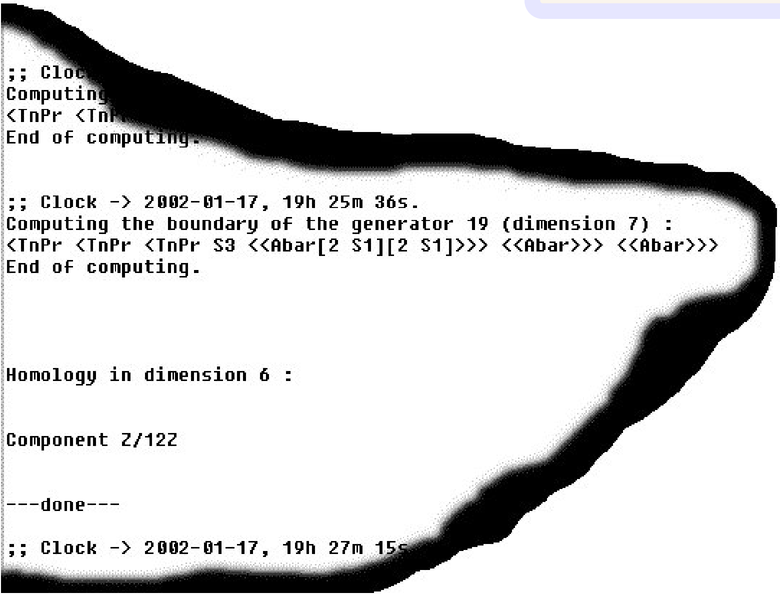
Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Simplicial Effective Homotopy



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EACA-2016, *Logroño*

1. Introduction.

"Duality" between Homology and Homotopy:

Homology Groups (H_*): Definition hard; calculation easy.

Homotopy Groups (π_*): Definition easy; calculation hard.

Example of the 2-sphere S^2:

$$H_*(S^2) = \{\mathbb{Z},\ 0,\ \mathbb{Z},\ 0,\ 0,\ 0,\ 0,\ \ldots\}.$$
$$\pi_*(S^2) = \{0,\ 0,\ \mathbb{Z},\ \mathbb{Z},\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ \mathbb{Z}/12,\ \ldots\}$$
$$\pi_{10}(S^2) = \mathbb{Z}/15 \qquad \pi_{14}(S^2) = \mathbb{Z}/4 + \mathbb{Z}/84$$

Effective Homology theory (1985) $\Rightarrow$

New homology groups calculated.

Example: $X = \Omega(\Omega(\Omega(P^\infty(\mathbb{R})/P^3(\mathbb{R})) \cup_4 D^4) \cup_2 D^3)$

Problem: $H_*(X) = ???$

Effective Homology calculation $\Rightarrow$

$H_0 = \mathbb{Z}$ $\quad$ $H_1 = \mathbb{Z}/2$

$H_2 = (\mathbb{Z}/2)^2 + \mathbb{Z}$ $\quad$ $H_3 = \mathbb{Z}/8$

$H_4 = (\mathbb{Z}/2)^{10} + \mathbb{Z}/4 + \mathbb{Z}^2$ $\quad$ $H_5 = (\mathbb{Z}/2)^{23} + \mathbb{Z}/8 + \mathbb{Z}/16$

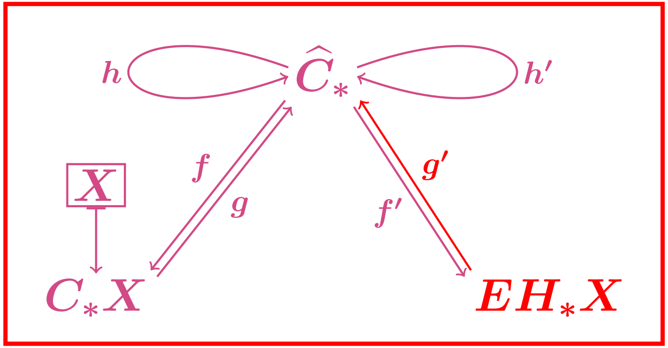
$H_6 = (\mathbb{Z}/2)^{52} + (\mathbb{Z}/4)^3 + \mathbb{Z}^3$

$H_7 = (\mathbb{Z}/2)^{113} + \mathbb{Z}/4 + (\mathbb{Z}/8)^3 + \mathbb{Z}/16 + \mathbb{Z}/32 + \mathbb{Z}$

Question: What about Effective Homotopy ??

