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On Double Eigenvalues of Hill's Operator*

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1. INTRODUCTION AND SUMMARY

Let us consider Hill's equation

$$-y''(x) + p(x)\, y(x) = \lambda y(x) \tag{1}$$

with periodic boundary conditions

$$y(x + 2\pi) = y(x), \tag{2}$$

where the potential p is in $L^2[0, \pi]$, extended periodically to all of $\mathbb{R}$. The numbers λ for which (1)–(2) has a solution are called periodic eigenvalues of $-d^2/dx^2 + p$. They form a sequence $(\lambda_k)_{k \geqslant 0}$ of real numbers, the periodic spectrum of p. This sequence is written in increasing order and with multiplicities. It satisfies the following inequalities: $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots$.

The following result is proved in this paper:

THEOREM. *The set of potentials p in $L^2[0, \pi]$ with the property that the Hill's operator $-d^2/dx^2 + p$ has only finitely many simple periodic eigenvalues is dense in the norm topology of $L^2[0, \pi]$.*

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This result has been conjectured by Novikov in 1974 [N] (cf. also [La]) and proved by various authors: Marchenko and Ostrovskii [M], Levitan [L], Garnett and Trubowitz [GT], and others (e.g., [Me]). All of them use the inverse spectral theory which has been extensively developed for Hill's operator. The main purpose of this paper is to give a proof of the theorem using no inverse spectral theory at all. By the same methods one can prove:

THEOREM′. *Let $N \geqslant 1$ be given. Then the set of potentials p in $C^\infty(\mathbb{R}/\pi\mathbb{Z})$ with the property that Hill's operator $-d^2/dx^2+p$ has N consecutive double eigenvalues is dense with respect to the uniform topology in C^∞ $(\mathbb{R}/\pi\mathbb{Z})$.*

The idea of the proof of the theorem is the following: Given p in $L^2[0, \pi]$ one looks at the perturbed operators $-d^2/dx^2+p+tp_{\alpha,\beta;n}$, where $n \geqslant 1$ is an integer, $\alpha=(\alpha_k)_{k\geqslant 0}$, $\beta=(\beta_k)_{k\geqslant 0}$ are sequences of real numbers in l^2 s.t. $\|\alpha\|^2+\|\beta\|^2=\sum_{k\geqslant 0}\alpha_k^2+\beta_k^2 \leqslant 1$, and $p_{\alpha,\beta;n}$ is given by $p_{\alpha,\beta;n}(x)=2\sum_{k\geqslant n}(a_{k-n}\cos 2kx+\beta_{k-n}\sin 2kx)$. By using the perturbation theory of linear operators one shows, as in [CdV1–CdV4; CCdV], an appropriate stability result to the effect that for $|t|>0$ arbitrarily small one can find $n \geqslant 1$ sufficiently large and sequences of real numbers s.t. the periodic spectrum $(\lambda_k)_{k\geqslant 0}$ of Hill's operator $-d^2/dx^2+p+tp_{\alpha,\beta;n}$ has the property $\lambda_{2k}=\lambda_{2k-1}$ $(k \geqslant n)$.

The paper is organized as follows: The theorem is proved in Section 4. Sections 2 and 3 are preparatory. In Section 3 the basic estimates for the proof are shown.

Before starting to work let us introduce the following notation and summarize some well known properties of Hill's operator which we will need later (cf. [MW] or [M]).

