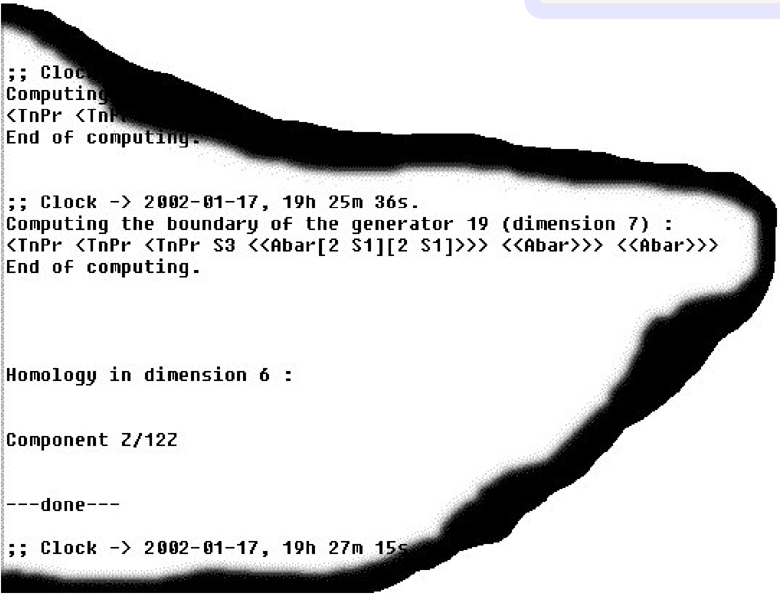
Discrete Vector Fields and Effective Homology



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Joint work with Ana Romero, Universidad de La Rioja
Jagiellonian University, Krakow, 12 May 2016

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, . . .
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, . . .
Green	=	**Solution, essential point,**
		mathematicians, . . .

Plan.

→ • **Introduction.**

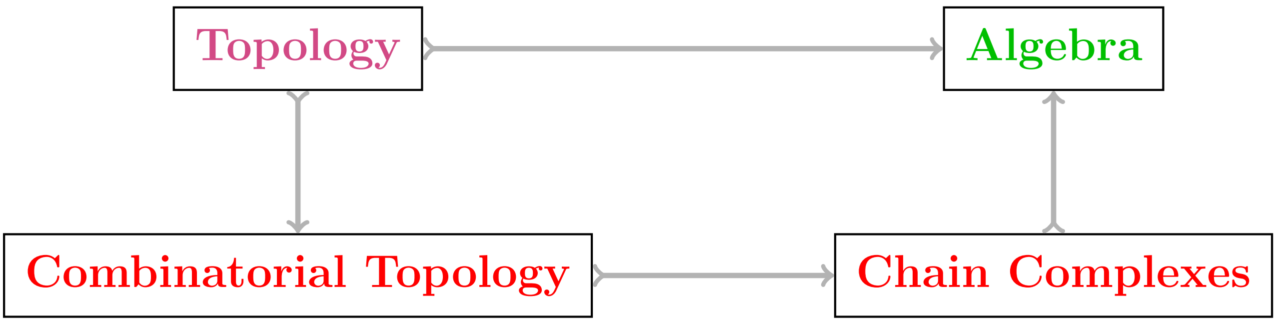
- **Discrete vector fields.**
- **Homological Reductions.**
- **Product problem in Combinatorial Topology.**
- **Discrete Vector Field for Products.**
- **Free generalization to twisted products.**
- **Effective Eilenberg-Moore spectral sequences.**

Introduction.

Algebraic Topology is a translator:

1. Introduction.

Algebraic Topology is a translator:



Serre (1950): Up to homotopy

any map can be transformed into a fibration.

Fibration = Twisted Product

Topology	$\longmapsto$	Algebra
Product	$\longmapsto$	Eilenberg-Zilber Theorem
Twisted product	$\longmapsto$	Serre Spectral Sequence

Discrete vector fields

$\Rightarrow$ New understanding of the Eilenberg-Zilber Theorem

$\Rightarrow$ An effective version of the Serre Spectral Sequence as a direct consequence of this version of Eilenberg-Zilber.

