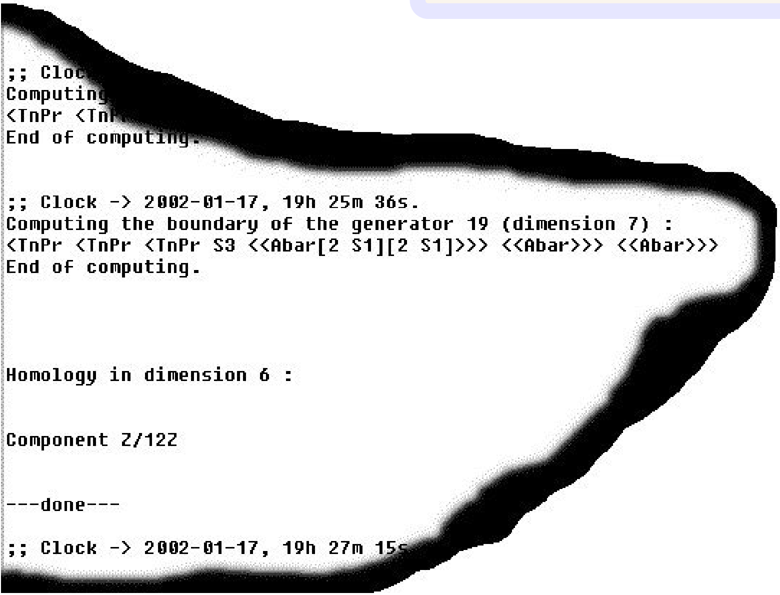
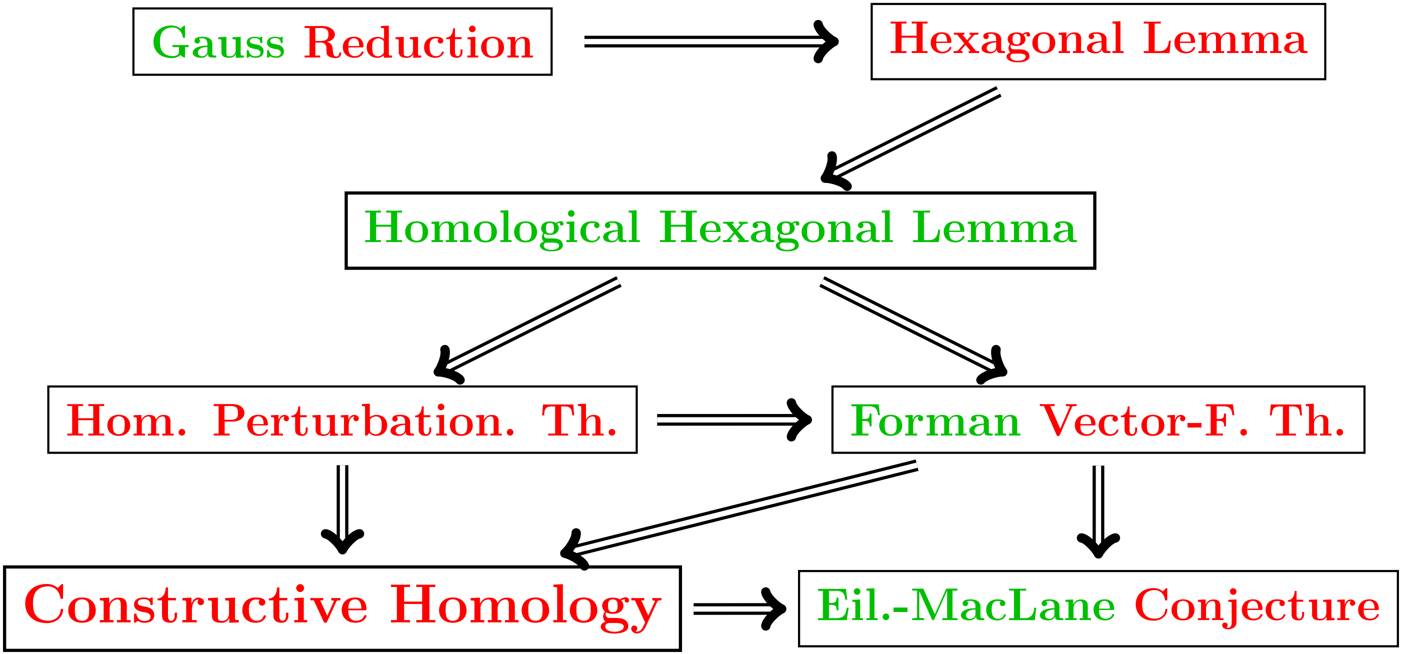
The Homological Hexagonal Lemma



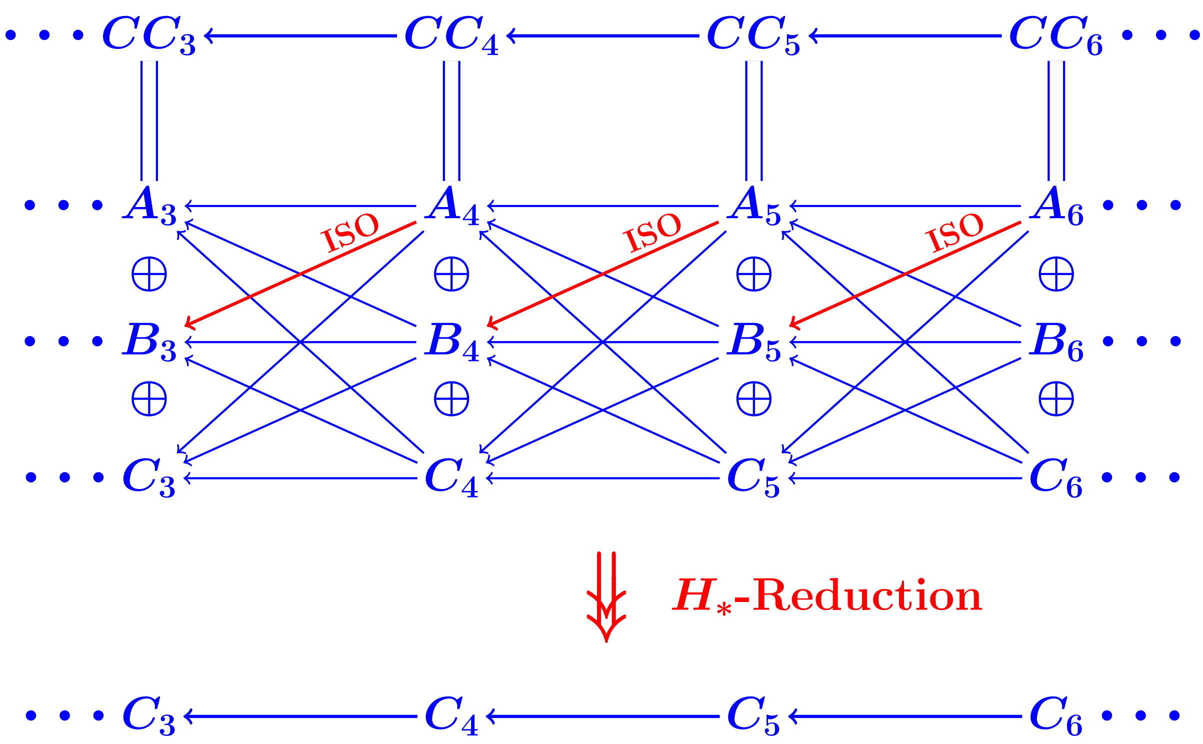
Francis Sergeraert, Institut Fourier, Grenoble

