White noises, random fields and pseudo-differential operators

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Abstract

Our aim in this note is to build quite general random fields with correlation distances given by a small parameter ε. It seems to be natural for that purpose to use $\varepsilon-$pseudo-differential operators. We will see how to compute the generalized *power spectrum* using Wigner measures. Using the previous tools, we will discuss natural statistics of waves, which we call "microcanonical".

1 White noises

Let $(\mathcal{H}, \langle .|. \rangle)$ be an Hilbert space. There exists a canonical *Gaussian random field* on it called the *white noise* and denoted by $w_{\mathcal{H}}$ (or simply w if there is no possible confusion). This random field is defined by the properties that:

- For all $\vec{e} \in \mathcal{H}$,
$$\mathbb{E}(\langle w|\vec{e}\rangle) = 0$$
- For all $\vec{e}, \vec{f} \in \mathcal{H}$,
$$\mathbb{E}(\langle w|\vec{e}\rangle \overline{\langle w|\vec{f}\rangle}) = \langle \vec{e}|\vec{f}\rangle$$

More concretely, if $(\vec{e_i})$ is an orthonormal basis of $\mathcal{H}$ and $w = \sum w_i \vec{e_i}$, we have $\mathbb{E}(w_i \overline{w_j}) = \delta_{ij}$ and hence $\mathbb{E}(\langle Aw|w\rangle) = \text{Trace}(A)$.

Unfortunately, w is not a random vector in $\mathcal{H}$ unless $\dim \mathcal{H} < \infty$, but only a random *Schwartz distribution.* If w were a vector in $\mathcal{H}$, we would have $w = \sum w_i \vec{e_i}$ and we see that
$$\mathbb{E}(\|w\|^2) = \sum \mathbb{E}(|w_i|^2) = \dim \mathcal{H} = \infty \ .$$

We have nevertheless the following usefull proposition:

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Proposition 1 *If A is an Hilbert-Schmidt operator on $\mathcal{H}$, the random field Aw is almost surely in $\mathcal{H}$.*

Proof.–

> $\mathbb{E}(\langle Aw|Aw\rangle) = \mathbb{E}(\langle A^\star Aw|w\rangle) = \mathrm{Trace}(A^\star A)$ which is finite, by definition, exactly for Hilbert-Schmidt operators.

□

2 Examples

Example 2.1 Stationnary noise on the real line: *let us take a random field on the real line which is given by the convolution product $f = F \star w$ with F smooth and compactly supported. Then f is stationnary: it means that the correlation kernel $K(t,t') = \mathbb{E}(f(t)\overline{f(t')})$ is a fonction of $t-t'$. On the level of Fourier transforms $\hat{f} = \hat{F}\hat{w}$, $\mathbb{E}(\hat{f}(\omega)\overline{\hat{f}(\omega')}) = |\hat{F}|^2(\omega)\delta(\omega-\omega')$ and the positive function $|\hat{F}|^2(\omega)$ is usually called the* power spectrum *of the stationnary noise.*

Example 2.2 *If X is a d-dimensional bounded domain. Let us denote the Sobolev spaces on X by $H^s(X)$. If $P : L^2(X) \to H^s(X)$ with $s > d/2$, Pw is in $L^2(X)$.*

Example 2.3 Brownian motions: *if $X = \mathbb{R}$, w is the derivative of the Brownian motion: if $b(t) = \int_0^t w(s)ds$, $b : [0,+\infty[\to \mathbb{R}$ is the Brownian motion which is in $L^2([0,T])$ for all finite T.*

Example 2.4 *If X is a smooth compact manifold or domain and P is smoothing, meaning that P is given by an integral smooth kernel*

$$Pf(x) = \int_X [P](x,y)f(y)|dy| \ ,$$

$F = Pw$ is a random smooth function. Its correlation kernel

$$C(x,y) := \mathbb{E}(F(x)\overline{F(y)})$$

is given by:

$$[PP^\star](x,y) = \int_X [P](x,z)\overline{[P](y,z)}|dz| \ .$$

Example 2.5 Random vector fields: *let us consider $\mathcal{H} = L^2(X, \mathbb{R}^N)$. For example, in the case of elasticity, X is a 3D domain and $N = 3$. The field here are just fields of infinitesimal deformations (a vector field).*

3 Modelling the noise using pseudo-differential operators

The main goal of the present section is to build natural random fields which are non homogeneous with small distances of correlation of the order of $\varepsilon \to 0$. The noise is non homogeneous in X, but could also be non isotropic w.r. to directions.

