A semi-classical inverse problem II: recovering the potential

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1 Introduction

This paper is the continuation of [6], where Victor Guillemin and I proved the following result: the Taylor expansion of a potential $V(x)$ ($x \in \mathbb{R}$) at a non degenerate critical point x_0 of V, satisfying $V'''(x_0) \neq 0$, is determined by the semi-classical spectrum of the associated Schrödinger operator near the corresponding critical value $V(x_0)$. Here, I prove results which are stronger in some aspects: the potential itself, without any analyticity assumption, but with some genericity conditions, is determined from the semi-classical spectrum. Moreover, my method gives an explicit way to reconstruct the potential.

Inverse spectral results for Sturm-Liouville operators are due to Borg, Gelfand, Levitan, Marchenko and others (see for example [12]). They need the spectra of the differential operator with two different boundary conditions in order to recover the potential. My results are different in several aspects:

- They are *local* using only the part of the spectrum included in some interval $]-\infty, E[$ in order to get V in the inverse image by V of this interval.
- They need only *approximate* spectra.
- They still apply if the operator is *essentially self-adjoint.*

After having completed the present work, I founded that similar methods were already used by David Gurarie [9] in order to recover a surface of revolution from the joint spectrum of the Laplace operator and the momentum operator. In the present paper, the genericity assumptions are weaker and more explicit:

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- David Gurarie assumes that the potential is a Morse function with pairwise different critical values, while I assume only a weaker non degeneracy condition (see Section 10.1.1).
- His argument for the separation of spectra associated to the different wells is less explicit than mine which uses the semi-classical trace formula (see Section 12.3).
- He does not say a word about the problem of a non generic symmetry defect and explicit non isomorphic potentials with the same semi-classical spectra (Section 7 and Assumption 3 in Theorem 5.1).

Semi classics has been used in inverse specral problems since the seventies; for a recent review, the reader could look at [14].

2 Motivation I: surfaces of revolution

Let us consider on a 2−sphere the metric of revolution

$$ds^2 = dx^2 + a^4(x)dy^2$$

with $x \in [0, L]$ and $y \in \mathbb{R}/2\pi\mathbb{Z}$. We assume that $a(0) = a(L) = 0$, $a(x) > 0$ for $0 < x < L$ and a is smooth. The volume element is given by $dv = a^2(x)|dxdy|$ and the Laplace operator by

$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{2a'}{a}\frac{\partial}{\partial x} - \frac{1}{a^4}\frac{\partial^2}{\partial y^2} \ .$$

Using the change of function $f = aF$, we get the operator $P = a\Delta a^{-1}$ which is formally symmetric w.r. to $|dxdy|$:

$$P = -\frac{\partial^2}{\partial x^2} + \frac{a''}{a} - \frac{1}{a^4}\frac{\partial^2}{\partial y^2} \ .$$

If $F(x, y) = \varphi(x)\exp(ily)$ with $l \in \mathbb{Z}$, we define Q_l as follows

$$PF = l^2(Q_l\varphi)e^{ily} \ ,$$

and putting $\hbar = l^{-1}$, we get

$$Q_\hbar\varphi = -\hbar^2\varphi'' + \left(a^{-4} + \hbar^2 W\right)\varphi$$

with $W = \frac{a''}{a}$. It implies that the knowledge of the joint spectrum of Δ and ∂_y is closely related to the spectra of $Q_\hbar$ for $\hbar = 1/l$ with $l \in \mathbb{Z} \setminus 0$. This relates our paper to Gurarie's result [9].

3 Motivation II: effective surface waves Hamiltonian

In our paper [3] Section 7, we started with the following acoustic wave equation[1]

$$\begin{cases} u_{tt} - \mathrm{div}(n\ \mathrm{grad} u) = 0 \\ u(\mathbf{x}, 0, t) = 0 \end{cases} \tag{1}$$

in the half space $X = \mathbb{R}^{d-1}_{\mathbf{x}} \times]-\infty, 0]_z$ where $n(z) : \mathbb{R}_- \to \mathbb{R}_+$ is a non negative function which satisfies

$$0 < n_0 := \inf n(z) < n_\infty := \liminf_{z \to -\infty} n(z) \ .$$

This equation describes the propagation of acoustic waves in a medium which is stratified: the variations of the density are on much smaller scales vertically than horizontally[2]. This equation admits solutions of the form $\exp(i(\omega t - \mathbf{x}\xi))v(z)$ provided that v is an eigenfunction of the operator L_ξ on the half line $z \leq 0$ defined as follows:

$$L_\xi v := -\frac{d}{dz}\left(n(z)\frac{dv}{dz} \right) + n(z)|\xi|^2 v \tag{2}$$

with Dirichlet boundary conditions and eigenvalue ω^2. These solutions are exponentially localized near the boundary provided that ω^2 is in the discrete spectrum of L_ξ contained in $J :=]n_0|\xi|^2, n_\infty|\xi|^2[$.

Let us denote by $\lambda_1(\xi) < \lambda_2(\xi) < \cdots < \lambda_j(\xi) < \cdots$ the spectrum of L_ξ in the interval J and $v_j(\xi, z)$ the associated normalized eigenfunctions. The unitary map from $L^2(\partial X)$ into $L^2(X)$ defined by

$$T_j(a) := (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \hat{a}(\xi) v_j(\xi, z) e^{i\mathbf{x}\xi} d\xi \ ,$$

with $\hat{a}(\xi) := \int_{\mathbb{R}^{d-1}} a(\mathbf{x}) e^{-i\mathbf{x}\xi} d\mathbf{x}$, satisfies:

$$PT_j = T_j \mathrm{Op}(\lambda_j) \ ,$$

where $P = -\mathrm{div}(n\ \mathrm{grad} u)$ with Dirichlet boundary conditions and $\mathrm{Op}(\lambda_j)$ is an elliptic pseudo-differential operator of degree 2 and of symbol λ_j. So that, for each $j = 1, \cdots$, we get an effective surface wave Hamiltonian with the Hamiltonian λ_j. The map $T : \oplus_{j=1}^\infty L^2(\partial X) \to L^2(X)$ given by $T = \oplus_{j=1}^\infty T_j$ is an injective isometry.

[1] $u = u(\mathbf{x}, z, t)$ is the pressure, $n = K/\rho$ with ρ the density and $K > 0$ the in-compressibility assumed to be a constant. The acoustic wave equation is a simplification of the elastic wave equation which holds if the medium is fluid.

[2] In [3], we took a more complicated function $n(\mathbf{x}, z) = N(\mathbf{x}, z/\varepsilon, z)$ with N smooth and ε small

We see that the high frequency surface waves are associated to the semi-classical spectrum of a Schrödinger type operator

$$\mathcal{L}_\hbar = -\hbar^2 \frac{d}{dz}\left(n(z) \frac{d}{dz} \right) + n(z) \ ,$$

with $\hbar = \|\xi\|^{-1}$.

One can try to recover $n(z)$ from the propagation of surface waves: this is equivalent to get the operator $\mathcal{L}_\hbar$ from its semi-classical spectrum.

4 Schrödinger operators and spectra

The following notations will be used everywhere in this paper.