Eff-Homology(X) = $H_*(X)$ + Rich relations $X \Leftrightarrow H_*(X)$

Eff-Homology(X) =

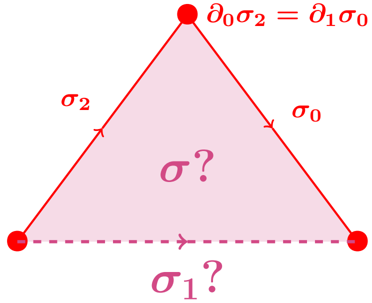


Eff-Homotopy(X) = ???

2. Kan spaces.

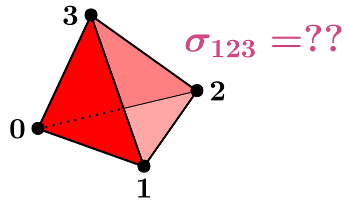
X = Simplicial set.

Kan condition:



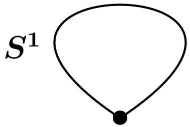
X Kan $\Leftrightarrow$ σ and σ_1 always findable +

the same for higher dimensions:

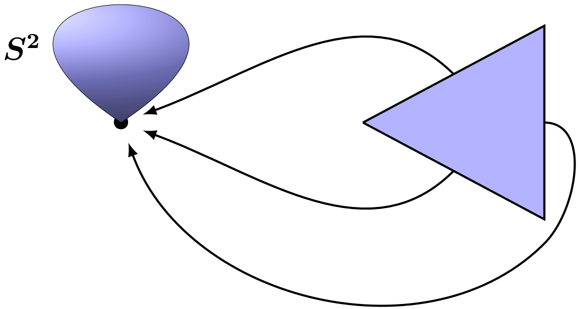


Spheres in Kan simplicial sets:

1-sphere:



2-sphere:

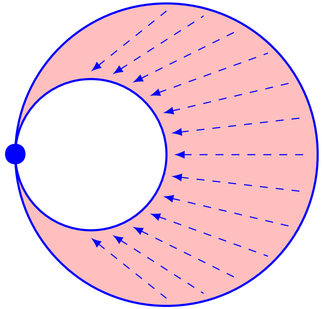


n-sphere $= \Delta^n/\partial\Delta^n$

$\{n\text{-sphere}\} =: S^n(X)$

X = Kan simplicial set.

A homotopy between two 1-spheres:

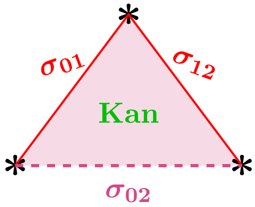


$$\pi_1(X) = S^1(X)/\text{homotopy} = \text{Poincaré group}$$

More generally:

$$\pi_n(X) := S^n(X)/\text{homotopy} = \text{Abelian group if } n > 1$$

Combinatorial group structure on the spheres.



$*$ = base point

Sphere(σ_{01}) $\circ$ Sphere(σ_{12}) $=:$ Sphere(σ_{02})

with σ_{02} given by the Kan property.

Obvious generalization for higher dimensions.

Composition compatible with homotopy $\Rightarrow$

Group structure on $\pi_n(X)$.

3. Effective Homotopy.

X = simply connected Kan simplicial set.