3.1 Pseudo-differential operators

Let us recall that an ε-pseudo-differential operator P (a ΨDO) on Z, a d-dimensional manifold, is given locally by

$$[P](z, z') = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} e^{i\langle z - z' | \zeta \rangle / \varepsilon} p(z, \zeta) |d\zeta|$$

where $p(z, \zeta)$, the so-called *symbol* of P, is smooth, compactly supported in z and of Schwartz class in ζ, and ε a small parameter. We are only interested into the asymptotic behaviours as $\varepsilon \to 0$. We will denote

$$P = \mathrm{Op}_\varepsilon(p) \ .$$

The kernel of P is then given by:

$$[P](z, z') = \varepsilon^{-d} \tilde{p} \left(z, \frac{z - z'}{\varepsilon} \right)$$

with $\tilde{p}$ the partial Fourier transform of $p(z, \zeta)$ w.r. to ζ. Very often, one is only able to compute the symbol mod $O(\varepsilon)$ which is called the *principal symbol* of P.

The most basic fact about ΨDO's is the fact they can be composed: if $P = \mathrm{Op}(p)$ and $Q = \mathrm{Op}(q)$, we have $PQ = \mathrm{Op}\left(pq + O(\varepsilon)\right)$.

3.2 Noises from pseudo-differential operators

It is therefore natural to take for noise on a manifold Z the image of an homogeneous white noise by a pseudo-differential operator N of smooth compactly supported symbol $n(z, \zeta)$. The correlation $C(z, z')$ will then be given as the Schwartz kernel of $NN^\star$ which is a ΨDO of principal symbol $|n|^2$. We have

$$C(z, z') = \varepsilon^{-d} \widetilde{|n|^2} \left(z, \frac{z - z'}{\varepsilon} \right) \ .$$

This construction gives smooth random fields which can be localized in some very small domains of the manifold Z, which are non isotropic and which have small distance of correlations. Moreover it will allow to use technics of microlocal analysis with the small parameter given by ε.

4 Power spectrum and Wigner measures

4.1 Wigner measures

If f_ε is a suitable family of functions, we define the *Wigner measures* W_{f_ϵ} which are signed measures on the phase space $T^\star Z$ defined by

$$\int a dW_{f_\epsilon} := \langle \mathrm{Op}_\varepsilon(a) f_\varepsilon | f_\varepsilon \rangle \ .$$

The measures dW_{f_ε} are the phase space density of energy of the functions f_ε. Wigner measures are not always ≥ 0, but they are ≥ 0 in the limit of small ε. It allows to replace our quantization by another one denoted Op^+ which satisfies the basic laws of ΨDO calculus, but also $\mathrm{Op}^+(a) \geq 0$ is $a \geq 0$. For more on Wigner measures, see the nice book [5].

4.2 Power spectrum

Let us give a:

Definition 1 *The* power spectrum *of a random field f_ε is the measure $\langle dW_{f_\varepsilon} \rangle$ on the phase space defined as the averaged Wigner measure:*

$$\langle dW_{f_\varepsilon} \rangle := \mathbb{E}(dW_{f_\varepsilon}) \ .$$

We can now compute the power spectra of our random fields $\mathrm{Op}_\varepsilon(n) w$ as follows:

Proposition 2 *If $f_\varepsilon = \mathrm{Op}_\varepsilon(n) w$, the power spectrum of f_ε, $\langle dW_{f_\varepsilon} \rangle$, converges as $\varepsilon \to 0$ to the measure $(2\pi\varepsilon)^{-d} |n|^2(x,\xi) |dx d\xi|$.*