The *interval* I is defined by $I =]a, b[$ with $-\infty \leq a < b \leq +\infty$. The *potential* $V : I \to \mathbb{R}$ is a smooth function with $-\infty < E_0 := \inf V < E_\infty = \liminf_{x\to\partial I} V(x)$.

The *Schrödinger operator* $\hat{H}$ is any self-adjoint extension of the operator $-\hbar^2 \frac{d^2}{dx^2} + V(x)$ defined on $C_o^\infty(I)$.

The *discrete spectrum* of $\hat{H}_\hbar$ will be denoted by

$$\lambda_1(\hbar) < \lambda_2(\hbar) < \cdots < \lambda_l(\hbar) < \cdots \ .$$

Lemma 4.1 *The discrete spectra below E_∞ are, modulo $O(\hbar^\infty)$, independent of the boundary conditions.*

This comes from the fact that the eigenfunctions are $O(\hbar^\infty)$ outside the wells; this is proved, using *semi-classical ellipticity,* in [13], Section 2.9.

Using the previous Lemma, we can assume the we work always with the Friedrichs extension with initial domain (the closure of) $C_o^\infty(I)$.

The *semi-classical limit* is associated to the classical Hamiltonian $H = \xi^2 + V(x)$ whose dynamics is given by the vector field

$$X_H = 2\xi \frac{\partial}{\partial x} - V'(x) \frac{\partial}{\partial \xi}$$

in the phase space $T^\star I$, the cotangent space of I with the canonical coordinates (x, ξ).

Definition 4.1 *Let us give E with $E_0 < E \leq E_\infty$ and a positive real number N. We say that a sequence $\mu_l(\hbar)$, $l = 1, \cdots$ is a* **semi-classical spectrum of $\hat{H}$ mod** $o(\hbar^N)$ **in** $]-\infty, E[$ *if, we have for the l's so that $\lambda_l(\hbar) < E$,* **uniformly** *on every compact $K \subset]-\infty, E[$,*

$$\lambda_l(\hbar) = \mu_l(\hbar) + o(\hbar^N) \ .$$

In the paper [6], it was enough to know the asymptotic expansions of the λ_l's for all l, but not uniformly in l in order to recover the Taylor expansion of V at the point x_0 where V reaches is minimum.

5 A Theorem for one well potentials

In what follows, E is given with $E_0 < E \leq E_\infty$.

Theorem 5.1 *Let us assume that the potential $V : I \to \mathbb{R}$ satisfies:*

1. **A single well below E:** *for any $y < E$, the sets $I_y := \{x | V(x) \leq y\}$ are compact intervals. There exists an unique x_0 so that $V(x_0) = E_0$ $(= \inf_{x\in I} V(x))$. For any y with $E_0 \leq y < E$, if we define the functions $f_\pm : [E_0, E[\to \mathbb{R}$ so that the intervals I_y are defined by $I_y = [f_-(y), f_+(y)]$, we have $V'(x) < 0$ for $f_-(E_-) < x < x_0$ and $V'(x) > 0$ for $x_0 < x < f_+(E_-)$.*

2. **A genericity hypothesis at the minimum:** *there exists $N \geq 2$ so that the N-th derivative $V^{(N)}(x_0)$ does not vanish.*

3. **A generic symmetry defect:** *if there exists $x_\pm$, satisfying $f_-(E_-) < x_- < x_0 < x_+ < f_+(E_+)$ and $\forall n \in \mathbb{N}$, $V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$, then V is globally even w.r. to $x_0 = (x_- + x_+)/2$ in the interval I_E. This is true for example if V is real analytic.*

Then the spectra modulo $o(\hbar^2)$ *in the interval* $]-\infty, E[$ *of the Schrödinger operators $\hat{H}_\hbar$, for a sequence $\hbar_j \to 0^+$, determine V in the interval I_E up to a symmetry-translation $V(x) \to V(c \pm x)$.*

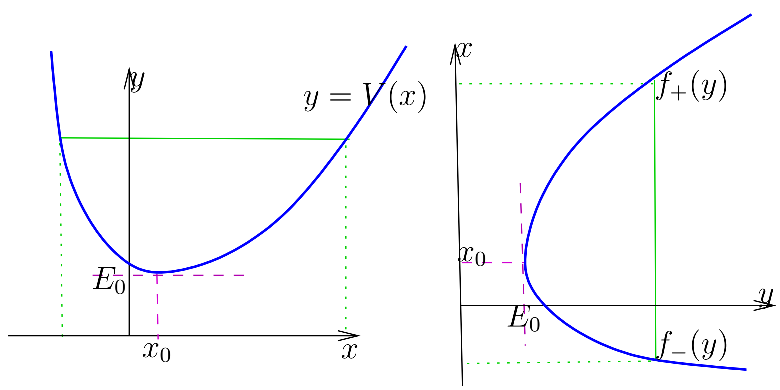

Figure 1: the potential V and the functions f_+ and f_-

6 One well potentials : Bohr-Sommerfeld rules and a "pseudo-differential" trace formula

It is a classical fact (see [4]) that the semi-classical spectrum (i.e. the spectrum up to $O(\hbar^\infty)$) of $\hat{H}_\hbar$ in the interval $]-\infty, E[$ is given by the "Bohr-Sommerfeld rules":

$$\Sigma(\hbar) = \{\mu_l(\hbar) \mid E_0 < \mu_l(\hbar) < E \text{ and } S(\mu_l(\hbar)) = 2\pi\hbar l\}$$

where, for $E_0 < y < E$, the function $S = S_\hbar(y) :]E_0, E[\to \mathbb{R}$ admits the formal series expansion

$$S(y) \equiv S_0(y) + \hbar\pi + \hbar^2 S_2(y) + \hbar^4 S_4(y) + \cdots \tag{3}$$

(the formal series S will be called the *semi-classical action* and the remainder term in the expansion is uniform in every compact sub-interval of $]E_0, E[$). We have

- $S_0(y) = \int_{\gamma_y} \xi dx = \int_{H(x,\xi)\leq y} |dx d\xi|$ with $\gamma_y = \{(x,\xi)|H(x,\xi) = y\}$ oriented according to the classical dynamics and

  $$\frac{dS_0}{dy}(y) = \int_{f_-(y)}^{f_+(y)} \frac{dx}{\sqrt{y - V(x)}}$$

  is the *period* $T(y)$ of the trajectory of energy y for the classical Hamiltonian H,

- If t is the time parametrization of γ_y (outside the caustic set $\{V(x) = y,\ \xi = 0\}$, we have $dt = dx/2\xi$),

  $$S_2(y) = -\frac{1}{12}\frac{d}{dy}\int_{\gamma_y} V''(x)|dt| \ ,$$

  which can be rewritten as:

  $$S_2(y) = -\frac{1}{12}\frac{d}{dy}\left(\int_{f_-(y)}^{f_+(y)} \frac{V''(x)dx}{\sqrt{y - V(x)}}\right) .$$

- For $j \geq 1$, $S_{2j}(y)$ is a linear combination of expressions of the form

  $$\left(\frac{d}{dy}\right)^n \int_{\gamma_y} P(V', V'', \cdots)|dt| \ ,$$

  where dt is the differential of the time on γ_y.