Effective Homotopy of X = $E\pi_*(X)$:

$$E\pi_*(X) = \{\pi_n, f_n, g_n, h_n\}_{n\geq 2}$$

with:

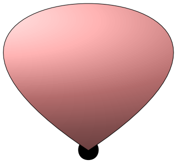
$\pi_n(X) = \pi_n(X)$ represented by its divisor sequence:

$$(\mathbb{Z}/2)^3 + (\mathbb{Z}/3)^2 + \mathbb{Z}/5 + \mathbb{Z} =: (2, 6, 30, 0)$$
$$= \mathbb{Z}/2 + \mathbb{Z}/6 + \mathbb{Z}/30 + \mathbb{Z}$$

$$\pi_n(X) \ni t = (0, 3, 17, -14)$$

$$E\pi_*(X) = \{\pi_n, f_n, g_n, h_n\}_{n\geq 2}$$

$$f_n : S^n(X) \to \pi_n$$

$f_2 : S^2(X) \ni$  $\longmapsto (0, 3, 17, -14) \in \pi_2$

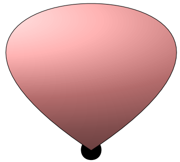
f_n must compute the homotopy class

of every sphere $\in S^n(X)$,

expressed as an element of the "abstract" π_n.

$$E\pi_*(X) = \{\pi_n, f_n, g_n, h_n\}_{n \geq 2}$$

$$g_n : \pi_n \longrightarrow S^n(X)$$

$$g_2 : \pi_2 \ni (0, 3, 17, -14) \longmapsto$$  $$\in S^2(X)$$

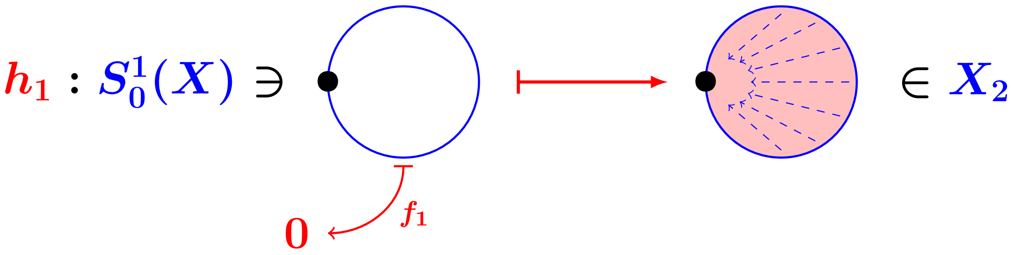
g_n must compute, for every $a \in \pi_n$,

a representative of a in $S^n(X)$:

$$f_n(g_n(a)) = a$$

$$E\pi_*(X) = \{\pi_n, f_n, g_n, h_n\}_{n\geq 2}$$

$$h_n : S^n_0(X) \longrightarrow X_{n+1}$$

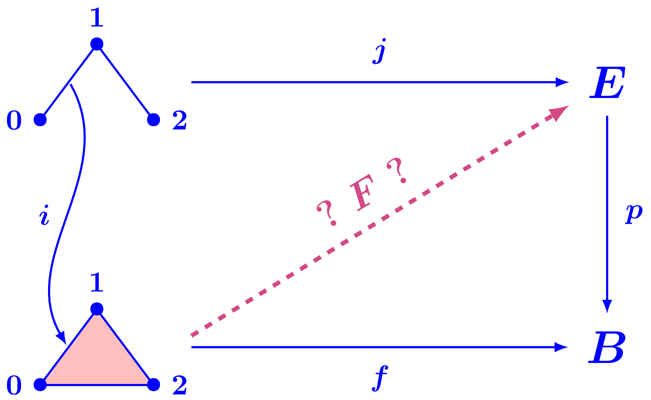


$$S^n_0(X) = \text{“ker”} f_n : S^n(X) \rightarrow \pi_n$$

h_n= algorithm producing a certificate of $s \in S^n_0(X)$

4. Main result of Effective Homotopy.

Definition: A **Kan fibration** is a **simplicial map** $p : E \to B$ satisfying the **Kan condition**:



+ Obvious generalization in higher dimensions.

Kan fibration: $[p : E \to B] \Rightarrow$ **fibre space** $F := p^{-1}(* \in B)$.

$\Rightarrow$ **diagram:**

$$F \xrightarrow{i} E \xrightarrow{p} B$$

$\Rightarrow$ **Long Serre exact sequence:**

$$\cdots \dashrightarrow \pi_{n+1}(F) \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \dashrightarrow \cdots$$

$$\cdots \dashrightarrow \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \dashrightarrow \cdots$$

Problem: Given $\pi_*(F)$ and $\pi_*(B)$, determine $\pi_*(E)$??