Proof.–

Let us put $N = \mathrm{Op}_\varepsilon(n)$, we have

$$\langle \mathrm{Op}_\varepsilon(a) N w | N w \rangle = \langle N^\star \mathrm{Op}_\varepsilon(a) N w | w \rangle$$

and $\mathbb{E}(\langle A w | w \rangle) = \mathrm{trace}(A)$. We get

$$\mathbb{E}(\int a dW_{f_\varepsilon}) = \mathrm{trace}(N^\star \mathrm{Op}_\varepsilon(a) N)$$

which can be evaluated using the ΨDO calculus as

$$\mathbb{E}(\int a dW_{f_\varepsilon}) \sim (2\pi\varepsilon)^{-d} \int a |n|^2 dx d\xi \ .$$

□

4.3 Space-time noises

If $Z = X \times \mathbb{R}$ is the space-time, we will take our noise as before $f = Nw$; we will assume the noise *homogeneous in time,* the symbol n of N is assumed to be given by $n(\omega; x, \xi)$.

In this case, the correlation is given by:

$$K(u; x, y) = [NN^\star](0, u; x, y) \tag{1}$$

which is the Schwartz kernel of a ΨDO of principal symbol $|n|^2(\omega; x, \xi)$.

4.4 The matrix case

If f_ε is a family of vector valued functions $f_\varepsilon : X \to \mathbb{C}^N$, the Wigner measures are as follows:

Definition 2

$$\int a dW_{f_\varepsilon} := \langle \mathrm{Op}(a) f_\varepsilon | f_\varepsilon \rangle$$

where $a : T^\star X \to \mathrm{Herm}(\mathbb{C}^N)$*.*

The power spectrum of the white noise associated to $L^2(X, \mathbb{C}^N)$ is

$$\int a \langle dW_w \rangle = (2\pi\varepsilon)^{-d} \int \mathrm{Trace}(a) |dx d\xi| \ .$$

5 Microcanonical white noise

5.1 General result

Let us give now a self-adjoint operator $\hat{H}$ on some Hilbert space $L^2(X, \mathbb{C}^N)$. For example, the Lamé operator of elastic waves is acting on $L^2(X, \mathbb{R}^3)$ where X is 3D manifold. Let us choose some intervall $I \subset \mathbb{R}$ and the Hilbert space $\mathcal{H}_I$ which is the image of $L^2(X, \mathbb{C}^N)$ by the spectral projector P_I of $\hat{H}$. In case of a discrete spectrum, $\mathcal{H}_I$ admits an orthonormal basis of eigenmodes with eigenvalues in I. The associated white noise w_I is called the *microcanical white noise.*

In the semi-classical limit, its power spectrum is given as follows:

Theorem 1 *Let* $H : T^\star X \to \mathrm{Herm}(\mathbb{C}^N)$ *be the classical limit (principal symbol) of* $\hat{H}$*, then the power spectrum* $\langle dW_I \rangle$ *admits the following asymptotic behaviour:*

$$\int A \langle dW_I \rangle \sim (2\pi\varepsilon)^{-d} \int \mathrm{Trace}(\chi_I(H) \circ A)(x, \xi) |dx d\xi| \ .$$

If $A = a\mathrm{Id}$*, we get*

$$\int A \langle dW_I \rangle \sim (2\pi\varepsilon)^{-d} \int \mathrm{Trace}(\chi_I(H)) a(x, \xi) |dx d\xi| \ .$$

Let us recall that, if M is an Hermitian matrix, $\chi(M)$ is the Hermitian matrix defined as $\chi(M) := \sum_j \chi(\mu_j)\Pi_j$ where the μ_j's are the eigenvalues of M and Π_j the associated spectral projectors.