In what follows, we will use only S_0 and S_2. It will be convenient to relate the semi-classical action to the spectra by using the following trace formula:

Theorem 6.1 (ΨDO trace formula) *Let $f \in C_o^\infty(]E_0, E[)$ and $F(y) := -\int_y^\infty f(u)du$, we have, with $Z = T^\star I$:*

$$\text{Trace} F(\hat{H}) = \frac{1}{2\pi\hbar}\left(\int_Z F(H)|dx d\xi| - \hbar^2 \int_{E_0}^{E} f(y)(S_2(y) + \hbar^2 S_4(y) + \cdots)dy\right) + O(\hbar^\infty) . \tag{4}$$

Corollary 6.1 *The functions $S_0,\ S_2 : [E_0, E[\to \mathbb{R}$ are determined by the semi-classical spectrum mod $o(\hbar^2)$ in $]-\infty, E[$.*

In fact, S_0 is already given from the Weyl asymptotics:

$$\#\{\lambda_l(\hbar) \leq y\} \sim_{\hbar \to 0} \frac{S_0(y)}{2\pi\hbar} \ .$$

Weyl asymptotic formula can easily be deduced from the trace formula (4).

Remark 6.1 *The previous trace formula can be seen as an extension to the semi-classical setting of the famous "heat trace" method introduced by Mark Kac in [11] and strongly developped by geometers as a tool in the inverse spectral problem for the Laplace-Beltrami operator (see [2]): puting $t = \hbar^2$ in the heat semi-group $\exp(-t\Delta)$, we get $\exp(-t\Delta) = F(\hbar^2\Delta)$ with F the exponential function. This way, we get an identification of the previous expansion in powers of $\hbar^2$ with the heat trace expansion in powers of t.*

We give now a proof of Theorem 6.1.
Proof.–

1. *The case where F is compactly supported in J:*
   Defining $F^\star(H)$ by $F(\hat{H}) = \mathrm{Op}_{\mathrm{Weyl}}(F^\star(H))$ we know (see [8] Lemma 4.2) that, with $z_0 = (x_0, \xi_0)$ and $H_0 = H(z_0)$,

   $$F^\star(H)(z_0) = F(H_0) + \frac{1}{2}F''(H_0)(H-H_0)^{\star 2}(z_0) + \frac{1}{6}F'''(H_0)(H-H_0)^{\star 3}(z_0) + O(\hbar^4) \ .$$

   Computing the Moyal powers of $H - H_0$ at the point z_0 mod $O(\hbar^4)$, we get

   $$F^\star(H) = F(H) - \hbar^2\left(\frac{1}{8}f'(H)\det(H'') + \frac{1}{24}f''(H)H''(X_H, X_H)\right) + O(\hbar^4) \ .$$

   Computing the trace of $F(\hat{H})$ as $1/2\pi\hbar$ the phase space integral of the symbol $F^\star(H)$, we get:

   $$\mathrm{Trace}(F(\hat{H})) = \frac{1}{2\pi\hbar}\Big[\int_Z F(H)|dxd\xi| - \hbar^2\Big(\int_J f'(y)\big(\int_{\gamma_y} \det(H'')|dt|\big)|dy| \cdots$$
   $$\cdots + \tfrac{1}{24}\textstyle\int_J f''(y)\big(\int_{\gamma_y} H''(X_H, X_H)|dt|\big)|dy|\Big)\Big] + O\big(\hbar^3\big)$$

   Using Stokes formula, we have

   $$\int_{\gamma_y} H''(X_H, X_H)|dt| = -2\int_{H \leq y} \det(H'')|dxd\xi| \ ,$$

and the final result for $S_2(y)$ using an integration by part:

$$S_2(y) = -\frac{1}{24}\frac{d}{dy}\int_{\gamma_y} \det(H'')|dt| \ .$$

2. *The harmonic oscillator case* $\Omega = -\frac{1}{2}\left(\frac{d^2}{dx^2} + x^2\right)$*:*

$$\text{Trace}F(\Omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{F}\left((n+\frac{1}{2})\hbar\right)$$

   with $\tilde{F}$ even and coinciding with F on the positive axis. We get with the Poisson summation formula:

$$\text{Trace}F(\Omega) = \frac{1}{2\pi\hbar}\int\int F\left(\frac{x^2+\xi^2}{2}\right)|dxd\xi| + O(\hbar^\infty) \ .$$

   Moreover, $\int_{\gamma_y} V''|dt|$ is independent of y because the classical period of the harmonic oscillator is constant.

   A more general argument holds if $H = \Omega$ in support $F \circ H$. It is based on the identity

$$F^\star(\Omega) = F(\Omega) - \frac{\hbar^2}{8}F''(\Omega) + O(\hbar^4) \ .$$

3. *A deformation argument:* Let us consider a new Hamiltonian K which coincides with H in $H^{-1}(J')$ where $J' \subset J$ and J' contains the support of f. Then we have $\text{Trace}(F(\hat{H})) \equiv \text{Trace}(F(\hat{K}))$. This is because the symbols $F^\star(H)$ and $F^\star(K)$ coincide with $F(E_0)$ in the domain bounded by $H^{-1}(J')$. Because the righthan-sides of Equation (4) also coincide, we can choose a suitable K in order to prove it. We will choose K which has globally a single well and which coincides with an harmonic oscillator near its minimum.

4. *The final step:* we can assume that K is as before with $E_0 = \inf K$, K is harmonic in the energy interval $[E_0, E_0+\alpha]$.

   We split $F = F_0 + F_1$ where F_1 is compactly supported in $]E_0, +\infty[$ and F_1 supported in $]-\infty, E_0+\alpha[$. Equation (4) is valid for F_0 (case studied in 1.) and for F_1 (case of the harmonic oscillator).

□

Theorem 6.1 is closely related to (but a bit stronger) than what is proved in my paper [4]. The trace formula contains implicitly the Maslov index: it is no more valid if we replace $\hbar\pi$ by another value in the expansion of the semi-classical action given in Equation (3).

7 Two potentials with the same semi-classical spectra

We introduced a genericity Assumption 3 on symmetry defects in Theorem 5.1. The Figure 2 shows two one well potentials with the same semi-classical spectra mod $O(\hbar^\infty)$. The fact that they have the same semi-classical spectra comes from the description of Bohr-Sommerfeld rules in Section 6.

It would be nice to prove that they do NOT have the same spectra!

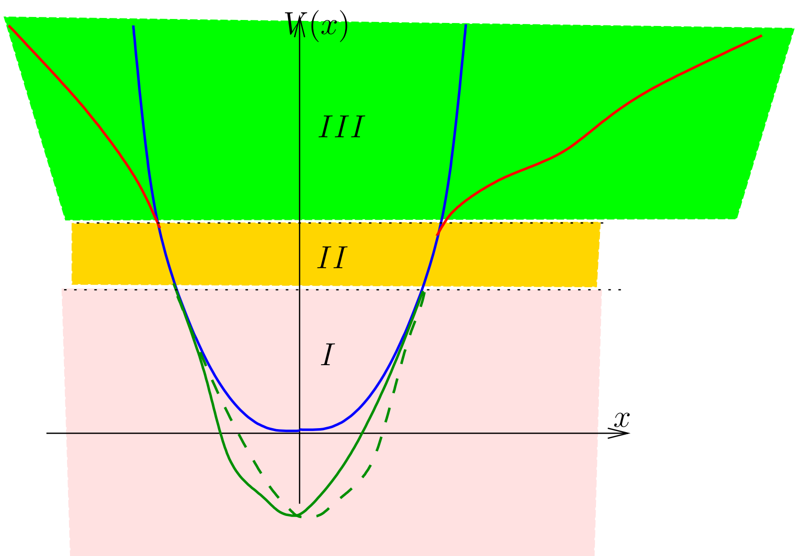

Figure 2: the (graphs of the) two potentials are the same in the sets II and III, they are mirror image of each other in I (green curve and dotted green curve), the potential is even in the set II.

