A Semi-classical calculus of correlations[*]

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Abstract

Français.

La méthode d'imagerie passive en sismologie a été développée récemment en vue d'imager la croûte terrestre à partir d'enregistrements du bruit sismique. Elle repose sur le calcul des fonctions de corrélation de ce bruit. Nous donnons dans cet article des formules explicites pour cette corrélation dans le régime "semi-classique". Pour cela, nous définissons le spectre de puissance d'un champ aléatoire comme l'espérance de sa mesure de Wigner, ce qui permet d'utiliser un calcul dans l'espace des phases : le calcul pseudo-différentiel et la théorie des "rays". Nous obtenons ainsi une formule pour la corrélation du bruit sismique dans le régime "semi-classique" avec une source de bruit qui peut être localisée et non homogène.

Nous montrons ensuite comment l'utilisation des ondes guidées de surface permet d'imager la croûte terrestre.

Mots clés : Imagerie passive ; semi-classique ; ondes de surfaces.

English.

The method of passive imaging in seismology has been developped recently in order to image the earth crust from recordings of the seismic noise. This method is founded on the computation of correlations of the seismic noise. In this paper, we give an explicit formula for this correlation in the "semi-classical" regime. In order to do that, we define the power spectrum of a random field as the ensemble average of its Wigner measure, this allows phase-space computations: the pseudo-differential calculus and the ray theory. This way, we get a formula for the correlation of the seismic noise in the semi-classcial regime with a source noise which can be localized and non homogeneous. After that, we show how the use of surface guided waves allows to image the earth crust.

Keywords : passive imaging, semi-classics, surface waves.

Introduction

Correlations of the noisy wave fields is used as a new tool in seismic imaging and monitoring, starting from the pioneering work of Campillo and Paul [3] (similar tools have been used in helio-seismology [11]) and followed by many works [7, 8, 17, 18, 19, 20, 21, 27, 28]. See also the review paper [14]. It has also been used in the monitoring of the deformations of volcanœs [2]. Because it is a very powerful method and, hopefully, in order to make it more efficient, it is quite challenging to give mathematical supports to this method, now called "passive imaging". This has been done in a rather great generality in [6, 5] using semi-classical analysis (see also [15, 17, 1, 14]).

Exact formulas for the correlations of the fields are known if the source noise is homogeneous (a white noise). This assumption is not satisfied in applications. It is therefore desirable to get formulate valid for more general source noises, in particular if the source noise is localized in some part of the domain. This turns out to be possible in the so-called semi-classical regime where the wavelengths are negligible with respect to the size of the propagation domain. The field correlation admits a general expression in terms of the Green's function and the source correlation (Equation (3)). The idea is to find the asymptotics of this expression in the semi-classical regime.

I will present in this paper approximate formulas which are valid in the range of high frequency wave propagation and for which the source noise is localized in some part of the domain of propagation. The correlation is explicitly given in term of the decomposition of the Green's function as a sum over rays and the (phase-space) power spectrum of the source noise. I can use ray theory if I assume that the source noise has a short correlation distance of the same order of magnitude than the wavelengths. This calculus can be presented in a very geometric way using rays propagation as well as a re-interpretation of the source correlation in terms of the phase space power spectra. I use the calculus of pseudo-differential operators in a very essential way. I will not reproduce the mathematical arguments which are presented in my paper [5], but I will try not only to give explicit formulas, but also to present the main ideas and tools.

Here is a more precise description of the content: the goal is to get the formula given in Theorem 4.1 which gives the modification of the correlation of the seismic noise induced by the non-homogeneity of the source noise. The modification is given in terms of the power spectrum of the source noise, the attenuation and the ray dynamics associated to the deterministic wave equation.

I first give a review of the pseudo-differential calculus (section 1): this allows to put the basic terminology of rays dynamics and to define power spectra of arbitrary random fields (section 2).

I then introduce the simplest mathematical model where the source noise is simply the right-handside of the wave equation (section 3) and I present our main formula in section 4. The interest of the result depends of the relation between

2 time scales discussed in section 5: the Ehrenfest time given in terms of the Lyapounov exponent and the attenuation time.