5.2 The power spectrum and the density of states

Definition 3 *The density of states $dE_I(x)$ is defined by $dE_I(x) := [P_I](x,x)|dx|$ where P_I is the spectral projector associated to the intervall I.*

Example 5.1 *If ϕ_j is an eigenmode orthonormal basis associated with eigenvalues λ_j, we have:*

$$dE_I(x) = (\sum_{\lambda_j \in I} |\varphi_j(x)|^2)|dx| \ .$$

Example 5.2 *If we start with the Laplace operator in $\mathbb{R}^d$, we have*

$$dE_I(x) = (2\pi)^{-d} \left(\int_{k^2 \in I} dk \right) |dx| \ ,$$

hence

$$dE_I(x) = \frac{b_d}{(2\pi)^d}(E_+^{d/2} - E_-^{d/2})|dx| \ .$$

Example 5.3 *If we have continuous spectrum described by scattered plane waves $e_k(x) = e^{ikx} + s_k(x)$, then*

$$dE_I(x) = (2\pi)^{-d} \left(\int_{k^2 \in I} |e_k(x)|^2 dk \right) |dx| \ .$$

Theorem 2 *The density of states is the projection of the power spectrum on the configuration space:*

$$\int_{T^\star X} a(x)\langle dW_I \rangle = \int_X \mathrm{trace}\,(a(x) \circ dE_I) \ .$$

It is well defined and not only defined in the semi-classical limit.

5.3 The case of body elastic waves

Recall the following simplified form of Lamé's equation:

$$u_{tt} + \hat{H}u = 0, \ \ -\hat{H}u = (\lambda + \mu)\mathrm{div}\ \mathrm{grad}u + \mu\Delta u \ .$$

The principal symbol of $\hat{H}$ is

$$H = (H_{ij}) = (\mu\|\xi\|^2\delta_{ij} + (\lambda + \mu)\xi_i\xi_j)$$

whose eigenvalues are $\lambda_P = (2\mu + \lambda)\|\xi\|^2$ with a 1d eigenspace E_P and eigenprojector π_P, and $\lambda_S = \mu\|\xi\|^2$ with a 2d eigenspace $E_S$and eigenprojector π_S. The corresponding speeds are $v_P = \sqrt{2\mu + \lambda}$ and $v_S = \sqrt{\mu}$.

The corresponding power spectrum is

$$\int A\langle dW_I\rangle = (2\pi\varepsilon)^{-3}\left(\int_{\lambda_P\in I}\text{Trace}(A_P)|dxd\xi| + 2\int_{\lambda_S\in I}\text{Trace}(A_S)|dxd\xi|\right)$$

where $A_P = \pi_P A \pi_P$ and $A_S = \pi_S A \pi_S$.

It leads to a natural equipartition of energy between $P-$ and $S-$waves given by ratio of the 2 terms in the previous formula:

$$\frac{\mathcal{E}_P}{\mathcal{E}_S} = \frac{\mu^{3/2}}{2(\lambda + 2\mu)^{3/2}} \ .$$

5.4 The problem of surface waves: wave guides

Let us consider now the case of scalar waves in a stratified medium as in [3]. More precisely, we consider on $X = \mathbb{R}^2_x \times \mathbb{R}^-_z$ the differential operator:

$$\hat{H} = -\text{div}N(z)\text{grad} = -N(z)\Delta_x - \partial_z(N(z)\partial z)$$

with $N > 0$, equal to 1 for $z << 0$ and Dirichlet boundary conditions. We have

$$\hat{H}u(x,z) = (2\pi)^{-2}\int e^{i\langle\xi|x-y\rangle}L^\xi u(y,z)dyd\xi$$

with $L^\xi := N(z)|\xi|^2 - \partial_z(N(z)\partial_z)$. The spectral theory of a Sturm-Liouville operator like L^ξ is reviewed in the Appendix.

Let us try to compute the density of states of the micro-canonical white noise associated to I.