8 One well potentials : the proof of Theorem 5.1

8.1 Some useful Lemmas

Lemma 8.1 *If V satisfies Assumption 2 in Theorem 5.1, we have:*

$$\lim_{y\to(E_0)_+}\int_{\gamma_y} V''(x)|dt| = \pi\sqrt{2V''(x_0)}\ .$$

This holds even if the minimum is degenerate[3].

[3] I do not know if this is still true without the genericity Assumption 2 in Theorem 5.1; it is the only place where I use it

The Lemma is clear if $V''(x_0) > 0$: the limit is then $V''(x_0)$ times the period of small oscillations of a pendulum which is $\pi\sqrt{2/V''(x_0)}$.

Let us consider the case of an isolated degenerate minimum with $V(x) = E_0 + a(x-x_0)^N(1+o(1))$ $(a > 0,\ N > 2)$, we can check that the integral to be evaluated is $O\left((y-E_0)^{\frac{3}{2}-\frac{3}{N}}\right) = o(1)$.

Lemma 8.2 *We have*

$$\lim_{y\to E_0}\left(\frac{1}{f_+'(y)} - \frac{1}{f_-'(y)}\right) = 0 \ .$$

Lemma 8.3 *If x_0 is the unique point where $V(x_0) = \inf V = E_0$, the first eigenvalue of $\hat{H}_\hbar$ satisfies $\lambda_1(\hbar) = E_0 + \hbar\sqrt{V''(x_0)/2} + o(\hbar)$*

This is well known if $V''(x_0) > 0$ and is still true otherwise by comparison: if $E_0 \leq V(x) \leq A(x-x_0)^2$ with $A > 0$, near x_0 then $E_0 < \lambda_1(\hbar) \leq 2\pi\hbar\sqrt{A}$.

8.2 Rewriting V using F and G

We will denote by $F = \frac{1}{2}(f_+ + f_-)$ and $G = \frac{1}{2}(f_+ - f_-)$.

- The function F is smooth on $]E_0, E[$, continuous on $[E_0, E[$ (smooth in the non degenerate case $V''(x_0) > 0$ as a consequence of the Morse Lemma), with $F(E_0) = x_0$, and is constant if and only if V is even w.r. to x_0. More generally, if F is constant on some interval, V is even on the inverse image of that interval. We call F the *parity defect.*

  Lemma 8.4 *Under the Assumption 3 in Theorem 5.1, the function F' is determined up to $\pm$ by its square.*

- The function G is smooth on $]E_0, E[$, continuous at $y = E_0$. We have $G(E_0) = 0$. It is clear that, from F and G, we can recover the restriction of V to I_E.

8.3 How to get V from S_0 and S_2

Let us consider, for $E_0 < y < E$,

$$I(y) := \int_{f_-(y)}^{f_+(y)} \frac{dx}{\sqrt{y - V(x)}}$$

and

$$J(y) = \int_{f_-(y)}^{f_+(y)} \frac{V''(x)dx}{\sqrt{y - V(x)}} \ .$$

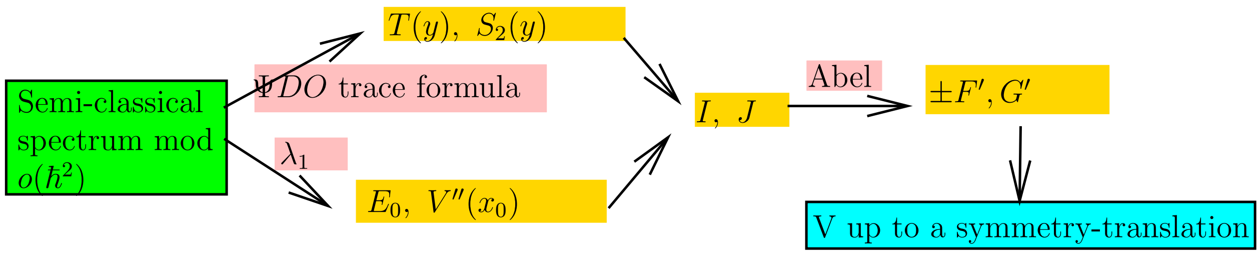

Figure 3: the scheme of the proof

We have $I(y) = dS_0(y)/dy$ and $S_2(y) = -(1/12)dJ(y)/dy$. This implies that S_0, S_2 and the limit $J(E_0)$ determine I and J. The limit $J(E_0)$ is determined by $V''(x_0)$ (Lemma 8.1) which is determined by the first semi-classical eigenvalue (Lemma 8.3). We can express I and J using F and G. Using the change of variables $x = f_+(u)$ for $x > x_0$ and $x = f_-(u)$ for $x < x_0$, we get:

$$I(y) = 2\int_{E_0}^{y} \frac{G'(u)du}{\sqrt{y-u}}$$

$$J(y) = \int_{E_0}^{y} \frac{d}{du}\left(\frac{1}{f'_+(u)} - \frac{1}{f'_-(u)}\right)\frac{du}{\sqrt{y-u}} \ .$$

Using Abel's result [1] (and Appendix), we can recover G' and

$$\frac{d}{dy}\left(\frac{1}{f'_+(y)} - \frac{1}{f'_-(y)}\right) = \frac{d}{dy}\left(\frac{2G'}{G'^2 - F'^2}\right) \ .$$

Using Lemma 8.2, we recover F'^2. The Assumption 3 implies that there exists an unique square root to F'^2 up to signs. From that we recover G' and $\pm F'$ and hence $\pm F$ and G modulo constants . This gives V up to change of x into $c \pm x$.

9 Taylor expansions

From the previous section, we see that the semi-classical spectra determine F'^2 and G even without assuming the hypothesis 3 of Theorem 5.1 on symmetry defect. It is not difficult to see that, if V satisfies the hypothesis 2 of Theorem 5.1, the parity defect F is a smooth function of $y^{2/N}$. We have the following:

Lemma 9.1 *Let us give two formal powers series $a = \sum_{j=0}^{\infty} a_j t^j$ and $b = \sum_{j=0}^{\infty} b_j t^j$ which satisfy $a^2 = b$. The equation $f^2 = b$ has exactly two solutions as formal powers series: $f = \pm a$.*

From this Lemma, we deduce the:

Theorem 9.1 *Under the Assumptions 1 and 2 of Theorem 5.1, but without Assumption 3, the Taylor expansion of V at a local minimum x_0 is determined (up to mirror symmetry) by the semi-classical spectrum modulo $o(\hbar^2)$ in a fixed neighborhood of E_0.*

In some aspects, this result is stronger than the one obtained in [6], but it requires the knowledge of the semi-classical spectrum in a fixed neighborhood of E_0, while, in [6], we need only N semi-classical eigenvalues in order to get $2N$ terms in the Taylor expansion.