Homological Perturbation Theory, Galway, December 2014

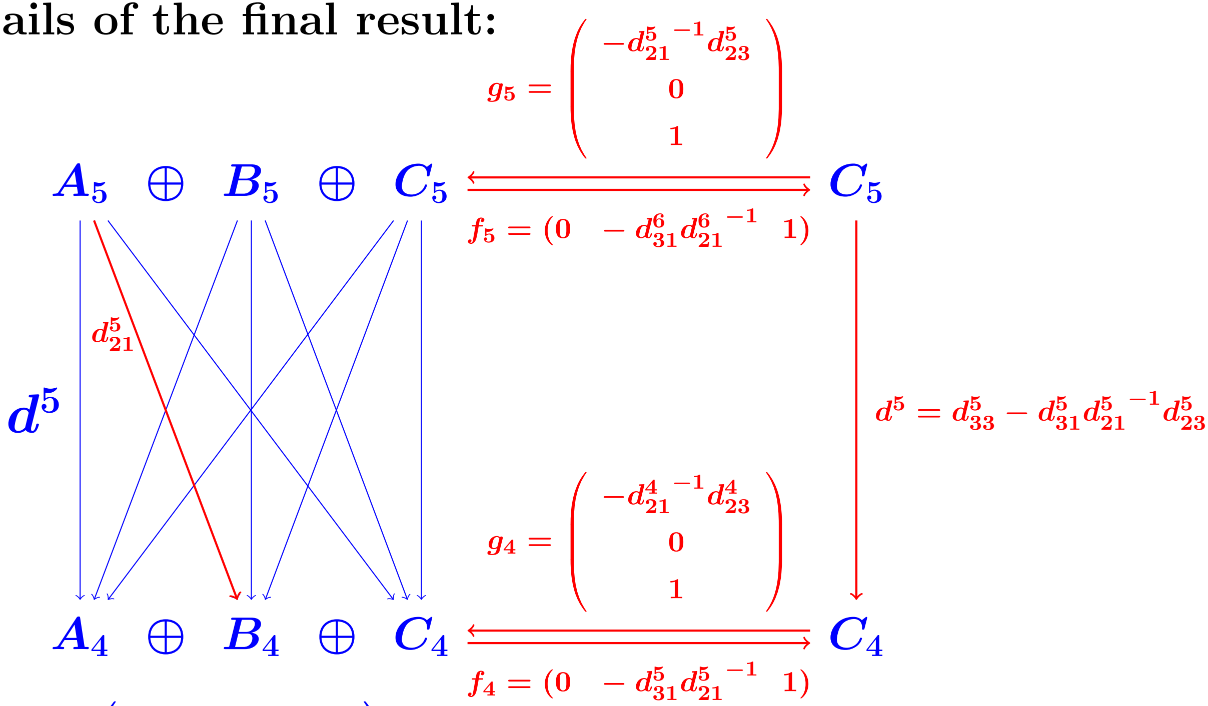
1/8. Introduction.



2/8. Homological Hexagonal Lemma.

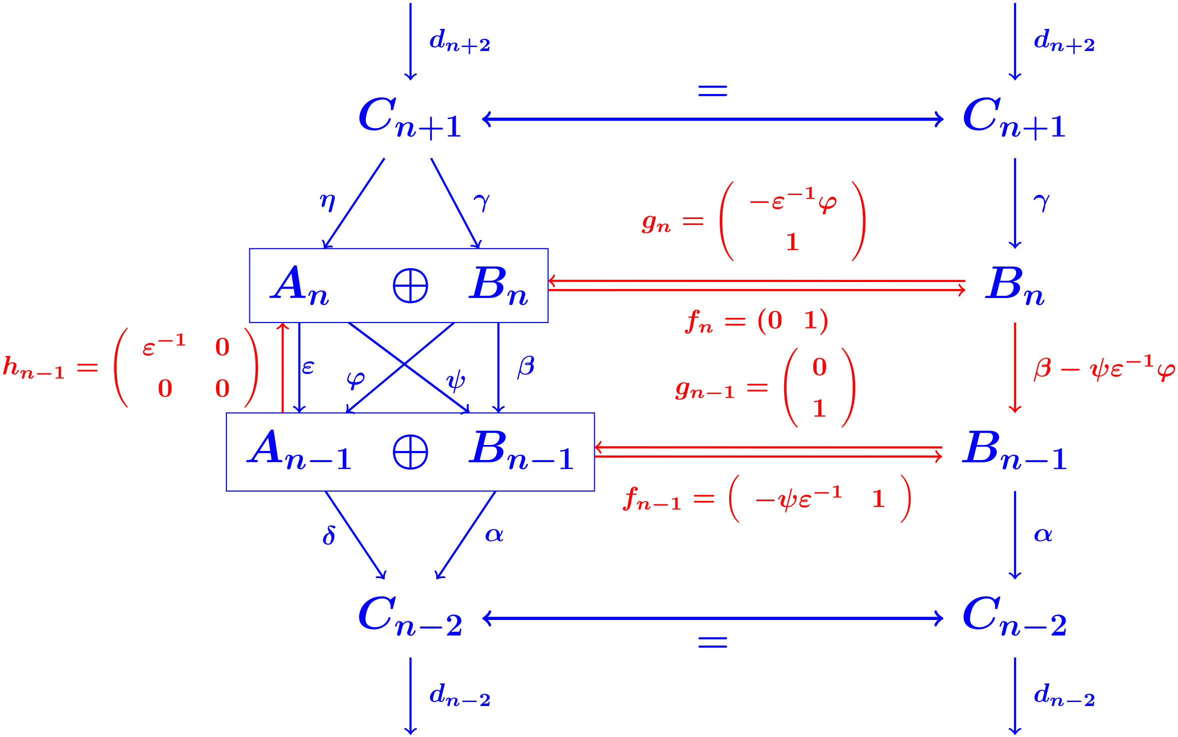


Details of the final result:

