The semi-classical spectrum and the Birkhoff normal form

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Introduction

The purposes of this note are

- To propose a direct and "elementary" proof of the main result of [3], nameley that the semi-classical spectrum near a global minimum of the classical Hamiltonian determines the whole semi-classical Birkhoff normal form (denoted the BNF) in the non-resonant case. I believe however that the method used in [3] (trace formulas) are more general and can be applied to any non degenerate non resonant critical point provided that the corresponding critical value is "simple".
- To present in the completely resonant case a similar problem which is NOT what is done in [3]: there, only the *non-resonant part* of the BNF is proved to be determined from the semi-classical spectrum!

1 A direct proof of the main result of [3]

1.1 The Theorem

Let us give a semi-classical Hamiltonian $\hat{H}$ on $\mathbb{R}^d$ (or even on a smooth connected manifold of dimension d) which is the Weyl quantization of the symbol $H \equiv H_0 + \hbar H_1 + \hbar^2 H_2 + \cdots$.

Let us assume that H_0 has a global non degenerate non resonant minimum E_0 at the point z_0: it means that after some affine symplectic change of variables

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$H_0 = E_0 + \frac{1}{2}\sum_{j=1}^{d}\omega_j(x_j^2+\xi_j^2)+\cdots$ where the ω_j's are >0 and independent over the rationals. We can assume that $0<\omega_1<\omega_2<\cdots<\omega_d$. We will denote $E_1 = H_1(z_0)$.

We assume also that

$$\liminf_{(x,\xi)\to\infty} H(x,\xi) > E_0 \ .$$

Let us denote by $\lambda_1(\hbar)<\lambda_2(\hbar)\leq\cdots\leq\lambda_N(\hbar)\leq\cdots$ the discrete spectrum of $\hat{H}$. This set can be finite for a fixed value of $\hbar$, but, if N is given, $\lambda_N(\hbar)$ exists for $\hbar$ small enough.

Definition 1.1 *The* semi-classical spectrum *of $\hat{H}$ is the set of all $\lambda_N(\hbar)$ ($N=1,\cdots$) modulo $O(\hbar^\infty)$.* **NO uniformity with respect to N in the $O(\hbar^\infty)$ is required.**

Definition 1.2 *The semi-classical Birkhoff normal form is the following formal series expansion in $\Omega=(\Omega_1,\cdots,\Omega_d)$ and $\hbar$:*

$$\hat{B} \equiv E_0 + \hbar E_1 + \sum_{j=1}^{d}\omega_j\Omega_j + \sum_{l+|\alpha|\geq 2} c_{l,\alpha}\hbar^l\Omega^\alpha$$

with $\Omega_j = \frac{1}{2}\left(-\hbar^2\partial_j^2 + x_j^2\right)$. The series $\hat{B}$ is uniquely defined as being the Weyl quantization of some symbol B equivalent to the Taylor expansion at z_0 of H by some automorphism of the semi-classical Weyl algebra (see [2]).

The main result is the

Theorem 1.1 ([3]) *Assume as before that the ω_j's are linearly independent over the rationals. Then the semi-classical spectrum and the semi-classical Birkhoff normal form determine each other.*

The main difficulty is that the spectrum of $\hat{B}$ is naturally labelled by $d-$uples $\mathbf{k}\in\mathbb{Z}_+^d$ while the semi-classical spectrum is labelled by $N\in\mathbb{N}$. We will denote by ψ the bijection $\psi : N\to\mathbf{k}$ of $\mathbb{N}$ onto $\mathbb{Z}_+^d := \{\mathbf{k}=(k_1,\cdots,k_d)|\forall j,\ k_j\in\mathbb{Z},\ k_j\geq 0\}$ given by ordering the numbers $\langle\omega|\mathbf{k}\rangle$ in increasing order: they are pair-wise distincts because of the non-resonant assumption.

2 From the semi-classical Birkhoff normal form to the semi-classical spectrum

We have the following result

Theorem 2.1 *The semi-classical spectrum is given by the following power series in $\hbar$:*

$$\lambda_N(\hbar) \equiv E_0 + \hbar \left(E_1 + \frac{1}{2}\langle \omega | \psi(N) + \frac{1}{2} \rangle \right) + \sum_{j=2}^{\infty} \hbar^j P_j(\psi(N)) \tag{1}$$

where the P_j's are polynomials of degree j given by

$$P_j(Z) = \sum_{l+|\alpha|=j} c_{l,\alpha} \left(Z + \frac{1}{2} \right)^{\alpha} .$$

This result is an immediate consequence of results proved by B. Simon [5] and B. Helffer-J. Sjöstrand [4] concerning the first terms, and by J. Sjöstrand in [6] (Theorem 0.1) where he proved a much stronger result.

3 From the semi-classical spectrum to the ω_j's

3.1 Determining the ω_j's

Because $E_0 = \lim_{\hbar \to 0} \lambda_1(\hbar)$, we can substract E_0 and assume $E_0 = 0$.