10 A Theorem for a potential with several wells

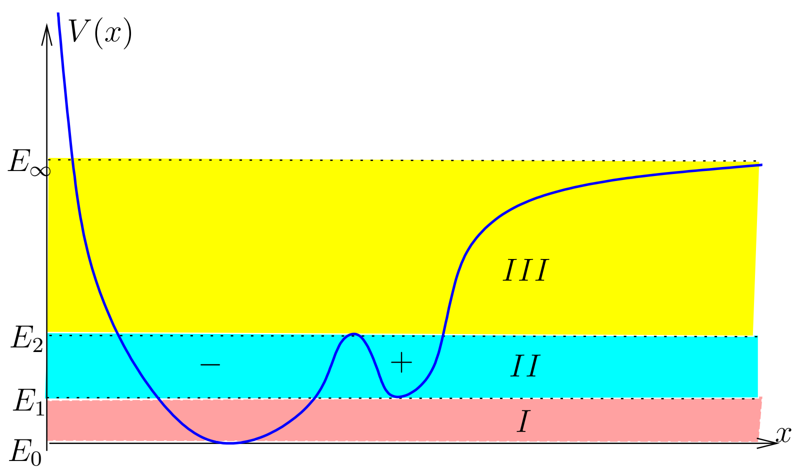

Figure 4: a 2 wells potential V

We will extend our main result to cases including that of Figure 4: a two wells potential with three critical values, $E_0 = 0$, E_1 and E_2. We can take any boundary condition at $x = 0$.

10.1 The genericity Assumptions

In what follows, we choose E so that $E_0 < E \leq E_\infty$ and define $I_E = \{x|V(x) < E\}$. The goal is to determine the restriction of V to I_E from the semi-classical spectrum in $]-\infty, E[$.

We need the following Assumptions which are generically satisfied. We introduce a:

Definition 10.1 *Two smooth functions $f, g : J \to \mathbb{R}$ are* weakly transverse *if, for every x_0 so that $f(x_0) = g(x_0)$, there exists an integer N such that the Nth derivative $(f - g)^{(N)}(x_0)$ does not vanish.*

10.1.1 Assumption on critical points

- for any point x_0 so that $V'(x_0) = 0$ and $V(x_0) < E$, there exists $N \geq 2$ so that, the N-th derivative $V^{(N)}(x_0)$ does not vanish.

- The *critical values* associated to different critical points are *distinct.*

The wells: Let us label the critical values of V below E_∞ as $E_0 < E_1 < \cdots < E_k < \cdots < E_\infty$ and the corresponding critical points by $x_0,\ x_1, \cdots$. The critical values can only accumulate at E_∞ because the critical points are isolated.

Let us denote, for $k = 1, 2, \cdots$ by $J_k =]E_{k-1}, E_k[$.

Definition 10.2 A well of order k *is a connected component of* $\{x \in I \mid V(x) < E_k\}$.

Let us denote by $N_k (\leq k)$ the number of wells of order k.

For any k, $H^{-1}(J_k)$ is an union of N_k topological annuli A_j^k and the map $H : A_j^k \to J_k$ is a fibration whose fibers $H^{-1}(y) \cap A_j^k$ are topological circles $\gamma_j^k(y)$ which are periodic trajectories of the classical dynamics: if $y \in J_k$, $H^{-1}(y) = \cup_{j=1}^{N_k} \gamma_j^k(y)$. We will denote by $T_j^k(y) = \int_{\gamma_j^k} |dt|$, the corresponding classical periods. We will often remove the index k in what follows.

We have the well known

Proposition 10.1 *The semi-classical spectrum in J_k is the union of N_k spectra which are given by Bohr-Sommerfeld rules associated to actions $S_j^k(y)$ given as in Section 6.*

This comes from the fact that the eigenfunctions are $O(\hbar^\infty)$ outside the wells. This is proved in [13] Section 2.9; see also [10] for much more precise results including estimates of the exponentially small "tunneling" effects.

10.1.2 A generic symmetry defect

If there exists $x_- < x_+$, satisfying $V(x_-) = V(x_+) < E$ and, $\forall n \in \mathbb{N},\ V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$, then V is globally *even* on I_E.

10.1.3 Separation of the wells

For any $k = 1, 2, \cdots$ and any j with $1 \leq j < l \leq N_k$, the *classical periods* $T_j(y)$ *and* $T_l(y)$ *are weakly transverse* in J_k.

10.2 Quartic potentials

If V is a polynomial of degree four with two wells like $V(x) = x^4 + ax^3 + bx^2$ with $b < 0$, the periods of the two wells (between E_1 and $E_2(= 0)$) are identical. This is because, on the complex projective compactification X_E (with $E < 0$) of $\xi^2 + V(x) = E$, the differential dx/ξ is holomorphic and the real part of X consists of 2 homotopic curves in X_E. One can check directly that all other actions $S_{2j},\ j \geq 1$ coincide; this is proved in [7] p. 191.

10.3 The statement of the result

Our result is:

Theorem 10.1 *Under the three Assumptions in Sections 10.1.1, 10.1.2 and 10.1.3, V is determined in the domain $I_E := \{x | V(x) < E\}$ by the semi-classical spectrum in $]-\infty, E[$ modulo $o(\hbar^{5/2})$ up to the following moves: I_E is an union of disjoint open intervals $I_{E,\alpha}$, each interval $I_{E,\alpha}$ is defined up to translation and the restriction of V to each $I_{E,\alpha}$ is defined up to $V(x) \to V(c - x)$.*

Remark 10.1 *We need $o(\hbar^{5/2})$ in the Theorem 10.1 while we needed only $o(\hbar^2)$ for the one well case in Theorem 5.1. This is due to the way we are able to separate the spectra associated to the different wells.*

11 The semi-classical trace formula

The semi-classical trace formula, also known as "Gutzwiller trace formula", is valid for a Schrödinger operator in any dimension (see [5] for a recent review). In the one dimensional case (and more generally in the "integrable" case), this formula can be derived from the Bohr-Sommerfeld rules, via the Poisson summation formula.

In this Section, we will derive the semi-classical trace formula in dimension 1 from the Bohr-Sommerfeld rules.

Let us start with the following application of the Poisson summation formula:

Lemma 11.1 *Let $S : J \to \mathbb{R}$ be a smooth function with $S' > 0$, then we have the following identity as Schwartz distributions in J, i.e. equality holds when applying both sides to a test function $\phi \in C_o^\infty(J)$,*

$$\sum_{l \in \mathbb{Z}} \delta(y - S^{-1}(2\pi\hbar l)) = \frac{1}{2\pi\hbar} \sum_{m \in \mathbb{Z}} e^{imS(y)/\hbar} S'(y) \ . \tag{5}$$

Let us insist that the identity (5) is valid for any fixed value of $\hbar$.

