ is a scalar ΨDO of principal symbol $|\chi(\mathbf{x}, \xi)|^2 \int_{-\infty}^0 \varepsilon^{-1} |\phi(\mathbf{x}, \xi, \frac{z}{\varepsilon})|^2 dz$ which is $\equiv 1$ on U.

The interlacing property is easily checked on exponentials and hence on microfunctions as a consequence of Lemma 1.

□

2.5 Speeds of propagation

To any dispersion relation $K(\mathbf{x},\xi)=\omega^2$ is associated a speed of propagation (the so-called "group-speed") defined by

$$\mathbf{v} := \frac{d\mathbf{x}}{dt} = \partial_\xi K/2\omega \ .$$

For the acoustic equation, we have $n(\mathbf{x},z)|\xi|^2=\omega^2$ and hence $\mathbf{v}=n\xi/\omega$ and $v:=|\mathbf{v}|=\sqrt{n}$. We see the known result that the speed is independent of the frequency.

Let us compute the speed for $\lambda(\mathbf{x},\xi)=\omega^2$: we have

$$\partial_\xi \lambda(x,\xi) = 2\left(\int_{-\infty}^0 N(\mathbf{x},0,Z)\varphi(Z)^2 dZ\right)\xi$$

and hence

$$v = \frac{|\xi|}{\omega}\left(\int_{-\infty}^0 N(\mathbf{x},0,Z)\varphi(Z)^2 dZ\right)$$

and using

$$\int_{-\infty}^0 N(\mathbf{x},0,Z)|\xi|^2\varphi(Z)^2 dZ < \lambda = \omega^2$$

we get

$$v < \frac{\omega}{|\xi|} < \sqrt{n(\mathbf{x},0,Z_0)} \ .$$

The speed of the surface waves is less than the speed of the body waves.

3 An inverse spectral problem

Recovering the vertical structure of earth crust from the dispersion relation of surface waves needs to solve the following inverse problem:

Let us assume that we know the discrete spectrum of $L_{\mathbf{x},\xi}$ at fixed $\mathbf{x}$ for all ξ's, can we recover the vertical profile $N(\mathbf{x},0,Z)$?

Let us show the

Theorem 3 *If $N(\mathbf{x},0,Z)$ is a* decreasing function *of Z from $N(\mathbf{x},0,Z_0)$ to $N(\mathbf{x},0,0)$, the asymptotics of the discrete spectra $\lambda_j(\mathbf{x},\xi)$, $1\le j\le k(\mathbf{x},\xi)$ as $\xi\to\infty$ determine the functions $N(\mathbf{x},0,Z)$.*

Proof.–

This is a consequence of Weyl's asymptotics: for $E<N(\mathbf{x},0,Z_0)$, let us introduce

$$N(\mathbf{x},\xi,E) = \#\{\lambda_j(\mathbf{x},\xi)\le E|\xi|^2\} \ .$$

We have the following (Weyl type) asymptotics as $\xi \to \infty$:

$$N(\mathbf{x},\xi,E) \sim \left(\frac{|\xi|}{2\pi}\right)^{\frac{1}{2}} \text{Area}\left(\{N(\mathbf{x},0,Z)(1+\zeta^2) \leq E\}\right) .$$

It implies that, from the asymptotics of $\lambda_j(\mathbf{x},\xi)$, we can recover

$$V(E) = \int_{z(E)}^{0} \sqrt{\frac{E}{N(Z)} - 1}\, dZ \ ,$$

with $N(z(E)) = E$. We can rewrite:

$$V(E) = \int_{N_0}^{E} \sqrt{E-u}\, \frac{z'(u)\, du}{\sqrt{u}}$$

and introducing

$$Kf(E) := \int_{N_0}^{E} \sqrt{E-u}\ f(u)\, du$$

we get

$$\frac{d^3}{dE^3}((K \circ K)f)(E) = 2f(E)$$

which allows to recover $z'(E)/\sqrt{E}$ as the third order derivative of $K(V)(E)$.

□

It is a quite interesting problem to try to extend the previous Theorem without the assumption that N is decreasing. In this case Weyl formula is no more enough, but more refined trace formulae like Gutzwiller trace formula may be usefull to give a general answer.

4 Bifurcations

What happens when the eigenvalue $\lambda_k(\mathbf{x},\xi)$ goes to the bottom of the continuous spectrum. It is an interesting place when there should be a mode conversion from body waves into surface waves or conversely. It could be interesting from the point of view of the study of earthquakes whose source is deep: how do they give birth to surface waves? It could also be interesting in order to study resonances created by surface waves into a bassin. We plan to study these (difficult?) questions in a future work.

References

[1] K. Aki & P. Richards, Quantitative Seismology I & II, *W.H. Freeman and Co,* New-York, 1980.

[2] Y. Colin de Verdière, Méthodes semi-classiques et Théorie spectrale. *Lecture Notes in Preparation [http://www-fourier.ujf-grenoble.fr/~ycolver/]*

[3] Y. Colin de Verdière, Mathematical models for passive imaging I: General Background. *In Preparation.*

[4] Y. Colin de Verdière, Mathematical models for passive imaging III: Noise located on the boundary. *In Preparation.*

[5] J. Duistermaat, Fourier Integral Operators. *Birkhäuser*, Boston, 1996.

[6] V. Guillemin & S. Sternberg, Geometric Asymptotics. *Amer. Math. Soc.,* 1977.

[7] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis. *Springer,* 2001.

[8] F. Trèves, Introduction to pseudodifferential and Fourier integral operators. *Plenum Press,* New York, 1980.