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SPECTRUM OF THE LAPLACE OPERATOR AND PERIODIC GEODESICS: THIRTY YEARS AFTER

by Yves COLIN de VERDIÈRE

Abstract. — What is called the "Semi-classical trace formula" is a formula expressing the smoothed density of states of the Laplace operator on a compact Riemannian manifold in terms of the periodic geodesics. Mathematical derivation of such formulas were provided in the seventies by several authors. The main goal of this paper is to state the formula and to give a self-contained proof independent of the difficult use of the global calculus of Fourier Integral Operators. This proof is close in the spirit of the first proof given in the authors thesis. It uses the time-dependent Schrödinger equation, some facts about the geodesic flow, the stationary phase approximation and the metaplectic representation as a computational tool.

Résumé. — On appelle "Formule de trace semi-classique" une formule exprimant la densité d'état régularisée du laplacien d'une variété riemannienne compacte en termes de ses géodésiques périodiques. Des preuves de telles formules ont été données par plusieurs auteurs dans les années 70. Le but principal de cet article est de présenter cette formule et d'en donner une preuve complète et indépendante du difficile calcul global des opérateurs intégraux de Fourier. Cette preuve est d'un esprit assez proche de celle de la thèse de l'auteur. Elle utilise l'équation de Schrödinger dépendant du temps, des propriétés des géodésiques, la méthode de la phase stationnaire et la représentation métaplectique comme outil de calcul.

Introduction

More[(1)] than 30 years after the first original works on the *semi-classical trace formulae* (SCTF), it is still not possible to find a complete presentation in textbooks. The standard reference [22] is quite difficult to read due to the massive use of the global symbolic calculus of Fourier Integral Operators. My own approach [52], [53] is closer to the intuition given from

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the Feynman path integral, but the mathematical analysis is still difficult to follow. In the present paper, I tried to present an optimized proof which works for Laplace operators on Riemannian manifolds and even for semi-classical Schrödinger operators.

The aim of SCTF is to derive properties of the eigenvalues of a *Schrödinger operator* $\widehat{H}_h$ in the *semi-classical regime* where h, the Planck constant, is very small. From the mathematical point of view, h will tend to 0^+. The formal idea is to compute the trace of a suitable function of $\widehat{H}_h$ in two ways

▷ $\mathrm{Trace}(f(\widehat{H}_h)) = \sum_j f(E_j(h))$ where $E_j(h)$ are the eigenvalues of $\widehat{H}_h$,

▷ $\mathrm{Trace}(f(\widehat{H}_h)) = \int_X [f(\widehat{H}_h)](x,x)|\mathrm{d}x|$ where $[A](x,y)$ denotes the Schwartz kernel of the operator A.

Then comes the hard part which is to find exact or approximate formulae for the kernel $[f(\widehat{H}_h)](x,y)$ using the methods of partial differential equations.

For example, if Δ_g is the Laplace operator on a compact Riemannian manifold, we start with $\widehat{H} = \frac{1}{2}h^2\Delta_g$. There are several possibilities for f:

▷ $f(E) = \mathrm{e}^{-tE}$ which corresponds to *heat equation*,

▷ $f(E) = \mathrm{e}^{-it\sqrt{2E}/h}$ which corresponds to the *wave equation*,

▷ $f(E) = \mathrm{e}^{-itE/h}$ which corresponds to the *Schrödinger equation*.

The trace of the heat kernel $\sum \exp(-\frac{1}{2}th^2\lambda_j)$ where λ_j are the eigenvalues of Δ_g admits as $h \to 0$ (or $t \to 0$ with $h = 1$) an asymptotic expansion whose coefficients are related to locally computable quantities like curvature and see nothing about the classical dynamics [10].

Wave equations give the most natural results for the Laplace operator, but are working only for the Laplace operator. Moreover they make a strong use of Schwartz's distributions and Fourier Integral Operators [22].

We will use Schrödinger equation. Our goal is to derive a rather elementary proof which is not using the full symbolic calculus of Fourier integral operators, but only the *metaplectic representation.*

We will start with a simpler type of operator which is the quantization of a *twist map* and then see how to reduce the Schrödinger case to the previous case.

There will be three main parts:

1) We will first write a trace formula for a Fourier Integral Operators U associated with a twist canonical transformation: the Schwartz kernel of such an operator admits a WKB form and the asymptotics of the traces $t_N := \mathrm{Trace}(U^N)$ can be derived from the stationary phase approximation.

The main result is a rather explicit asymptotic expansion involving the fixed points of χ^N.

2) From the previous formula, we will derive a trace formula for *Schrödinger operators* in the semi-classical limit and the expression of the smoothed density of states known as *Gutzwiller formula*. Gutzwiller formula expresses the smoothed density of states in terms of periodic orbits of the classical limit which is an Hamiltonian system. We will need the short time asymptotic of the propagator which goes back to Lax and Hörmander [39], [33].

3) One of the main difficulties is the computation of the stationary phase expansions. We will show how the calculus of determinants can be achieved using the metaplectic representation. In particular, we will give formulae for some *determinants of discrete Sturm-Liouville operators* (Jacobi type matrices) which goes back to a formula of Levit and Smilansky [40] in the continuous case (see also [38]).

1. About the history

SCTF has several origines : on one side, Selberg's trace formula [50] is an exact summation formula concerning the case of locally symmetric spaces; this formula was interpreted by H. Huber [36] as a formula relating eigenvalues of the Laplace operator and lengths of closed geodesics (also called the "lengths spectrum") on a closed surface of curvature -1.

On the other side, around 1970, two groups of physicists developed independently *asymptotic* trace formulae:

▹ M. Gutzwiller [29] for the Schrödinger operator, using the quasi-classical approximation of the Green function; it is interesting to note that the word "trace formula" is not written, but Gutzwiller instead speaks of a new "quantization method" (the old one being "EBK" or "Bohr-Sommerfeld rules").

▹ R. Balian and C. Bloch [4], [5], [6] for the eigenfrequencies of a cavity used what they call the "multiple reflection expansions". They asked about a possible application to Kac's problem.

At the same time, under the influence of Mark Kac's famous paper [37] "Can one hear the shape of a drum?", mathematicians became quite interested into inverse spectral problems, mainly using heat kernel expansions (for the state of the art around 1970, see [10]).

The SCTF was put into its final mathematical form for the Laplace operator on compact Riemannian manifolds without boundary by three groups:

▹ In my thesis [52], [53] inspired from the work of Balian and Bloch, I used the short time expansion of the Schrödinger kernel and an approximate Feynman path integral. As a corollary, I proved that the spectrum of the Laplace operator determines generically the lengths of closed geodesics.

▹ Soon after my lectures in Nice, J. Chazarain [20] saw how to derive the qualitative form of the trace for the wave kernel using FIO's.

▹ Independently, H. Duistermaat and V. Guillemin [22], using the full power of the symbolic calculus of FIO's, were able to compute the main term of the singularity from the Poincaré map of the closed orbit. Their paper became, at least for mathematicians, the canonical reference on the subject.

We will see in section 7 how the Schrödinger trace is related to the wave trace.

After that, people were able to extend SCTF to:

▹ General semi-classical Hamiltonians [16], [43].

▹ Manifolds with boundary [27].

▹ Surfaces with conical singularities and polygonal billiards [13], [31].

▹ Several operators commuting operators [19].

▹ Around critical points of the Hamiltonian [17].

▹ In the presence of a finite symmetry group [18].

Recently, people [26], [60], [61] become able to say something about the non principal terms in the singularities expansion which come from the semi-classical Birkhoff normal form.

2. The trace formula for a quantized twist

2.1. Twist maps

Definition 1. — *A* twist map *is a canonical diffeomorphism $\chi : V \to W$ where $V_{(y,\eta)}$ and $W_{(x,\xi)}$ are open sets of the cotangent bundle $T^\star X$ of a smooth d-manifold X, which satisfies: the projection $\pi : \Gamma_\chi \to X \times X$, (defined by $\pi\left((y,\eta),(x,\xi)\right) = (x,y)$), where Γ_χ is the graph of χ, is a diffeomorphism of Γ_χ onto an open subset Ω of $X \times X$.*

The twist map χ will be called exact *if the closed form α_χ, which is the restriction of the the Liouville 1-form $\xi \mathrm{d}x - \eta \mathrm{d}y$ of $T^\star(X \times X)$ to Γ_χ is exact.*

We will call $S : \Omega \to \mathbb{R}$ any function which satisfies $\mathrm{d}S = \alpha_\chi$. Such a function S is called a generating function *of χ.*

A locally twist map *is a canonical diffeomorphism* $\chi : V \to W$ *which is locally a twist map.*

If χ is a twist map with generating function S, we have

$$\chi\Big(y, -\frac{\partial S}{\partial y}\Big) = \Big(x, \frac{\partial S}{\partial x}\Big).$$

Also, if $A := \chi'(z_0)$ is the derivative of χ at the point z_0 and if χ is twist at z_0, A is a twist linear map whose generating function is the Hessian of the generating function S of χ at the point (x_0, y_0) corresponding to z_0.

Example 2.1 (Twist maps of the annulus). — This is a much studied example since Poincaré and Birkhoff (see [9]). We take $X = \mathbb{R}/\mathbb{Z}$:

▷ The *Poincaré map* P of an elliptic periodic orbit of an Hamiltonian system with 2 degrees of freedom: the map P admits in the generic case a so-called Birkhoff normal form

$$P : (\theta, \rho) \longmapsto \big(\theta + F(\rho) + 0(\rho^N), \rho + 0(\rho^N)\big)$$

and hence is a twist map of the annulus for ρ close to 0 if $F'(0) \neq 0$.