We will now develop semi-classical approximations of the identity (5). Let us start with the

Definition 11.1 *Let $D_\hbar$ be an $\hbar-$dependent distribution on the interval J. We will write $D_\hbar = o(\hbar^N)$ if for any $\hbar-$pseudo-differential operator $P = \mathrm{Op}_\hbar(p)$ with $p \in C_o^\infty(T^\star J)$, we have*

$$\|PD_\hbar\|_{L^2(J)} = o(\hbar^N) \ .$$

With the previous definition, we get

Lemma 11.2 *Let us give 2 sequences $\lambda_l(\hbar)$ and $\mu_l(\hbar)$ in J so that*

- *$\mu_l(\hbar) = \lambda_l(\hbar) + o(\hbar^N)$ uniformly on every compact of J*
- *$\#\{\lambda_l(\hbar) \in K\} = O(1/\hbar)$ for any K compact subset of J,*

then

$$\sum_{l\in\mathbb{Z}} \delta(y-\mu_l(\hbar)) - \delta(y-\lambda_l(\hbar)) = o(\hbar^{N-\frac{5}{2}}) \ .$$

Proof.–

Let us consider first the operator Q with symbol $a(\eta)\chi(y)$ with $a \in C_o^\infty(\mathbb{R})$ and $\chi \in C_o^\infty(J)$. The L^2 norm of Qu is equal to the L^2 norm of $a(\eta)$ times the $\hbar-$Fourier transform of χu. In our case, this is the L^2 norm of

$$a(\eta)\frac{1}{\sqrt{2\pi\hbar}}\sum_{l\in\mathbb{Z}}\left(\chi(\lambda_l(\hbar))e^{-i\eta\lambda_l(\hbar)/\hbar} - \chi(\mu_l(\hbar))e^{-i\eta\mu_l(\hbar)/\hbar}\right)$$

which is $o(\hbar^{N-\frac{5}{2}})$. Any other P is of the form $P = PQ$ for some suitable Q. The conclusion follows by the $\hbar-$uniform L^2 continuity of P.

□

Let us give another

Definition 11.2 *The* L^2-Microsupport *of a family of distributions $T_\hbar$ in the interval I is the closed subset of $T^\star I$ denoted* $\mathrm{MS}(T_\hbar)$ *given by*

$$((x,\xi)\notin \mathrm{MS}(T_\hbar)) \ \text{if and only if} \ (\exists p \in C_o^\infty(T^\star I), p(x,\xi)\neq 0 \text{ and } \mathrm{Op}_\hbar(p)T_\hbar = o(1)) \ .$$

We get the following statement of the semi-classical trace formula (for the general statement:

Theorem 11.1 *As distributions on J_k, we have, if $\mu_l(\hbar)$ is a semi-classical spectrum modulo $o(\hbar^{5/2})$,*

$$\sum_{l\in\mathbb{Z}}\delta(y-\mu_l(\hbar))) = \frac{1}{2\pi\hbar}\sum_{j=1}^{N_k}\sum_{m\in\mathbb{Z}}(-1)^m e^{imS_0^j(y)/\hbar}T_j(y)\left(1+im\hbar S_2^j(y)\right)+o(1) \ . \quad (6)$$

This means that $\sum_{l\in\mathbb{Z}} \delta(y-\mu_l(\hbar))$ is mod($o(1)$) a (locally finite in the cotangent space) sum of the WKB functions

$$Z_{m,j} = \frac{1}{2\pi\hbar}(-1)^m e^{imS_0^j(y)/\hbar} T_j(y)\left(1 + im\hbar S_2^j(y)\right)$$

associated to the Lagrangian manifolds (the micro-support of $Z_{m,j}$)

$$L_{m,j} := \{(y, mT_j(y)) | y \in I\} \ .$$

Proof.–

> The trace formula is a consequence of Equation (5) applied to the spectra given by the Bohr-Sommerfeld rules (see Section 6) and Lemma 11.2.

□

12 The case of several wells: the proof of Theorem 10.1

12.1 What can be read from the Weyl's asymptotics?

Lemma 12.1 *Under the Assumption 10.1.1, the singular (non smooth) points of the function $y \to A(y) = \int_{H(x,\xi)\leq y} |dxd\xi|$ in $]-\infty, E[$ are exactly the critical values $E_0 < E_1 < \cdots (< E)$ of V. Moreover,*

- *the function $A(y)$ in smooth on $]E_k - c, E_k]$, with $c > 0$, if and only if x_k is a local minimum of V,*
- *from the singularity of $A(y)$ at E_k, on can read the value of $V''(x_k)$.*

The function $A(y)$ is determined by the semi-classical spectrum mod $o(1)$, this is a consequence of the Weyl asymptotics:

$$\#\{\lambda_l(\hbar) \leq y\} \sim_{\hbar\to 0} \frac{A(y)}{2\pi\hbar} \ .$$

12.2 The scheme of the reconstruction

The proof is by "induction" on E.

We start by constructing the piece of V where $V(x) < E_1$ using Theorem 5.1.

We want then to construct V where $E_1 \leq V(x) < E_2$.

There are two cases:

1. x_1 *is not an extremum:* we know then V in the interval $\{V(x) \leq E_1\}$ by continuity. We can then extend the proof of Theorem 5.1 using the fact that we know, using Section 12.4, the limits of $\int_{\gamma_y} V''(x)|dt|$ and $f'_\pm(y)$ as $y \to E_1^+$. We can reduce to an Abel transform starting from E_1 using, for $E_1 < y < E_2$,
$$\int_{V(x)\leq y} = \int_{V(x)\leq E_1} + \int_{E_1 \leq V(x) \leq y}$$
where the first part is known from the knowledge of $V(x)$ in $\{x|V(x) \leq E_1\}$.

2. x_1 *is a local minimum:* using the separation of spectra (Section 12.3) and Theorem 5.1, we can construct the 2 wells of order 2 if we know $V''(x_1)$ (Lemma 12.1).

We then proceed to the interval $[E_2, E_3]$. A new case arises when x_2 *is a local maximum.* Then we need to glue together the wells of order 2. This case works then as before.

12.3 Separation of spectra

The main input of the proof of Theorem 10.1 is the fact that the Assumption 10.1.3 allows to split the semi-classical trace formulas in the interval J_k into the contributions of the N_k wells: from the spectra mod $o(\hbar^{5/2})$ in J_k, we will recover the WKB functions $Z_{1,j}$ for $j = 1, \cdots, N_k$.