How to use all of this in imaging problems? I do that (section 6) in the case of seismology using the effective wave equation for the guided surface waves. The final problem turns out to be an inverse spectral problem whose mathematical solution is known.

Finally, I discuss in section 7 a related issue, namely the calculus of the correlations of plane waves scattered by an obstacle or an inhomogeneity viewed as random waves: the direction of the waves is supposed to be random and uniform. This way, I show that the result of [19] is completely general.

1 A short review of the pseudo-differential calculus and Wigner measures

For the mathematics of pseudo-differential operators, see [9, 12, 13, 24].

The *pseudo-differential operators* (ΨDO's) were introduced in the sixties by Calderon, Zygmund, Nirenberg, Hörmander and others as a tool in the study of linear partial differential equations with non constant coefficients. They provide also the geometrical extension of Hamiltonian formalism of *classical mechanics* to *wave mechanics* (see [10]). In applications to physics, it is often called the *ray theory* (see [16]). The same tools apply to the study of the semi-classical limit of quantum mechanics and to the high frequency limit of wave equations (acoustic, electromagnetic or seismic waves).

There is a **small parameter** $\varepsilon > 0$ in the theory which is the Planck "constant" $\hbar$ in quantum mechanics and the wave length or more precisely the dimensionless ratio between the wave length and the size of the propagation domain for wave equations. Most results are only valid in the limit $\varepsilon \to 0$, but, for simplicity, the reader can think of ε as a fixed, small enough, number.

1.1 ΨDO's

A pseudo-differential operator (ΨDO) on $\mathbb{R}^d$ is a linear operator on functions $f : \mathbb{R}^d \to \mathbb{C}$, $A_\varepsilon := \mathrm{Op}_\varepsilon(a)$, defined using a suitable function defined on the phase space, $a(x, \xi) : \mathbb{R}_x^d \oplus \mathbb{R}_\xi^d \to \mathbb{C}$, called the *symbol* of A_ε, by the formula (Weyl quantization)

$$A_\varepsilon(f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \varepsilon\xi\right) f(y) dy d\xi \ .$$

The function a is assumed to be smooth and homogeneous near infinity in ξ. The Schwartz kernel[1] $[A_\varepsilon](x, y)$ of A_ε is located near the diagonal $x = y$ and is of the

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[1] The "Schwartz kernel" of a linear operator A is the "continuous matrix" of A, we will denote it by $[A](x, y)$ and it is characterized by $Af(x) = \int_X [A](x, y) f(y) dy$.

form
$$[A_\varepsilon](x,y) \cong_{\varepsilon \to 0} k(x,(x-y)/\varepsilon)$$
where $k(x,z)$ is a smooth function outside $z = 0$ going to 0 as $z \to \infty$.

Simple examples are

- $\text{Op}_\varepsilon(1) = \text{Id}$ by the Fourier inversion formula

- $\text{Op}_\varepsilon(\xi_j) = \frac{\varepsilon}{i}\frac{\partial}{\partial x_j}$

- $\text{Op}_\varepsilon(x_j)$ is the multiplication by x_j

- If χ is a positive function with bounded support, the operator $\text{Op}_\varepsilon(\chi(\xi))$ is a frequency filter

- $\text{Op}_\hbar(|\xi|^2 + V(x)) = -\hbar^2\Delta + V(x)$: the Schrödinger operator

- $\text{Op}_\varepsilon(n(x)|\xi|^2) = -\varepsilon^2 \text{div}\,(n(x)\,\text{grad})$: the acoustic wave operator.

The main properties are the following ones which hold as $\varepsilon \to 0$:

- Composition:
$$\text{Op}_\varepsilon(a) \circ \text{Op}_\varepsilon(b) \approx \text{Op}_\varepsilon(ab)$$

- Brackets:
$$[\text{Op}_\varepsilon(a), \text{Op}_\varepsilon(b)] \approx \frac{\varepsilon}{i}\text{Op}_\varepsilon\{a,b\}$$

  where
$$\{a,b\} = \sum_{j=1}^{d} \left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} \right)$$

  is the Poisson bracket. This last property is very important because it relates the algebra of ΨDO's to the geometry of the phase space given by the Poisson bracket.

