Essential self-adjointness for combinatorial Schrödinger operators

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1 The Dirichlet problem at infinity

We will use in this section the distance d_p defined using the weights $p_{x,y} = c_{x,y}^{-\frac{1}{2}}$.

Let us consider the quadratic form

$$Q(f) = \langle (\Delta_{\omega,c} + 1)f \mid f \rangle_\omega = \sum_{\{x,y\}\in E} c_{xy}(f(x) - f(y))^2 + \sum_{x\in V} \omega_x^2 f(x)^2 \ .$$

We have

Lemma 1.1 *For any $f : V \to \mathbb{R}$ so that $Q(f) < \infty$ and for any $a, b \in V$, we have $|f(a) - f(b)| \leq \sqrt{Q(f)} d_p(a, b)$.*

Proof.–

For any $\{x, y\} \in E$, $|f(x) - f(y)| \leq \sqrt{Q(f)}/\sqrt{c_{xy}}$. For any path γ from a to b, defined by the vertices $x_1 = a, x_2, \cdots, x_N = b$, we have $|f(a) - f(b)| \leq \sqrt{Q(f)}\text{length}(\gamma)$. Taking the infimum of the righthandside w.r. to γ we get the result.

□

Lemma 1.1 implies that any function f with $Q(f) < \infty$ extends to $\hat{V}$ as a Lipschitz function $\hat{f}$. We will denote by f_∞ the restriction of $\hat{f}$ to V_∞.

Theorem 1.1 *Let $f : V \to \mathbb{R}$ with $Q(f) < \infty$, then there exists an unique continuous function $F : \hat{V} \to \mathbb{R}$ which satisfies*

- $(F - f)_\infty = 0$
- $(\Delta_{\omega,c} + 1)(F_{|V}) = 0$

Proof.–

We will denote by A_f the affine space of continuous functions $G : \hat{V} \to \mathbb{R}$ which satisfy $Q(G) < \infty$ and $(G - f)_\infty = 0$.

Q is lower semi-continuous for the pointwise convergence on V as defined by $Q = \sup Q_\alpha$ with $Q_\alpha(f) =$ sum of a finite number of terms in Q.

Let $Q_0 := \inf_{G\in A_f} Q(G)$ and G_n be a corresponding minimizing sequence. The G_n's are equicontinuous and pointwise bounded. From Ascoli's Theorem, this implies the existence of a locally uniformly convergent subsequence $G_{n_k} \to F$. Using semi-continuity, we have $Q(F) = Q_0$.

If $x \in V$ and δ_x is the Dirac function at the vertex x, we have

$$\frac{d}{dt}_{|t=0} Q(F + t\delta_x) = 2(\Delta_{\omega,c} + 1)F(x)$$

and this is equal to 0, because F is the minimum of Q restricted to A_f.

Uniqueness is proved using a maximum principle.

□

2 Non essentially self-adjoint Laplacians

Theorem 2.1 *Let $\Delta_{\omega,c}$ be a Laplacian and assume that*

- *(V, d_p) with $p_{xy} = c_{xy}^{-\frac{1}{2}}$ is NOT complete*
- *There exists a function $f : V \to \mathbb{R}$ with $Q(f) < \infty$ and $f_\infty \neq 0$*

then $\Delta_{\omega,c}$ is not ESA.

Proof.–

Because $\Delta_{\omega,c}$ is ≥ 0 on $C_0(V)$, it is enough (see Theorem X.26 in [R-S]) to build a non zero function $F : V \to \mathbb{R}$ which is in $l^2_\omega(V)$ and satisfies

$$(\Delta_{\omega,c} + 1)F = 0 \ . \tag{1}$$

The function F will be the solution of Equation (1) whose limit at infinity is f_∞.

□

Remark 2.1 *The assumptions of the Theorem are satisfied if (G, d_p) is non complete and $\sum \omega_y^2 < \infty$: it is enough to take $f \equiv 1$.*

They are already satisfied if G has a non complete "end" of finite volume.

Remark 2.2 *Theorem 2.1 is not valid for the Riemannian Laplacian: if X is a closed Riemannian manifold of dimension ≥ 4, $x_0 \in X$ and $Y = X \setminus x_0$, the Laplace operator on Y is ESA (see [CdV1]) and Y has finite volume (what is wrong in (the extension of) the proof?).*

Remark 2.3 *In Theorem 2.1, what is the deficiency index of $\Delta_{\omega,c}$ in terms of the geometry of the weighted graph?*

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