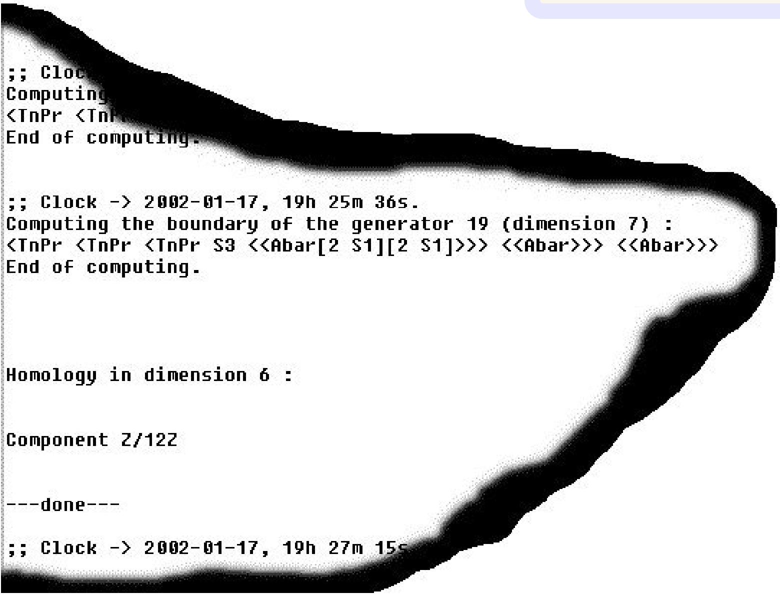
Discrete Vector Fields and Fundamental Algebraic Topology



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Université de Poitiers, January 2015

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Plan.

→ • **Introduction.**

- **Discrete vector fields.**
- **Homological Reductions.**
- **Product problem in Combinatorial Topology.**
- **Discrete Vector Field for Products.**
- **Free generalization to twisted products.**
- **Effective Eilenberg-Moore spectral sequences.**

Introduction.

Algebraic Topology is a translator:

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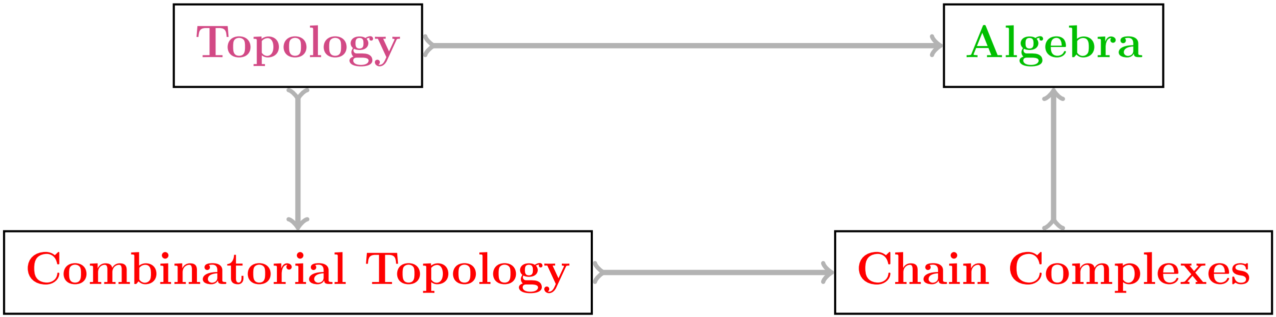


Serre (1950): Up to homotopy

any map can be transformed into a fibration.

Fibration = Twisted Product

Topology	$\longmapsto$	Algebra
Product	$\longmapsto$	Eilenberg-Zilber Theorem
Twisted product	$\longmapsto$	Serre Spectral Sequence

Discrete vector fields

$\Rightarrow$ New understanding of the Eilenberg-Zilber Theorem

$\Rightarrow$ An effective version of the Serre Spectral Sequence

as a direct consequence of this version of Eilenberg-Zilber.

Example: Rubio-Morace homotopy for Eilenberg-Zilber:

$$RM : C_*(X \times Y) \to C_{*+1}(X \times Y)$$

$$RM(x_p \times y_p) \;=\; \sum_{\substack{0 \le r \le p-1 \\ 0 \le s \le p-r-1 \\ (\eta,\eta') \in \mathrm{Sh}(s+1,r)}} (-1)^{p-r-s}\, \varepsilon(\eta,\eta') \ldots$$

$$\ldots (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p)$$

with $\mathrm{Sh}(p,q) = \{(p,q)\text{-shuffles}\} = \{(\eta_{i_{p-1}}\cdots\eta_{i_0}, \eta_{j_{q-1}}\cdots\eta_{j_0})\}$

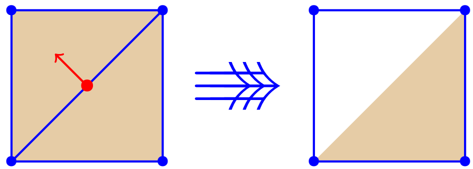
for $0 \le i_0 < \cdots < i_{p-1} \le p+q-1$

and $0 \le j_0 < \cdots < j_{q-1} \le p+q-1$

and $\{i_0,\ldots,i_{p-1}\} \cap \{j_0,\ldots,j_{q-1}\} = \emptyset$.

and $\uparrow^k (\eta_\alpha\eta_\beta\cdots) = \eta_{\alpha+k}\eta_{\beta+k}\cdots$ $\quad$ ($\uparrow^k$ = k-shift operator.)