Example: Rubio-Morace homotopy for Eilenberg-Zilber:

$$RM : C_*(X \times Y) \to C_{*+1}(X \times Y)$$

$$RM(x_p \times y_p) = \sum_{\substack{0\leq r\leq p-1 \\ 0\leq s\leq p-r-1 \\ (\eta,\eta')\in \mathrm{Sh}(s+1,r)}} (-1)^{p-r-s}\, \varepsilon(\eta,\eta') \ldots$$

$$\ldots (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1} y_p)$$

with $\mathrm{Sh}(p,q) = \{(p,q)\text{-shuffles}\} = \{(\eta_{i_{p-1}}\cdots\eta_{i_0}, \eta_{j_{q-1}}\cdots\eta_{j_0})\}$

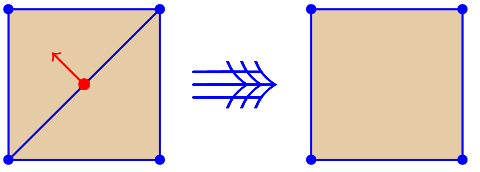
for $0 \leq i_0 < \cdots < i_{p-1} \leq p+q-1$

and $0 \leq j_0 < \cdots < j_{q-1} \leq p+q-1$

and $\{i_0,\ldots,i_{p-1}\} \cap \{j_0,\ldots,j_{q-1}\} = \emptyset$.

and $\uparrow^k(\eta_\alpha\eta_\beta\cdots) = \eta_{\alpha+k}\eta_{\beta+k}\cdots$ $\quad$ ($\uparrow^k$ = k-shift operator.)

Simpler:



once the notion of discrete vector field is understood.

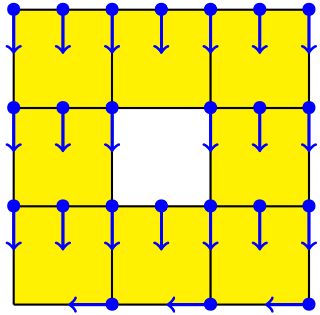
2. Discrete Vector Fields.

Definition: **A Discrete Vector Field is a pairing:**

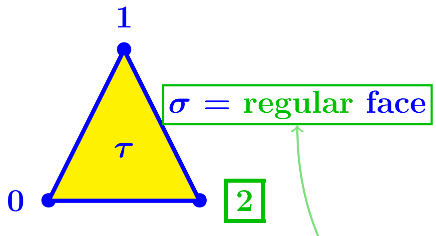
$$\boldsymbol{V = \{(\sigma_i, \tau_i)\}_{i \in v}}$$

satisfying:

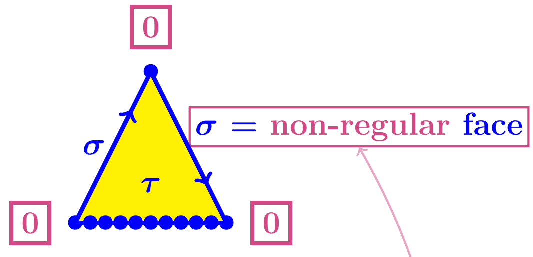
- $\forall \boldsymbol{i} \in \boldsymbol{v}$, $\boldsymbol{\tau_i}$ = **some** $\boldsymbol{k_i}$**-cell and** $\boldsymbol{\sigma_i}$ = **some** $\boldsymbol{(k_i - 1)}$**-cell.**
- $\forall \boldsymbol{i} \in \boldsymbol{v}$, $\boldsymbol{\sigma_i}$ **is a regular face of** $\boldsymbol{\tau_i}$.
- $\forall \boldsymbol{i} \neq \boldsymbol{j} \in \boldsymbol{v}$, $\boldsymbol{\{\sigma_i, \tau_i\} \cap \{\sigma_j, \tau_j\} = \emptyset}$.
- **The vector field** $\boldsymbol{V}$ **is admissible.**



Geometrical example of non-regular face:



Triangle

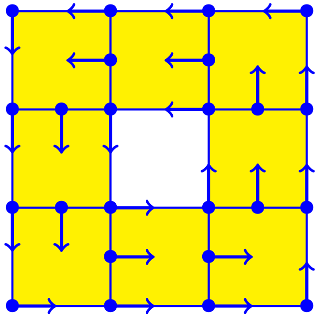


$P^2\mathbb{R}$

$$C_*(\text{Triangle}) = \{0 \longleftarrow \mathbb{Z}^3 \xleftarrow[\begin{bmatrix}-1 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & \boxed{1}\end{bmatrix}]{} \mathbb{Z}^3 \xleftarrow[\begin{bmatrix}\boxed{1}\\ -1\\ 1\end{bmatrix}]{} \mathbb{Z} \longleftarrow 0\}$$