For p in $L^2[0, \pi]$ the periodic eigenvalues $\lambda_k=\lambda_k(p)$ $(k \geqslant 0)$ satisfy the asymptotics $\lambda_{2k-1}, \lambda_{2k}=k^2+O(1)$, or equivalently, $\sqrt{\lambda_{2k-1}}, \sqrt{\lambda_{2k}}=k+O(1/k)$. The error terms $O(1)$ and $O(1/k)$ are uniform for bounded sets of potentials p. Moreover $(\lambda_{2k}-\lambda_{2k-1})_{k\geqslant 1}$ is in l^2 and if the potential p is in $C^\infty(\mathbb{R}/\pi\mathbb{Z})$ then $\lambda_{2k}-\lambda_{2k-1}=O(1/k^d)$ for any $d \geqslant 1$. Denote by $f_n(x)=f_n(x, p)$ $(n \geqslant 0)$ a system of orthonormal eigenfunctions corresponding to $(\lambda_n)_{n\geqslant 0}$. By $E_n=E_n(p)$ we denote the linear subspace of $L^2[0, 2\pi]$, generated by f_{2n} and f_{2n-1}. E_n has an orthonormal basis of the form $(\sin nx)/\sqrt{\pi}+O(1/n)$, $(\cos nx)/\sqrt{\pi}+O(1/n)$, where the error terms $O(1/n)$ are uniform for $0 \leqslant x \leqslant 2\pi$ and for bounded sets of potentials.

For potentials of the form $p+tp_{\alpha,\beta;n}$ we will write $\lambda_k=\lambda_k(t)$, $f_k(x):=f_k(x, t)$, and $E_k=E_k(t)$. Finally, let us denote by $Q(E_n)[Q_0(E_n)]$ the space of quadratic forms on E_n [of quadratic forms q with $\mathrm{Tr}\, q=0$].

2. Quadratic Forms

Fix p in $L^2[0, \pi]$. Then introduce for real numbers α, β and $n \geqslant 1$ the quadratic form $q_{\alpha,\beta;n}$ on $E_n = E_n(p)$ given by (f, g in E_n)

$$q_{\alpha,\beta;n}(f, g) := 2 \int_0^{2\pi} (\alpha \cos 2nx + \beta \sin 2nx) f(x) g(x) \, dx.$$

The matrix representation of $q_{\alpha,\beta;n}$ with respect to an orthonormal basis of E_n of the form $(\cos nx)/\sqrt{\pi} + O(1/n)$, $(\sin nx)/\sqrt{\pi} + O(1/n)$ is given by

$$\begin{pmatrix} \alpha\left(1 + O\left(\frac{1}{n}\right)\right) & \beta\left(1 + O\left(\frac{1}{n}\right)\right) \\ \beta\left(1 + O\left(\frac{1}{n}\right)\right) & -\alpha\left(1 + O\left(\frac{1}{n}\right)\right) \end{pmatrix} + \begin{pmatrix} \beta O\left(\frac{1}{n}\right) & \alpha O\left(\frac{1}{n}\right) \\ \alpha O\left(\frac{1}{n}\right) & \beta O\left(\frac{1}{n}\right) \end{pmatrix}$$

and thus the norm $\|q_{\alpha,\beta;n}\|$ of $q_{\alpha,\beta;n}$ can be computed to be $\sqrt{\alpha^2 + \beta^2}\,(1 + O(1/n))$. Observe that $\operatorname{Tr} q_{\alpha,\beta;n} = \alpha O(1/n) + \beta O(1/n)$. We have thus proved

Lemma 1. *For n sufficiently large $B_n: \mathbb{R}^2 \to Q_0(E_n)$ is a linear isomorphism s.t. $\|B_n(\alpha, \beta)\|^2 = (\alpha^2 + \beta^2)(1 + O(1/n))$, where $B_n(\alpha, \beta) := q^0_{\alpha,\beta;n} := q_{\alpha,\beta;n} - \frac{1}{2}(\operatorname{Tr} q_{\alpha,\beta;n}) \operatorname{Id}_{E_n}$.*

Let us denote by V_n the subspace of $L^2[0, 2\pi]$ generated by the family of eigenspaces E_k $(k \geqslant n)$. Thus for g in V_n one can find g_k in E_k s.t. $g = \sum_{k \geqslant n} g_k$. By $Q_0(V_n)$ we denote the space of all quadratic forms $q: V_n \times V_n \to \mathbb{R}$ which can be written as $q(g, g) = \sum_{k \geqslant n} q_k(g_k, g_k)$ s.t. q_k is in $Q_0(E_k)$ and $\|q\|^2 := \sum_{k \geqslant n} \|q_k\|^2 < \infty$, where q_k denotes the restriction of q to $E_k \times E_k$. For such a q we write conveniently $q = \sum_{k \geqslant n} q_k$. Clearly $Q_0(V_n)$ with the norm introduced above is a Hilbert space. To make notation easier, denote by $q_{\alpha,\beta;n;k}$ the quadratic form $q_{\alpha_{k-n},\beta_{k-n};k}$. Applying Lemma 1 we get