Simpler:



once the notion of discrete vector field is understood.

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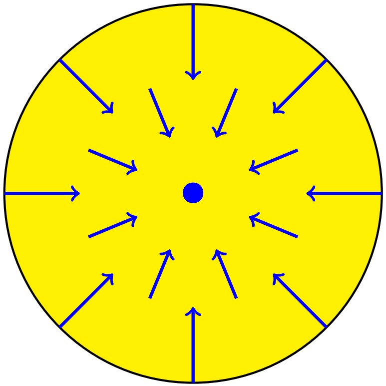
Discrete vector fields

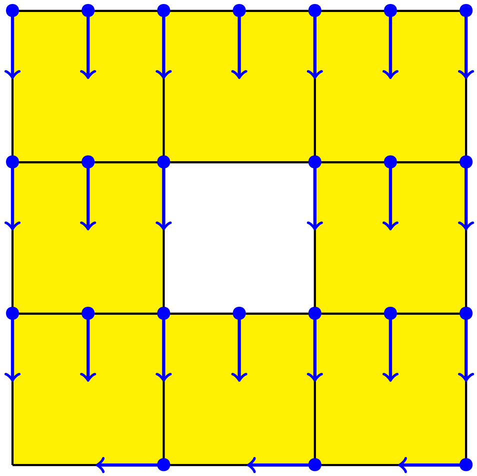
Ordinary vector fields → **Discrete vector fields** → **Algebraic vector fields**

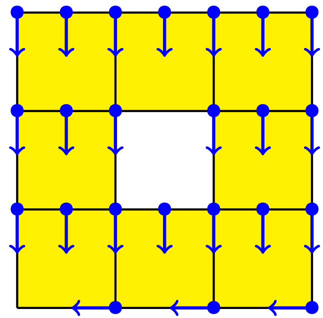
Ordinary vector field:

$D^2 =$  $\Rightarrow$ $D^2 \sim \bullet$

Discrete vector field in a cellular complex.

Example for a cubical complex.





Definition:

A Discrete Vector Field is a pairing:

$$\boldsymbol{V = \{(\sigma_i, \tau_i)\}_{i \in v}}$$

satisfying:

- $\forall i \in v, \quad \tau_i =$ **some** k_i**-cell and** $\sigma_i =$ **some** $(k_i - 1)$**-cell.**
- $\forall i \in v, \quad \sigma_i$ **is a regular face of** τ_i**.**
- $\forall i \neq j \in v, \quad \sigma_i \neq \sigma_j \neq \tau_i \neq \tau_j$**.**
- **The vector field** V **is admissible.**

Definition: A(n algebraic) cellular chain complex C_*

is a triple $C_* = \{C_p, \beta_p, d_p\}_{p \in \mathbb{Z}}$ satisfying:

- β_p is a distinguished basis

  of the free $\mathbb{Z}$-module $C_p = \mathbb{Z}[\beta_p]$.

- $d_p : C_p \to C_{p-1}$ is a differential ($d^2 = 0$).

Examples: Chain complexes coming from:

- Simplicial complexes, cubical complexes,

  simplicial sets, CW-complexes...

- Digital images.

- Chain complex defining some Koszul homology ($\mathbb{Z} \mapsto \mathfrak{k}$).

- ... ...

$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = **Cellular chain complex.**

Definition: A p-**cell** is an element of β_p.

Definition: If $\tau \in \beta_p$ and $\sigma \in \beta_{p-1}$,

then $\varepsilon(\sigma, \tau)$:= **coefficient** of σ in $d\tau$

is called the **incidence number** between σ and τ.

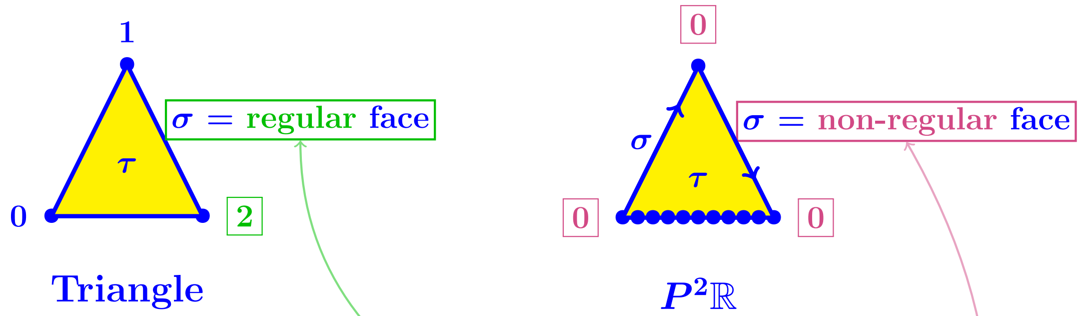
Definition: σ is a **face** of τ if $\varepsilon(\sigma, \tau) \neq 0$.

Definition: σ is a **regular face** of τ if $\varepsilon(\sigma, \tau) = \pm 1$.