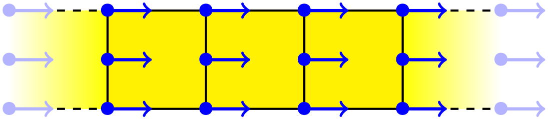
$$C_*(P^2\mathbb{R}) = \{0 \longleftarrow \mathbb{Z} \xleftarrow[\begin{bmatrix}\boxed{0}\end{bmatrix}]{} \mathbb{Z} \xleftarrow[\begin{bmatrix}\boxed{2}\end{bmatrix}]{} \mathbb{Z} \longleftarrow 0\}$$

Typical examples of non-admissible vector fields.



???!!!

$\sim \qquad \emptyset$



???!!!

$\sim \qquad \emptyset$

$\mathbb{R} \times \boldsymbol{I}$

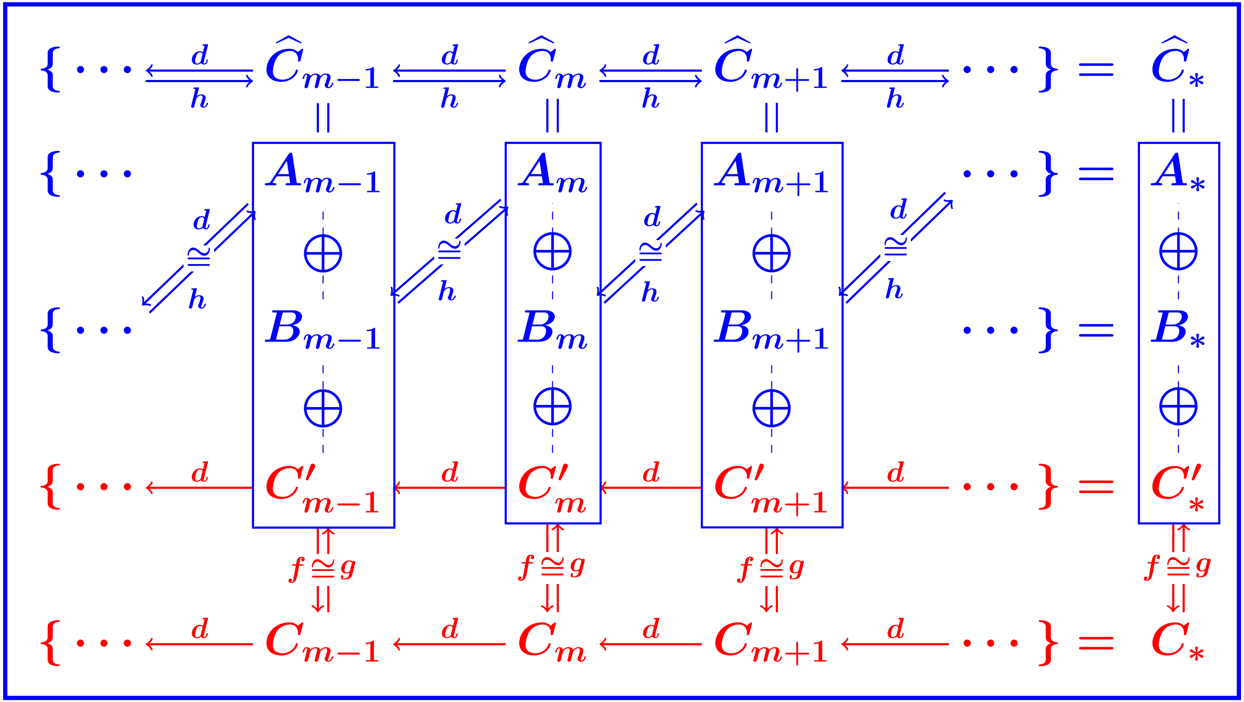
3. Homological Reductions.

Definition: A (homological) reduction is a diagram:

$$\rho: \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree $+1$).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.