Lemma 2. *For n sufficiently large $A_n: l^2 \to Q_0(V_n)$ is a linear isomorphism s.t. $\|A_n(\alpha, \beta)\|^2 = (\sum_{k \geqslant 0} \alpha_k^2 + \beta_k^2)(1 + O(1/n))$, where $A_n(\alpha, \beta) = \sum_{k \geqslant n} q^0_{\alpha,\beta;n;k}$.*

3. Perturbation Theory for Quadratic Forms

Let us fix p in $L^2[0, \pi]$ and denote by L the operator $-d^2/dx^2 + p$. For arbitrary $n \geqslant 1$ and α, β in l^2 with $\|\alpha\|^2 + \|\beta\|^2 = \sum_{k \geqslant 0} \alpha_k^2 + \beta_k^2 \leqslant 1$ let us consider $L + tp_{\alpha,\beta;n}$. Denote by $P_k(t)$ the orthogonal projection on $E_k(t)$. It follows from [K] that $P_k(t)$ is analytic in t. By $U_k(t)$ we denote the linear map on $L^2[0, \pi]$ determined by $U_k'(t) = P_k'(t) U_k(t)$ with the initial value $U_k(0) = P_k(0)$. Here means d/dt.

$U_k(t)$ is analytic in t and the restriction $U_k(t)\,|_{E_k(0)}$ is an isometry from $E_k(0)$ to $E_k(t)$ (cf. [K]). We now consider the quadratic form

$$Q_k(t)(f, g) := \langle U_k(t) f, (L + tp_{\alpha,\beta;n}) U_k(t) g \rangle.$$

Expanding $Q_k(t)(f, g)$ in t one gets

$$\begin{aligned} Q_k(t)(f, g) &= Q_k(0)(f, g) + t \frac{d}{dt}\bigg|_{t=0} Q_k(t)(f, g) \\ &\quad + \int_0^t (t-s) \frac{d^2}{ds^2} Q_k(s)(f, g)\, ds. \end{aligned}$$

To simplify notation denote by $'$ the derivative d/dt. Then

$$\begin{aligned} Q_k'(t)(f, g) &= \langle U_k'(t) f, (L + tp_{\alpha,\beta;n}) U_k(t) g \rangle \\ &\quad + \langle U_k(t) f, p_{\alpha,\beta;n} U_k(t) g \rangle \\ &\quad + \langle U_k(t) f, (L + tp_{\alpha,\beta;n}) U_k'(t) g \rangle. \end{aligned}$$

Using that $P_k(t) P_k'(t) P_k(t) = 0$ and that $U_k'(t) = P_k'(t) U_k(t)$ one concludes that $P_k(t) U_k'(t) = 0$. Now $P_k(t)$ commutes with $L + tp_{\alpha,\beta;n}$ and $U_k(t) = P_k(t) U_k(t)$. This implies that

$$Q_k'(t)(f, g) = \langle U_k(t) f, p_{\alpha,\beta;n} U_k(t) g \rangle.$$

Moreover

$$\begin{aligned} Q_k(t)(f, g) &= 2\langle U_k(t) f, p_{\alpha,\beta;n} U_k'(t) g \rangle \\ &= 2\langle U_k(t) f, p_{\alpha,\beta;n} P'(t)\, U_k(t) g \rangle. \end{aligned}$$

Thus the expansion of $Q_k(t)$ in t can be written as

$$\begin{aligned} Q_k(t)(f, g) &= \langle P_k(0) f, L P_k(0) g \rangle \\ &\quad + t\langle P_k(0) f, p_{\alpha,\beta;n} P_k(0) g \rangle \\ &\quad + \int_0^t (t-s) 2\langle U_k(s) f, p_{\alpha,\beta;n} P_k'(s) U_k(s) g \rangle\, ds. \end{aligned}$$

The last term can be estimated by

$$t^2 \sup_{0 \leqslant s \leqslant t} |\langle U_k(s) f, p_{\alpha,\beta;n} P_k'(s) U_k(s) g \rangle|.$$

The aim of this section is to provide an estimate for this expression, which is needed for the proof of the theorem.