[**More generally if** $\mathbb{Z} \mapsto R$,

regular face $\Leftrightarrow$ $\varepsilon(\sigma, \tau)$ **invertible**]

Geometrical example of non-regular face:



$$C_*(\text{Triangle}) = \{0 \longleftarrow \mathbb{Z}^3 \xleftarrow[\begin{bmatrix}-1 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & \boxed{1}\end{bmatrix}]{} \mathbb{Z}^3 \xleftarrow[\begin{bmatrix}\boxed{1}\\ -1\\ 1\end{bmatrix}]{} \mathbb{Z} \longleftarrow 0\}$$

$$C_*(P^2\mathbb{R}) = \{0 \longleftarrow \mathbb{Z} \xleftarrow[\begin{bmatrix}\boxed{0}\end{bmatrix}]{} \mathbb{Z} \xleftarrow[\begin{bmatrix}\boxed{2}\end{bmatrix}]{} \mathbb{Z} \longleftarrow 0\}$$

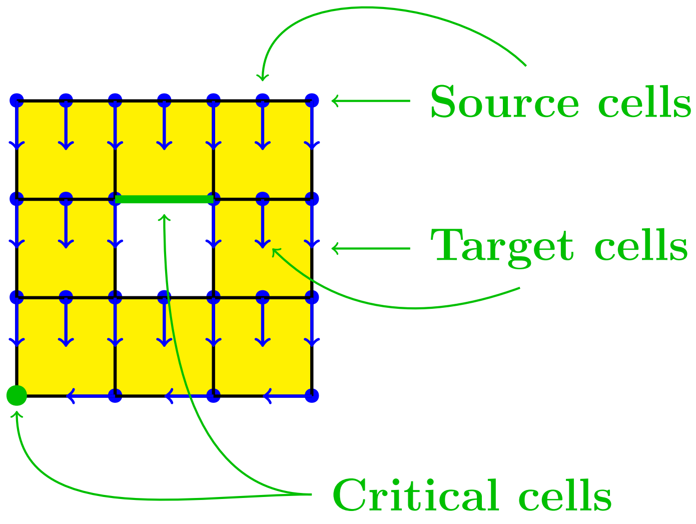
$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = **Cellular chain complex.**

$V = \{(\sigma_i, \tau_i)\}_{i\in v}$ = **Vector field.**

Definition: **A critical p-cell is an element of β_p which does not occur in V.**

Other cells divided in source cells and target cells.

Example:



$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)\}_{i\in v}$ = Vector field.

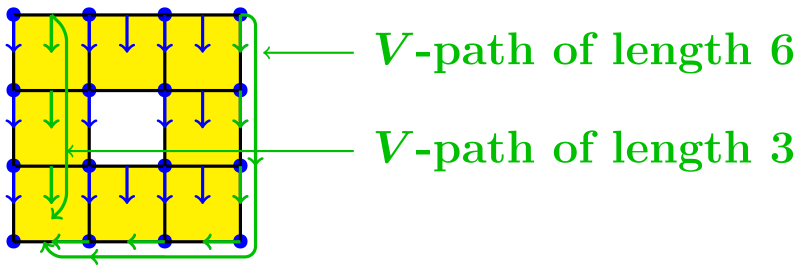
Definition: V-path = sequence $(\sigma_{i_1}, \tau_{i_1}, \sigma_{i_2}, \tau_{i_2}, \ldots, \sigma_{i_n}, \tau_{i_n})$

satisfying:

1. $(\sigma_{i_j}, \tau_{i_j}) \in V$.
2. σ_{i_j} face of $\tau_{i_{j-1}}$.
3. $\sigma_{i_j} \neq \sigma_{i_{j-1}}$.

Remark: σ_{i_j} not necessarily regular face of $\tau_{i_{j-1}}$.

Examples:



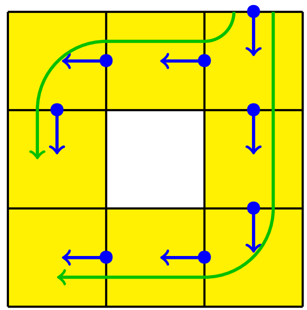
Definition: **A vector field is admissible if**

for every source cell $\boldsymbol{\sigma}$,

the length of any path starting from $\boldsymbol{\sigma}$

is bounded by a fixed integer $\boldsymbol{\lambda(\sigma)}$.

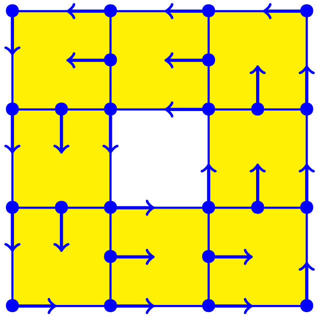
Example of two different paths with the same starting cell.



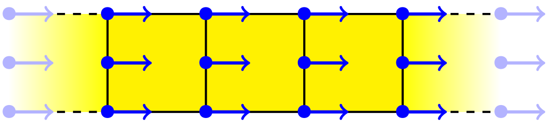
Remark: **The paths from a starting cell**

are not necessarily organized as a tree.

Typical examples of non-admissible vector fields.