$A_* = \ker f \cap \ker h$ $\quad$ $B_* = \ker f \cap \ker d$ $\quad$ $C'_* = \mathrm{im}(g)$

$$\widehat{C}_* = A_* \oplus B_* \text{exact} \oplus C'_* \cong C_*$$

Fundamental Theorem:

Given: $C_* = (C_p, \beta_p, d_p)_{p\in\mathbb{Z}}$ = Cellular **chain complex.**

$V = (\sigma_i, \tau_i)_{i\in v}$ = Admissible **Discrete Vector Field.**

$\Rightarrow$ **Canonical Reduction:**

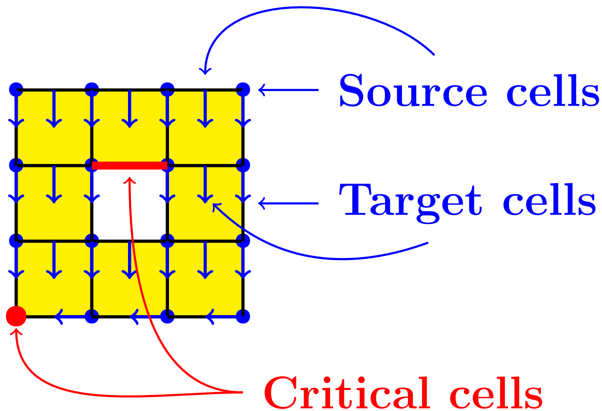
$$\rho_V = \boxed{h \circlearrowleft (C_p, \beta_p, d_p)_{p\in\mathbb{Z}} \underset{f}{\overset{g}{\leftrightarrows}} (C_p^c, \beta_p^c, d_p^c)_{p\in\mathbb{Z}}}$$

$$\boxed{\text{Initial Complex}} \overset{\rho_V}{\Rrightarrow} \boxed{\text{Critical complex}}$$

Proof: Homological Perturbation Theorem.

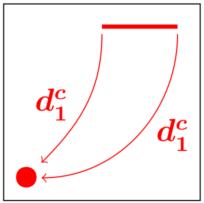
Toy Example.

$C_* =$



Fundamental Reduction Theorem $\Rightarrow$

$\rho : C_* \Rrightarrow C_*^c =$



$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \textbf{Circle}$

$$\textbf{Rank}(C_*) = (16, 24, 8) \quad \text{vs} \quad \textbf{Rank}(C_*^c) = (1, 1, 0)$$

4. Product problem in Combinatorial Topology.

1. **Simplicial organisation necessary**

   for example for Eilenberg-MacLane spaces.

2. **$\Rightarrow$ Elementary models $= \boldsymbol{\Delta}^n$ for $\boldsymbol{n} \in \mathbb{N}$.**

3. **Fact:**

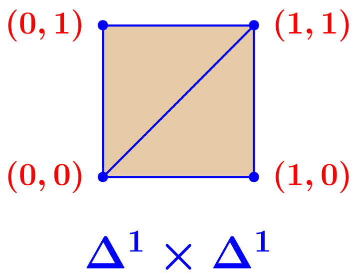
   No direct simplicial structure for a product $\boldsymbol{\Delta}^p \times \boldsymbol{\Delta}^q$.

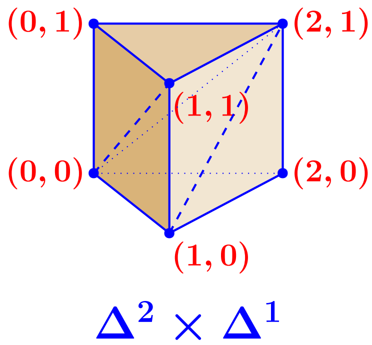
4. **What about twisted products = Fibrations ??**

5. **Classical solution = Eilenberg-Zilber + Kan + RM**

   + Serre and Eilenberg-Moore Spectral sequences.