▷ The *Frenkel-Kontorova* map $\chi(y, \eta) = (y + \alpha(y) + \eta, \alpha(y) + \eta)$ is a twist canonical map which is exact iff $\alpha(y)$ is the derivative of a periodic function $f(y)$. In the latter case, we can take

$$S(x, y) = \tfrac{1}{2}(x - y)^2 + f(y).$$

▷ The *billiard map*: if B is a smooth convex billiard plane table with a boundary ∂B, the return map, $T(s, \sin\theta) = (s', \sin\theta')$ where $s \in \mathbb{R}/L\mathbb{Z}$ is the arc length on ∂B and θ the incidence angle on the boundary at the point $m(s)$, contains the main part of the dynamics and is a twist map [38]. It admits the distance $S(s, s') = d(m(s), m(s'))$ as a generating function.

Example 2.2 (Small time flow of a regular Lagrangian). — Let us start with a Lagrangian $L(x, v) : TX \to \mathbb{R}$.

Definition 2. — *The Lagrangian L is said to be* regular *if the* Legendre transform

$$\mathrm{Leg} : (x, v) \longrightarrow \Big(x, \frac{\partial L}{\partial v}\Big)$$

is a global diffeomorphism from TX onto $T^\star X$.

Let φ_t be the associated Hamiltonian flow: $\varphi_t : T^\star X \to T^\star X$. If $V \subset T^\star X$ is an open *bounded* set, the flow φ_t restricted to V is a twist map for $t \neq 0$ small enough.

Proof. — Consider the expansion

$$\varphi_t(y,\eta) = \Big(y + t\frac{\partial H}{\partial \xi}(y,\eta) + O(t^2), \eta - t\frac{\partial H}{\partial x}(y,\eta) + O(t^2)\Big)$$

and apply inverse function theorem using the fact that $\partial^2 H/\partial \xi^2$ is invertible. □

Let us consider the open set $\Omega \subset X \times X$ as before, for any $(x,y) \in \Omega$ there exists an unique extremal curve $\gamma_{xy} : [0,t] \to X$ from y to x so that $\mathrm{Leg}(y, \dot\gamma(0)) \in V$. The action integral

$$S(x,y) := \int_0^t L\big(\gamma_{xy}(s), \dot\gamma_{xy}(s)\big)\,\mathrm{d}s$$

is a generating function for $\varphi_t : V \to W$.

A more specific example:

Example 2.3. — (X,g) is a Riemannian manifold, $Q : X \to \mathbb{R}$,

$$L = \tfrac{1}{2}\|v\|^2 - Q(x), \quad H = \tfrac{1}{2}\|\xi\|^2 + Q(x)$$

and $Q_\infty = \liminf_{x\to\infty} Q(x)$. We can take $E < Q_\infty$ and $V = \{H(y,\eta) < E\}$.

If $Q \equiv 0$, we have $S(x,y) = d^2(x,y)/2t$ where d is the Riemannian distance and we can take $\Omega = \{(x,y) \mid d(x,y) < \rho\}$ where ρ is the injectivity radius of (X,g). In this case, the map $(y,\eta) \mapsto x(\varphi_t(y,\eta))$ is the exponential map: $x = \exp(y, t\eta)$. The flow φ_t is locally twist near (y,η) with $x = \exp(y,t\eta)$ if x and y are not conjugate points along the geodesic $\gamma(s) = \exp(y, st\eta)$, $0 \leqslant s \leqslant 1$. Given E, this is the case for $t \neq 0$ and small enough.

2.2. Quantizations of a twist map

We will consider, for $h > 0$, an operator $U_h : L^2(X) \to L^2(X)$ whose (Schwartz) kernel is

$$[U_h](x,y) = (2\pi i h)^{-\frac{1}{2}d} a(x,y) s(x,y)\, \mathrm{e}^{iS(x,y)/h} |\mathrm{d}x\,\mathrm{d}y|^{\frac{1}{2}}$$

with $a \in C_0^\infty(\Omega)$ and $s(x,y) = |\det(\partial^2_{x,y} S)|^{\frac{1}{2}}$. The function s is a normalization which makes U_h unitary in case $a = 1$ and S is quadratic (see Section 11.2). The operator U is smoothing, with compactly supported kernel and is called a *quantization of* χ. It would be nice to know about its spectrum in the semi-classical limit!

It is interesting to note that in general U_h is not a *normal operator*. It would imply, by stationary phase approximation, that, if $\tilde a(y,\eta) = a(x,y)$

with $(x,\xi)=\chi(y,\eta)$, then $|\tilde{a}|$ is constant along the trajectories of χ which is impossible if they escape to infinity! An important exception will be $\psi(\widehat{H})\exp(-it\widehat{H}/h))$ with $\widehat{H}$ a semi-classical Schrödinger operator and the support of ψ is contained in $]-\infty,E_\infty[$ (see Section 3).

2.3. Discrete Feynman path integrals

One way to get some information on the eigenvalues of U is to derive "trace formulae" expressing the asymptotic behavior of $t_N:=\mathrm{Trace}(U^N)$ as $h\to 0$.

We can formally write $[U^N]$ and t_N as discrete Feynman integrals: a path γ of length N will be a sequence $(y=x_0,\dots,x=x_N)$ with $(x_k,x_{k+1})\in\Omega$. We will denote by $P^N_{x,y}$ the set of such paths with the measure $|\mathrm{d}\gamma|=|\mathrm{d}x_1\cdots\mathrm{d}x_{N-1}|$. The action of the path $\gamma=(y=x_0,\dots,x=x_N)$ will be $S(\gamma):=S(x,x_{N-1})+\cdots+S(x_1,y)$. We denote by

$$A(\gamma):=\prod_{j=0}^{N-1}a(x_j,x_{j+1})s(x_j,x_{j+1}).$$

We have

$$[U^N](x,y)=(2\pi ih)^{-\frac12 Nd}\int_{P^N_{x,y}}\mathrm{e}^{iS(\gamma)/h}A(\gamma)|\mathrm{d}\gamma|, \tag{2.1}$$

and

$$t_N=(2\pi ih)^{-\frac12 Nd}\int_X|\mathrm{d}x|\Big(\int_{P^N_{x,x}}\mathrm{e}^{iS(\gamma)/h}A(\gamma)|\mathrm{d}\gamma|\Big). \tag{2.2}$$

Defining P^N as the set of periodic sequences $(x_0,x_1,\cdots,x_{N-1},x_0)$ and using Fubini, so that for all i, $(x_i,x_{i+1})\in\Omega$, we get

$$t_N=(2\pi ih)^{-\frac12 Nd}\int_{P^N}\mathrm{e}^{iS(\gamma)/h}A(\gamma)|\mathrm{d}\gamma|.$$

We will evaluate the previous expressions by the stationary phase (see Section 11.1).

Lemma 1. — ▹ *The critical points of $S:P^N_{x,y}\to\mathbb{R}$ are sequences $(y,x_1,\cdots,x_{N-1},x)$ which are the projections of orbits $\chi^k(y,\eta)$:*

$$\chi^k(y,\eta)=(x_k,\xi_k),\quad k=0,\dots,N-1.$$

▹ *The critical points of $S:P^N\to\mathbb{R}$ are the projections $(x_0,x_1,\cdots,x_{N-1},x_0)$ of closed orbits sitting in V.*

Proof. — Let us define $\xi_0 = -\partial_2 S(x_1, y)$ and $\xi_k = \partial_1 S(x_k, x_{k-1})$, for $k = 1, \dots, N$. The criticality condition is

$$\partial_2 S(x_{k+1}, x_k) = -\partial_1 S(x_k, x_{k-1})$$

for $k = 1, \dots, N-1$, which is equivalent to say that we have

$$(x_{k+1}, \xi_{k+1}) = \chi(x_k, \xi_k).$$

The same argument extends to the cyclic case. □

2.4. Non degeneracy

There are several possible non degeneracy assumptions. They can be formulated “à la Morse-Bott” (critical point of action integrals) or purely symplectically.

Definition 3 (Clean intersections). — *Two sub-manifolds Y and Z of X intersect* cleanly *iff $Y \cap Z$ is a manifold whose tangent space is the intersection of the tangent spaces of Y and Z.*

A diffeomorphism F admits a clean manifold of fixed points *if the graph of F intersects cleanly the diagonal.*

Definition 4 (Morse-Bott non degenerate critical points). — *A manifold W of critical points of a function $f : M \to \mathbb{R}$ is* Morse-Bott ND *if the Hessian of f is non degenerate transversely to W.*

One can reformulate the Morse-Bott non degeneracy condition in terms of clean intersection as follows: W is a Morse-Bott ND critical manifold of f if $Y = \{(x, f'(x)) \mid x \in M\}$ and $Z = \{(x, 0) \mid x \in M\}$ intersect cleanly as submanifolds of $X = T^\star M$.

We will need the

Lemma 2. — *Using the previous notations, χ^N admits a clean manifold W of fixed points if and only if the action $S(x_0, x_1, \dots, x_{N-1}, x_0)$ admits a Morse-Bott ND critical manifold.*

The following statement will be explained in Section 11.4:

Lemma 3 (Canonical measures). — *If W is a clean manifold of fixed points of a symplectic map, W is equipped with a canonical smooth measure* $\mathrm{d}\mu_W$.