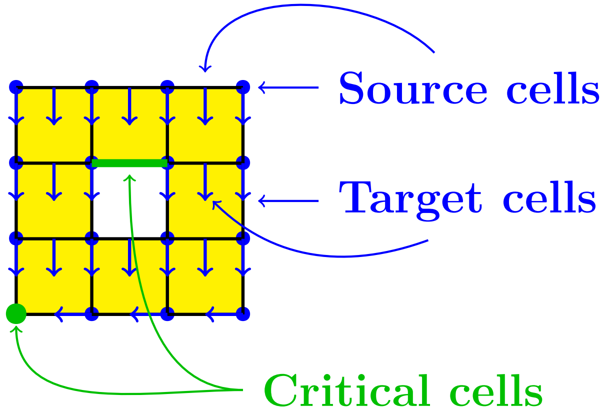
???!!! $\sim$ $\emptyset$



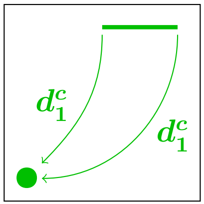
???!!! $\sim$ $\emptyset$

$\mathbb{R} \times \boldsymbol{I}$

Main motivation.

$C_* =$ 

Fundamental Reduction Theorem $\Rightarrow$

$\rho : C_* \Rrightarrow C_*^c =$ 

$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} =$ Circle

$\mathbf{Rank}(C_*) = (16, 24, 8)$ **vs** $\mathbf{Rank}(C_*^c) = (1, 1, 0)$

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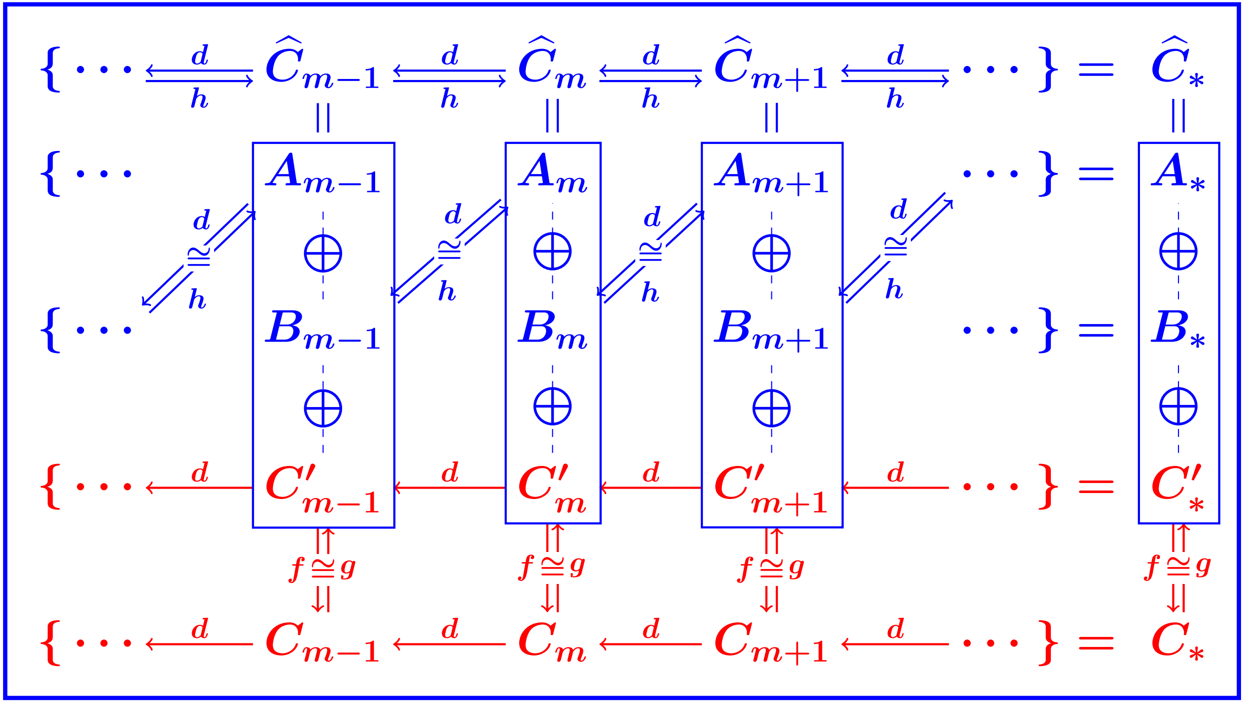
Homological Reductions.

Definition: A (homological) reduction is a diagram:

$$\rho: \quad h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*$$

with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree $+1$).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.