6. **Other solution = Discrete Vector Fields.**





Two Δ^2 **in** $\Delta^1 \times \Delta^1$**:** $(0,0) < (0,1) < (1,1)$

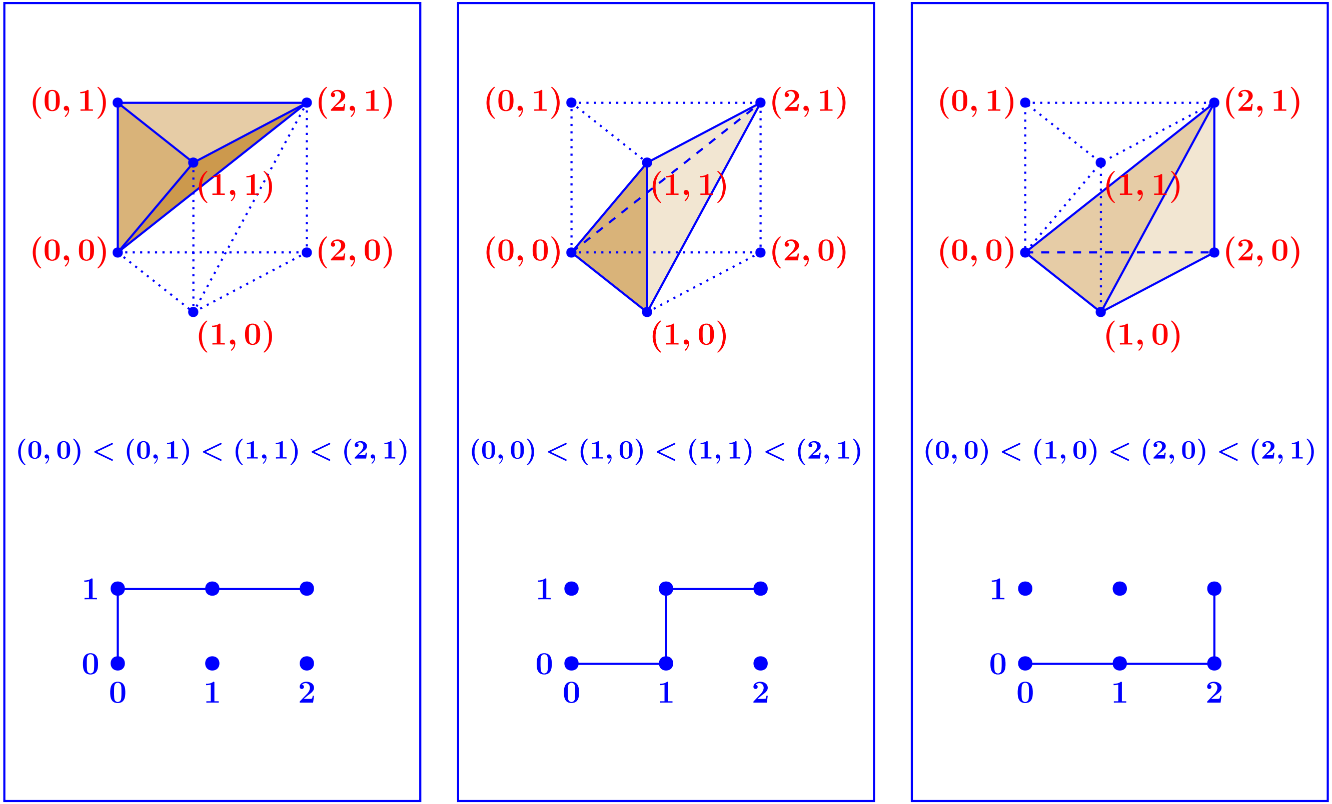
$(0,0) < (1,0) < (1,1)$

Three Δ^3 **in** $\Delta^2 \times \Delta^1$**:** $(0,0) < (0,1) < (1,1) < (2,1)$

$(0,0) < (1,0) < (1,1) < (2,1)$

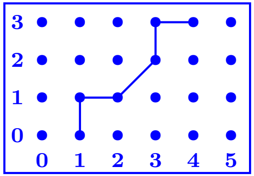
$(0,0) < (1,0) < (2,0) < (2,1)$

Rewriting the triangulation of $\Delta^2 \times \Delta^1$.



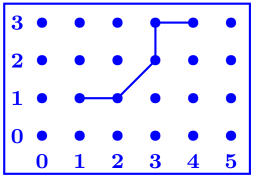
Increasing chain in the lattice ⟷ Simplex in the Product

"Seeing" the triangulation of $\Delta^5 \times \Delta^3$.