1.2   Wigner functions

Wigner functions define the localization of energy in the phase space $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ for a wave function $u = u(x)$. They involve the scale ε. The Wigner function $W_u^\varepsilon(x,\xi)$ of u is the function on the phase space defined by the identities

$$\forall a \in C_0^\infty\left(\mathbb{R}^{2d}\right),\ \int_{\mathbb{R}^{2d}} a(x,\xi)W_u^\varepsilon(x,\xi)dxd\xi = \langle \text{Op}_\varepsilon(a)u|u\rangle\ ,$$

where $\langle u|v\rangle = \int u(x)\bar{v}(x)dx$, or

$$W_u^\varepsilon(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-iv\cdot\xi}u\left(x + \frac{\varepsilon v}{2}\right)\bar{u}\left(x - \frac{\varepsilon v}{2}\right)dv\ .$$

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I have

$$\int_{\mathbb{R}^d} W_u^\varepsilon(x,\xi)d\xi = |u(x)|^2, \quad \int_{\mathbb{R}^d} W_u^\varepsilon(x,\xi)dx = |\mathcal{F}_\varepsilon u(\xi)|^2 \ ,$$

where $\mathcal{F}_\varepsilon u(\xi) =$ is the $\varepsilon-$Fourier transform of u given by

$$\mathcal{F}_\varepsilon u(\xi) = \frac{1}{(2\pi\varepsilon)^{d/2}} \int e^{-ix.\xi/\varepsilon} u(x)dx \ .$$

This means that the marginals of the Wigner measure $W_u^\varepsilon(x,\xi)dxd\xi$ are $|u(x)|^2dx$ and $|\mathcal{F}_\varepsilon u(\xi)|^2d\xi$.

1.3   Hamiltonian dynamics and ray method

Let us consider the wave equation $u_{tt} - \mathcal{L}u = 0$ where $\mathcal{L}$ is an elliptic ΨDO like the acoustic operator $\mathcal{L} = \text{div}(n \text{ grad})$. The symbol, usually called the dispersion relation, of this equation is $\omega^2 - n(x)\|\xi\|^2 = 0$. To this relation is associated a dynamics called the ray dynamics given by the Hamilton equations:

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j}, \ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j} \tag{1}$$

with $H = \sqrt{n}\|\xi\|$. The main result (Theorem 4.1 below) uses the "Hamiltonian flow" Φ_t: $\Phi_t(x,\xi)$ is the value at time t of the previous differential system (1) with data (x,ξ) at the time $t = 0$. In the case of an homogeneous medium, $n = n_0 =$constant, I have

$$\Phi_t(x,\xi) = (x + t\sqrt{n_0}\xi/\|\xi\|, \xi) \ .$$

In the ray theory, this correspond to the group velocity of waves $\sqrt{n_0}$.

The mathematical theory of rays is called the theory of Fourier Integral Operators and has been developed in the seventies by Hörmander and Duistermaat (see [10]) following some pioneering work of Lax and Maslov. A presentation more adapted to physicists is given in [16]. Unfortunately, the geometric background is rather sophisticated and cannot be presented in a few pages. However, explicit formulas in terms of oscillatory functions and oscillatory integrals are available.

In what follows, I will use the fact that the Green's function $G(t,x,y)$ of wave equation admits, in the semi-classical regime (short wave-length), a decomposition as a sum of contributions of rays γ going from y to x in time t: $G = \sum_\gamma G_\gamma$.

2   Random fields: power spectra and correlations

Let $f = f(x)$, $x \in \mathbb{R}^d$, be a random complex-valued field with zero mean value. Let us denote by $\mathbb{E}$ the expectation or ensemble average.