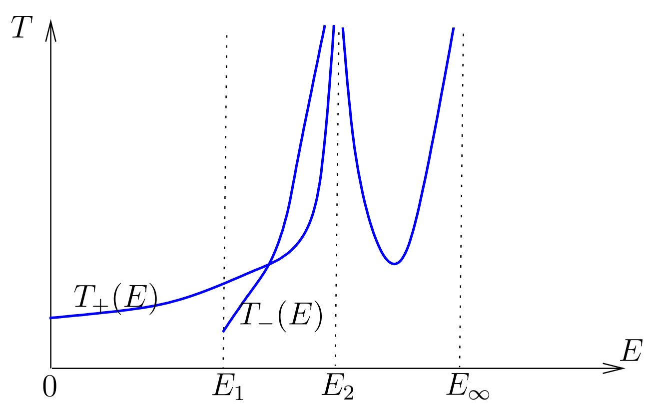

Figure 5: the primitive periods as functions of y for the Example of Figure 4

Let the distributions $D_\hbar = \sum_l \delta(y - \mu_l(\hbar))$ be given modulo $o(1)$ in the interval $J = J_k$ by Equation (6). The distributions $D_\hbar$ are determined mod $o(1)$ by the semi-classical spectra mod $o(\hbar^{5/2})$. Let us denote by B the set defined by
$$B := \{y \in J_k \mid \exists j \neq l,\ T_j(y) = T_l(y)\} \ ;$$

using Assumption 10.1.3, we see that B is discrete subset of J_k. Let us denote by $Z_\hbar$ the finite sum defined by the r.h.s of Equation (6) restricted to $m=1$, i.e.

$$Z_\hbar = \sum_{j=1}^{N_k} Z_{1,j} \ .$$

We have the

Lemma 12.2 *Using the weak transversality assumption of Section 10.1.3, the set B and the distributions $Z_\hbar$ mod $o(1)$ are determined by the distributions $D_\hbar$ mod $o(1)$.*

Proof.–

The difficulty is that there are possible cancellations in the trace formula: we do not assume the weak transversality of the non primitive periods mT_j.

Let us denote by $\tau_1(y) = \inf_j T_j(y)$; the function τ_1 is piecewise smooth. The non smooth points belong to B. Let us take a maximal (open) interval K where τ_1 is smooth. On K, $\tau_1 = T_{j_0}$ for an unique j_0 and $D_\hbar = Z_{1,j_0} + o(1)$ near the graph L_1 of τ_1, meaning that

$$\mathrm{MS}(D_\hbar - Z_{1,j_0}) \cap L_1 = \emptyset \ .$$

This is clear because the Lagrangian curves $L_{m,j}$ for $j \neq j_0$ and for $j = j_0,\ m \neq 1$ are disjoint from L_1. So, we can recover L_1 as

$$L_1 = \{(y,\eta) \mid \eta = \inf\{\eta' > 0,\ (y,\eta') \in \mathrm{MS}(D_\hbar)\}\} \ .$$

From Z_{1,j_0}, we recover, for any $m \in \mathbb{Z}$, the Z_{m,j_0}'s and we introduce a new distribution $D_\hbar^1$ in K defined by

$$D_\hbar^1 = D_\hbar - \sum_{m\in\mathbb{Z}} Z_{m,j_0} \ .$$

We do the same constructions with $D_\hbar^1$ in K: this allows to split again K into sub-intervals separated by points of B where $T_j(y) = T_l(y)$ for some $j \neq l,\ j \neq j_0,\ l \neq j_0$ and to get a function τ_2 and distributions $D_\hbar^2$. After a finite number of steps the new distributions $D_\hbar^N$ is $o(1)$. We have $N_k = N$ and B is the union of all points of non smoothness of the τ_j's.

□

We will need the:

Lemma 12.3 *There is an unique splitting of $Z_\hbar$ as a sum*

$$Z_\hbar(y) = \frac{1}{2\pi\hbar}\sum_{j=1}^{N_k}(a_j(y) + \hbar b_j(y))e^{iS_j(y)/\hbar} + o(1) ,$$

where the S_j's are smooth and the a_j's do not vanish.

Proof.–

> The L^2-Microsupport of $Z_\hbar$ is the union of the graphs of the S'_j: this decomposition is unique due to Assumption 10.1.3. Hence the decomposition of $Z_\hbar$ as a finite sum of smooth WKB functions is unique.

□

From the two previous Lemmas, it follows that, with Assumption 10.1.3, the spectrum in J_k modulo $o(\hbar^{5/2})$ determine the actions S_j and $S_{j,2}(y)$.

12.4 Limit values of some integrals

Using the trick of Section 8.3, we can use Abel's result (Section 13.4) once we know the following limits (or asymptotic behaviors) as $y \to E_j^+$ ($j = 0, 1, \cdots$):

- $f_\pm^j(y)$
- $\int_{H^{-1}(y)} V''(x)|dt|$ where $H = \xi^2 + V(x)$ is the classical Hamiltonian.
- $f_\pm'^j(y)$

All of them are determined by the knowledge of V in the set $\{x|V(x) \leq E_j\}$.

It is clear, except for the second one; we have:

Lemma 12.4 *Let us assume that V satisfies Assumption 1 of Section 10.1. If E_j is a critical value of V which is not a local minimum and $\tau(z) := \int_{H^{-1}(E_j+z)} V''(x)|dt| - \int_{H^{-1}(E_j-z)} V''(x)|dt|$, then $\lim_{z\to 0^+}\tau(z) = 0$.*

Proof.–

> We cut the integrals into pieces. One piece near each critical point and another piece far from them. Far from the critical points, the convergence is clear.
>
> - *Local maximum:* let us take a critical point where $V(x) = E_j - A(x - x_0)^{2N}(1 + o(1))$ with $N \geq 1$ and $A > 0$. We use a smooth

change of variable $x = \psi(y)$ with $\psi(0) = x_0$ so that $V(\psi(y)) = E_j - y^{2N}$. We are reduced to check that

$$\lim_{\varepsilon \to 0^+} \left(\int_0^1 \frac{W(y)dy}{\sqrt{\varepsilon + y^{2N}}} - \int_{\varepsilon^{1/2N}}^1 \frac{W(y)dy}{\sqrt{y^{2N} - \varepsilon}} \right) = 0 ,$$

assuming that $W(y) = O(y^{2N-2})$.

- *Other critical points:* let us take a critical point where $V(x) = E_j + A(x - x_0)^{2N+1}(1 + o(1))$ with $N \geq 1$ and $A > 0$. We use the same method.

□

13 Extensions to other operators

13.1 The statement

Let us indicate in this Section how to extend the previous results to the operator

$$\mathcal{L}_\hbar = -\hbar^2 \frac{d}{dx} \left(n(x) \frac{d}{dx} \right) + n(x)$$

which was introduced in Section 3. We want to recover the function $n(x)$. Let us sketch the one well case for which we will get:

Theorem 13.1 *Assuming that*

- *the function* $n(x)$ *admits a* non degenerate minimum $n(x_0) = E_0 > 0$*,*
- *the function* $n(x)$ *has* no critical values in $]E_0, E_1[$ *with* $E_1 \leq \liminf_{x \to \partial I} n(x)$*,*
- *the function* $n(x)$ *has* a generic symmetry defect *as in Theorem 5.1,*

then the function n *is determined in* $\{x | n(x) \leq E_1\}$ *by the semi-classical spectrum of* $\mathcal{L}_\hbar$ *modulo* $o(\hbar^2)$*.*

The proof works along the same lines as that of Theorem 5.1 except that we get an integral transform which is not exactly Abel's transform.