By looking at the limits, as $\hbar \to 0$, $\mu_N := \lim \lambda_N(\hbar)/\hbar$ (N fixed), we know the set of all $E_1 + \sum_{j=1}^d \omega_j (k_j + \frac{1}{2})$, $(k_1, \cdots, k_d) \in \mathbb{Z}_+^d$.

Let us give 2 proofs that the μ_N's determine the ω_j's.

1. **Using the partition function:** from the μ_N's, we know the meromorphic function
$$Z(z) := \sum e^{-z\mu_N} .$$
$$Z(z) := e^{-z(E_1 + \frac{1}{2}\sum_{j=1}^d \omega_j)} \sum_{\mathbf{k} \in \mathbb{Z}_+^d} e^{-z\langle \omega | \mathbf{k} \rangle} ,$$
We have
$$Z(z) = e^{-z(E_1 + \frac{1}{2}\sum_{j=1}^d \omega_j)} \Pi_{j=1}^d (1 - e^{-z\omega_j})^{-1} ,$$
The poles of Z are $\mathcal{P} := \cup_{j=1,\cdots,d} \{ \frac{2\pi i \mathbb{Z}}{\omega_j} \}$. The set of ω_j is hence determined up to a permutation. We fix now $\omega = (\omega_1, \cdots, \omega_d)$ with $\omega_1 < \omega_2 < \cdots$.
From the knowledge of the ω_j's, we get the bijection ψ.

2. **A more elementary proof:** substract $\mu_1 = E_1 + \frac{1}{2}\sum \omega_j$ from the whole sequence and denote $\nu_N = \mu_N - \mu_1$. Then $\omega_1 = \nu_2$. Then remove the multiples of ω_1. The first remaining term is ω_2. Remove all integer linear combinations of ω_1 and ω_2, the first remaining term is ω_3, $\cdots$

3.2 Determining the $c_{l,\alpha}$'s

Let us first fix N: from Equation (1) and the knowledge of λ_N mod $O(\hbar^\infty)$ we know the $P_j(\psi(N))$'s for all j's.

Doing that for all N's and using ψ determine the restriction of the P_j's to $\mathbb{Z}_+^d$ and hence the P_j's.

4 A natural question in the resonant case

4.1 The context

For simplicity, we will consider the completely resonant case $\omega_1 = \cdots = \omega_d = 1$ and work with the Weyl symbols. Let us denote by $\Sigma = \frac{1}{2}\sum(x_j^2 + \xi_j^2)$.

The (Weyl symbol of the) QBNF is then of the form

$$B \equiv \Sigma + \hbar P_{0,1} + \sum_{n=2}^{\infty} \sum_{j+l=n} \hbar^j P_{2l,j}$$

where $P_{2l,j}$ is an homogeneous polynomial of degree $2l$ in (x, ξ), Poisson commuting with Σ: $\{\Sigma, P_{2l,j}\} = 0$[1].

For example, the first non trivial terms are:

- for $n = 2$: $P_{4,0} + \hbar P_{2,1} + \hbar^2 P_{0,2}$
- for $n = 3$: $P_{6,0} + \hbar P_{4,1} + \hbar^2 P_{2,2} + \hbar^3 P_{0,3}$.

The semi-classical spectrum splits into clusters C_N of $N+1$ eigenvalues in an interval of size $O(\hbar^2)$ around each $\hbar(N + \frac{1}{2}d + P_{0,1})$ with $N = 0, 1, \cdots$.

The whole series B is however NOT unique, contrary to the non-resonant case, but defined up to automorphism of the semi-classical Weyl algebra commuting with Σ.

Let G be the group of such automorphisms (see [2]). The natural question is roughly:

Is the QBNF determined modulo G from the semi-classical spectrum, i.e. from all the clusters?

4.2 The group G

The linear part of G is the group M of all A's in the symplectic group which commute with $\hat{H}_2$, i.e. the unitary group $U(d)$.

We have an exact sequence of groups:

$$0 \to K \to G \to M \to 0 \; .$$

[1]The Moyal bracket of any A with H_2 reduces to the Poisson bracket

Let us describe K (the "pseudo-differential" part):

Let $S = S_3 + \cdots$ in the Weyl algebra (the formal power series in $(\hbar, x, \xi)$ with the Moyal product and the usual grading degree$(\hbar^j x^\alpha \xi^\beta) = 2j + |\alpha| + |\beta|$)

$$g_S(H) = e^{iS/\hbar} \star H \star e^{-iS/\hbar}$$

preserves Σ iff $\{S_n, \Sigma\} = 0$. This implies that n is even and S_n is a polynomial in $z_j \overline{z_k}$ $(z_j = x_j + i\xi_j)$. Then K is the group of all g_S's with $\{S, \Sigma\} = 0$.

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