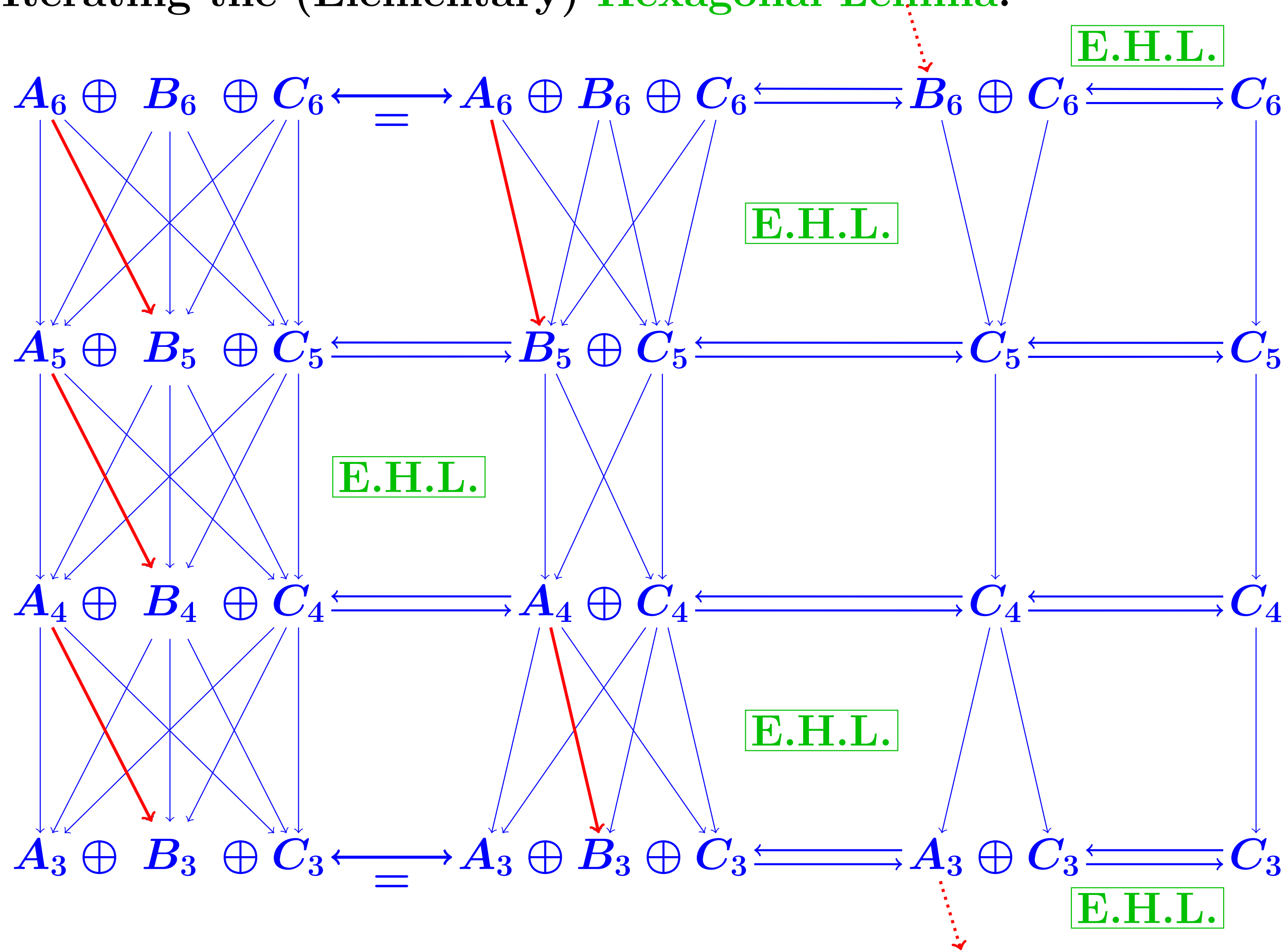


$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

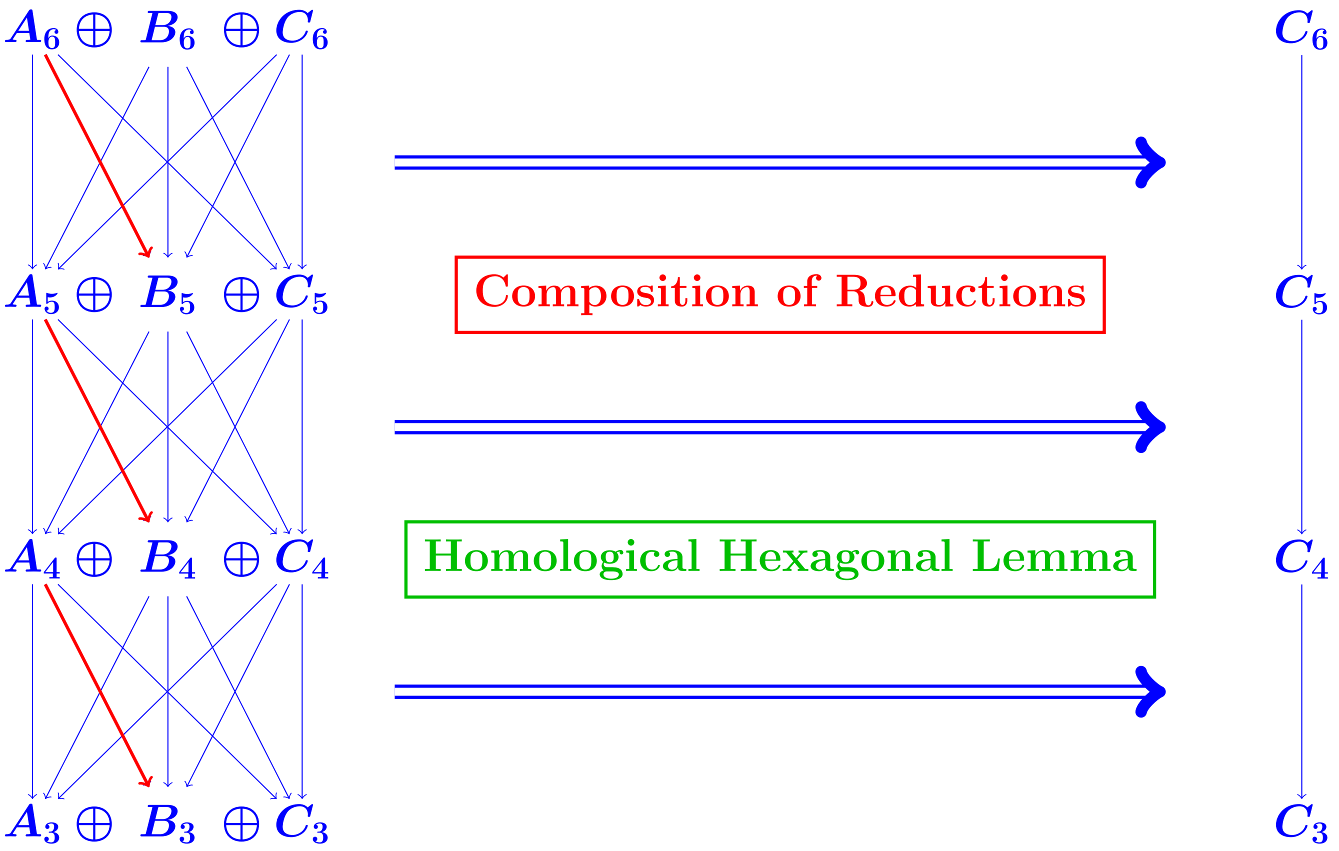
3/8. Elementary Hexagonal Lemma.



Iterating the (Elementary) Hexagonal Lemma:



Iterating the (Elementary) Hexagonal Lemma:

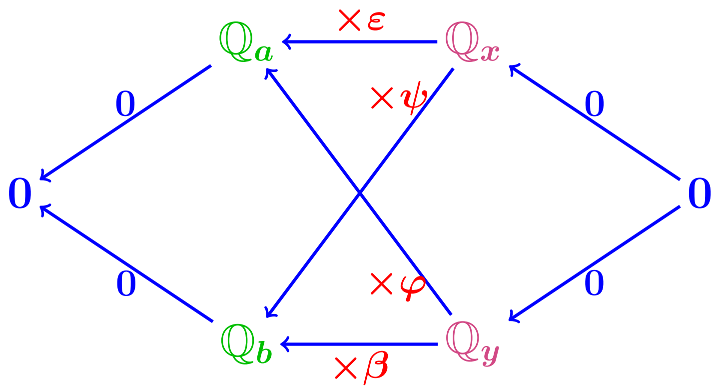


4/8. Gauss reduction ⟼ Hexagonal Lemma

Giving the elementary linear system

a homological hexagonal shape:

$$\begin{array}{rcl} \varepsilon x + \varphi y & = & a \\ \psi x + \beta y & = & b \end{array} \longmapsto$$



Gauss Reduction ⟼ **Elementary Hexagonal Lemma**

$\boldsymbol{R}$ = **Unitary ring**

$\boldsymbol{\varepsilon, \varphi, \psi, \beta \in R}$ **with** $\boldsymbol{\varepsilon}$ **invertible.**

Gauss discussion of (1) + (2):

$$\text{(1)} \qquad \boldsymbol{\varepsilon x + \varphi y = a}$$

$$\text{(2)} \qquad \boldsymbol{\psi x + \beta y = b}$$

$\boldsymbol{(2) - \psi\varepsilon^{-1}(1) \;\Rightarrow}$

$$\text{(2}'\text{)} \qquad \boldsymbol{(\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a)}$$

$\Rightarrow$ **(1) + (2) has a solution** $\Leftrightarrow$

$$\boldsymbol{(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; y = \cdots}$$

$$\boldsymbol{\Rightarrow\; x = \varepsilon^{-1}a - \varepsilon^{-1}\varphi y}$$

Matrix translation:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Leftrightarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} x + \varepsilon^{-1}\varphi y \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}}_{\begin{pmatrix} a \\ -\psi\varepsilon^{-1}a + b \end{pmatrix}}$$

$\Leftrightarrow$

$$\varepsilon(x + \varepsilon^{-1}\varphi\, y) = a$$
$$(\beta - \psi\varepsilon^{-1}\varphi)\, y = (b - \psi\varepsilon^{-1}a)$$

$\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; \ldots$$

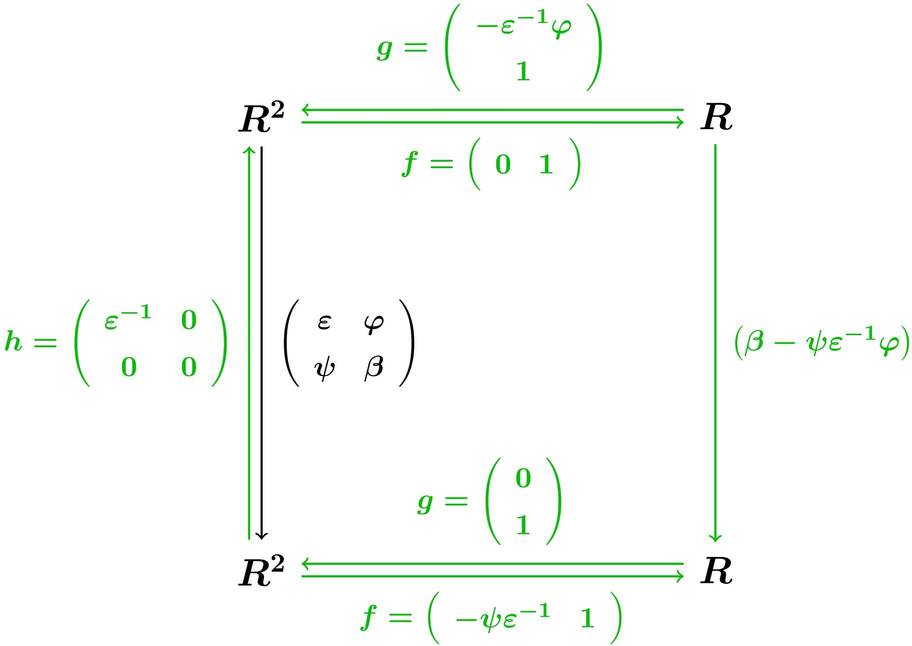
Diagram translation:

$$
\begin{array}{ccc}
 & \begin{pmatrix} \mathbf{1} & -\varepsilon^{-1}\varphi \\ \mathbf{0} & \mathbf{1} \end{pmatrix} & \\
\boldsymbol{R}^2 & \underset{\longrightarrow}{\longleftarrow} & \boldsymbol{R}^2 \\
 & \begin{pmatrix} \mathbf{1} & \varepsilon^{-1}\varphi \\ \mathbf{0} & \mathbf{1} \end{pmatrix} & \\
\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \Big\downarrow & & \Big\downarrow \begin{pmatrix} \varepsilon & \mathbf{0} \\ \mathbf{0} & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} \\
 & \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \psi\varepsilon^{-1} & \mathbf{1} \end{pmatrix} & \\
\boldsymbol{R}^2 & \underset{\longrightarrow}{\longleftarrow} & \boldsymbol{R}^2 \\
 & \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\psi\varepsilon^{-1} & \mathbf{1} \end{pmatrix} &
\end{array}
$$