$\Rightarrow$ **Short exact sequence:**

$$0 \to \operatorname{coker} \partial \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \ker \partial \to 0$$

$=$ **Extension problem !!**

$$0 \to \operatorname{coker} \partial \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \ker \partial \to 0$$

Fact: $\pi_n(E)$ **determined by** $\chi \in H^2(\ker \partial; \operatorname{coker} \partial)$.

Problem: **How to compute** χ **??**

$$0 \rightarrow \operatorname{coker} \partial \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \ker \partial \rightarrow 0$$

Theorem: $[p : E \rightarrow B] = \boxed{\text{effective}}$ Kan fibration $\Rightarrow$

a direct algorithm produces

the characteristic class $\chi \in H^2(\ker \partial; \operatorname{coker} \partial)$.

$\Rightarrow$

Algorithm: $\quad E\pi_*(F) + E\pi_*(B) \longmapsto E\pi_*(E)$.

More generally:

Constructor: $\boldsymbol{X_1, \ldots, X_m} \xmapsto{c} \boldsymbol{Y}$

$\Rightarrow$

Algorithm c_π:

$$\boldsymbol{E\pi_*(X_1)} + \cdots + \boldsymbol{E\pi_*(X_m)} \xmapsto{c_\pi} \boldsymbol{E\pi_*(Y)}$$

Main application:

Adams spectral sequence:

$$H_*(X) \text{ “}\Longrightarrow\text{” } \pi_*(X)$$

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$$\boldsymbol{H_*(X)} \text{ “}\Longrightarrow\text{” } \boldsymbol{\pi_*(X)}$$

$$\downarrow$$

$$\boldsymbol{EH_*(X)} \boxed{\Longrightarrow}^{*} \boldsymbol{E\pi_*(X)}$$

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Adams spectral sequence:

$$H_*(X) \text{ “} \Longrightarrow \text{”} \pi_*(X)$$

$$\big\downarrow \textbf{ Ana's ten years work}$$

$$EH_*(X) \boxed{\Longrightarrow}^* E\pi_*(X)$$

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$\downarrow$

Concrete computer program

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$$EH_*(X) \boxed{\Longrightarrow}^* E\pi_*(X)$$

$\downarrow$ **ten years work ??**

Concrete computer program

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↓ Ana's ten years work

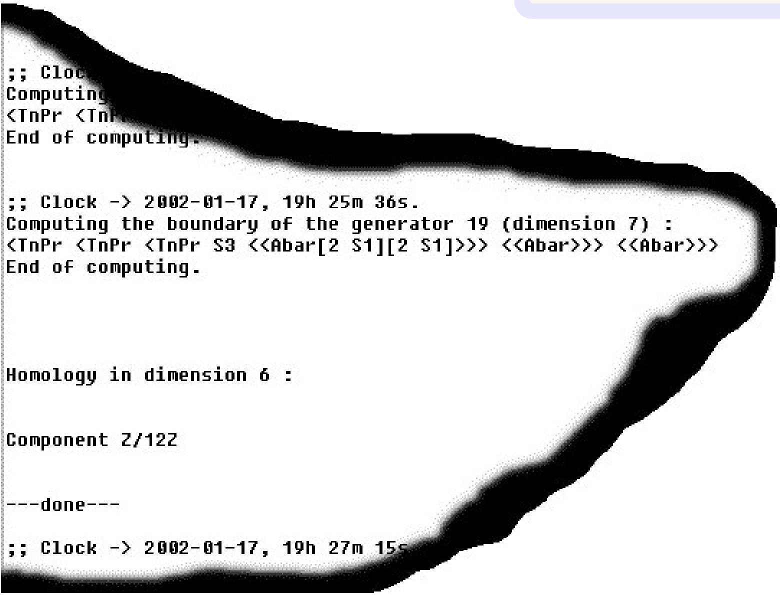
$$EH_*(X) \boxed{\Longrightarrow}^* E\pi_*(X)$$

↓ ten years work ??

Concrete computer program

* **Just published Found. of Comput. Math., May 2016.**

The END



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