13.2 The Weyl symbol and the actions

The Weyl symbol l of $\mathcal{L}$ can be computed, using the Moyal product, as $l = \xi \star n \star \xi + n$. We get:

$$l(x, \xi) = n(x)(1 + \xi^2) + \frac{\hbar^2}{4} n''(x) .$$

The action S_0 satisfies:

$$\frac{dS_0}{dy}(y) = T(y) = \int_{n(x)\leq y} \frac{dx}{\sqrt{n(x)(y-n(x))}} \ .$$

The action S_2 is given from [4] by

$$S_2(y) = -\frac{1}{12}\frac{d}{dy}\int_{\gamma_y} \left(yn'' - 2\left(\frac{y}{n}-1\right)n'^2\right)|dt| - \frac{1}{4}\int_{\gamma_y} n''|dt| \ ,$$

which we rewrite:

$$S_2(y) = -\frac{1}{12}\frac{d}{dy}J(y) - \frac{1}{4}K(y) \ .$$

As in Section, using n instead of V, we introduce the fucntions $f_\pm$, F and G.

13.3 Recovering G

We have

$$T(y) = 2\int_{E_0}^{y} \frac{G'(z)}{\sqrt{z}}\frac{dz}{\sqrt{y-z}} \ .$$

So that T is the Abel transform, starting from E_0, of the continuous function $G'(z)/\sqrt{z}$ (E_0 is > 0). Using the inversion of Abel transform, we get G.

13.4 Recovering $\pm F$

- **The integral J:**

$$J(y) = \int_{x_-(y)}^{x_+(y)} \left(yn'' - 2\left(\frac{y}{n}-1\right)n'^2\right)\frac{dx}{\sqrt{n(y-n)}}$$

Using $x = f_\pm(y)$ as in Section 5 and

$$\Phi(y) = \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \ ,$$

we get $J(y) = (\mathcal{J}\Phi)(y)$, with

$$(\mathcal{J}\Phi)(y) = \int_{E_0}^{y} \left(y\Phi'(u) - 2\left(\frac{y}{u}-1\right)\Phi(u)\right)\frac{du}{\sqrt{u(y-u)}} \ .$$

- **The integral K:**

$$K(y) = \int_{E_0}^{y} \Phi'(u)\frac{du}{\sqrt{u(y-u)}}$$

and

$$K(y) = 2\frac{d}{dy}\int_{E_0}^{y} \Phi'(u)\frac{\sqrt{y-u}\ du}{\sqrt{u}}$$

which is rewritten as:

$$K(y) = 2\frac{d}{dy}(\mathcal{K}\Phi)(y)\ .$$

An integral transform

Lemma 13.1 *If $0 < E_0 < E_1$, the kernel of $A := \mathcal{J} + 6\mathcal{K}$ on the space of continuous function on $[E_0, E_1]$ at most two dimensional and all functions in this kernel are smooth.*

Proof.–

we have

$$A\Phi(y) = \int_{E_0}^{y}\left((7y-6u)\Phi'(u) - 2\left(\frac{y}{u}-1\right)\Phi(u)\right)\frac{du}{\sqrt{u(y-u)}}\ . \quad (7)$$

We compute $T \circ A$ with the operator T defined by $T\psi(y) = \int_{E_0}^{y}\frac{\psi(u)du}{\sqrt{y-u}}$. We will need the easy:

Lemma 13.2 *We have:*

$$\int_{E_0}^{y}\frac{udu}{\sqrt{y-u}}\int_{E_0}^{u} f(t)\frac{dt}{\sqrt{u-t}} = \frac{\pi}{2}\int_{E_0}^{y}(t+y)f(t)dt\ ,$$

and

$$\int_{E_0}^{y}\frac{du}{\sqrt{y-u}}\int_{E_0}^{u} f(t)\frac{dt}{\sqrt{u-t}} = \pi\int_{E_0}^{y} f(t)dt\ ,$$

Applying the previous formulas, we get:

$$T\circ A\ (\Phi)(y) = \frac{\pi}{2}\int_{E_0}^{y}[(t+y)(7\Phi'(t)-2\frac{\Phi(t)}{t})+2(-6t\Phi'(t)+2\Phi(t))]\frac{dt}{\sqrt{t}}\ .$$

Taking two derivatives:

$$\frac{\pi}{y^{3/2}}\frac{d^2}{dy^2}((T\circ A)\ \Phi)(y) = y^2\Phi''(y) + 4y\Phi'(y) - \Phi(y)\ .$$

From S_2 and $A\Phi(E_0)$, we get $A\Phi$, then we get $P(\Phi)$ where $P\phi = y^2\phi'' + 4y\phi' - \phi$ is a non singular linear differential equation (remind that $E_0 > 0$). So, if we know also $\Phi(E_0)$ and the asymptotic behavior of $\Phi'(E_0)$, we can get Φ. Let us assume $n''(x_0) = a > 0$. Then we have:

- $A\Phi(E_0) = 2\pi\sqrt{aE_0}$
- $\Phi(E_0) = 0$
- $\Phi'(y) \sim 4\sqrt{a}/\sqrt{y-E_0}$.

□

Appendix: Abel's result

Let us consider the linear operator T which acts on continuous functions on $[E_0, E[$ defined by:

$$Tf(x) = \int_{E_0}^{x} \frac{f(y)dy}{\sqrt{x-y}} \ .$$

Then $T^2 f(x) = \pi \int_{E_0}^{x} f(y)dy$. This implies that T is injective! This is the content of [1].

References

[1] Niels Abel. Auflösung einer mechanichen Aufgabe. *Journal de Crelle* **1***:153-157 (1826).*

[2] Marcel Berger, Paul Gauduchon & Edmond Mazet. Le spectre d'une variété riemannienne comapcte. *Springer Lecture Notes in Maths, Vol.* **194** *(1971).*

[3] Yves Colin de Verdière. Semi-classical analysis and passive imaging. *Nonlinearity* **22***:45–75 (2009).*

[4] Yves Colin de Verdière. Bohr-Sommerfeld rules to all orders. *Ann. Henri Poincaré* **6***:925-936 (2005).*

[5] Yves Colin de Verdière. Spectrum of the Laplace operators and periodic geodesics: thirty years after. *Ann. Inst. Fourier* **57***:2429–2463 (2007).*

[6] Yves Colin de Verdière & Victor Guillemin. A semi-classical inverse problem I: Taylor expansions. *ArXiv 0802.1605 .*

[7] Eric Delabaere & Frédéric Pham. Unfolding the quartic oscillator. *Annals of Physics* **261***:180–218 (1997).*

[8] Alfonso Gracia-Saz. The symbol of a function of a pseudo-differential operator. *Ann. Inst. Fourier* **55***:2257–2284 (2005).*

[9] David Gurarie. Semi-classical eigenvalues and shape problems on surface of revolution. *J. Math. Phys.* **36***:1934–1944 (1995).*

[10] Bernard Helffer & Johannes Sjöstrand. Multiple wells in the semi-classical limit I, *Commun. in PDE,* **9***(4):337-408 (1984).*

[11] Mark Kac. Can one hear the shape of a drum? *Amer. Math. Monthly,* **73***(4):1–23 (1966).*

[12] B. M. Levitan & M. G. Gasymov. Determination of a differential equation by two of its spectra. *Russian Math. Surveys* **19***(2):1–63 (1964).*

[13] André Martinez. An Introduction to semi-classical Theory. *Springer (2002).*

[14] Steve Zelditch. The inverse spectral problem. *Surveys in Differential Geometry* **IX***:401–467 (2004).*