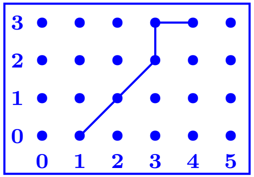
Example of 5-simplex : 

$= \sigma \in (\Delta^5 \times \Delta^3)_5$

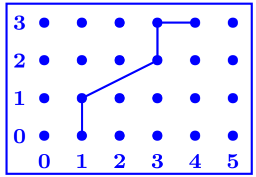
$\Rightarrow$ 6 faces:



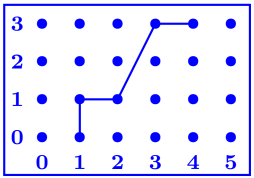
$\partial_0\sigma$



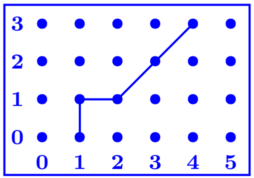
$\partial_1\sigma$



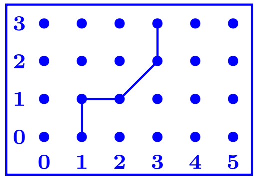
$\partial_2\sigma$



$\partial_3\sigma$

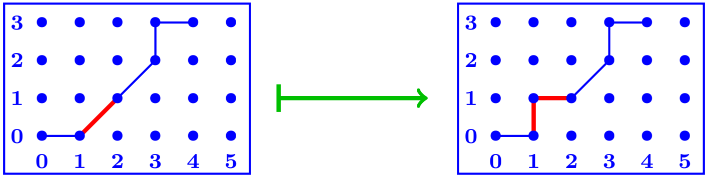


$\partial_4\sigma$

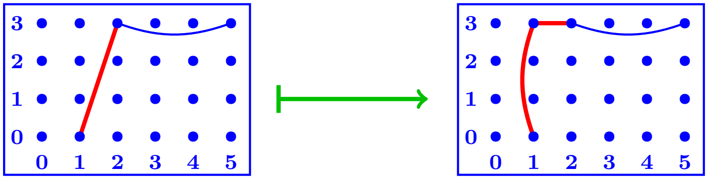


$\partial_5\sigma$

$\Rightarrow$ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

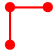


Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

<u>Recipe</u>: First "event" = Diagonal step =  $\Rightarrow$ Source cell.

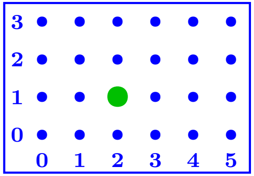
= (-90°)-bend =  $\Rightarrow$ Target cell.

Critical cells ??

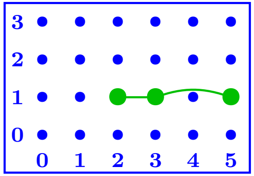
Critical cell = cell without any "event"

= without any diagonal or $-90°$-bend.

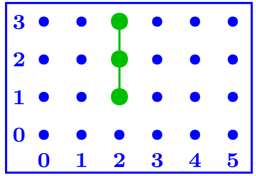
Examples.



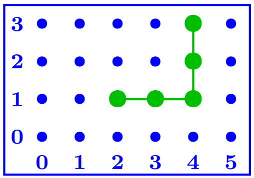
$\Delta^0_2 \otimes \Delta^0_1$



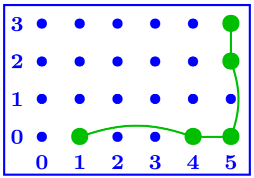
$\Delta^2_{2,3,5} \otimes \Delta^0_1$



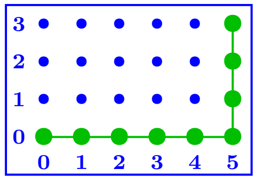
$\Delta^0_2 \otimes \Delta^2_{1,2,3}$



$\Delta^2_{2,3,4} \otimes \Delta^2_{1,2,3}$



$\Delta^2_{1,4,5} \otimes \Delta^2_{0,2,3}$



$\Delta^5_{0,1,2,3,4,5} \otimes \Delta^3_{0,1,2,3}$

Conclusion:

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

Fundamental theorem of vector fields ⇒

Canonical Homological Reductions:

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

More generally: X and Y = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$.