2.5. Van Vleck's formula

THEOREM 1. — *Let us give $x, y \in X$ so that χ^N is a locally twist map near each (y, η) so that $\chi^N(y, \eta) = (x, .)$ (which is true for almost all pairs (x, y) thanks to Sard's theorem), we have*

$$[U^N](x,y) \sim (2\pi i h)^{-\frac{1}{2}d}\Big(\sum_\alpha |\det \partial^2_{xy} S_\alpha|^{\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2} i\mu_\alpha \pi}\, \mathrm{e}^{iS_\alpha/h} \sum_{j=0}^{\infty} B_j^\alpha h^j\Big)$$

where

▷ *α labels the solutions of $\chi^N(y, \eta_\alpha) = (x, .)$ We will denote by*

$$\gamma_\alpha = (x_0^\alpha = y, x_1^\alpha, \dots, x_N = x)$$

the projection of the trajectories: $\chi^k(y, \eta_\alpha) = (x_k^\alpha, \xi_k^\alpha)$;

▷ *μ_α is the Morse index*[2] *of $S(\gamma) : P^N_{xy} \to \mathbb{R}$ for the critical point γ_α;*

▷ *$S_\alpha(x,y) = \sum_{k=0}^{N-1} S(x_{k+1}^\alpha, x_k^\alpha)$, with $x_N^\alpha = x$, $x_0^\alpha = y$ (a local generating function for χ^N);*

▷ *$B_0^\alpha = \Pi_{k=0}^{N-1} a(x_{k+1}^\alpha, x_k^\alpha)$.*

Proof. — This formula is obtained by applying the method of stationary phase (Theorem 11) to the integral given in Equation (2.1) and using the formula for the determinant of a Jacobi matrix (Theorem 13) in order to evaluate the large determinant which we need to evaluate. □

2.6. Trace formulae

We get:

THEOREM 2. — *Let us assume that χ admits (a finite number of)* clean connected *manifolds W_α of periodic points, i.e. points satisfying $\chi^N(z) = z$. We get the following asymptotic expansion of the trace:*

$$t_N := \mathrm{Trace}(U^N) \sim \sum_\alpha (2\pi i h)^{-\frac{1}{2}\nu_\alpha}\, \mathrm{e}^{-\frac{1}{2} i\iota_\alpha \pi}\, \mathrm{e}^{iS(W_\alpha)/h}\Big(\sum_{j=0}^{\infty} A_j^\alpha h^j\Big),$$

with

▷ $$A_0^\alpha = \int_{W_\alpha} \prod_{j=1}^{N} a\big(x(\chi^j(z)), x(\chi^{j-1}(z))\big)\, \mathrm{d}\mu_\alpha,$$

[2] The Morse index of a critical point x of a smooth function $F : X \to \mathbb{R}$ is the maximal dimension of a subspace of the tangent space $T_x X$ on which the hessian of F is < 0.

where $\mathrm{d}\mu_\alpha$ *is the canonical measure on* W_α*;*

▷ $\nu_\alpha = \dim W_\alpha$*;*

▷ ι_α *is the Morse index of the action* $S(\gamma)$ *on the periodic path* $\gamma_\alpha = (x_0, x_1, \dots, x_N = x_0)$.

Proof. — The proof is by application of the stationary phase for ND critical manifolds (Section 11.1) to the integral given in Equation (2.2) and using the computation of the canonical measure (Section 11.5) in order to evalauate the principal contribution. □

In general, it is difficult to extract precise information on the asymptotic behavior of the eigenvalues of U_h from the trace formula. We will now see that the situation is much better in case of a flow $U(t)$.

3. Schrödinger operators

3.1. The Laplace operator

Let (X, g) be a smooth compact connected Riemannian manifold of dimension d. Let us denote by $|\mathrm{d}x| = |g| \cdot |\mathrm{d}x_1 \cdots \mathrm{d}x_d|$, with $|g| = \sqrt{\det(g_{ij})}$, the canonical volume element on (X, g) and $L^2(X, |\mathrm{d}x|)$ the associated Hilbert space. There exists a canonical symmetric differential operator of order 2, the *Laplace operator* on (X, g), denoted Δ_g, which is given in local coordinates by:

$$\Delta_g = -|g|^{-1}\partial_i |g|\, g^{ij}\partial_j.$$

Because X is compact, Δ_g admits an unique self-adjoint extension and $L^2(X, |\mathrm{d}x|)$ admits an orthonormal basis of (smooth) eigenfunctions. It will be convenient to label such a basis by integers $(\varphi_j, j = 1, \dots)$ so that $\Delta_g \varphi_j = \lambda_j \varphi_j$ and

$$\lambda_1 = 0 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$$

with $\lambda_j \to \infty$. Note that the sequence (λ_j) is uniquely defined, but the φ_j are not unique, only the eigenspaces are.

With the exception of a very few cases (compact rank 1 symmetric spaces, tori) the eigenvalues are not explicitly computable.

It is known (Weyl's law) that $\lambda_j \sim c_d \operatorname{vol}(X)^{-2/d} j^{2/d}$ with $c_d = 4\pi^2 b_d^{-d/2}$ where b_d is the volume of the unit ball in $\mathbb{R}^d$.

The goal of the SCTF is the description in terms of the geodesic flow of some asymptotic properties as $j \to \infty$ (the classical limit) of the eigenvalues of Δ_g.

3.2. Semi-classical Schrödinger operators

If (X, g) is a (possibly non compact) Riemannian manifold and $Q : X \to \mathbb{R}$ a smooth function which satisfies $\liminf_{x\to\infty} Q(x) = E_\infty > -\infty$, the differential operator $\widehat{H} = \frac{1}{2}h^2\Delta + Q$ is semi-bounded from below and admits self-adjoint extensions. For all those extensions, the spectrum is discrete in the interval $]-\infty, E_\infty[$ and the eigenfunctions $\widehat{H}\varphi_j = E_j\varphi_j$ are localized in the domain $Q \leqslant E_j$. If X is compact and $Q = 0$, we recover the case of the Laplace operator with $E_j = \frac{1}{2}h^2\lambda_j$.

We will denote by

$$\inf Q < E_1(h) < E_2(h) \leqslant \cdots \leqslant E_j(h) \leqslant \cdots < E_\infty$$

this part of the spectrum.

4. Classical dynamics

Most results are in principle well known. Some possible references are [2] and [1].

4.1. Geodesics

The geodesic flow G_t of (X, g) is defined as the following 1-parameter group of diffeomorphisms on the tangent bundle TX: $G_t(x_0, v_0) = (x_t, v_t)$ were $t \mapsto x_t$ is the geodesic of X so that $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$ and v_t is the speed at time t of the previous geodesic. It is well known that the geodesic flow can also be described using the Hamiltonian formulation. On $T^\star X$, we consider the Hamiltonian $H_0(x, \xi) = \frac{1}{2}\|\xi\|_g^2$ and we will still denote by G_t the geodesic flow on $T^\star X$.

4.2. Newton flows

The Euler-Lagrange equations for the Lagrangian $L(x, v) := \frac{1}{2}\|v\|_g^2 - Q(x)$ admit an Hamiltonian formulation on $T^\star X$ whose energy is given by

$$H = \tfrac{1}{2}\|\xi\|_g^2 + Q(x).$$

We will denote by X_H the Hamiltonian vector field

$$X_H := \sum_j \frac{\partial H}{\partial \xi_j}\partial_{x_j} - \frac{\partial H}{\partial x_j}\partial_{\xi_j}.$$

Preservation of H by the dynamics shows immediately that the Hamiltonian flow Φ_t restricted to $H < E_\infty$ is complete.

4.3. Closed orbits

Definition 5. — *A* closed orbit (γ, T) *of the Hamiltonian* H *consists of an orbit* γ *of* X_H *which is homeomorphic to a circle and a nonzero real number* T *so that* $\Phi_T(z) = z$ *for all* $z \in \gamma$*.* T *will be called the period of* γ*.*

We will denote by $T_0(\gamma) > 0$ *(the primitive period) the smallest* $T > 0$ *for which* $\Phi_T(z) = z$*.*

▷ *The (linear) Poincaré map* Π_γ *of a closed orbit* (γ, T) *with* $H(\gamma) = E$. We restrict the flow to $S_E := \{H = E\}$ and take an hypersurface Σ inside S_E transversely to γ at the point z_0. The associated return map P_γ is a local diffeomorphism fixing z_0. Its linearization $\Pi_\gamma := P'_\gamma(z_0)$ is the linear Poincaré map, an invertible (symplectic) endomorphism of the tangent space $T_{z_0}\Sigma$.

▷ *The Morse index* $\iota(\gamma)$. — Closed orbits (γ, T) are critical point of the action integral $\int_0^T L(\gamma(s), \dot\gamma(s))ds$ on the manifold $C^\infty(\mathbb{R}/T\mathbb{Z}, X)$. They have always a finite Morse index (see [46]) which is denoted by $\iota(\gamma)$. For general Hamiltonian systems, the Morse index is replaced by the Conley-Zehnder index [44].

▷ *The nullity index* $\nu(\gamma)$ is the dimension of the space of infinitesimal deformations of the closed orbit γ by closed orbits. It is the dimension of the kernel of the map $(\delta t, \delta z) \mapsto \Phi'_T(z_0)\delta z - \delta z + \delta t X_H$. We have always $\nu(\gamma) \geqslant 2$.

Example 4.1 (Geodesic flows). — ▷ *Manifold with* < 0 *sectional curvature*: in this case, we have for all closed geodesics $\iota(\gamma) = 0$, $\nu(\gamma) = 2$.

▷ *Flat tori of dimension* d: we have then $\iota(\gamma) = 0$ and $\nu(\gamma) = d + 1$.

▷ *Sphere of dimension* 2 *with constant curvature*: if γ_n is the nth iterate of the great circle, we have $\iota(\gamma_n) = 2|n|$ and $\nu(\gamma_n) = 4$.

It is a beautiful result of J.-P. Serre [51] that any pair of points (a, b) on a closed Riemannian manifold are the end points of infinitely many geometrically distincts geodesics. Counting geometrically distinct closed geodesics is much harder especially for simple manifold like the spheres. It is now known that every closed Riemannian manifold admits infinitely many geometrically distinct closed geodesics (at least in some cases for a generic metric, [11, Chap. V]). Concerning more general Hamiltonian systems a lot is known [32].