LEMMA 3. *There exists $K>0$ s.t. for $k \geqslant K$*

$$\|P_k'(t) P_k(t) g\|_{L^2} \leqslant 16\pi \left[\sum_{\substack{j \geqslant 0 \\ j \neq 2k, 2k-1}} \sup_{\zeta \in C_k} \frac{1}{|\lambda_j(t) - \zeta|^2} \right]^{1/2} \|P_k(t) g\|_{L^2},$$

where C_k denotes a circle in $\mathbb{C}$ with center $(\lambda_{2k} + \lambda_{2k-1})/2$ and radius $\lambda_{2k} - \lambda_{2k-1}$. K can be chosen independently of $0 \leqslant |t| \leqslant 1$, $n \geqslant 1$, and α, β with $\|\alpha\|^2 + \|\beta\|^2 \leqslant 1$.

Remark. The proof gives a slightly better estimate.

Proof. Recall from [K] that $P_k(t)$ can be represented by $P_k(t) = -(1/2\pi i) \int_{\Gamma_k} R(\zeta, t)\, d\zeta = -(1/2\pi i) \int_{C_k} R(\zeta, t)\, d\zeta$, where $R(\zeta, t)$ denotes the resolvent $(L + tp_{\alpha,\beta;n} - \zeta)^{-1}$ and Γ_k denotes a circle in $\mathbb{C}$ with center k^2 and radius N where N is chosen such that for $k \geqslant N$, the circles Γ_k are all disjoint and $\lambda_{2k}(t)$ and $\lambda_{2k-1}(t)$ are the only eigenvalues of $L + tp_{\alpha,\beta;n}$ inside Γ_k for $0 \leqslant |t| \leqslant 1$, $n \geqslant 1$, and α, β with $\|\alpha\|^2 + \|\beta\|^2 \leqslant 1$.

Observe that Γ_k is independent of t and thus for $k \geqslant N$

$$P_k'(t) = -\frac{1}{2\pi i} \int_{\Gamma_k} R'(\zeta, t)\, d\zeta = \frac{1}{2\pi i} \int_{C_k} R(\zeta, t)\, p_{\alpha,\beta;n} R(\zeta, t)\, d\zeta,$$

where we have used that $R'(\zeta, t) = -R(\zeta, t)\, p_{\alpha,\beta;n} R(\zeta, t)$. Now $R(\zeta, t)$ is given by $R(\zeta, t)\, g = \sum_{j \geqslant 0} (1/(\lambda_j - \zeta)) \langle f_j, g \rangle f_j$, thus in particular

$$R(\zeta, t) P_k(t)\, g = \frac{1}{\lambda_{2k} - \zeta} \langle f_{2k}, P_k g \rangle f_{2k} + \frac{1}{\lambda_{2k-1} - \zeta} \langle f_{2k-1}, P_k g \rangle f_{2k-1}.$$

This together with $P_k(t)\, P_k'(t) P_k(t) \equiv 0$ yields

$$P_k'(t) P_k(t)\, g = \frac{1}{2\pi i} \int_{C_k} d\zeta\, \frac{\langle f_{2k}, P_k g \rangle}{\lambda_{2k} - \zeta} \sum_{\substack{j \geqslant 0 \\ j \neq 2k, 2k-1}} \frac{1}{\lambda_j - \zeta} \langle f_j, p_{\alpha,\beta;n} f_{2k} \rangle f_j$$
$$+ \frac{1}{2\pi i} \int_{C_k} d\zeta\, \frac{\langle f_{2k-1}, P_k g \rangle}{\lambda_{2k-1} - \zeta} \sum_{\substack{j \geqslant 0 \\ j \neq 2k, 2k-1}} \frac{1}{\lambda_j - \zeta} \langle f_j, p_{\alpha,\beta;n} f_{2k-1} \rangle f_j.$$