$A_* = \ker f \cap \ker h$ $B_* = \ker f \cap \ker d$ $C'_* = \mathrm{im}(g)$

$\widehat{C}_* = A_* \oplus B_* \text{exact} \oplus C'_* \cong C_*$

Fundamental Theorem:

Given: $C_* = (C_p, \beta_p, d_p)_{p\in\mathbb{Z}}$ = **Cellular chain complex.**

$V = (\sigma_i, \tau_i)_{i\in v}$ = **Admissible Discrete Vector Field.**

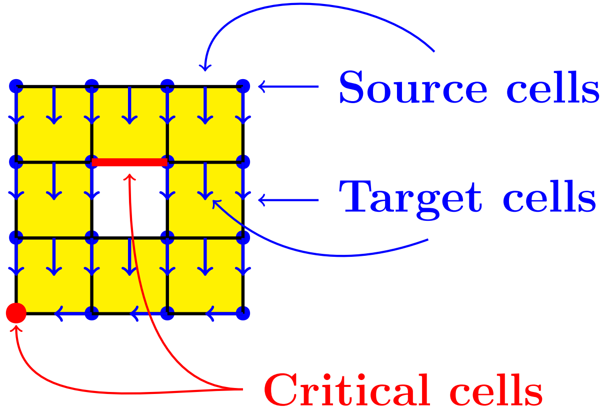
$\Rightarrow$ **Canonical Reduction:**

$$\rho_V = \boxed{h \circlearrowleft (C_p, \beta_p, d_p)_{p\in\mathbb{Z}} \underset{f}{\overset{g}{\leftleftarrows\!\!\rightrightarrows}} (C_p^c, \beta_p^c, d_p^c)_{p\in\mathbb{Z}}}$$

$$\boxed{\text{Initial Complex}} \overset{\rho_V}{\Rrightarrow} \boxed{\text{Critical complex}}$$

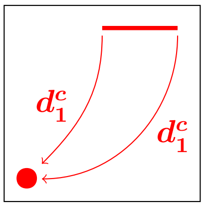
Toy Example.

$$C_* =$$



Fundamental Reduction Theorem $\Rightarrow$

$$\rho : C_* \Rrightarrow C_*^c =$$



$$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \text{Circle}$$

$$\boxed{\text{Rank}(C_*) = (16, 24, 8) \quad \text{vs} \quad \text{Rank}(C_*^c) = (1, 1, 0)}$$

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Product problem in Combinatorial Topology.

1. **Simplicial organisation necessary**

   for example for Eilenberg-MacLane spaces.

2. **$\Rightarrow$ Elementary models $= \boldsymbol{\Delta}^{n}$ for $n \in \mathbb{N}$.**

3. **Fact:**

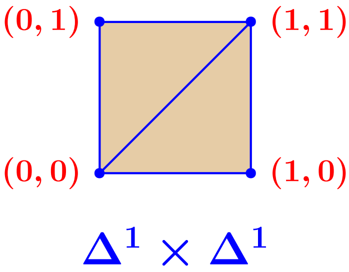
   No direct simplicial structure for a product $\boldsymbol{\Delta}^{p} \times \boldsymbol{\Delta}^{q}$.

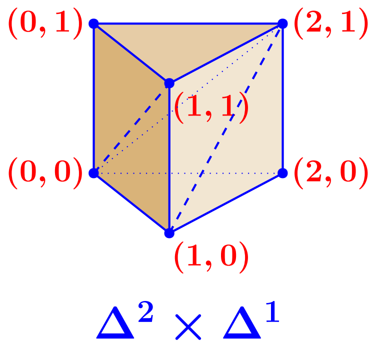
4. **What about twisted products = Fibrations ??**

5. **Classical solution = Eilenberg-Zilber + Kan + RM**

   + Serre and Eilenberg-Moore Spectral sequences.