4.4. Non degeneracy and orbits cylinders

Definition 6. — *A closed orbit γ will called* weakly non degenerate *(WND) if* 1 *is not an eigenvalue of the Poincaré map* Π_γ.

Remark 1. — If γ is WND, it does not imply that the iterates are WND, because some non trivial roots of unity could be eigenvalues of Π_γ.

Lemma 4. — *If γ_{E_0} is a WND orbit with $H(\gamma) = E_0$, there exists for E close to E_0 a closed orbit γ_E contained in $H = E$ smoothly depending on E.*

Definition 7. — *The smooth family γ_E with E close to E_0 will be called an* orbit cylinder.

For an orbit cylinder γ_E we will denote by $T(E)$ the period of γ_E which is a smooth function of E.

Definition 8. — *A closed orbit γ will called* strongly non degenerate *(SND) if γ is weakly non degenerate and $T'(E_0) \neq 0$.*

Example 4.2. — In the case of the Riemannian geodesics, both non degeneracy assumptions coincide. They are true if, for example, the metric has < 0 curvature. If X is fixed, all closed geodesics are (SND) for a generic (in the Baire sense) Riemannian metric.

More general ND assumptions can be introduced in order to cover for example the case of completely integrable systems. In this case they recover the usual ND assumptions of KAM theories.

Proposition 1. — *If γ is a WND orbit of period T, there exists an (unique) symplectic splitting of the tangent space to any point z of γ so that the differential of the flow at time T is given by*

$$\Phi'_T(z) = \begin{pmatrix} 1 & -\mathrm{d}T/\mathrm{d}E & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Pi_\gamma \end{pmatrix}.$$

The first piece of the splitting is the 2d tangent space to the orbit cylinder.

4.5. Actions

Definition 9. — *If (γ, T) is a periodic orbit, we define the following quantities which are both called* action *of γ:*

$$\widehat{S}(\gamma) = \int_\gamma \xi \,\mathrm{d}x, \quad S(\gamma) = \int_\gamma (\xi \,\mathrm{d}x - H \,\mathrm{d}t) = \int_0^T L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t.$$

We get

$$\hat{S}(\gamma) - S(\gamma) = H(\gamma)T.$$

If γ is SND, we have two smooth functions $\widehat{S} = \widehat{S}(E)$ and $S = S(T)$ associated to any orbit cylinder. They satisfies

$$\frac{\mathrm{d}\widehat{S}}{\mathrm{d}E} = T(E), \quad \frac{\mathrm{d}S}{\mathrm{d}T} = -E(T).$$

We can summary the previous properties as follows:

Theorem 3. — *$\widehat{S}(E)$ and $-S(T)$ are Legendre transforms of each other in the symplectic space* $(\mathbb{R}^2, dE \wedge dT)$.

5. Statement of SCTF for Schrödinger operators

5.1. Playing with spectral densities: the pre-trace formula

We will define "regularized spectral densities". The general idea is as follows: we want to study a sequence of real numbers $E_j(h)$ (a spectrum) in some interval $[a, b]$ depending of a small parameter h. We introduce a non negative function $\rho \in \mathcal{S}(\mathbb{R})$ which satisfy $\int \rho(t)\mathrm{d}t = 1$ and introduce

$$D_{\rho,\varepsilon,h}(E) = \sum \frac{1}{\varepsilon}\rho\Big(\frac{E - E_j(h)}{\varepsilon}\Big).$$

It gives the analysis of the spectrum at the scale ε. Of course, we will adapt the scaling ε to the small parameter h. If the scaling is smaller than the size of the mean spacing of the spectrum, we will get a very precise resolution of the spectrum.

The general philosophy is:

▷ If h is the semi-classical parameter of a semi-classical Hamiltonian, the mean spacing $\overline{\Delta}E$ is of order h^d (Weyl's law). The trace formula gives the asymptotic behaviour of $D_{\rho,\varepsilon,h}(E)$ for $\varepsilon \sim h$ (and hence $\varepsilon\gamma\overline{\Delta}E$ except if $d = 1$) . This behaviour is not universal and thus contains a lot of information's (in our case, on closed trajectories)

▷ Better resolution of the spectrum needs the use of the long time behaviour of the classical dynamics and is conjectured "universal"

From now on, we choose two smooth functions:

▷ $\rho(\tau)$ which is equal to 1 near 0 and so that $\hat{\rho}(t) = \int \mathrm{e}^{-i\tau t}\rho(\tau)\mathrm{d}\tau$ is compactly supported

▷ $\psi \in C_0^\infty(\,]-\infty, E_\infty[\,)$.

We define:

▷ The *regularized density of states*

$$D(E) := \sum_{j=1}^{\infty} \psi(E_j)\, \frac{1}{h} \rho\Big(\frac{E - E_j(h)}{h}\Big).$$

If $I = [a, b] \subset]-\infty, E_\infty[$, $a < E < b$ and $\psi \equiv 1$ near E, we have

$$D(E) = \sum_{a<E_j<b} \frac{1}{h} \rho\Big(\frac{E - E_j(h)}{h}\Big) + O(h^\infty).$$

▷ The *partition function* given by the (finite) sum

$$Z(t) = \sum_{j=1}^{\infty} \psi(E_j)\, \mathrm{e}^{-itE_j/h} = \mathrm{Trace}(U_\psi(t)) \tag{5.1}$$

with $U_\psi(t) = \psi(\widehat{H}) \exp(-it\widehat{H}/h)$.

Duistermaat-Guillemin's trick relates the behaviour of $D(E)$ to the behaviour of $Z(t)$ thanks to the fundamental *pre-trace formula*

$$D(E) = \frac{1}{2\pi h} \int_{\mathbb{R}} \mathrm{e}^{itE/h} \widehat{\rho}(t) Z(t) \mathrm{d}t. \tag{5.2}$$

The idea is then to start from a semi-classical approximation of the propagator $[U_\psi(t)](x, y)$ and to insert $Z(t) = \int_X [U_\psi(t)](x, x) |\mathrm{d}x|$ into Equation (5.2).

5.2. Formal series and WKB expansions

An usefull notation: if $\sum a_j h^j$ is a formal power series in h, we will write $A(h) = \sum a_j h^j$ as a way to say that $A(h)$ is a function of h (h small) admitting the asymptotic expansion $\sum a_j h^j$. Such a function is usually obtained by a so-called summation process.

Definition 10. — *A function $u(x, h)$ where $x \in X$, a smooth manifold, and $h \to 0$, a real parameter, admits a WKB expansion (in short, u is a WKB function), if u admits an asymptotic expansion*

$$u(x, h) = \mathrm{e}^{iS(x)/h} \sum_{j=0}^{\infty} a_j(x) h^j$$

with $S : X \to \mathbb{R}$ smooth and $a_j : X \to \mathbb{C}$ smooth.

More precisely, for all N, α,

$$D_x^\alpha\big(\mathrm{e}^{-iS(x)/h} u(x, h) - \sum_{j=0}^{N-1} a_j(x) h^j\big) = O(h^N)$$

uniformly on every compact set in X.

5.3. The Schrödinger trace

Our goal is now to describe the asymptotic behaviour of $D(E)$ around E_0. The main results can be expressed as follows:

THEOREM 4. — *Let us assume that $E_0 < E_\infty$ is not a critical value of H and all periodic trajectories contained in $H = E_0$ and of periods in the support of $\widehat{\rho}$ are WND, we have, for E close enough to E_0:*

▷ *$D(E) = D_{\mathrm{Weyl}}(E) + \sum_\gamma D_\gamma(E) + O(h^\infty)$ where γ is one of the (finitely many) periodic trajectories in $H = E$ and periods in* Support$(\widehat{\rho})$*;*

▷ *with $a_0(E) = \int_{H=E} |\mathrm{d}x\,\mathrm{d}\xi/\mathrm{d}H|$,*

$$D_{\mathrm{Weyl}}(E) = (2\pi h)^{-d}\Big(\sum_{j=0}^{\infty} a_j(E)h^j\Big); \tag{5.3}$$

▷ *$D_\gamma(E)$ is a WKB function whose principal part is, if γ is SND:*

$$\frac{\varepsilon}{2\pi h}\,\mathrm{e}^{-\frac{1}{2}i\iota(\gamma)\pi}\,\mathrm{e}^{i\int_\gamma \xi \mathrm{d}x/h} a_\gamma(E) \tag{5.4}$$

with $a_\gamma(E) = \rho(T(E))T_0(E)/|\det(\mathrm{Id} - \Pi_\gamma)|^{\frac{1}{2}}$, $T_0(E)$ the primitive period, and

$$\varepsilon = \begin{cases} -i & \text{if } T'(E) > 0, \\ 1 & \text{if } T'(E) < 0. \end{cases}$$

If γ is only WND, the formula is the same except that $\varepsilon\,\mathrm{e}^{-\frac{1}{2}i\iota(\gamma)\pi}$ is now (the exponential of) a Maslov index.

The SCTF can also be derived the same way for Schrödinger operators with magnetic field ... One can even extend it to Hamiltonian systems which are not obtained by Legendre transform from a regular Lagrangian. In this case, Morse indices have to be replaced by the more general Maslov indices.

5.4. The Weyl term

Choose $a < E_\infty$ and let $t_0(a) > 0$ be the smallest period of a non trivial closed trajectory γ with $H(\gamma) \leqslant a$ (see [59]). In this section, we fix ρ so that Support$(\widehat{\rho})$ is contained in $]-t_0, t_0[$. If E is not a critical value of H, and, for $\psi \in C_0^\infty(]-\infty, a[)$, we have, insisting on the dependence w.r. to ψ:

$$D_\psi(E) = D_{\text{Weyl}}(E) \sim (2\pi h)^{-d}\Big(\sum_{j=0}^{\infty} a_j(E,\psi)h^j\Big),$$

with $a_0(E,\psi) = \int_{H=E} \psi(H)|\mathrm{d}L\,\mathrm{d}H|$.