Thus $\|P_k'(t)P_k(t)g\|_{L^2}\leqslant I+II$, where

$$I=\Bigg(\sum_{\substack{j\geqslant 0\\ j\neq 2k,2k-1}}\sup_{\zeta\in C_k}\frac{1}{|\lambda_j-\zeta|^4}\Bigg)^{1/4}\Bigg(\sum_{j\geqslant 0}|\langle f_j, p_{\alpha,\beta;n}f_{2k}\rangle|^4\Bigg)^{1/4}$$

$$\times|\langle f_{2k}, P_k g\rangle|\,\frac{1}{2\pi}\int_{C_k}\frac{1}{|\lambda_{2k}-\zeta|}\,d|\zeta|$$

and II is a similar expression as I, which can be estimated in the same way. To estimate I we observe

$$\Bigg(\sum_{j\geqslant 0}|\langle f_j, p_{\alpha,\beta;n}f_{2k}\rangle|^4\Bigg)^{1/4}\leqslant\|p_{\alpha,\beta;n}f_{2k}\|_{L^2}\leqslant\|p_{\alpha,\beta;n}\|_{L^2}\,\|f_{2k}\|_{L^\infty}.$$

But $\|p_{\alpha,\beta;n}\|_{L^2}\leqslant 2\pi$ and $\|f_{2k}\|_{L^\infty}\leqslant 2$ for k sufficiently large as f_{2k} can be written as a linear combination $a_1((\sin kx)/\sqrt{\pi}+O(1/k))+a_2((\cos kx)/\sqrt{\pi}+O(1/k))$ with $a_1^2+a_2^2=1$.

Clearly $|\langle f_{2k}, P_k g\rangle|\leqslant\|P_k g\|_{L^2}$ and $(1/2\pi)\int_{C_k}(1/|\lambda_{2k}-\zeta|)\,d|\zeta|\leqslant 2$. Putting all this together one gets

$$I\leqslant\Bigg(\sum_{\substack{j\geqslant 0\\ j\neq 2k,2k-1}}\sup_{\zeta\in C_k}\frac{1}{|\lambda_j-\zeta|^4}\Bigg)^{1/4}8\pi\,\|P_k g\|_{L^2}$$

for k sufficiently large. As the same estimate holds for II, Lemma 3 follows.

A straightforward application of Lemma 3 proves

LEMMA 4. *Let K and C_k be as in Lemma 3. Then*

$$|\langle U_k(t)f, p_{\alpha,\beta;n}P_k'(t)\,U_k(t)\,g\rangle|$$

$$\leqslant 64\pi^{3/2}\Bigg(\sum_{\substack{j\geqslant 0\\ j\neq 2k,2k-1}}\sup_{\zeta\in C_k}\frac{1}{|\lambda_j-\zeta|^2}\Bigg)^{1/2}\|P_k(0)\,f\|_{L^2}\,\|P_k(0)\,g\|_{L^2}.$$

LEMMA 5. *There exist $N>0$ and $M>0$ s.t.*

$$\sum_{k\geqslant N}\sum_{\substack{j\geqslant 0\\ j\neq 2k,2k-1}}\sup_{\zeta\in C_k}\frac{1}{|\lambda_j(t)-\zeta|^2}\leqslant M.$$

N and M are independent of $0\leqslant|t|\leqslant 1$, $n\geqslant 1$, and $\|\alpha\|^2+\|\beta\|^2\leqslant 1$.