Combined with an obvious reduction:

$$
\begin{array}{ccccc}
R^2 & \underset{\begin{pmatrix}1 & \varepsilon^{-1}\varphi\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & -\varepsilon^{-1}\varphi\\ 0 & 1\end{pmatrix}}{\leftrightarrows}} & R^2 & \underset{\begin{pmatrix}0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}0\\ 1\end{pmatrix}}{\rightleftarrows}} & R \\
\begin{pmatrix}\varepsilon & \varphi\\ \psi & \beta\end{pmatrix}\Big\downarrow & & {\color{red} h = \begin{pmatrix}\varepsilon^{-1} & 0\\ 0 & 0\end{pmatrix}\Big\uparrow}\Big\downarrow\begin{pmatrix}\varepsilon & 0\\ 0 & \beta-\psi\varepsilon^{-1}\varphi\end{pmatrix} & & \Big\downarrow(\beta-\psi\varepsilon^{-1}\varphi) \\
R^2 & \underset{\begin{pmatrix}1 & 0\\ -\psi\varepsilon^{-1} & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & 0\\ \psi\varepsilon^{-1} & 1\end{pmatrix}}{\leftrightarrows}} & R^2 & \underset{\begin{pmatrix}0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}0\\ 1\end{pmatrix}}{\rightleftarrows}} & R
\end{array}
$$

$\Rightarrow$ Canonical reduction induced by ε invertible



The same is valid with

$\boldsymbol{R}^2 = \boldsymbol{R} \oplus \boldsymbol{R}$ **replaced by** $A_n \oplus B_n = C_n$

or by $A_{n-1} \oplus B_{n-1} = C_{n-1}$

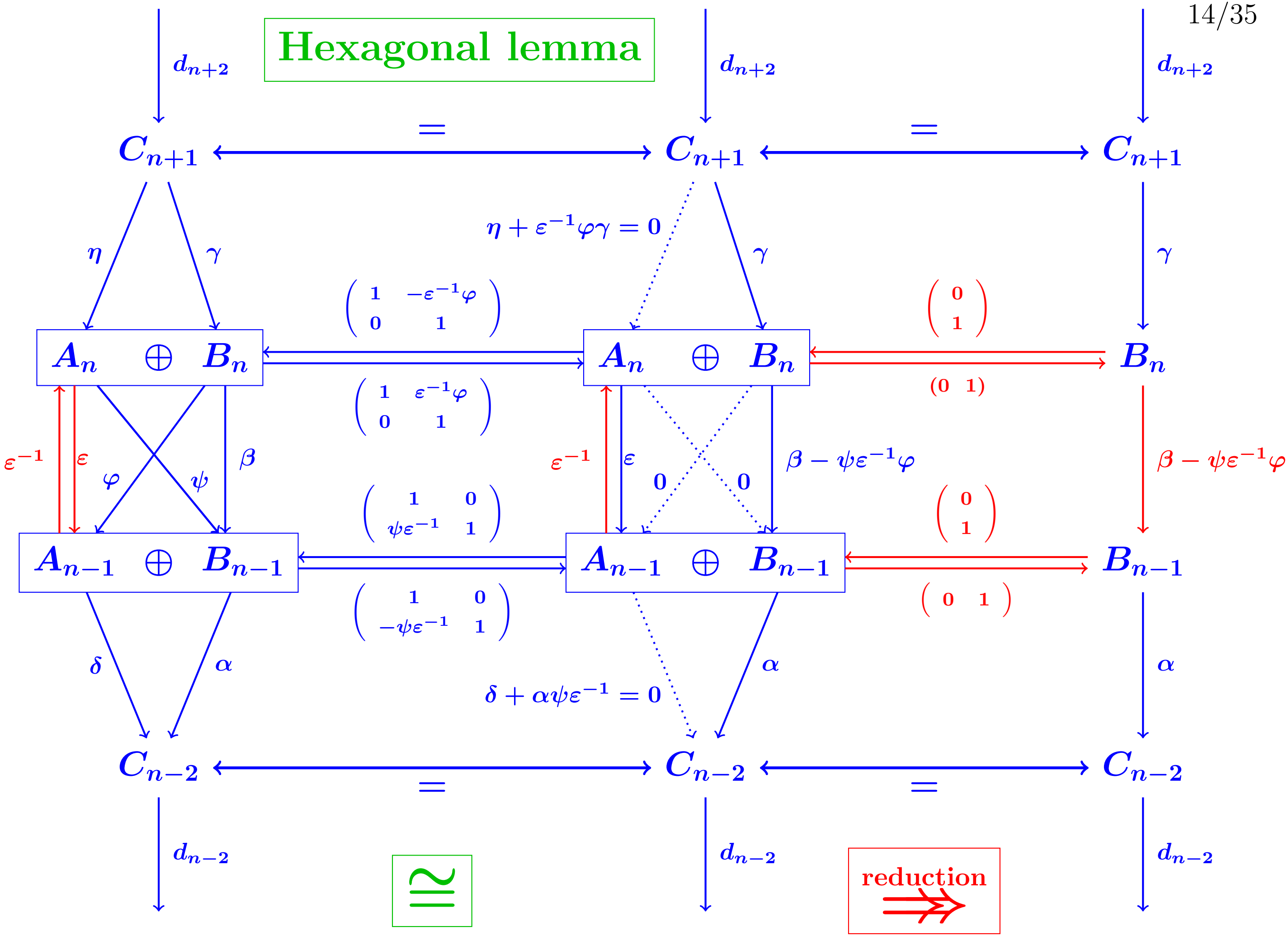
and:

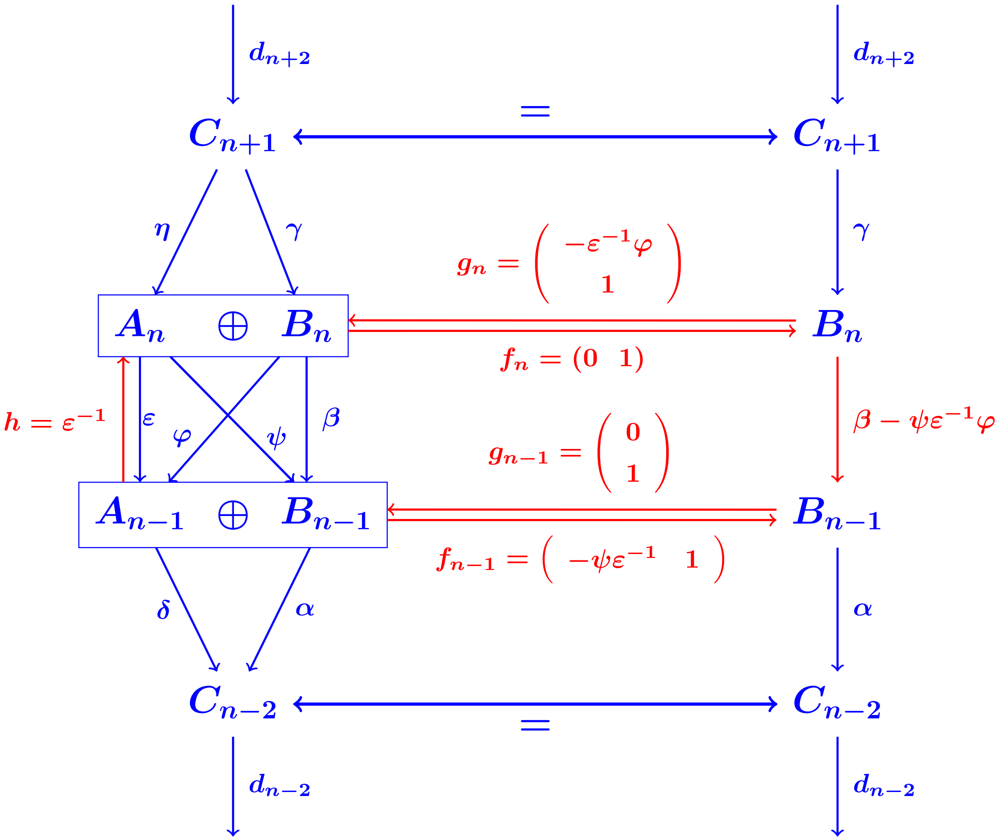
$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} : A_n \oplus B_n \to A_{n-1} \oplus B_{n-1}$$