6. **Other solution = Discrete Vector Fields.**





Two Δ^2 in $\Delta^1 \times \Delta^1$: $(0,0) < (0,1) < (1,1)$

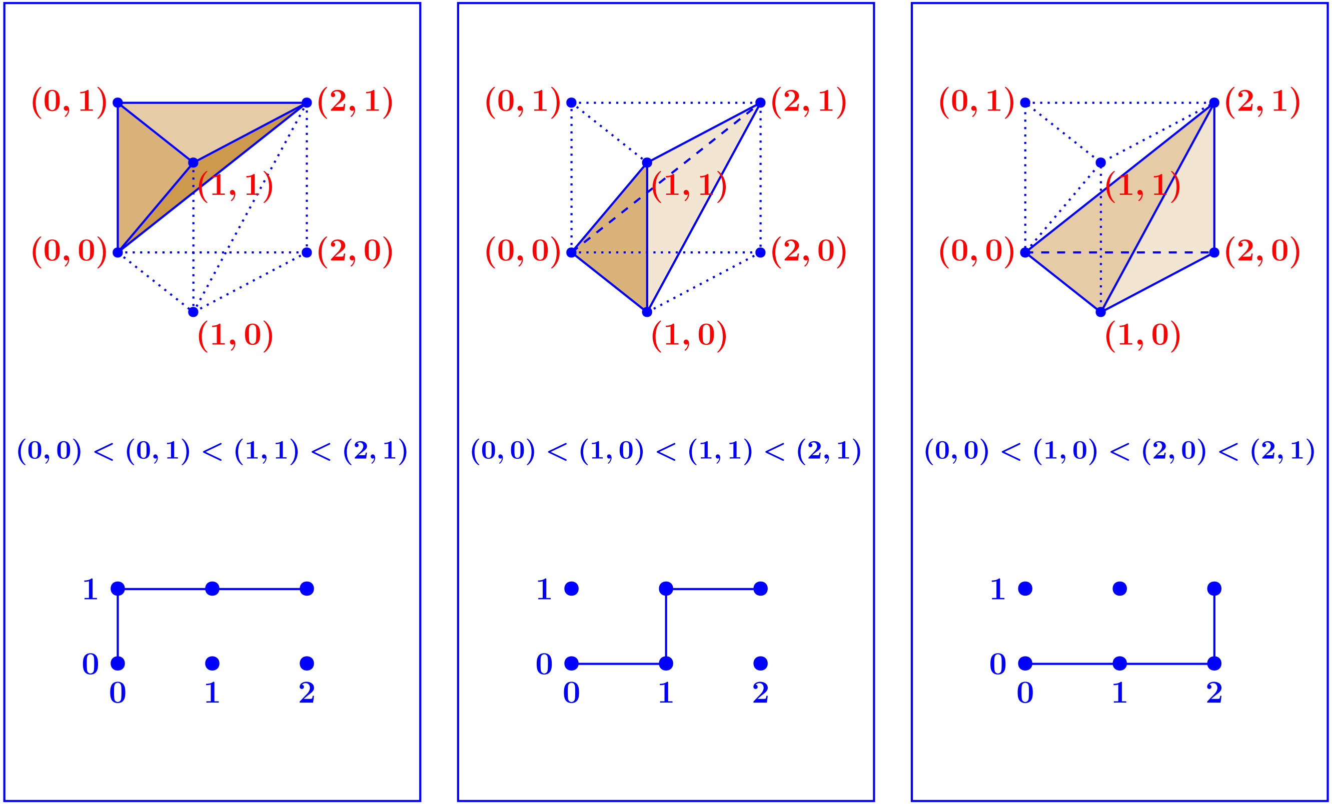
$(0,0) < (1,0) < (1,1)$

Three Δ^3 in $\Delta^2 \times \Delta^1$: $(0,0) < (0,1) < (1,1) < (2,1)$

$(0,0) < (1,0) < (1,1) < (2,1)$

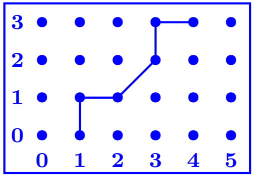
$(0,0) < (1,0) < (2,0) < (2,1)$

Rewriting the triangulation of $\Delta^2 \times \Delta^1$.



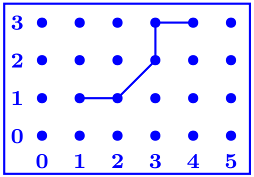
Increasing chain in the lattice $\longleftrightarrow$ Simplex in the Product

“Seeing” the triangulation of $\Delta^5 \times \Delta^3$.

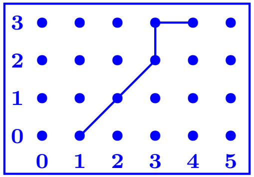
Example of 5-simplex : 

$= \sigma \in (\Delta^5 \times \Delta^3)_5$

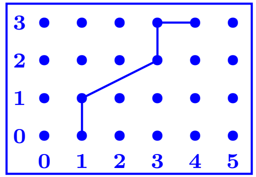
$\Rightarrow$ 6 faces:



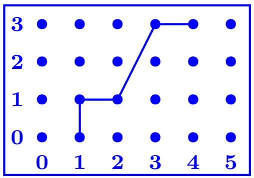
$\partial_0\sigma$



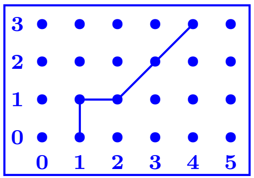
$\partial_1\sigma$



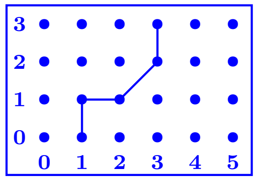
$\partial_2\sigma$



$\partial_3\sigma$



$\partial_4\sigma$



$\partial_5\sigma$

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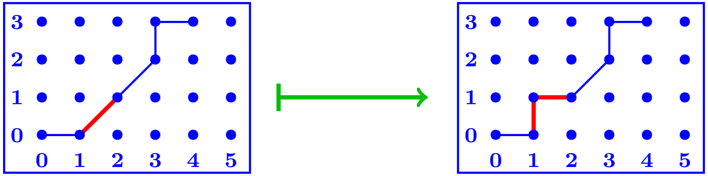
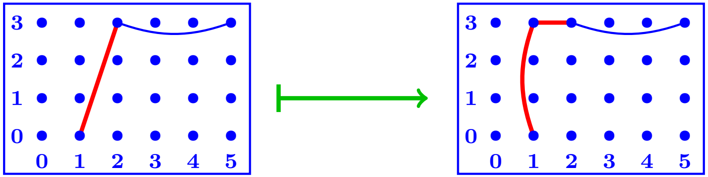
⇒ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

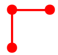


Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

Recipe: First "event" = Diagonal step =  ⇒ Source cell.

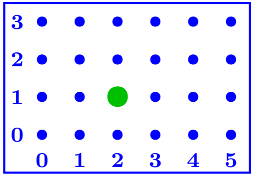
= (-90°)-bend =  ⇒ Target cell.

Critical cells ??

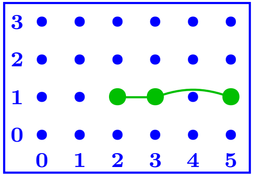
Critical cell = cell without any "event"

= without any diagonal or $-90°$-bend.