Proof. By the definition of C_k as given in the statement of Lemma 3 it suffices to estimate for N large enough

$$III = \sum_{k \geqslant N} \sum_{j \geqslant 2k+1} \left(\frac{1}{\lambda_j - \left[\lambda_{2k} + \frac{\lambda_{2k} - \lambda_{2k-1}}{2} \right]} \right)^2$$

and

$$IV = \sum_{k \geqslant N} \sum_{j \leqslant 2(k-1)} \left(\frac{1}{\left(\lambda_{2k-1} - \frac{\lambda_{2k} - \lambda_{2k-1}}{2} \right) - \lambda_j} \right)^2.$$

Without loss of any generality we may assume that $\lambda_0(t) \geqslant 1$ (all $n \geqslant 1$, $0 \leqslant |t| \leqslant 1$, and α, β with $\|\alpha\|^2 + \|\beta\|^2 \leqslant 1$). Then we can write

$$\lambda_j - \left(\lambda_{2k} + \frac{\lambda_{2k} - \lambda_{2k-1}}{2} \right)$$
$$= \left(\sqrt{\lambda_j} - \sqrt{\lambda_{2k} + \frac{\lambda_{2k} - \lambda_{2k-1}}{2}} \right) \left(\sqrt{\lambda_j} + \sqrt{\lambda_{2k} + \frac{\lambda_{2k} + \lambda_{2k-1}}{2}} \right).$$

Using the asymptotics for $\sqrt{\lambda_j}$ one can find $N \geqslant 1$ independent of t, α, β, n s.t. for $k \geqslant N$, $j \neq k$

$$\left| \sqrt{\lambda_{2j}} \mp \sqrt{\lambda_{2k} + \frac{\lambda_{2k} - \lambda_{2k-1}}{2}} \right| \geqslant |j \mp k| - \frac{1}{2}$$

$$\left| \sqrt{\lambda_{2j-1}} \mp \sqrt{\lambda_{2k} + \frac{\lambda_{2k} - \lambda_{2k-1}}{2}} \right| \geqslant |j \mp k| - \frac{1}{2}$$

and Lemma 5 easily follows.

Taking the results of the previous lemmata together one obtains

LEMMA 6. *There exist $N > 0$ and $M > 0$ s.t.*

$$\sum_{k \geqslant N} \| \langle U_k(t) \cdot, p_{\alpha, \beta; n} P'_k(t) U_k(t) \cdot \rangle \|^2 \leqslant M$$

for $0 \leqslant |t| \leqslant 1$, $n \geqslant 1$, α, β *with* $\|\alpha\|^2 + \|\beta\|^2 \leqslant 1$.

4. Proof of Theorem

Let p be a given potential. Without loss of generality we may assume that p is in $C^\infty(\mathbb{R}/\pi\mathbb{Z})$. Choose a positive and decreasing sequence $(t_n)_{n\geq 1}$, converging to 0 and satisfying $t_n > c/n^d$ for some $c>0$ and $d>0$. Introduce for $k\geq n$ the quadratic forms $\phi_{\alpha,\beta;k} = \phi_{\alpha,\beta;n;k}$ and $\psi_{\alpha,\beta;k} = \psi_{\alpha,\beta;n;k}$ on $E_k(0)$:

$$\phi_{\alpha,\beta;k}(f,g) := \left\langle P_k(0)f, \left[\frac{1}{t_n}(L-\lambda_{2k-1}(0)) + p_{\alpha,\beta;n}\right] P_k(0)g\right\rangle$$

$$\psi_{\alpha,\beta;k}(f,g) := \left\langle U_k(t)f, \left[\frac{1}{t_n}(L-\lambda_{2k-1}(0)) + p_{\alpha,\beta;n}\right] U_k(t)g\right\rangle.$$

Now define the quadratic forms

$$\Phi_{\alpha,\beta;n} = \sum_{k\geq n} \phi^0_{\alpha,\beta;n} \qquad \text{and} \qquad \Psi_{\alpha,\beta;n} = \sum_{k\geq n} \psi^0_{\alpha,\beta;k},$$

where $\phi^0_{\alpha,\beta;k} := \phi_{\alpha,\beta;k} - \frac{1}{2}(\mathrm{Tr}\,\phi_{\alpha,\beta;k})\,\mathrm{Id}_{E_k(0)}$ and $\psi^0_{\alpha,\beta;k} := \psi_{\alpha,\beta;k} - \frac{1}{2}(\mathrm{Tr}\,\phi_{\alpha,\beta;k})\,\mathrm{Id}_{E_k(0)}$ are evidently trace free quadratic forms, thus in $Q_0\ (E_k(0))$.