Definition 2.1 *The **correlation** of the random field f is the 2-points function given by*

$$C(x, y) := \mathbb{E}(f(x)\bar{f}(y))$$

*The **power spectrum** of the random field f is the function on the phase space given by the expectation of the Wigner functions*

$$p_\varepsilon := \mathbb{E}(W_f^\varepsilon) \ .$$

The power spectrum and the correlation contain the same information:

- The correlation $C(x, y)$ is ($(2\pi\varepsilon)^d$ times) the operator kernel of $Op_\varepsilon(p)$ or

$$C(x, y) = \int e^{i\langle x - y | \xi \rangle / \varepsilon} p_\varepsilon \left(\frac{x + y}{2}, \xi \right) d\xi \ .$$

- p_ε is ($(2\pi\varepsilon)^{-d}$ times) the symbol of the operator whose integral kernel is C.

Example 2.1 *: the **white noise***

$C = \delta(x - y)$, $p_\varepsilon = 1/(2\pi\varepsilon)^d$.

Example 2.2 *: a **stationary noise** on $\mathbb{R}$ with $\varepsilon = 1$, $C(s, t) = F(s - t)$ and $p_1(s, \omega)$ is the Fourier transform $\mathcal{F}(F)(\omega)$.*

3   A mathematical model

I will now discuss the basic mathematical model: it consists of 2 parts :

- A *deterministic wave equation* which could be the elastic wave equation or more simply here the acoustic wave equation. Because the source of noise will be permanent, some attenuation in the equation is needed.

- A *source noise* assumed to be stationary and ergodic in time. In seismology, this source is usually created by the interaction of the fluids surrounding the earth crust (atmosphere or ocean) with the crust itself. This source is modeled by a random field which I put on the right-handside of the equation.

For simplicity, I will discuss only the case of a scalar acoustic wave equation on some domain in $\mathbb{R}^d_x$ with a random source field $f = f(x, t)$ (t is the time):

$$u_{tt} + a(x)u_t - \mathcal{L}u = f \tag{2}$$

where

- The field $u = u(x,t)$ is scalar

- a, the attenuation, is a smooth > 0 function. I will assume for simplicity that a is time independent, but it is not really necessary

- $\mathcal{L}$ is a self-adjoint pseudo-differential operator of symbol $-\varepsilon^{-2}l_0^2(x,\xi)$. Usually, l_0 is homogeneous of degree 1 which makes $\mathcal{L}$ independent of ε. This will not be the case for dispersive waves like surface waves. Typical examples are the Laplace-Beltrami operator of a Riemannian metric on X with $l_0(x,\xi) = \sqrt{g^{ij}(x)\xi_i\xi_j}$ and the acoustic wave operator $\mathrm{div}\,(n(x)\,\mathrm{grad}))$ with $l_0(x,\xi) = \sqrt{n(x)}\|\xi\|$. I introduce $L_0 := \mathrm{Op}_\varepsilon(l_0) = \varepsilon\sqrt{-\mathcal{L}}$

- $f = f(x,t)$ is a stationary and ergodic (in time) random field with correlation $\mathbb{E}(f(s,x)f(s',y)) = \delta(s - s')\Gamma(x,y)$ and power spectrum $p(x,\xi)$; I assume that $p(x,\xi)$ has bounded support and that f is real valued and hence that $p(x,\xi)$ is even w.r. to ξ. I assume that p is independent of ε, this implies that the correlation is $\varepsilon-$ dependent: in particular, $\Gamma(x,y) << |x - y|/\varepsilon$. The source noise decorrelates rapidly as $|x - y| >> \varepsilon$.

The Green's function is the integral kernel G giving the causal solution of Equation (2) in terms of f:

$$u(x,t) = \int_0^\infty ds \int_X G(s,x,y)f(t - s,y)dy \ .$$

Our goal is to compute the correlation

$$C_{A,B}(\tau) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T u(A,t)u(B,t - \tau)dt \ .$$

Lemma 3.1 *The following relation holds: $C_{A,B}(-\tau) = C_{B,A}(\tau)$.*

Hence I can (and will!) restrict ourselves to $\tau > 0$.