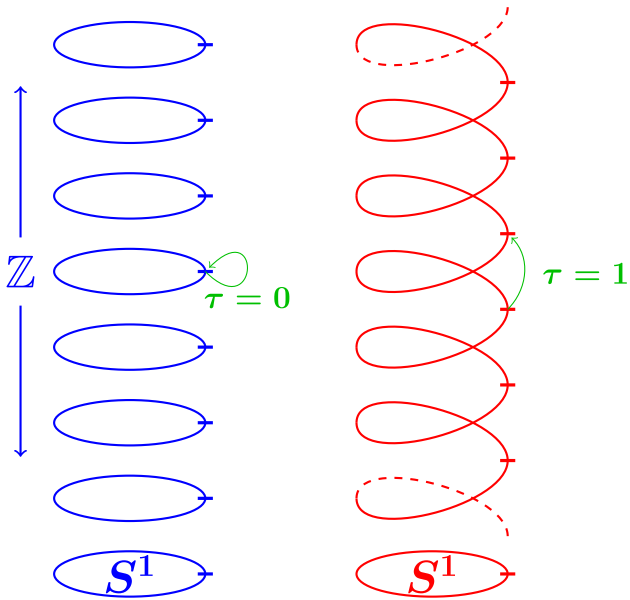
Eilenberg-Zilber Theorem: Canon. homological reduction:

$$\rho_{EZ} : C_*(X \times Y) \Rrightarrow C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

5. Extension to twisted product.

Simplest example: $\mathbb{Z} \times \boldsymbol{S}^1$ **vs** $\mathbb{Z} \times_{\tau} \boldsymbol{S}^1 = \mathbb{R}$**:**



General notion of twisted product: B = base space.

F = fibre space.

G = structural group.

Action $G \times F \rightarrow F$.

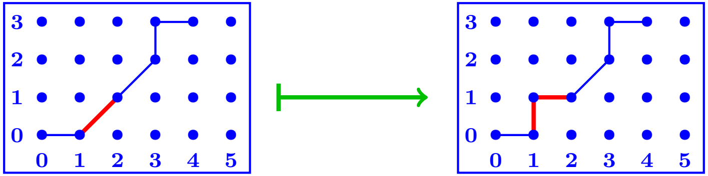
$\tau : B \rightarrow G$ = Twisting function.

Structure of $F \times_\tau B$:

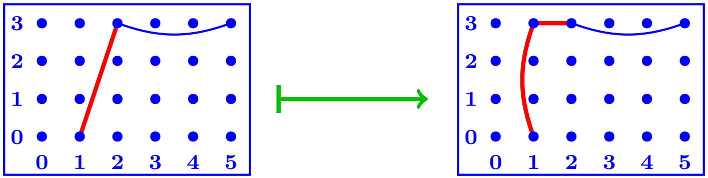
$$\partial_i(\sigma_f, \sigma_b) = (\partial_i\sigma_f, \partial_i\sigma_b) \quad \text{for } i > 0$$
$$\partial_0(\sigma_f , \sigma_b) = (\tau(\sigma_b).\partial_0\sigma_f , \partial_0\sigma_b)$$

$\Rightarrow$ Only the **0-face** is modified in the twisted product.

Reminder about the EZ-vector field of $\Delta^5 \times \Delta^3$.



Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

The vector field is concerned by faces ∂_i only if $i > 0$.

1. The twisting function τ modifies only $\boxed{0}$-faces.

2. The EZ-vector field V_{EZ} of $X \times Y$

uses only $\boxed{i}$-faces with $i \geq 1$.

$\Rightarrow$ V_{EZ} is defined and admissible as well on $X \times_{\boxed{\tau}} Y$.

Fundamental theorem of admissible vector fields $\Rightarrow$

$$C_*(X \times Y) \qquad\qquad C_*(X \times_{\boxed{\tau}} Y)$$

$$V_{EZ} \Rightarrow \Downarrow\!\Downarrow \qquad\qquad V_{EZ} \Rightarrow \Downarrow\!\Downarrow$$

$$C_*(X) \otimes C_*(Y) \qquad\qquad C_*(X) \otimes_{\boxed{t}} C_*(Y)$$

Known as the twisted Eilenberg-Zilber Theorem.

Corollary: Base B 1-reduced $\Rightarrow$ Algorithm:

$$[(F, C_*(F), EC_*^F, \varepsilon_F) + (B, C_*(B), EC_*^B, \varepsilon_B) + G + \tau]$$
$$\longmapsto (F \times_\tau B, C_*(F \times_\tau B), EC_*^{F\times_\tau B}, \varepsilon_{F\times_\tau B}).$$

Version of F with effective homology

+ Version of B with effective homology

+ $G + \tau$ describing the fibration $F \hookrightarrow F \times_\tau B \to B$

$\Rightarrow$ Version with effective homology of the total space $F \times_\tau B$.