If we define the distributions

$$A_j(\psi) = \int_{-\infty}^{+\infty} a_j(E,\psi)\mathrm{d}E$$

we get

$$\sum \psi(E_j) \sim (2\pi h)^{-d}\Big(\sum_{j=0}^{\infty} A_j(\psi)h^j\Big),$$

Moreover, the previous asymptotic works even if they are critical values of H inside the support of ψ. The A_j's are, for $j \geqslant 1$ of the following form:

$$A_j(\psi) = \int_{T^\star X} \sum_{l\geqslant 2}^{N_j} \psi^{(\ell)}(H(x,\xi))P_{j,\ell}(x,\xi)|\mathrm{d}x\,\mathrm{d}\xi|$$

and the $P_{j,\ell}(x,\xi)$'s are "locally computable" from the Hamiltonian $H(x,\xi)$ and its derivatives [56].

Remark 2. — If $H(x,-\xi) = H(x,\xi)$, we have the "transmission property" and the A_{2j+1} vanish.

In the case of the Laplace operator and $\chi(E) = \mathrm{e}^{-E}$, the A_j's are called the "heat invariants" [10].

From that kind of formula, it is possible to deduce the following estimates on the remainder term in Weyl's law:

THEOREM 5. — *If a, b are not critical values of H:*

$$\#\big\{j \mid a \leqslant E_j(h) \leqslant b\big\} = (2\pi h)^{-d}\,\text{volume}\,\big(\{a \leqslant H \leqslant b\}\big)\big(1 + O(h)\big).$$

This remainder estimate is optimal and was first shown in rather great generality by Hörmander [33].

5.5. The contributions of periodic orbits

The proof involves the study, for $t \neq 0$, of $Z(t) = \text{Trace}(U_\psi(t))$ which is given by:

Proposition 2. — ▷ *If t is not the period of a trajectory γ with $H(\gamma) \subset \mathrm{Support}(\psi)$, then $Z(t) = O(h^\infty)$*

▷ *If γ is an SND closed trajectory of period T, $Z_\gamma(t)$ is near T a WKB function whose principal symbol is*

$$\frac{1}{(2\pi i h)^{\frac{1}{2}}}\, \mathrm{e}^{-\frac{1}{2} i\iota(\gamma)\pi}\, \mathrm{e}^{iS(T)/h} b_0(T), \tag{5.5}$$

with $b_0(T) = T_0 |\mathrm{d}E/\mathrm{d}T|^{\frac{1}{2}} \cdot |\det(\mathrm{Id} - \Pi_\gamma)|^{-\frac{1}{2}}$.

It is then enough to put the previous asymptotic formula in Equation (5.2) and to apply once more a stationary phase argument.

5.6. Formal derivation from the Feynman path integral

5.6.1. The Feynman integral

R. Feynman (see [24], [48]) found a *geometric* representation of the propagator, i.e. the kernel $p(t, x, y)$, with $t \neq 0$, of the unitary group $\exp(-it\widehat{H}/h)$ using an *integral* (FPI := Feynman path integral) on the manifold $\Omega_{t,x,y}$ of paths from y to x in the time t; if $L(\gamma, \dot\gamma)$ is the Lagrangian, we define

$$J_t(\gamma) := \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, \mathrm{d}s$$

and we have, for $t > 0$:

$$p(t, x, y) = \int_{\Omega_{t,x,y}} \mathrm{e}^{\frac{i}{h} J_t(\gamma)} |\mathrm{d}\gamma|,$$

where $|\mathrm{d}\gamma|$ is a "Riemannian measure" on the manifold $\Omega_{t,x,y}$ with the natural Riemannian structure.

There is no justification FPI as a flexible tool, but as we will see FPI is a very efficient tool in semi-classics.

5.6.2. The trace and loop manifolds

Let us try a formal calculation of the partition function and its semi-classical limit. We get:

$$Z(t) = \int_X |\mathrm{d}x| \int_{\Omega_{x,x,t}} \mathrm{e}^{\frac{i}{h} J_t(\gamma)} |\mathrm{d}\gamma|.$$

If we denote by Ω_t the manifold of paths $\gamma : \mathbb{R}/t\mathbb{Z} \to X$, (loops) and we apply Fubini (sic!), we get:

$$Z(t) = \int_{\Omega_t} \mathrm{e}^{\frac{i}{h} J_t(\gamma)} |\mathrm{d}\gamma|.$$

5.6.3. The semi-classical limit

We want to apply stationary phase in order to get the asymptotic expansion of $Z(t)$; critical points $J_t : \Omega_t \to \mathbb{R}$ are the closed orbits of the Euler-Lagrange flow and hence of the Hamiltonian flow of period t. We need

1) *the ND assumption (Morse-Bott)*,
2) *the Morse index*,
3) *the determinant of the Hessian.*

1) The ND assumption is the original Morse-Bott one in Morse theory: we have smooth manifolds of critical points and the Hessian is transversely ND.

2) The Morse index is the Morse index of the action functional J_t on closed loops.

3) The Hessian is associated to a periodic Sturm-Liouville operator for which many regularization's have already been proposed (Levit-Smilanski [40] and CdV [55], Zeta regularization, see [49]).

We get that way a sum of contributions given by the components $W_{j,t}$ of W_t:

▷ $Z_j(t) = (2\pi i h)^{-\frac{1}{2}} \mathrm{e}^{\frac{i}{h} S(\gamma_j)} A_j(h)$ with $A_j(h) \sim \sum_{l=0}^{\infty} a_{j,\ell} h^{\ell}$ and

▷ $a_{j,0} = \mathrm{e}^{-\frac{1}{2} i\mu\pi} / |\delta|^{\frac{1}{2}}$ where μ is the Morse index and δ is a regularized determinant.

6. A proof of SCTF for Schrödinger operators

We will prove SCTF using the fact that the short time propagator is given as a quantization of the short time flow which is a twist map, so that we can apply Theorem 2.

6.1. The short time propagator

The purpose of this section is to derive an asymptotic formula for the kernel $P_\chi(t,x,y)$ of $U_\chi(t) = \chi(\widehat{H}) \exp(-it\widehat{H}/h)$ with $\chi \in C_0^\infty$ for t small enough.

We will start with a finite atlas (Ω_α) of X and a partition of unity $a_\alpha \in C_0^\infty(\Omega_\alpha)$. We will also choose $b_\alpha \in C_0^\infty(\Omega_\alpha)$ so that b_α is equal to 1 near the support of a_α. We will choose $\delta > 0$ so that the balls of radius δ around $\mathrm{Supp}(b_\alpha)$ are contained in Ω_α. Let us denote by $t_0 > 0$ so that if $\phi_t(y,\eta) = (x,\xi)$, $|t| \leqslant t_0$ and $H(x,\xi) \in \mathrm{Supp}(\chi)$, we have $\mathrm{dist}(x,y) \leqslant \delta$. We will start from

$$f = \sum b_\alpha a_\alpha f \quad \text{and} \quad a_\alpha f(x) = (2\pi h)^{-d} \int \mathrm{e}^{ix\xi/h} \widehat{a_\alpha f}(\xi) \mathrm{d}\xi$$

so that it is enough to have the:

Lemma 5. — *The solution of the Cauchy problem*

$$\frac{h}{i} u_t + \widehat{H} u = 0, \quad u(0,x) = b_\alpha(x) \mathrm{e}^{i\langle x|\xi\rangle/h}$$

is given for $|t| \leqslant t_0$ *by*

$$u(t,x) = \mathrm{e}^{i\Sigma_\alpha(t,x,\xi)/h} \Big(\sum_{j=0}^{\infty} b_{\alpha,j}(t,x,\xi) h^j \Big) + O(h^\infty).$$

Here:

▷ Σ_α *is the solution of the Hamilton-Jacobi equation* $\Sigma_t + H(x,\Sigma_x) = 0$ *with* $\Sigma(0,x,\xi) = \langle x \mid \xi \rangle$*;*

▷ $b_{\alpha,0}(0,x,\xi) = b_\alpha(x)$.

This lemma goes back essentially to Lax [39].

Sketch of proof. — The proof uses only the formulae giving the action of a ΨDO on a WKB function. The first term vanishes because of the Hamilton-Jacobi equation. The next one then vanishes if $b_{\alpha,0}(t,x,\xi)$ satisfies a suitable differential equation called the "transport equation". We get that way a complete asymptotic expansion solving the Cauchy problem up to $O(h^\infty)$. It remains to prove that the computed asymptotic expansion is the asymptotic expansion of the solution which uses only the unitarity of $U(t)$. □

From the lemma, we get the small time propagator as follows:

Theorem 6. — *For* $|t| \leqslant t_0$*, we have*

$$P_\chi(t,x,y) = (2\pi h)^{-d} \sum_\alpha \int \mathrm{e}^{i(\Sigma_\alpha(t,x,\xi) - \langle y|\xi\rangle)} a_{\alpha,h}(t,x,y,\xi) \mathrm{d}\xi$$

with $a_{\alpha,h}$ *a compactly supported symbol and*

$$a_{\alpha,0}(0,x,y,\xi) = b_\alpha(x) a_\alpha(y) \chi\big(H(x,\xi)\big).$$

Proof. — Let us start with the Fourier inversion formula:

$$f(x) = (2\pi h)^{-d} \sum_\alpha \int b_\alpha(x)\, \mathrm{e}^{i\langle x|\xi\rangle/h} \widehat{a_\alpha f}(\xi)\mathrm{d}\xi.$$

Putting inside the formula given in Lemma 5 and using the fact that $\chi(\widehat{H})$ is a ΨDO of principal symbol $\chi(H)$ gives the proof. □

From Theorem 6, we get, using the stationary phase approximation, the following WKB approximation for the propagator:

Theorem 7. — *We have, for $0 < t \leqslant t_0$, the following WKB expansion:*

$$P_\chi(t,x,y) = (2\pi i h)^{-\frac{1}{2}d} \sum_\alpha \mathrm{e}^{iS_\alpha(t,x,y)/h} \Big(\sum_{j=0}^{\infty} A_{\alpha,j}(t,x,y) h^j\Big)$$

whera $S_\alpha(t,x,y)$ is a generating function for the flow at time t given by the action integral and

$$A_{\alpha,0}(t,x,y) = \det(\partial^2_{xy} S_\alpha)^{\frac{1}{2}} a_\alpha(y) \chi\big(H(x,\xi)\big).$$

Proof. — The proof is by applying the stationary phase to the formula of Theorem 6. It allows to identify S as the value of $\Sigma(t,x,\xi) - \langle y \mid \xi\rangle$ at the critical point $y = \Sigma_\xi$. Moreover the Morse index is clearly d from the expansion $\Sigma(t,x,\xi) = \langle x|\xi\rangle - tH(x,\xi) + O(t^2)$. The value of $A_{\alpha,0}(t,x,y)$ is derived from the unitarity of $U(t)$. □

6.2. The proof of SCTF

Using the approximation of the propagator $P_\chi(t,x,y)$ given by Theorem 7, we see that the proof is completed using the trace formula for quantized twist maps as given in Theorem 2.