with $\varepsilon : A_n \to A_{n-1}$ **isomorphism.**

$\Rightarrow$ **Hexagonal lemma.**

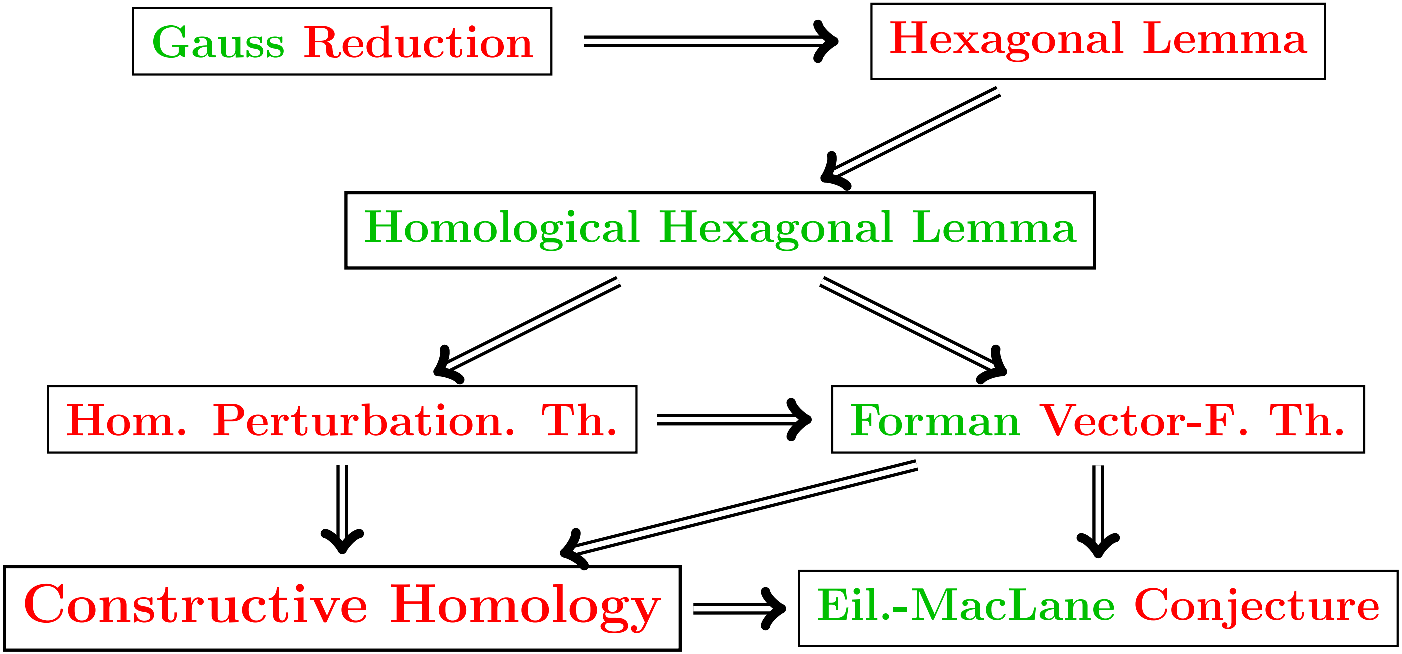
Hexagonal lemma





Hexagonal lemma

1/8. Introduction.



5/8. Homological Reductions and HP theorem.

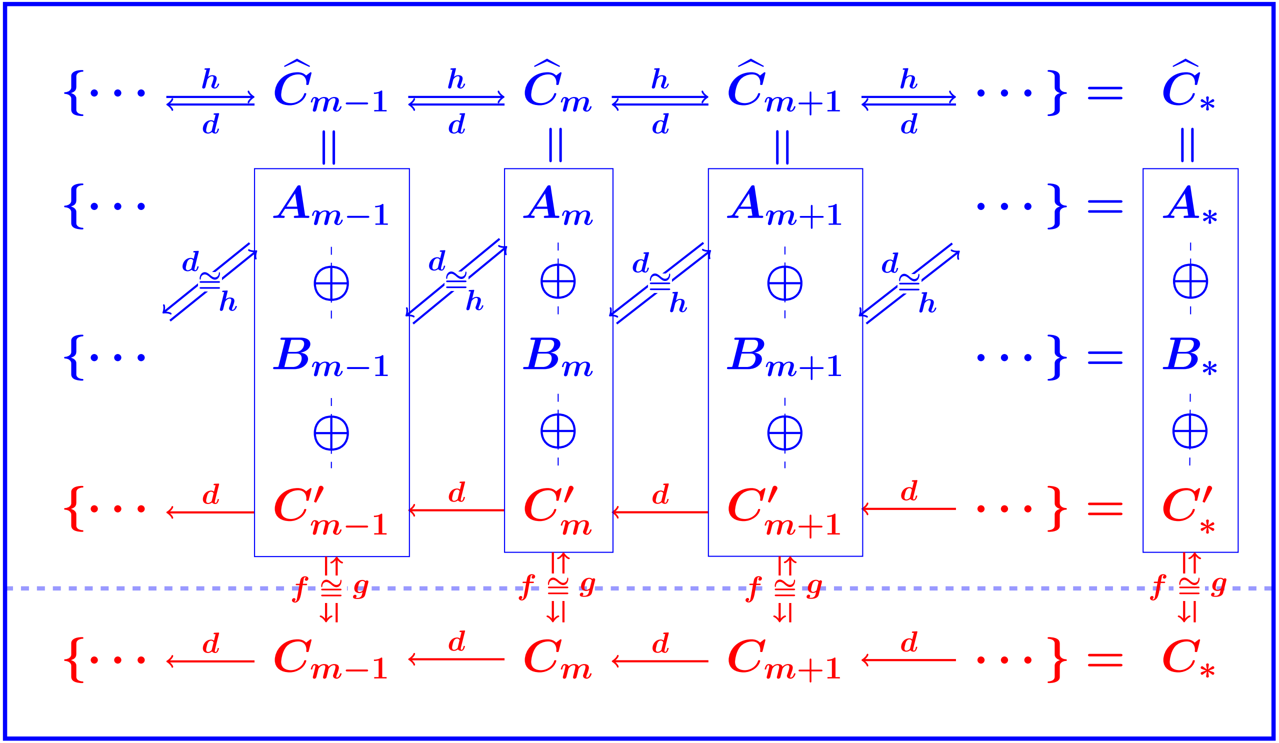
Definition: A (homological) reduction is a diagram:

$$\rho = (f, g, h) = \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$$

with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree +1).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.

Meaning = Reduction Diagram:



Homological Perturbation Theorem (HPT)

<u>Definition</u>: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d{+}\delta).$$

Let $\rho : \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$ be a given reduction