= Version with effective homology

of the Serre Spectral Sequence

6. The Eilenberg-Moore spectral sequence.

Key results:

G = Simplicial group $\Rightarrow$ BG = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

X = Simplicial set $\Rightarrow$ KX = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

Analogous process $\Rightarrow$ Algorithms:

$$(G, C_*G, EC_*^G, \varepsilon_G) \mapsto (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(X, C_*X, EC_*^X, \varepsilon_X) \mapsto (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

More generally:

$[\alpha : E \to B] + [\alpha' : E' \to B] + [\alpha \text{ fibration}]$

$\Rightarrow$ **algorithm:** $(B_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E \times_B E')_{EH}$.

$$\begin{array}{ccc} E' \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \alpha \\ E' & \xrightarrow[\alpha']{} & B \end{array}$$

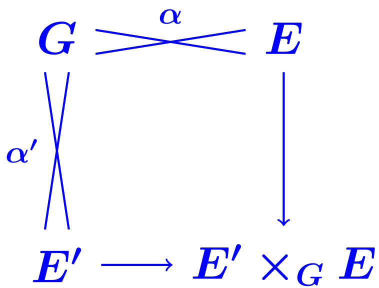
= Version with effective homology

of Eilenberg-Moore spectral sequence I.

Also:

$[G \text{ simplicial group}] + [\alpha : G \times E \to E] + [\alpha' : E' \times G \to E'] + [\alpha \text{ principal fibration}]$

$\Rightarrow$ algorithm: $(G_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E' \times_G E)_{EH}$.



$(\alpha'(x', g), x) \sim (x', \alpha(g, x))$

= Version with effective homology of Eilenberg-Moore spectral sequence II.

Integrating the Vector Field technology

in the Kenzo program

$\Rightarrow$ **Faster program!**

<u>**Example:**</u> $\pi_5(\Omega(S^3) \cup_2 D^3) = ??$

Integrating the Vector Field technology

in the Kenzo program

$\Rightarrow$ Faster program!

Example: $\boldsymbol{\pi_5(\Omega(S^3) \cup_2 D^3) = (\mathbb{Z}/2)^4}$

On the same machine:

Old version $\Rightarrow$ 1h32m

New version $\Rightarrow$ 0h05m

with the same result !

Computing time divided by 18.

Most recent calculation:

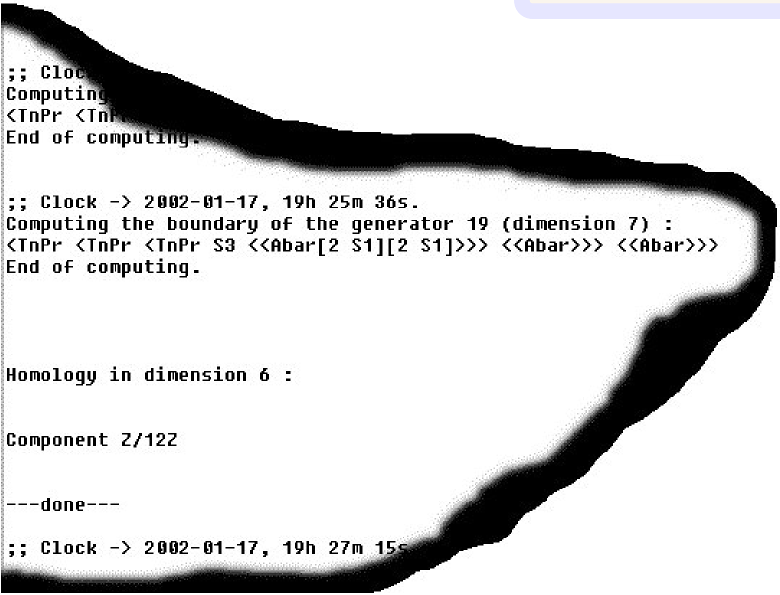
$$\pi_6(\Omega(S^3) \cup_2 D_3) = \quad ??$$

14 days long calculation on an idle machine of our cryptography team (1To RAM):

$$\pi_6(\Omega(S^3) \cup_2 D_3) = (\mathbb{Z}/2)^5 + \mathbb{Z}$$

(14 days ×18 = 252 days)

The END



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Joint work with Ana Romero, Universidad de La Rioja
Jagiellonian University, Krakow, 12 May 2016