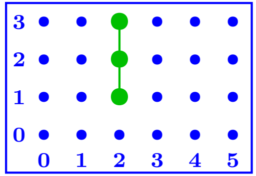
Examples.



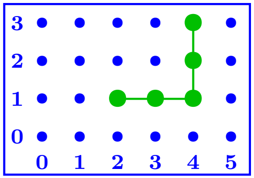
$\Delta^0_2 \otimes \Delta^0_1$



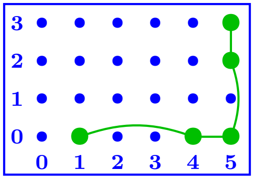
$\Delta^2_{2,3,5} \otimes \Delta^0_1$



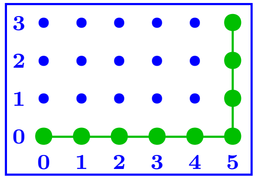
$\Delta^0_2 \otimes \Delta^2_{1,2,3}$



$\Delta^2_{2,3,4} \otimes \Delta^2_{1,2,3}$



$\Delta^2_{1,4,5} \otimes \Delta^2_{0,2,3}$



$\Delta^5_{0,1,2,3,4,5} \otimes \Delta^3_{0,1,2,3}$

Conclusion:

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

Fundamental theorem of vector fields ⇒

Canonical Homological Reductions:

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

More generally: X and Y = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C_*^c(X \times Y) = C_*(X) \otimes C_*(Y)$.

Eilenberg-Zilber Theorem: Canon. homological reduction:

$$\rho_{EZ} : C_*(X \times Y) \Rrightarrow C_*^c(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

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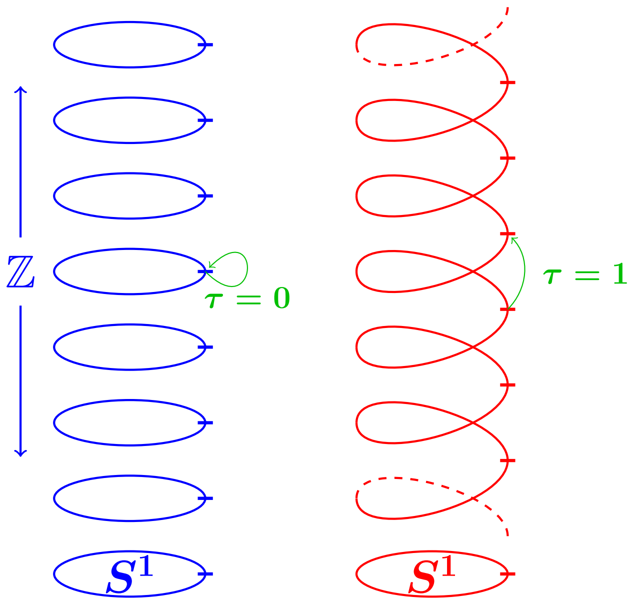
Notion of twisted product.

Simplest example: $\mathbb{Z} \times \boldsymbol{S^1}$ **vs** $\mathbb{Z} \times_{\boldsymbol{\tau}} \boldsymbol{S^1} = \mathbb{R}$**:**



General notion of twisted product: B = base space.

F = fibre space.

G = structural group.

Action $G \times F \to F$.

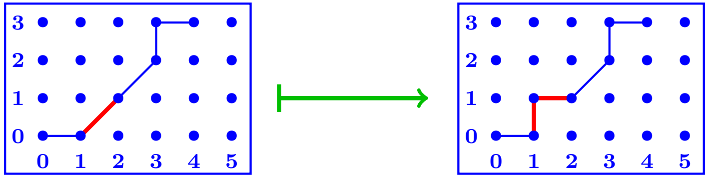
$\tau : B \to G$ = Twisting function.

Structure of $F \times_\tau B$:

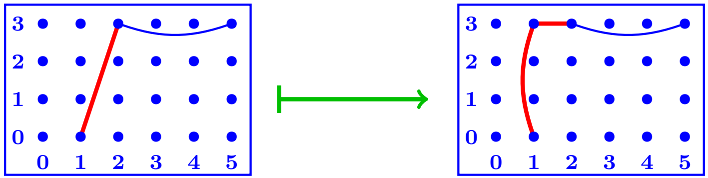
$$\partial_i(\sigma_f, \sigma_b) = (\partial_i\sigma_f, \partial_i\sigma_b) \quad \text{for } i > 0$$
$$\partial_0(\sigma_f\ ,\ \sigma_b) = (\tau(\sigma_b).\partial_0\sigma_f\ ,\ \partial_0\sigma_b)$$

$\Rightarrow$ **Only the 0-face is modified in the twisted product.**

Reminder about the EZ-vector field of $\mathbf{\Delta}^5 \times \mathbf{\Delta}^3$.



Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

The vector field is concerned by faces ∂_i only if $i > 0$.

1. The twisting function τ modifies only $\boxed{0}$-faces.

2. The EZ-vector field V_{EZ} of $X \times Y$ uses only $\boxed{i}$-faces with $i \geq 1$.

$\Rightarrow$ V_{EZ} is defined and admissible as well on $X \times_{\boxed{\tau}} Y$.