and $\widehat{\delta}$ a perturbation of $\widehat{d}$

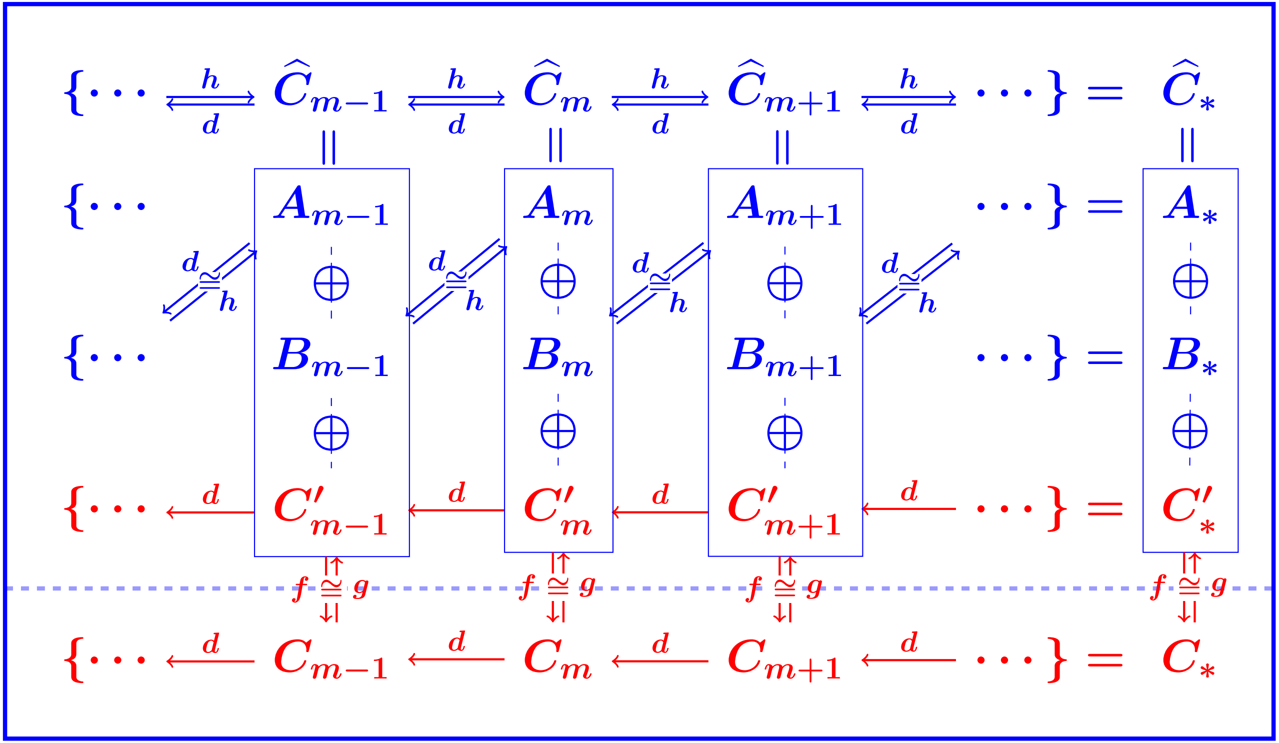
satisfying $h\widehat{\delta}$ pointwise nilpotent.

<u>Theorem</u>: The HPT determines a new reduction:

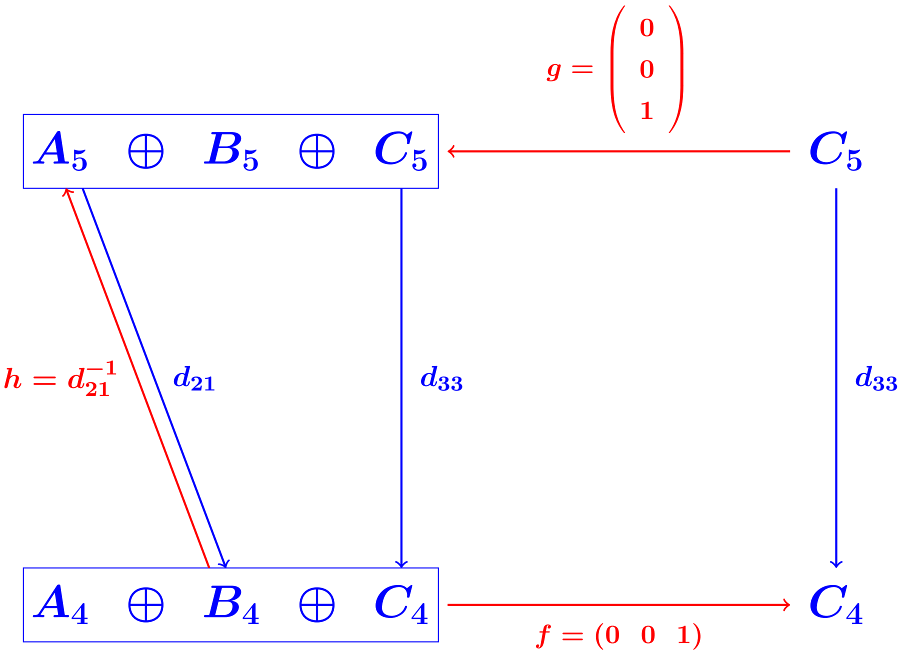
$$\rho' : \boxed{h + \delta_h \circlearrowright (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \underset{f + \delta_f}{\overset{g + \delta_g}{\leftrightarrows}} (C_*, d_* + \delta_{d*})}$$

Proof:

Reduction Diagram:

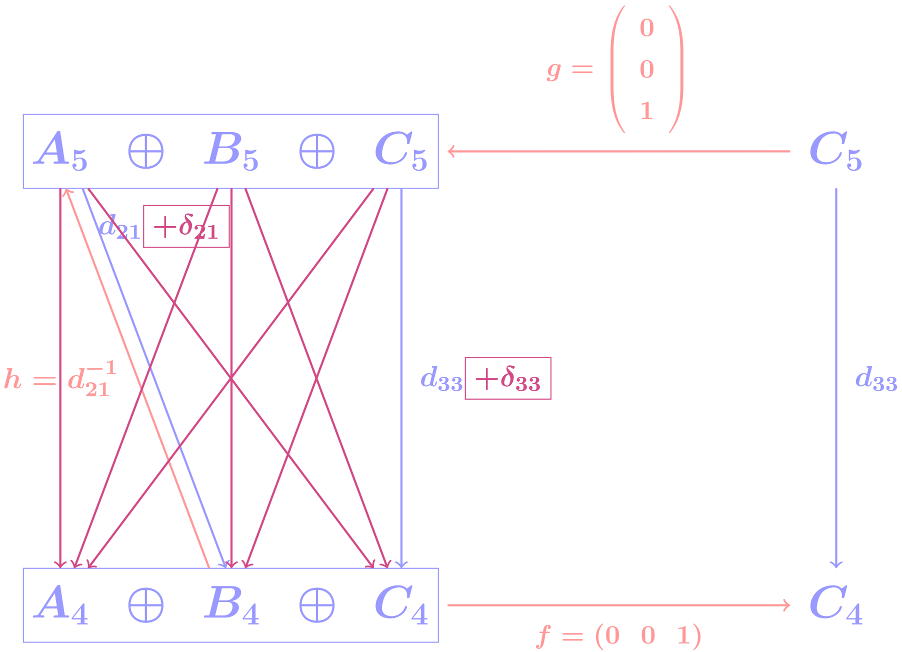


Main part:



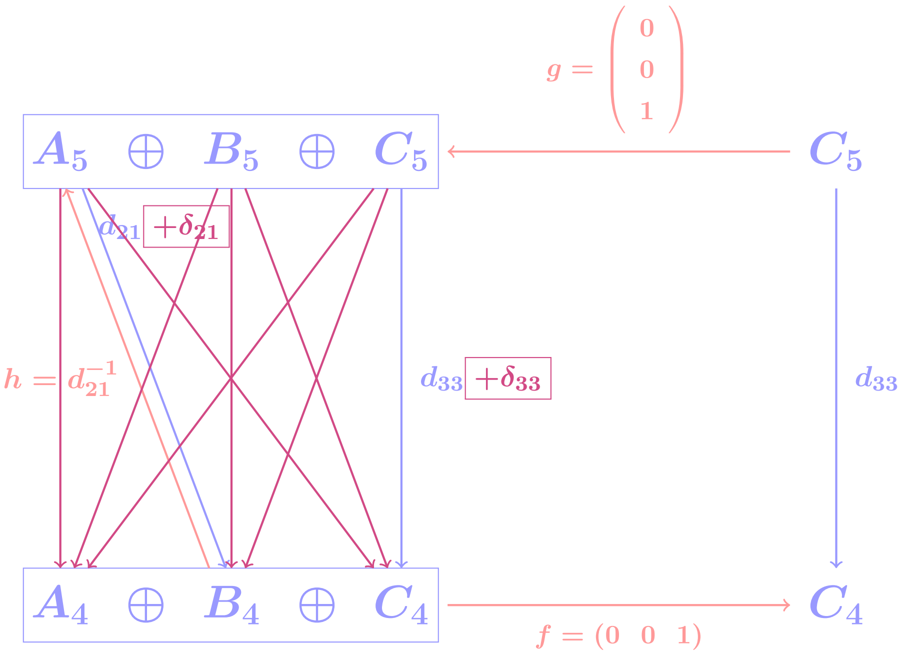
with d_{21} = isomorphism.

Perturbation $= \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$:



Question: $(\boldsymbol{d_{21}} + \boldsymbol{\delta_{21}})$ again isomorphism?

(applying the Global Hexagonal Theorem possible ?)