7. From the Schrödinger trace to the wave trace

Let (X,g) be a compact Riemannian manifold and

$$\lambda_1 = \mu_1^2 \leqslant \cdots \leqslant \lambda_j = \mu_j^2 \leqslant \cdots$$

the eigenvalues of Δ_g. The solution of the Cauchy problem for the wave equation

$$\partial_t^2 u + \Delta_g u = 0$$

is given in terms of the unitary group $U^W(t)=\exp(-it\sqrt{\Delta})$ whose trace $Z^W(t)=\sum_j \mathrm{e}^{-it\mu_j}$ is a Schwartz distribution which is called the *wave trace.* We can rewrite in a semi-classical way

$$U^W(t)=\exp(-it\widehat{K}/h)$$

with $\widehat{K}:=h\sqrt{\Delta}$ and the classical Hamiltonian is then $K(x,\xi)=\|\xi\|$ which is homogeneous of degree 1. The associated flow is the geodesic flow with speed $\equiv 1$. A closed orbit for K, even if it satisfies WND, is never SND, because periods of $K-$closed orbits are independent of energy!

One can pass from the Schrödinger SCTF to the wave trace quite formally because the eigenvalues $F_j=h\mu_j$ of $h\sqrt{\Delta}$ are related to the eigenvalues $E_j=\frac{1}{2}h^2\lambda_j$ of $\widehat{H}=\frac{1}{2}h^2\Delta$ by $F_j=\sqrt{2E_j}$.

The wave trace expansion can be derived from Theorem 4 as applied to $h\sqrt{\Delta}$. One can also pass directly from the trace formula for $\widehat{H}$ to a trace formula for $\Phi(\widehat{H})$ where Φ is a diffeomorphism near the energy E; here $\Phi(E)=\sqrt{2E}$.

Theorem 8. — *Let us assume that $\widehat{\rho}(t)\equiv 1$ near T_0 and $\psi(E)\equiv 1$ near E_0. The expansion corresponding to the periodic orbit γ (for both Hamiltonians H and $\Phi(H)$) of new period $T_0'=T_0/\phi'(E_0)$ and new energy $F_0=\Phi(E_0)$ is obtained just by the change of variables $F=\Phi(E)$ in the smoothed measures $D_{\rho,\psi}(E)|\mathrm{d}E|$.*

Proof. — Let us fix some interval $I=]a,b[$ with $a<E_0<b$ and consider the two h-dependent measures

$$Z(E)=\sum_{E_j\in I}\delta(E_j)\quad\text{and}\quad W(F)=\sum_{F_j\in\Phi(I)}\delta(F_j).$$

The measure W is the push-forward of Z by the diffeomorphism Φ which we will consider as the Fourier integral operator

$$\Phi_\star : z(E)\mathrm{d}E\longmapsto z\big(\Psi(F)\big)\Psi'(F)\mathrm{d}F$$

with $\Psi=\Phi^{-1}$. We will now re-interpret $D_{\rho,\psi}(E)dE$ as the image of $Z(E)$ by the pseudo-differential operator $P_{\rho,\psi}$ of Schwartz kernel $\frac{1}{h}\rho(\frac{1}{h}(E-E'))\psi(E')$ which is compactly supported in $T^\star I$ and of principal symbol $\widehat{\rho}(t)\psi(E)$. Moreover $P_{\rho,\psi}$ is equal to Id micro-locally near (E_0,T_0). We have

$$\Phi_\star(P_{\rho,\psi}Z)=\Phi_\star P_{\rho,\psi}(\Phi_\star)^{-1}W,$$

from which we conclude by using Egorov Theorem. More precisely, the pseudo-differential operator $\Phi_\star P_{\rho,\psi}(\Phi_\star)^{-1}$ is the identity micro-locally near (F_0,T_0'). □

From the Schrödinger trace $D(\mu) := D_{\rho,\psi}(h, E)$, with $h = \mu^{-1}$, $E = \frac{1}{2}$ and $\psi \in C_0^\infty(]0,1[)$, equal to 1 near $\frac{1}{2}$, we get the asymptotic expansion of $\sum \rho(\mu - \mu_j)$ which is the content of the formula of Duistermaat-Guillemin [22]:

$$\sum \rho(\mu - \mu_j) \underset{\mu \to +\infty}{\sim} \mu^{-1} D\Big(\frac{1}{\mu}, \frac{1}{2}\Big),$$

where $\sim$ means that the asymptotic expansions are the same.

Remark 3. — The previous observation does not apply near $E = 0$ where Φ is no longer a diffeomorphism. As a result, the singularity of the wave trace at the origin is more complicated than the singularity of the (complex) heat trace. The former may contain logarithmic terms as explained in details in [22].

8. Degenerate cases

The trace formula can be extended to much more degenerated cases. The non trivial contributions to $D(E)$ will come from oscillating integrals with phases whose critical points are bijectively associated with closed orbits. It implies that general results on stationary phase expansions depending on the theory of singularities can be applied (see for example [3], [41]).

8.1. The integrable case

As observed by Berry-Tabor [12], the trace formula in this case comes from the Poisson summation formula. Asymptotics of the eigenvalues to any order can then be given in the so-called quantum integrable case by Bohr-Sommerfeld rules.

Using (semi-classical) action-angle coordinates, we start with the Hamiltonian on the torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$ defined by

$$\widehat{H} \exp\big(i\langle \nu \mid x\rangle\big) = H(h\nu) \exp\big(i\langle \nu \mid x\rangle\big)$$

and compute the trace $Z_a(t)$ of $a(hD_x)\exp(-it\widehat{H}/h)$ using the expression of the eigenvalues $H(h\nu)$:

$$Z_a(t) = \sum_{\nu \in \mathbb{Z}^d} a(h\nu)\,\mathrm{e}^{-itH(h\nu)/h}.$$

We will apply Poisson summation formula as well as the approximation of the h-Fourier transform of $a(\xi)\exp(-itH(\xi))$ given from stationary phase.

The ND condition for stationary phase will be that $\xi \to \nabla H(\xi)$ is a local diffeomorphism on the support of a. After some calculations we get, for $t \neq 0$:

$$Z(t) \sim \frac{1}{(2\pi t h)^{\frac{1}{2}d}} \sum_{\gamma \in \mathbb{Z}^d} \mathrm{e}^{itA(\gamma)/h} (2\pi)^d \cdot \left| \det \left(H''_{\xi\xi}(\xi_\gamma) \right) \right|^{-\frac{1}{2}}$$

with $t\nabla H(\xi_\gamma) = \gamma$ and $tA(\gamma)$ the action of the closed orbit γ.

8.2. The maximally degenerated case

Let us assume that (X, g) is a compact Riemannian manifold for which all geodesics have the same smallest period $T_0 = 2\pi$. Then we have the following clustering property [57], [54], [47]:

Theorem 9. — *There exists some constant C and some integer α so that:*

1) *the spectrum of Δ is contained in the union of the intervals*

$$I_k = \left[\left(k + \tfrac{1}{4}\alpha\right)^2 - C, \left(k + \tfrac{1}{4}\alpha\right)^2 + C \right];$$

2) $N(k) = \#\mathrm{Spectrum}(\Delta) \cap I_k$ *is a polynomial function of k for k large enough.*

Property 2) is consequence of the trace formula [54].