Fundamental theorem of admissible vector fields $\Rightarrow$

$$C_*(X \times Y) \qquad\qquad C_*(X \times_{\boxed{\tau}} Y)$$

$$V_{EZ} \Rightarrow \Downarrow\!\!\Downarrow\!\!\Downarrow \qquad\qquad V_{EZ} \Rightarrow \Downarrow\!\!\Downarrow\!\!\Downarrow$$

$$C_*(X) \otimes C_*(Y) \qquad\qquad C_*(X) \otimes_{\boxed{t}} C_*(Y)$$

Known as the twisted Eilenberg-Zilber Theorem.

Corollary: **Base B 1-reduced** $\Rightarrow$ **Algorithm:**

$$[(F, C_*(F), EC_*^F, \varepsilon_F) + (B, C_*(B), EC_*^B, \varepsilon_B) + G + \tau]$$
$$\longmapsto (F \times_\tau B, C_*(F \times_\tau B), EC_*^{F \times_\tau B}, \varepsilon_{F \times_\tau B}).$$

Version of F with effective homology

+ Version of B with effective homology

+ $G + \tau$ describing the fibration $F \hookrightarrow F \times_\tau B \to B$

$\Rightarrow$ Version with effective homology of the total space $F \times_\tau B$.

= Version with effective homology

of the Serre Spectral Sequence

Plan.

- ✓ • **Introduction.**
- ✓ • **Discrete vector fields.**
- ✓ • **Homological Reductions.**
- ✓ • **Product problem in Combinatorial Topology.**
- ✓ • **Discrete Vector Field for Products.**
- ✓ • **Free generalization to twisted products.**
- → • **Effective Eilenberg-Moore spectral sequences.**

Analogous result for the Eilenberg-Moore spectral sequence.

Key results:

G = Simplicial group $\Rightarrow$ BG = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

X = Simplicial set $\Rightarrow$ KX = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

Analogous process $\Rightarrow$ Algorithms:

$$(G, C_*G, EC_*^G, \varepsilon_G) \mapsto (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(X, C_*X, EC_*^X, \varepsilon_X) \mapsto (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

More generally:

$[\alpha : E \rightarrow B] + [\alpha' : E' \rightarrow B] + [\alpha \text{ fibration}]$

$\Rightarrow$ **algorithm:** $(B_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E \times_B E')_{EH}$.

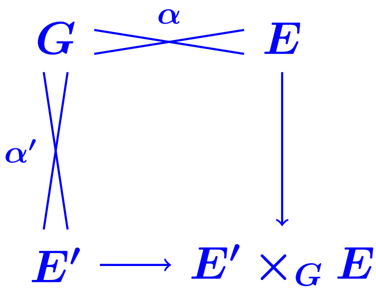
$$\begin{array}{ccc} E' \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \alpha \\ E' & \xrightarrow[\alpha']{} & B \end{array}$$

= **Version with effective homology**

of Eilenberg-Moore spectral sequence I.

Also:

$[G \text{ simplicial group}] + [\alpha : G \times E \to E] +$
$[\alpha' : E' \times G \to E'] + [\alpha \text{ principal fibration}]$
$\Rightarrow$ **algorithm:** $(G_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E' \times_G E)_{EH}$.



= **Version with effective homology**

of Eilenberg-Moore spectral sequence II.

Integrating the Vector Field technology

in the Kenzo program

⇒ Faster program!

Example: $\pi_5(\Omega(S^3) \cup_2 D^3) = ??$

On the same machine:

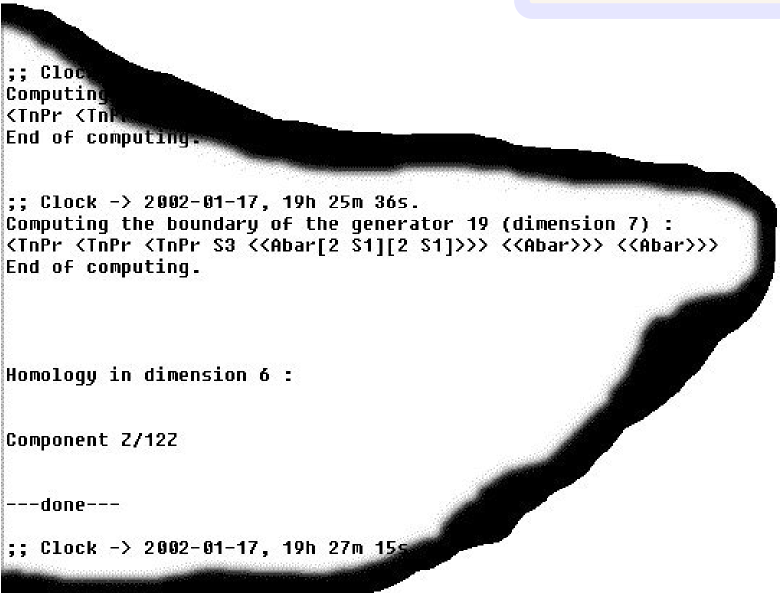
Old version ⇒ 1h32m

New version ⇒ 0h05m

with the same result !

Computing time divided by 18.

The END



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