Using the fact that the source noise is ergodic and stationary, I get the following result

Theorem 3.1 *The field correlation is given by Equation (3) only in terms of the Green's function G and the correlation Γ of the source noise*

$$C_{A,B}(\tau) = \int_0^\infty ds \int_{X\times X} dxdy\, G(s + \tau, A, x)G(s, B, y)\Gamma(x,y). \qquad (3)$$

All the work is now concentrated to get a more explicit and more geometric expression: this will be done using an expression of the Green's function as a sum over rays going from B to A in time τ and using the power spectrum p of f which is a semi-classical expression of the correlation of the source noise.

4 The main formula

Let us denote by $\Omega_\pm(t)$ the "one-parameter groups" of linear operators generated by $\pm i L_0 - \varepsilon a/2$: $\Omega_+(t)u_0$ is the solution of the differential equation $\dot{u} = (\frac{i}{\varepsilon}L_0 - a/2)u$ with $u(0) = u_0$, and similarly for $\Omega_-(t)$. The use of $\Omega_\pm(t)$ is a way to split the Green function of the wave equation usually given by some "sinus" function into 2 exponentials: this way, I reduce the wave equation from an equation with of second order in time to a diagonal system of first order in time.

$$\Omega_\pm(t) = e^{t\left(\pm \frac{i}{\varepsilon}L_0 - a/2\right)} = e^{t\left(\pm i\sqrt{-\mathcal{L}} - a/2\right)} .$$

I will express the result in terms of operators instead of expressing them in terms of their kernels (matrices). This gives a much more compact expression! The symbol $\circ$ means the composition of operators while $\hat{C}(\tau)$ is the operator whose integral kernel (matrix) is $C_{A,B}(\tau)$:

$$(\hat{C}(\tau)u)(A) = \int_X C_{A,B}(\tau)u(B)dB .$$

The main result is

Theorem 4.1 *The correlation is given, for $\tau > 0$, as ε goes to 0, by*

$$\hat{C}(\tau) \cong [\Omega_+(\tau) + \Omega_-(\tau)] \circ \Pi , \tag{4}$$

with $\Pi = \mathrm{Op}_\varepsilon(\overline{\pi})$ and

$$\overline{\pi}(x,\xi) = \frac{\varepsilon^2}{4l_0^2} \int_{-\infty}^0 e^{-\int_t^0 a(\Phi_s(x,\xi))ds} p(\Phi_t(x,\xi))dt , \tag{5}$$

and if $a = a_0$ is constant

$$\overline{\pi}(x,\xi) = \frac{\varepsilon^2}{4l_0^2} \int_{-\infty}^0 e^{-a_0|t|} p(\Phi_t(x,\xi))dt .$$

I will compare our result (Equations (4) and (5)) to the Green's function. In the semi-classical regime, i.e. as $\varepsilon \to 0$, I have

$$G(t, A, B) \cong \frac{\varepsilon}{2i}\left[(\Omega_+(t) - \Omega_-(t)) \circ L_0^{-1}\right](A, B) ,$$

Let us now compute the τ-derivative of $C_{A,B}(\tau)$:

$$\frac{d}{d\tau}C(\tau) \cong -\frac{\varepsilon}{i}(\Omega_+(\tau) - \Omega_-(\tau)) \circ L_0 \circ \Pi .$$

In the case of white noise and constant attenuation a_0, I know (see for example [5] Section 5.1 for a derivation)) that

$$\frac{d}{d\tau}C(\tau) = -\frac{1}{2a_0}G(\tau) \tag{6}$$

which is consistent with the previous semi-classical formula.