9. An example: rational harmonic oscillators

Let us consider the harmonic oscillator

$$\widehat{H} = \tfrac{1}{2} h^2 \Delta + \tfrac{1}{2}(x^2 + 4y^2)$$

in $(\mathbb{R}^2, \mathrm{Eucl})$ whose spectrum is $E_N = h(N + \frac{3}{2})$ with multiplicities $a_N = \#\{2k + \ell = N \mid k, \ell \geqslant 0\}$. One can check that

$$a_N = \tfrac{1}{2}\left(N + \tfrac{3}{2}\right) + \tfrac{1}{4}(-1)^N. \tag{9.1}$$

Let us consider the SCTF with a function ρ so that the compactly supported Fourier transform $\widehat{\rho}$ satisfies $\sum_{\ell \in \mathbb{Z}} \widehat{\rho}(t - 2\pi\ell) \equiv 1$. We have then by applying Equation (5.2) with $E = 1,\ h = (N_0 + \frac{3}{2})^{-1}$:

$$D(1) = \frac{N_0 + \frac{3}{2}}{2\pi} \sum_{N=0}^{\infty} a_N \int_{\mathbb{R}} \widehat{\rho}(t) \psi\Big(\frac{N + \frac{3}{2}}{N_0 + \frac{3}{2}} \Big) \mathrm{e}^{it(N_0 - N)} \mathrm{d}t$$

which is equal to $(N_0 + \frac{3}{2})a_{N_0}$ and hence

$$a_N = \big(N + \tfrac{3}{2}\big)^{-1}(D_{\mathrm{Weyl}} + D_\gamma) + o(1).$$

We easily compute, using formulas (5.3) and (5.4)

$$\begin{aligned} D_{\mathrm{Weyl}} &= \Big(\frac{N+\frac{3}{2}}{2\pi}\Big)^2 \Big(\frac{\mathrm{d}}{\mathrm{d}E}\Big)_{E=1} \mathrm{vol}\,\big\{\tfrac{1}{2}(\xi^2+\eta^2) + \tfrac{1}{2}(x^2+4y^2) \leqslant E\big\} \\ &= \tfrac{1}{2}\big(N+\tfrac{3}{2}\big)^2 + O(1), \end{aligned}$$

D_γ for the "short" orbit

$$\gamma(t) = \big(x=0,\ \xi=0;\ y=\cos 2t/\sqrt{2},\ \eta = -2\sin 2t/\sqrt{2}\big),$$

with $T_0 = \pi$, $\Pi_\gamma = -\mathrm{Id}$, $A(\gamma) = 2\pi$:

$$D_\gamma = \big(N + \tfrac{3}{2}\big)\tfrac{1}{4}(-1)^N$$

and deduce an equality

$$a_N = \tfrac{1}{2}(N + \tfrac{3}{2}) + \tfrac{1}{4}(-1)^N + \varepsilon_N,$$

with $\varepsilon_N \to 0$. Because ε_N is an integer, $\varepsilon_N \equiv 0$ for N large. The previous type of result can be extended to any rational harmonic oscillator (see the contribution of B. Zhilinskii in [45, pp. 126–136]).

10. Applications to the inverse spectral problem

We will now restrict ourselves to the case of the Laplace operator on a compact Riemannian manifold (X, g). The following result is a corollary of SCTF:

THEOREM 10 (see [52], [53]). — *If X is given, there exists a generic subset $\mathcal{G}_X$, in the sense of Baire category, of the set of smooth Riemannian metrics on X, so that, if $g \in \mathcal{G}_X$, the* length spectrum *of (X, g) can be recovered from the Laplace spectrum. The set $\mathcal{G}_X$ contains all metric with <0 sectional curvature and (conjecturally) all metrics with $\leqslant 0$ sectional curvature.*

We can take the set of metrics for which all closed geodesics are non degenerate and the length spectrum is simple.

11. Sturm-Liouville determinants and the metaplectic representation

11.1. The stationary phase approximation

Let us consider the following integral

$$I(h) = (2\pi i h)^{-\frac{1}{2}N} \int_{\mathbb{R}^N} \mathrm{e}^{iS(x)/h} a(x)|\mathrm{d}x|$$

where $S : \mathbb{R}^N \to \mathbb{R}$ is smooth and $a \in C_0^\infty(\mathbb{R}^N)$.

Theorem 11 (stationary phase). — ▷ *If S has no critical point in the support of a, $I(h) = O(h^\infty)$*

▷ *If the critical points of S in the support of a belongs to a non degenerate connected critical manifold W of dimension n,*

$$I(h) = (2\pi i h)^{-\frac{1}{2}n}\, \mathrm{e}^{-\frac{1}{2}i\nu\pi}\, \mathrm{e}^{iS(W)/h} \Big(\sum_{k=0}^{\infty} c_j h^j \Big) + O(h^\infty)$$

with

$$c_0 = \int_W a(y)\mathrm{d}\mu_W$$

where ν is the Morse index of S along W and $\mathrm{d}\mu_W$ is the quotient of the measure $|\mathrm{d}x|$ by the "Riemannian measure" on the normal bundle to W associated to the Hessian of S:

$$\mathrm{d}\mu_W := \frac{|\mathrm{d}x|}{|\det(\partial^2_{\alpha\beta} S)|^{\frac{1}{2}} |\mathrm{d}z|}$$

where $z = (z_\alpha)$ are coordinates on the normal bundle.

Proof. — Using Morse lemma, the integral can be reduced to the case where there are local coordinates (y, z) so that $S(y, z) = \frac{1}{2}Q(z)$ with Q a non degenerate quadratic form. The proof then works by first integrating with respect to z and using an elegant argument due to Hörmander (see [34, Sec. 7.7]) for the case of a non degenerate critical point. □

Remark 4. — As suggested by Don Zagier, we can reformulate the stationary phase formula as follows: let us consider the case of an ND manifold W of critical points of dimension n and the Schwartz distributions $T_h = (2\pi i h)^{\frac{1}{2}(n-N)}\, \mathrm{e}^{iS(x)/h}|\mathrm{d}x|$. Then the weak limit of T_h is the Radon measure $\mathrm{e}^{-\frac{1}{2}i\nu\pi}\mu_W$.

11.2. The metaplectic representation

Good references for the metaplectic representation are [25, Chap. 4] and [35, Section 18.5].

In what follows, ε will denote a number in the set $\{\pm 1, \pm i\}$.

A symplectic linear map $\chi : T^\star\mathbb{R}^d \to T^\star\mathbb{R}^d$ defined by

$$[\,\chi\,] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is a *linear* twist symplectic map if and only if β is invertible; χ admits then an unique quadratic generating function $q(x,y) = \frac{1}{2}\langle Ax|x\rangle + \langle Bx|y\rangle + \frac{1}{2}\langle Cy|y\rangle$ with A, C symmetric matrices and B an invertible matrix. We have:

$$[\,\chi\,] = \begin{pmatrix} -B^{-1}C & -B^{-1} \\ {}^tB - AB^{-1}C & -AB^{-1} \end{pmatrix}.$$

We will define four operators $\widehat{\chi}$ on $L^2(\mathbb{R}^d)$ by their Schwartz kernels

$$[\,\widehat{\chi}\,](x,y) = \varepsilon(2\pi i h)^{-\frac{1}{2}d}\,\mathrm{e}^{iq(x,y)/h}\big|\det(B)\big|^{\frac{1}{2}}.$$

Using unitarity of Fourier transforms, we see that $\widehat{\chi}$ is an unitary map of $L^2(\mathbb{R}^d)$. Moreover, if $\chi_1,\ \chi_2,\ \chi_2\circ\chi_1$ are twist maps, we have

$$\widehat{\chi}_2 \cdot \widehat{\chi}_1 = \varepsilon\, \widehat{\chi_2 \circ \chi_1}$$

as follows from the calculus of Fresnel integrals. The closure of all $\widehat{\chi}$'s, with χ linear twist maps, is a Lie subgroup $\mathrm{M}(d)$ of the Hilbert unitary group $U(L^2(\mathbb{R}^d))$. The mapping $\hat{\chi} \to \chi$ extends to a group morphism of $\mathrm{M}(d)$ onto the symplectic group $\mathrm{Sp}(d)$ whose kernel is $\mathbb{Z}/4\mathbb{Z}$. The connected component of the identity of $\mathrm{M}(d)$ is a two-fold covering of the symplectic group called the *metaplectic group* $\mathrm{Mp}(d)$ and the previous recipe gives a natural unitary representation of $\mathrm{Mp}(d)$ into $L^2(\mathbb{R}^d)$ called the *metaplectic representation.*

11.3. Metaplectic traces

Metaplectic maps are not trace class, but they admit traces in the sense of distribution as follows:

Let us consider, for χ a linear symplectic map, the distribution I_χ on $\mathbb{R}^{2d}$ defined by

$$I_\chi(p) = \mathrm{Trace}\left(\widehat{\chi}\,\mathrm{Op}(p)\right)$$

where $\mathrm{Op}(p)$ is the Weyl quantization of $p \in C_0^\infty(T^\star \mathbb{R}^d)$ defined by

$$[\mathrm{Op}(p)](x,y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{i\langle x-y|\xi\rangle/h} p\big(\tfrac{1}{2}(x+y),\xi\big)|\mathrm{d}\xi|.$$

Let us denote by $F_\chi = \ker(\chi - \mathrm{Id})$ the space of fixed points of χ and $n = \dim F_\chi$.

Theorem 12. — *The distributional trace admits the following asymptotic behaviour* $I_\chi(p) \sim \varepsilon(2\pi i h)^{-\frac{1}{2}n} \int_{F_\chi} p \mathrm{d}\mu_\chi$ *where* $\mathrm{d}\mu_\chi$ *is a Lebesgue measure on* F_χ*. Moreover* $\mathrm{d}\mu_\chi$ *is a purely symplectic invariant of* χ*: if* $\chi_2 = \chi^{-1}\chi_1\chi$*, then* $\mathrm{d}\mu_{\chi_2} = \chi^\star(\mathrm{d}\mu_{\chi_1})$.

Proof. — Let us prove first the second assertion: from the *exact* Egorov theorem (see [35, p. 158]), for any ψ in $\mathrm{Sp}(d)$, we have

$$\widehat{\psi}\, \mathrm{Op}(p) \widehat{\psi}^{-1} = \mathrm{Op}(p \circ \psi).$$

We deduce that for any χ and ψ in $\mathrm{Sp}(d)$, we have

$$I_{\psi\chi\psi^{-1}}(p) = I_\chi(p \circ \psi).$$

Using ψ so that $\psi\chi\psi^{-1}$ is a twist map, the first assertion comes from a direct use of the stationary phase approximation. $\square$

Remark 5. — If χ is twist, one can reduce to $p = p(x)$ and use stationary phase on $\mathbb{R}^d$ instead $\mathbb{R}^{2d}$.