It follows from Section 3 that there exists $N\geq 1$ s.t. $\Phi_{\alpha,\beta;n}$ and $\Psi_{\alpha,\beta;n}$ are in $Q_0(V_n)$ for $n\geq N$. Moreover

$$\left(\sum_{k\geq N} \|\phi_{\alpha,\beta;k} - \psi_{\alpha,\beta;k}\|^2\right)^{1/2} \leq Mt_n$$

for a certain $M>0$ which might be chosen independently of $n\geq 1$ and α, β with $\|\alpha\|^2 + \|\beta\|^2 \leq 1$. With A_n defined as in Lemma 2 one concludes that

$$\lim_{n\to\infty} A_n^{-1}\Psi_{\alpha,\beta;n} = \lim_{n\to\infty} A_n^{-1}\Phi_{\alpha,\beta;n}.$$

Finally, $\lim_{n\to\infty} A_n^{-1}\Phi_{\alpha,\beta;n} = (\alpha,\beta)$ as one can see by representing $\phi_{\alpha,\beta;k}$ with respect to the orthonormal basis $(\cos kx)/\sqrt{\pi} + O(1/k)$ and $(\sin kx)/\sqrt{\pi} + O(1/k)$ to get

$$\begin{pmatrix} \alpha_{n-k} & \beta_{n-k} \\ \beta_{n-k} & -\alpha_{n-k} \end{pmatrix} + \begin{pmatrix} O\left(\frac{1}{k}\right) & O\left(\frac{1}{k}\right) \\ O\left(\frac{1}{k}\right) & O\left(\frac{1}{k}\right) \end{pmatrix} \|p_{\alpha,\beta;n}\|$$

$$+\frac{1}{t_n}(\lambda_{2k}(0) - \lambda_{2k-1}(0)) \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}$$

and using that p is in $C^\infty(\mathbb{R}/\pi\mathbb{Z})$ and thus $(\lambda_{2k}(0) - \lambda_{2k-1}(0)) = 0(1/k^l)$ for any $l\geq 1$.

We are thus in a position to apply an elementary topolgical lemma [CCdV], which is stated below for the convenience of the reader. Define $F_n(\alpha, \beta) := A_n^{-1} \psi_{\alpha,\beta;n}$. Then $F_n: B \to l^2$ is continuous $(n \geqslant 1)$, where B denotes the unit ball in l^2. From the considerations above it follows that, uniformly for (α, β) in B,

$$F(\alpha, \beta) = \lim_{n \to \infty} F_n(\alpha, \beta),$$

where F is the identity. We thus conclude from Lemma 7 that for sufficiently large n there exist $\alpha = (\alpha_k)_{k \leqslant 0}$ and $\beta = (\beta_k)_{k \geqslant 0}$ with $\sum_{k \geqslant 0} \alpha_k^2 + \beta_k^2 \leqslant 1$s.t. $A_n^{-1} \psi_{\alpha,\beta;k}$ is identically zero. Interpreting this we conclude that $\psi_{\alpha,\beta;k}$ has a double eigenvalue for $k \geqslant n$ and thus the theorem is proved.

LEMMA 7 [CCdV]. *Let B be the open unit ball in l^2 and denote by $(F_n)_{n \geqslant 1}$ a sequence of continuous applications from B to l^2 s.t. $(F_n)_{n \geqslant 1}$ converges, uniformly on B, to a homeomorphism $F: B \to C$ where C is a bounded open subset of l^2, which contains the origin. Then, for n sufficiently large, the origin is in the image $F_n(B)$.*

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