I can now give a more concrete formula:

Corollary 4.1 *Writing $G(\tau, A, B)$ as a sum $\sum_\gamma G_\gamma$ of contributions of rays $\gamma(s)$ with $\gamma(0) = (B, \xi_B)$ and $\gamma(\tau) = \Phi_\tau(B, \xi_B) = (A, \xi_A)$, I get*

$$\frac{d}{d\tau} C(\tau, A, B) \cong \sum_\gamma M_\gamma G_\gamma \ ,$$

with

$$M_\gamma = -\frac{1}{2} \int_{-\infty}^0 e^{-\int_t^0 a(\gamma(s))ds} p(\gamma(t)) dt \ .$$

In the case of the white noise $p = 1$ and $a = a_0$, I recover the formula

$$M_\gamma = -1/2a_0 \ . \tag{7}$$

Let us also remark that, if there is an unique trajectory from B to A in time τ, the prefactor M_γ applies to the Green's function itself. It is the case, if I work with wave equations with constant coefficients in $\mathbb{R}^n$.

The previous formula is consistent with the observations of the paper [22]: the correlation $C_{A,B}(\tau)$ is not always an even function of τ as it is if the source is a white noise. The evenness is valid only up to scaling of $C_{A,B}(\tau)$:

$$C_{B,A}(\tau) = C_{A,B}(-\tau) \sim k C_{A,B}(\tau) \ .$$

The factor k is the ratio of the integrals giving M_γ for the ray $\gamma(t)$ going from B to A and $\gamma(-t)$ going from A to B.

5 Time scales

As I see from the general expression of the correlation given in Equation (3), the proof of the main theorem 4.1 involves the knowledge of the Green's function at large times. This is a well known difficulty and the semi-classical expansions of the Green's functions are valid up to to the so-called *Lyapounov time* which involves the Lyapounov exponent measuring the rate of instability of the ray dynamics. Roughly speaking, the Lyapounov exponent is the smallest number λ so that the distance between any to rays $\gamma_1(t)$ and $\gamma_2(t)$ satisfies the estimates

$$d(\gamma_1(t), \gamma_2(t)) \le Ce^{\lambda t} d(\gamma_1(0), \gamma_2(0))$$

with C independent of $\gamma_1(0)$ and $\gamma_2(0)$. There is an associated time scale $T_{\text{Lyap}} = 1/\lambda$. On the other hand there is an attenuation time scale for the wave dynamics expressed in terms of the decay of the Green's function

$$|G(t, x, y)| \le Ce^{-T/T_{\text{att}}} \ .$$

T_{att} satisfies the estimate $T_{\text{att}} \ge 2/\inf a$. The approximation given in Theorem 4.1 is better when $T_{\text{att}} >> T_{\text{Lyap}}$. In particular, this condition is necessary in order to get point-wise convergence (i.e. convergence for A and B fixed).

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6 The use of surface waves for passive imaging

A remarkable application of the previous tool is to the imaging of the earth crust [15, 25, 26, 3, 20, 21]. This is done using the part of the Green's function associated to the surface waves: the earth crust acts as a wave guide on elastic waves and these waves follow an effective wave equation. The effective Hamiltonian is described now: let us start with the acoustic wave equation $u_{tt} - \text{div}(n \, \text{grad} u) = 0$ with the function n coming from a stratified medium $n = n(\mathbf{x}, z)$ (here $z = 0$ is the surface) where n is weakly dependent of $\mathbf{x}$ (this can be formalized as $n(\mathbf{x}, z) = N(\varepsilon \mathbf{x}, z)$ with N smooth and ε small). Using the adiabatic separation of variables $u \sim U(\varepsilon x, z)e^{i\langle \mathbf{x} | \xi \rangle}$ with U weakly dependent of $\mathbf{x}$, I can operate as if n was independent of $\mathbf{x}$ and I get the reduced equation

$$U_{tt} + \text{Op}_1(\lambda(\mathbf{x}, \xi))U = 0$$

where $\lambda(\mathbf{x}, \xi)$ is an eigenvalue of the Sturm-Liouville operator

$$L_{\mathbf{x}, \xi} = -\frac{d}{dz}n(\mathbf{x}, z)\frac{d}{dz} + n(\mathbf{x}, z)\|\xi\|^2$$

with appropriate boundary conditions at $z = 0$.