But d_{21} invertible with $d_{21}h = 1 \quad \Rightarrow$

$$d_{21} + \delta_{21} = d_{21} + d_{21}h\delta_{21} = d_{21}(1 + h\delta_{21})$$

$\Rightarrow \quad d_{21} + \delta_{21}$ **invertible** $\Leftrightarrow (1 + h\delta_{21})$ **invertible.**

A sufficient condition is $h\delta_{21}$ nilpotent, in which case:

$$(1 + h\delta_{21})^{-1} = \sum_{i=0}^{\infty}(-1)^i(h\delta_{21})^i$$

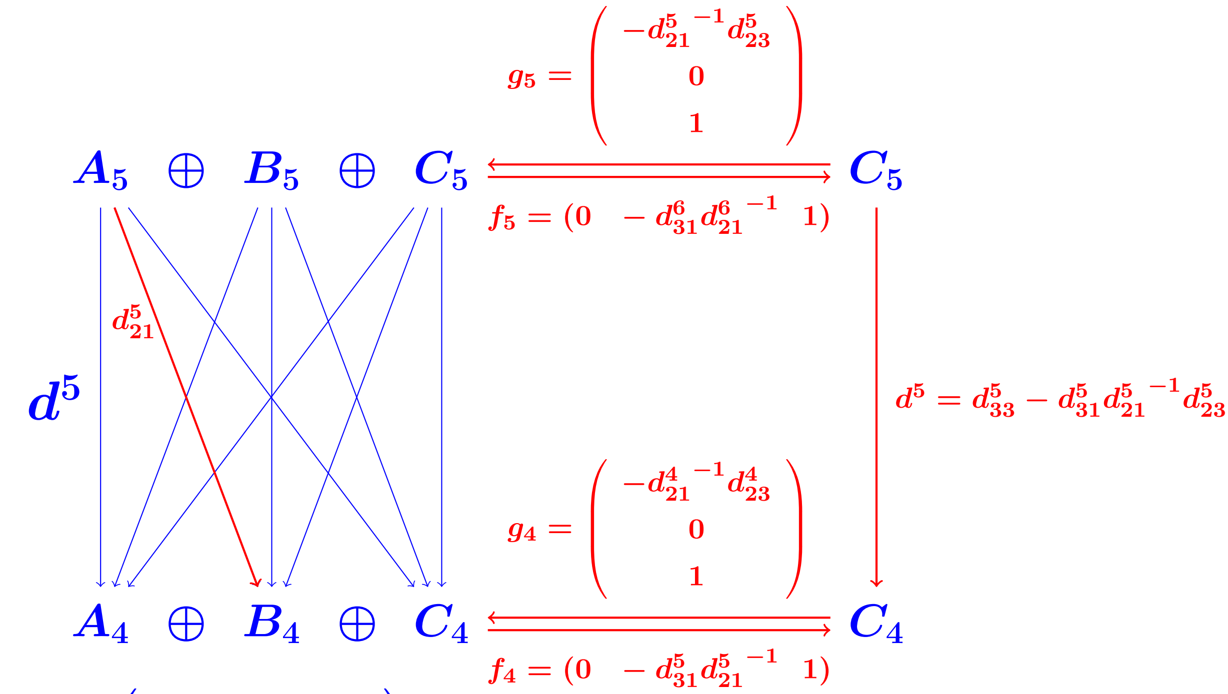
Then:

$$(d_{21} + \delta_{21})^{-1} =: h' := \left(\sum_{i=0}^{\infty}(-1)^i(h\delta_{21})^i\right) h$$

Remark:

$$\left(\sum_{i=0}^{\infty}(-1)^i(h\delta_{21})^i\right) h = \left(\sum_{i=0}^{\infty}(-1)^i(h\delta)^i\right) h$$

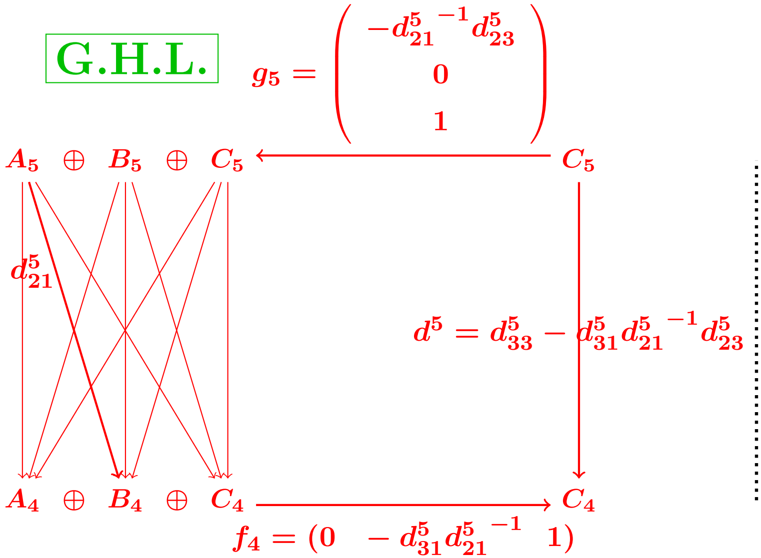
Global Hexagonal Theorem:



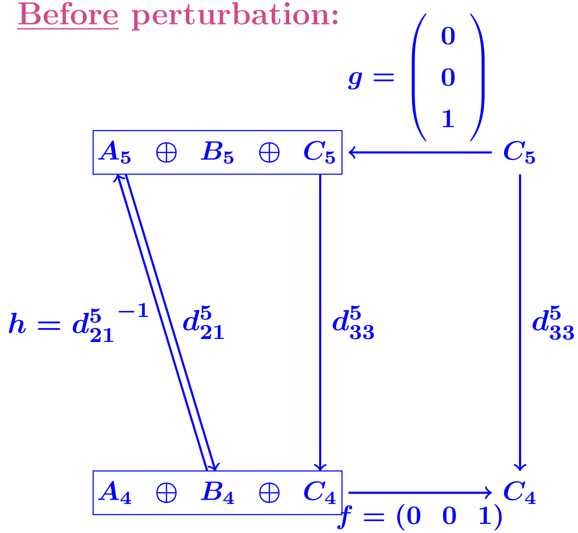
$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Applying to our situation:

G.H.L.



Before perturbation:



$$d^5_{21} \mapsto d^5_{21} + \delta^5_{21} =: d^5_{21}$$

$$h = {d^5_{21}}^{-1} \mapsto \left(\sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h =: h'$$

$$g \mapsto (1 - h'\delta) g =: g'$$

$$f \mapsto f(1 - \delta h') =: f'$$

$$d_{33} \mapsto (d_{33} + \delta_{33}) - f\delta h' \delta g$$

$$= d_{33} + f\delta g - f\delta h'\delta g =: d'_{33}$$

$=$ **Homological Perturbation Theorem** **QED**

6/8. The topological case.

Corollary: The HPT can easily be extended

to topological situations.

Example 1: Banach situations:

$$\|h\delta_{21}\| < 1 \Rightarrow (1 + h\delta_{21}) \text{ invertible } \Rightarrow \text{OK.}$$

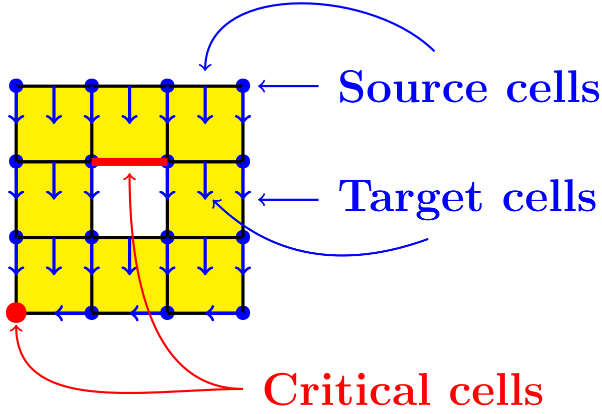
Example 2: Frechetic situations:

The Nash-Moser-Schwartz technology

often allows to prove $(1 - h\delta_{21})$ is invertible $\Rightarrow$ OK.

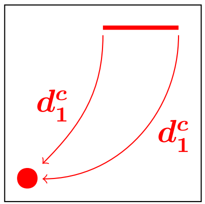
7/8. Forman Theorems.