11.4. Measures on fixed point sets

Using the fact that $d\mu_\chi$ is invariant by conjugacy, we can reduce the computations to suitable normal forms:

Example 11.1. — ▷ If $F = 0$, then $\mathrm{d}\mu_\chi = |\det(\mathrm{Id} - \chi)|^{-\frac{1}{2}}\delta(0)$.
▷ If $[\chi] = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ in a symplectic basis $(\partial_x, \partial_\xi)$ with $a \neq 0$, then

$$\mathrm{d}\mu_\chi = |a|^{-\frac{1}{2}}|\mathrm{d}x|.$$

Both examples can be proved using twist maps and evaluating the trace by stationary phase from a generating function.

▷ For the first example, we can take:

$$q(x,y) = \tfrac{1}{2}\langle Ax \mid x\rangle + \langle Bx \mid y\rangle + \tfrac{1}{2}\langle Cy \mid y\rangle$$

with B invertible, and check the identity:

$$\det(\mathrm{Id} - \chi)\det(B) = \det(A + B +^t B + C).$$

▷ For the second one take
$$q(x,y) = \frac{1}{2a}(x-y)^2.$$
It is easy to extend the previous construction to the case of a symplectic diffeomorphism with a clean manifold of fixed points W getting a measure μ_W on W.

11.5. Applications to discrete Sturm-Liouville with Dirichlet or periodic boundary conditions

We assume that E, a d-dimensional real vector space, is equipped with a Lebesgue measure $|\mathrm{d}x|$. We will consider a "Jacobi matrix" $[L]$ on $E^{\oplus N+1}$ given by
$$[L] := \begin{pmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 \\ {}^tB_0 & A_1 & B_1 & \cdots & 0 & 0 \\ 0 & {}^tB_1 & A_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & A_{N-1} & B_{N-1} \\ 0 & 0 & \cdots & \cdots & {}^tB_{N-1} & 0 \end{pmatrix},$$
and denote by $Q = Q_L$ the associated quadratic form
$$Q(x_0,\dots,x_N) = \tfrac{1}{2}\langle Lx \mid x\rangle.$$
Our goal is to compute the determinants of L, of the restriction L_0 of L to $x_0 = x_N = 0$ and of the "restriction" L_{per} of L to $x_0 = x_N$:
$$[L_{\mathrm{per}}] := \begin{pmatrix} A_0 & B_0 & 0 & \cdots & 0 & B_{N-1} \\ {}^tB_0 & A_1 & B_1 & \cdots & 0 & 0 \\ 0 & {}^tB_1 & A_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^tB_{N-1} & 0 & \cdots & \cdots & {}^tB_{N-2} & A_{N-1} \end{pmatrix}.$$
Let us denote by $b_i = \det(-B_i)$, by χ_i the canonical transformation generated by $q_i(u,v) = \frac{1}{2}\langle A_i u \mid u\rangle + \langle B_i v \mid u\rangle$ and
$$\chi = \chi_{N-1} \circ \cdots \circ \chi_0.$$
We have
$$\chi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

THEOREM 13. — *We have the following formulae:*

$$\det(L) = b_0 \cdots b_{N-1} \det(\gamma), \tag{11.1}$$

$$\det(L_0) = b_0 \cdots b_{N-1} \det(\beta), \tag{11.2}$$

$$\det(L_{\mathrm{per}}) = (-1)^d b_0 \cdots b_{N-1} \det(\mathrm{Id} - \chi). \tag{11.3}$$

Proof. — We will do the proof in the case where χ is twist. We have the following expressions of $[\widehat{\chi}](x,y)$ with $b := b_0 \cdots b_{N-1}$:

$$[\widehat{\chi}](x,y) = \varepsilon(2\pi i h)^{-\frac{1}{2}Nd} \int \mathrm{e}^{\frac{i}{h}Q(x,x_1,\dots,x_{N-1},y)} |b|^{\frac{1}{2}} |\mathrm{d}x_1 \cdots \mathrm{d}x_{N-1}|$$

and

$$[\widehat{\chi}](x,y) = (2\pi i h)^{-\frac{1}{2}d} |\det(B)|^{\frac{1}{2}} \mathrm{e}^{\frac{i}{h}(\frac{1}{2}\langle Ax|x\rangle + \langle Bx|y\rangle + \frac{1}{2}\langle Cy|y\rangle)}.$$

The proof is as follows:

▷ For formula (11.1), we compute by stationary phase the integral $\int [\widehat{\chi}](x,y) \mathrm{d}x\, \mathrm{d}y$ using the two expressions of $[\hat{\chi}](x,y)$.

▷ For formula (11.2), we compute by stationary phase in the first expression the integral $[\widehat{\chi}](0,0)$ and compare with the kernel given in the second one.

▷ For formula (11.3), we compute by stationary phase the integral $\int [\widehat{\chi}](x,x) \mathrm{d}x$ using the tow expressions of $[\widehat{\chi}](x,y)$.

It is then enough to check the signs $\pm$. □

From the previous formulae, we can get the Van Vleck formula (Theorem 1) and the trace formula (Theorem 2) for a twist map with isolated periodic points.

11.6. Regularized Determinants of continuous Sturm-Liouville operators

Let us consider the scalar differential operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x), \quad 0 \leqslant x \leqslant T$$

which we will discretize as follows: let $\varepsilon = T/N$, and $L_\varepsilon : \mathbb{R}^{N+1} \to \mathbb{R}^{N-1}$ defined by

$$(L_\varepsilon x)_j = \frac{2x_j - x_{j-1} - x_{j+1}}{\varepsilon^2} + q(j\varepsilon)x_j, \quad 1 \leqslant j \leqslant N.$$

We will consider the quadratic form q_ε on $\mathbb{R}^{N+1}$ defined by

$$q_\varepsilon(x_0, \cdots, x_N) = \tfrac{1}{2}\langle Lx \mid x\rangle_{\mathbb{R}^{N-1}}.$$

We introduce the following operators:

▷ $L_{\varepsilon,\mathrm{Dir}}$ the restriction of L_ε to $x_0 = x_{N+1} = 0$,

▷ $L_{\varepsilon,\mathrm{Neu}}$ the operator on $\mathbb{R}^{N+1}$ associated to q_ε,

▷ $L_{\varepsilon,\mathrm{Per}}$ the operator on $\mathbb{R}^N$ associated to the restriction of q_ε to $x_0 = x_{N+1}$

and $\chi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ the symplectic map defined by

$$\chi_\varepsilon\Big(x_0, \frac{x_1 - x_0}{\varepsilon}\Big) = \Big(x_N, \frac{x_N - x_{N-1}}{\varepsilon}\Big),$$

with $L_\varepsilon x = 0$ and the matrix of χ_ε:

$$[\chi_\varepsilon] = \begin{pmatrix} \alpha_\varepsilon & \beta_\varepsilon \\ \gamma_\varepsilon & \delta_\varepsilon \end{pmatrix}.$$

From Theorem 13, we get:

▷ $\det(L_{\varepsilon,\mathrm{Dir}}) = \varepsilon^{-N}\gamma_\varepsilon$,

▷ $\det(L_{\varepsilon,\mathrm{Neu}}) = \varepsilon^{-N}\beta_\varepsilon$,

▷ $\det(L_{\varepsilon,\mathrm{Per}}) = -\varepsilon^{-N}\det(\mathrm{Id} - \chi_\varepsilon)$.

As $\varepsilon \to 0$, we get:

▷ For $k = 1, \cdots$, the eigenvalue $\lambda_k(\varepsilon)$ of $L_{\varepsilon,\mathrm{Dir}}$ (resp. $L_{\varepsilon,\mathrm{Neu}}$, resp. $L_{\varepsilon,\mathrm{Per}}$) converges to the the eigenvalue λ_k of the corresponding boundary value operator associated to L

▷ χ_ε converges to the map $\chi : (x(0), x'(0)) \to (x(T), x'(T))$ with $Lx = 0$.

It is then possible [55] to deduce the result of Levit-Smilansky [40]:

Theorem 14. — *Let us consider the Dirichlet eigenvalues λ_k^i of L_{q_i}, $i = 1, 2$, we have*

$$\prod_{k=1}^{\infty} \frac{\lambda_k^1}{\lambda_k^2} = \frac{\beta_1}{\beta_2},$$

where β_i are the corresponding entries of the matrices of χ_i.

Similar results holds for the two other boundary value problems.

12. Recent progress

▷ Birkhoff normal forms: Zelditch [60], [61] and Guillemin [26] were able to use an extension of the classical Birkhoff normal form associated to a closed geodesic to a semi-classical one. From these formulae, it is rather clear that it is possible to deduce the full asymptotic expansion of the contribution of this geodesic to the SCTF.

▷ Long time trace formulae: using an extension of the Sternberg Theorem due to Delatte, F. Faure [23] were recently able to get a SCTF for a quantum hyperbolic map were the summation include all closed trajectories of periods $O(\log h)$.

13. Open problems

▷ Are they isospectral Riemannian manifolds with different length spectra and conversely?

Zoll manifolds provides example of manifolds for which the geodesic flow is conjugated to the geodesic flow on the round 2-sphere. On the other hand, it is known that manifolds isospectral to the round 2-sphere is a round 2-sphere.

It is very unlikely that there exists isospectral manifolds with different length spectra. It is however possible that the WKB expansion associated to two closed geodesics of the same length cancel, such making invisible that length in the SCTF.

▷ What are the implications of SCTF for eigenvalue statistics? It has been conjectured since a long time [14], [12] that at small scale, the eigenvalue statistics of the Laplace operator on a manifold with Anosov geodesic flow obey to laws derived from random matrix theory. At the moment SCTF provides only information on a much marger scale (not universal) and I see no ways to go further from SCTF contrary to the belief of many physicists!

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