From the correlation, I get the ray dynamics of the surface waves and hence the effective Hamiltonians $\lambda(\mathbf{x}, \xi)$. The inverse problem to be solved is the following inverse spectral problem: from the fundamental mode (or any other available mode) of $L_{\mathbf{x}_0, \xi}$ in some range of wave numbers $|\xi|$, recover $n(\mathbf{x}_0, z)$. This is the kind of well posed inverse problem for which analytical/numerical method can be used (see [4]).

7 A formula for the scattering of random plane waves

I have seen an exact formula for the correlation of the wave field when the attenuation a is constant and the source noise is a white noise. I will see another exact formula in the context of wave scattering by a perturbation sitting in a bounded domain of $\mathbb{R}^d$ (see [6]). This formula is very general and applies in all situations of wave scattering (scalar or elastic waves), i.e. for any medium which is homogeneous near infinity: non-homogeneity's lies at finite distances or there is a scattering by a bounded obstacle. This calculus was motivated by the result of [19], showing that this result is completely general.

Let us consider for example an acoustic wave equation (2) with $n = n_0$ outside a bounded set of $\mathbb{R}^d$. I will consider scattering solutions of the stationary wave equation

$$\text{div}(n \, \text{grad} u) - \omega^2 u = 0 \tag{8}$$

which are of the following form: let us define, for $\mathbf{k} \in \mathbb{R}^d$, the plane wave
$$e_0(x, \mathbf{k}) = e^{i\mathbf{k}.x} \ .$$
I am looking for solutions
$$e(x, \mathbf{k}) = e_0(x, \mathbf{k}) + e^s(x, \mathbf{k})$$
of equation (8) in $\mathbb{R}^d$, with $n_0 k^2 = \omega^2$([2]), where e^s, the scattered wave, satisfies the so-called Sommerfeld radiation condition:
$$e^s(x, \mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left(e^\infty \left(\frac{x}{|x|}, \mathbf{k} \right) + O \left(\frac{1}{|x|} \right) \right), \ x \to \infty \ .$$
The complex function $e^\infty(\hat{x}, \mathbf{k})$ is usually called the *scattering amplitude* and is a signature of the inhomogeneities. The functions $e(x, \mathbf{k})$ are deformed exponentials and allow to write an explicit spectral decomposition of our wave operator, which is a "deformation" of the Fourier transform.

Let us look at $e(x, \mathbf{k})$ as a random wave with $k = \omega/\sqrt{n_0}$ fixed. The point-point correlation of such a random wave $C^{\text{scatt}}_\omega(x, y)$ is given by:
$$C^{\text{scatt}}_\omega(x, y) = \int_{k\sqrt{n_0} = \omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} d\sigma(\hat{k}).$$

It is proved in [6], section 8, that
$$C^{\text{scatt}}_\omega(x, y) = -\frac{2^{d+1} \pi^{d-1} n_0^{d/2}}{\omega^{d-2}} \Im(G(\omega + i0, x, y)) \ ,$$
where $G(\omega, x, y)$ is the stationary Green's function, i.e. the Schwartz kernel of $(\omega^2 + \text{div}(n \ \text{grad}))^{-1}$.

8   Conclusions

I hope to have convinced the reader, even if he is not very much involved in mathematics, that it is possible to derive rather explicit asymptotic formulas for the correlation $C_{A,B}(\tau)$ of seismic noise. The main conclusion is that, in the semi-classical regime, even if the source noise is not homogeneous, the field correlation is very close to the Green's function; in many cases, there is only a prefactor which I computed and which introduces no phase shift. This prefactor vanishes if the support of the source noise does not meet the rays from B to A.

Many other ideas and applications remains to be exploited:

Is it possible to use the previous tools in order to get informations on the source noise? Can I extend the previous calculus to the case where the source noise is located on a surface? Can I do something similarly in other regimes of propagation, in particular in non-smooth media? Can I get applications of the general formula to monitoring?

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[2]As often, I denote $k := |\mathbf{k}|$ and $\hat{\mathbf{k}} := \mathbf{k}/k$

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