$C_* =$



Forman Reduction Theorem $\Rightarrow$

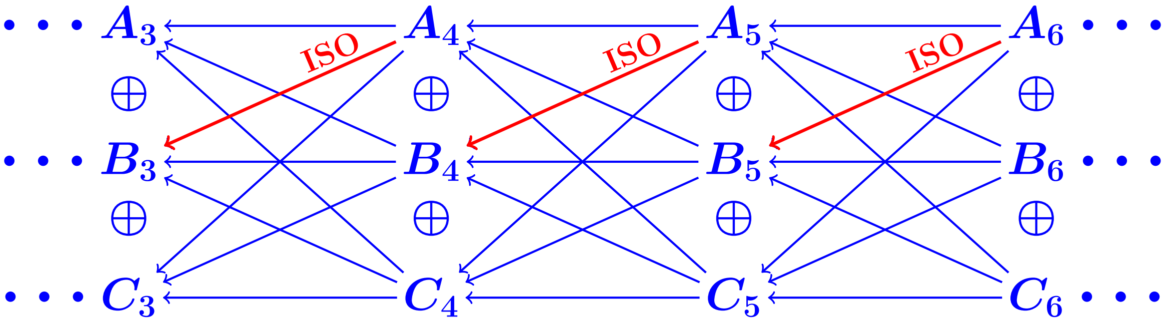
$\rho : C_* \Rrightarrow C_*^c =$



$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} =$ Circle

Homological Hexagonal Lemma

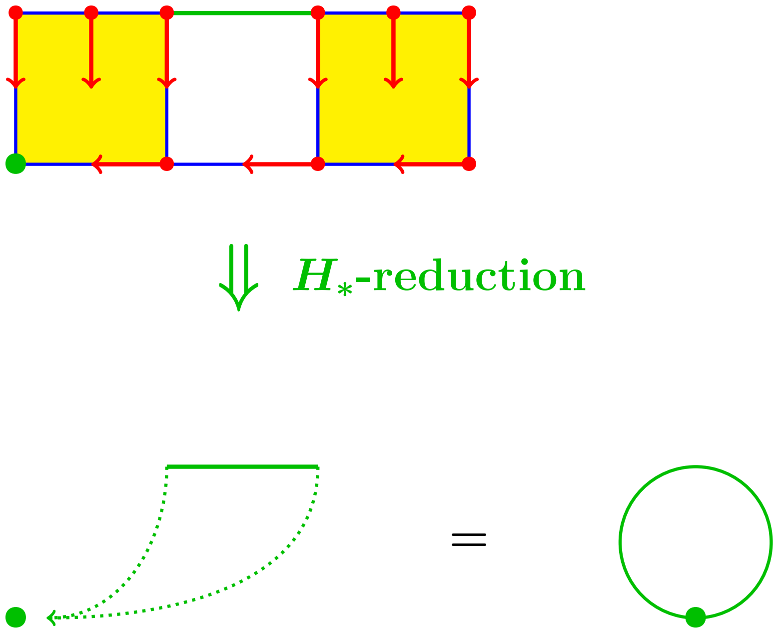
$\Rightarrow$ Forman Reduction Theorem:



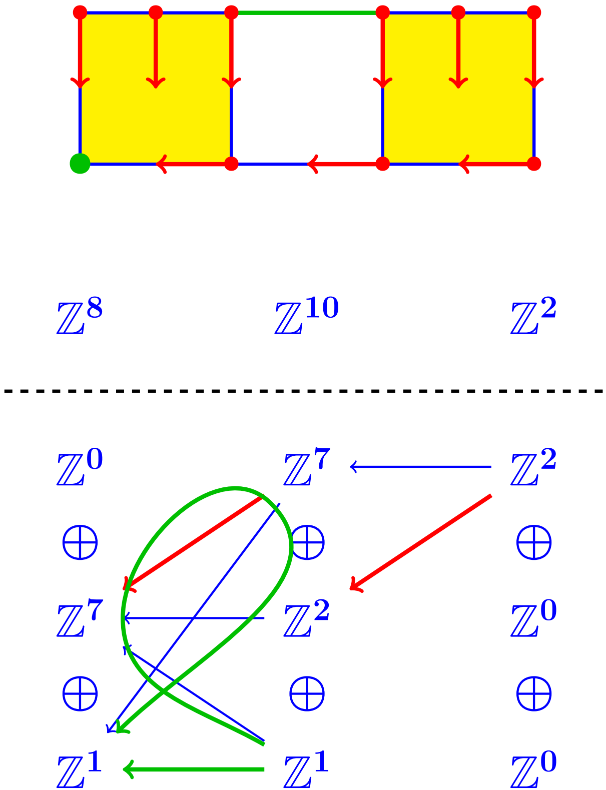
Forman Theorem = Particular case where:

ISO = Triangular Unimodular Invertible Matrix.

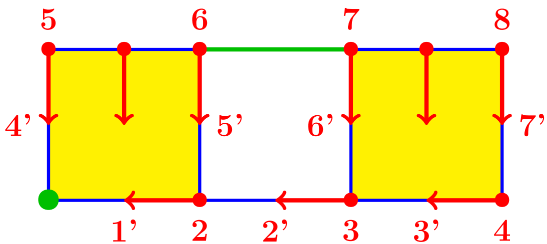
Toy example:



Toy example:

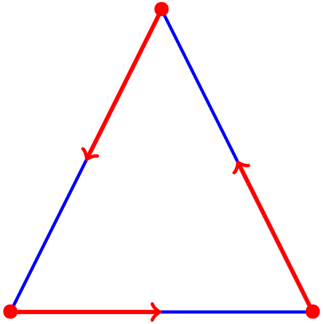

$\mathbb{Z}^8$ $\mathbb{Z}^{10}$ $\mathbb{Z}^2$
$\mathbb{Z}^0$ $\mathbb{Z}^7$ $\mathbb{Z}^2$
$\oplus$ $\oplus$ $\oplus$
$\mathbb{Z}^7$ $\mathbb{Z}^2$ $\mathbb{Z}^0$
$\oplus$ $\oplus$ $\oplus$
$\mathbb{Z}^1$ $\mathbb{Z}^1$ $\mathbb{Z}^0$

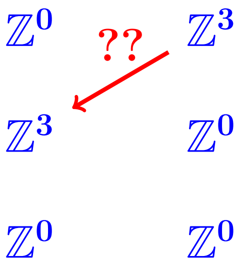
Toy example:



$$d_{21} = \begin{array}{c|ccccccc} & 1' & 2' & 3' & 4' & 5' & 6' & 7' \\ \hline 2 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 3 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Other example:





$$d_{21} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Not invertible!!

8/8. Eilenberg-MacLane conjecture (1953):

From Eilenberg-MacLane = Annals of Maths, 1953, vol.58, pp.55-106:

20. The main theorem

THEOREM 20.1. *For any commutative and augmented R-complex R, the graded ∂-ring homomorphism* $g: B_N(R_N) \to W_N(R)$ *is a reduction, in the sense of* §13.

We shall first draw some corollaries, postponing the proof of the theorem itself to the next sections. We conjecture that g is not only a reduction, but also the injection of a contraction, in the sense of §12.

First proof = Pedro Real's thesis, 1993.

Discrete vector fields + New understanding of Eilenberg-Zilber

⇒ Totally different simple new proof

⇒ Very efficient new algorithms

in computational Algebraic Topology.

Given G = reduced simplicial group,

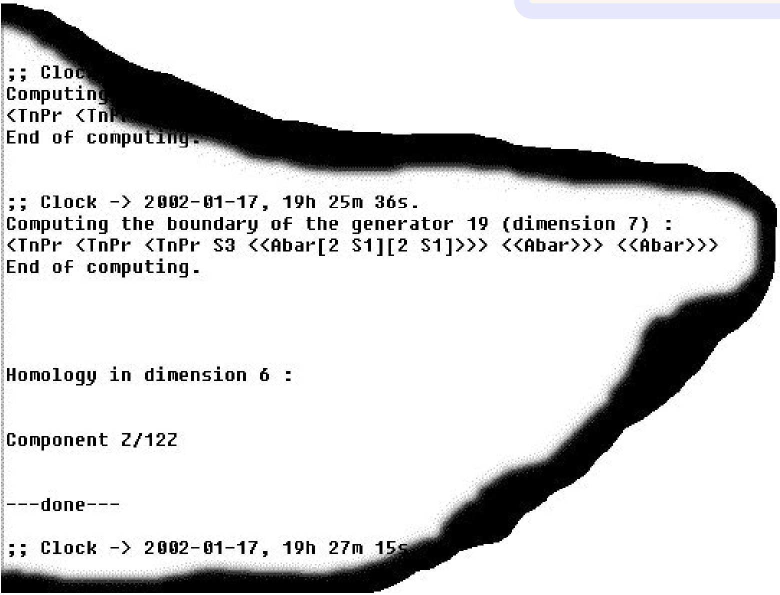
there exists a canonical reduction:

$$C_*(BG) \Rrightarrow \text{Bar}(C_*(G))$$

Proved by discrete vector fields and immediately implemented in 2012.

Application: Given $X := \Omega S^3 \cup_2 D^3$:

$$\begin{aligned}
\pi_2 X &= \mathbb{Z}/2 \\
\pi_3 X &= \mathbb{Z}/2 \\
\pi_4 X &= \mathbb{Z}/4 + \mathbb{Z} \\
\pi_5 X &= (\mathbb{Z}/2)^4 && (1998) \\
\pi_6 X &= (\mathbb{Z}/2)^5 + \mathbb{Z} && (2014)
\end{aligned}$$



The END

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MAP 2014, IHP Paris, May 2014