We get:

$$dE_I = (2\pi)^{-2}\left(\int[\Pi^\xi_I](z,z)d\xi\right)|dxdz| \ .$$

where Π^ξ_I is the spectral projector of L^ξ. $[\Pi^\xi_I]$ splits into the (finite) discrete part and the continuous part. We get

$$[\Pi^\xi_I](z,z) := \sum_{\lambda_j(x,\xi)\in I}|\phi_j|(z)^2 + d^\xi_I(z) \ ,$$

with

$$d^\xi_I(z) = \frac{1}{4\pi}\int_{k^2+|\xi|^2\in I}|e^\xi_k(z)|^2 dk$$

with $e^\xi_k(z) \sim e^{ikz} + r(k)e^{-ikz}$ as $z \to -\infty$ is the "scattered" plane wave. For $z << 0$, we recover the usual body waves while near $z = 0$, both terms contributes in a **non universal** way.

5.5 The case of surface waves: toy model of Rayleigh waves

Let us take on $\mathbb{R}^2_x \times \mathbb{R}^-_z$, $H = -\Delta$ with boundary conditions $\frac{\partial u}{\partial z} - cu = 0$ on ∂X. We will assume that $c > 0$. We can perform the same computation as before in a more explicit way. Our operator L^ξ is now $-d^2/dz^2 + |\xi|^2$ with boundary condition $u'(0) - cu(0) = 0$.

The spectrum consist of the "Rayleigh" mode $\sqrt{2c}e^{cz}$ with eigenvalue $|\xi|^2 - c^2$ and the continuous spectrum $[|\xi|^2, +\infty[$ with generalized eigenfunction $e_k(z) = e^{ikz} + r(k)e^{-ikz}$ and $r(k) = (ik - c)/(ik + c)$. We have $L^\xi e_k(z) = (|\xi|^2 + k^2)e_k(z)$. We get

$$dE_I = (2\pi)^{-2}\left(2ce^{2cz}\text{Area}(|\xi|^2 - c^2 \in I) + \frac{1}{4\pi}\int_{k^2+|\xi|^2\in I}|e_k(z)|^2 dk d\xi\right)|dxdz| \ .$$

We can check the large energy asymptotics for $z < 0$:

$$dE_{[0,K^2]}(x,z) \sim \frac{K^3}{6\pi^2}|dxdz| \ .$$

6 Dynamical equipartition

In this section, I will present 2 mathematical results on the case of quantum dynmaical systems whose classical limit is "chaotic".

6.1 Shnirelman semi-classical ergodic Theorem

Let us consider the Schrödinger operator $\hat{H} := -h^2\Delta + V(x)$ where we assume that V is smooth, the following statement, known as the "semi-classical ergodic Theorem", was initially stated by Shnirelman [8] and proved in various context in [11, 1]:

Theorem 3 *Let $I = [E_0, E_1]$ be so that $V^{-1}(I)$ is compact and the classical flow of $H = |\xi|^2 + V$ is* ergodic *on each energy surface $H^{-1}(E)$ with $E \in I$. Let us choose, for each value of h an eigenbasis $(\varphi_j^h, \lambda_j^h)$, $\lambda_j^h \in I$, with $j \in A_h$ of $P_I(L^2)$ where P_I is the spectral projector. Then there exists subsets $B_h \subset A_h$ so that the Wigner measures of the ϕ_j^h with $j \in B_h$ and $\lambda_j^h \to E$ converge to the Liouville measure on $H^{-1}(E)$ and*

$$\lim_{h\to 0}\frac{\#B_h}{\#A_h} = 1 \ .$$

6.2 Dynamical equipartition of a state which is localised at time 0

Let us take ϕ_h so that W_{ϕ_h} converges to $\delta(x,\xi)$. Then we expect that there exists a windows of time of the order of $|\log h|$ (Ehrenfest time) so that the Wigner measures of $U(t)\phi_h$ converges to the Liouville measure.

This result has been proved in [4] for the quantum cat map and a slighly different version starting with WKB states has been proved in [9] for hyperbolic flows.

It is expected that that such results are still valid for the elastic wave equation in presence of interfaces. On the other hand, the quantum ergodic Theorem is probably not valid for general matrix Hamiltonian even assuming the classical limit to be ergodic as a classical random walk. It is clearly non valid for quantum graphs as a result of explicit calculation for star graphs [7].

Appendix: review on spectral theory of some Sturm-Liouville operators

Let us consider, on the half-line $z \leq 0$, the formally symmetric differential operator $L = -\frac{d}{dz}\left(N(z)\frac{d}{dz}\right) + V(z)$ with $N > 0$, $N \equiv 1$ as $z << 0$ and $V \equiv a$ as $z << 0$. Let us take some *self-adjoint boundary conditions at* $z = 0$*:*

$$(\star)\ u'(0) = cu(0) \text{ or } u(0) = 0\ .$$

The self-adjoint operator $(L, \star)$ admits a finite discrete spectrum

$$\lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots < \lambda_N < a$$

with L^2-normalized eigenfunctions φ_j and a continuous spectrum $[a, +\infty[$. The eigenfunctions corresponding to the continuous spectrum are the functions $e_k(z),\ k \in \mathbb{R} \setminus \{0\}$ which satisfy

- $Le_k = (k^2 + a)e_k$
- $e_k(z) = e^{ikz} + r(k)e^{-ikz}$ for $z << 0$; $r(k)$ is called the *reflexion coefficient* and satisfies:
$$|r(k)| \equiv 1 \text{ and } r(-k) = \overline{r(k)}$$
- e_k satisfies the boundary condition $(\star)$ at $z = 0$.

A typical example is $N \equiv 1$, $V \equiv 0$ and $u'(0) = 0$. We have then $e_k(z) = e^{ikz} + e^{-ikz} = 2\cos kz$.

The spectral projector Π_I is given by

$$[\Pi_I](z, z') = \sum_{\lambda_j \in I} \varphi_j(z)\varphi_j(z') + \frac{1}{4\pi}\int_{k^2+a \in I} e_k(z)\overline{e_k(z')}dk\ .$$

The *density of states* of L is then given by

$$dE_I(z) = \left(\sum_{\lambda_j \in I} \varphi_j^2(z) + \frac{1}{4\pi} \int_{k^2+a\in I} |e_k(z)|^2 dk \right) |dz| \ .$$

For *large values* of z, the φ_j's are small and we get

$$dE_I(z) \sim \frac{1}{4\pi} \left(\int_{k^2+a\in I} |e^{ikz} + r(k) e^{-ikz}|^2 dk \right) |dz| \ .$$

For *large values of z and k*, we get $dE_{[0,k^2]}(z) \sim \frac{k|dz|}{\pi}$.

For large values of k and any z, we get, by what is called the *local Weyl law,*

$$dE_{[0,k^2]}(z) \sim \frac{1}{\pi} \sqrt{\frac{k^2 - V(z)}{N(z)}} |dz| \ .$$

References

[1] Y. Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Commun. Math. Phys.* 102:497-502 (1985).

[2] Y. Colin de Verdière. Mathematical models for passive imaging I: general background. http://fr.arxiv.org/abs/math-ph/0610043/

[3] Y. Colin de Verdière. Mathematical models for passive imaging II: surface waves. http://fr.arxiv.org/abs/math-ph/0610044/

[4] F. Faure & S. Nonnenmacher. On the maximal scarring for quantum cat map eigenstates *Commun. Math. Phys.* 245, 201–214 (2004).

[5] G. Folland. Harmonic analysis in phase space. *Princeton Univ. Press,* 1989.

[6] I.M. Gelfand & N.Y. Vilenkin. Les distributions IV : applications de l'analyse harmonique. *Dunod (Paris),* 1967.

[7] J.P. Keating, J. Marklof & B. Winn. Value distribution of the eigenfunctions and spectral determinant of quantum star graphs. *Commun. Math. Phys.* 241:421–452 (2003).

[8] A. Shnirelman. Ergodic properties of eigenfunctions. *Usp. Math. Nauk.* 29:181-182 (1974).

[9] R. Schubert. Semiclassical behaviour of expectation values in time evolved states fo large times *Commun. Math. Phys.* 256:239-254 (2005).

[10] L. Schwartz. Radon measures on arbitrary topological spaces and cylindrical measures. *Oxford University Press,* 1973.

[11] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.* 55:919-941 (1987).