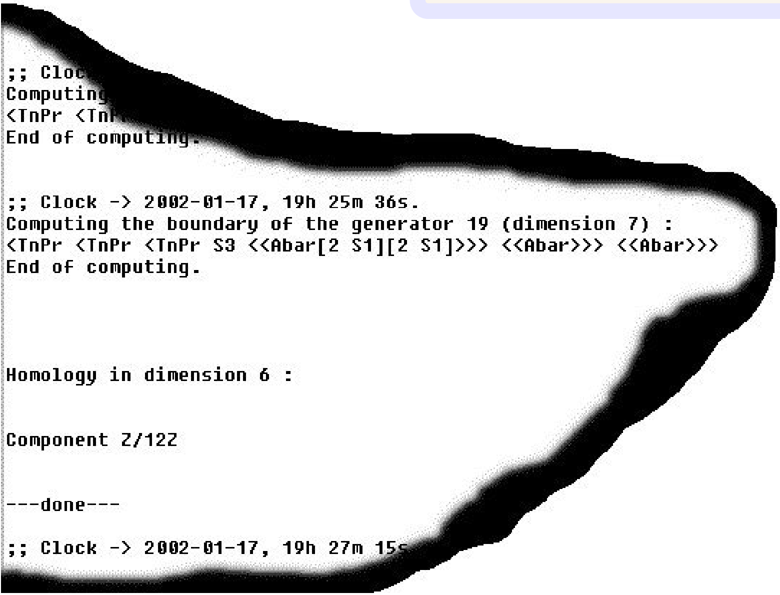
Spectral sequences downgraded to Elementary Gauss reductions



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MAP 2014, IHP Paris, May 2014

Semantics of colours:

Blue	=	**"Standard" Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, ...
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, ...
Green	=	**Solution, essential point,**
		mathematicians, ...

Plan.

1. **Introduction.**
2. **Connection HPT $\Rightarrow$ SS.**
3. **Homological Reductions and HPT statement.**
4. **Hexagonal Lemma.**
5. **Algebraic HPT Proof.**
6. **The topological case.**

1/6. Introduction.

SEMSS = Serre or Eilenberg-Moore Spectral Sequence

HPT = Homological Perturbation Theorem

= Perturbation Lemma.

GR = Elementary Gauss Reduction

(known by the Babylonians).

Theorem: GR $\Rightarrow$ HPT $\Rightarrow$ SEMSS.

2/6. Connection HPT $\Rightarrow$ SSS = Serre Spectral Seq.

SSS = Info connecting H_*F, H_*B and H_*E

when E = total space of a fibration

of base space B and fibre space F.

$$E = F \times_\tau B \qquad (\tau = \text{twisting function})$$

$$S^3 = S^1 \times_h S^2 \qquad (h = \text{Hopf twisting function})$$

Proof of HPT $\Rightarrow$ SSS:

1. **Start with τ_0 = trivial twisting = no twisting at all**

   Eilenberg-Zilber Theorem $\Rightarrow$ $[H_*B + H_*F \Rightarrow H_*E_0]$

2. **HPT = Implicit Function Theorem $\Rightarrow$**

   $[\tau$ close to $\tau_0] \Rightarrow [H_*E_0 + \text{HPT} \Rightarrow H_*E]$.

Remark: SSS is not **an algorithm**

$$H_*B + H_*F \Rightarrow H_*(F \times_\tau B).$$

HPT is **an algorithm** $H_*B + H_*F \Rightarrow H_*(F \times_\tau B)$

Ana Romero's thesis $\Rightarrow$ **[HPT** $\Rightarrow$ **SSS].**

Theorem: GR $+$ **[(1 − x) invertible if** $|x| < 1$**]** $\Rightarrow$ **HPT.**

3/6. Homological Reductions and HPT Statement.

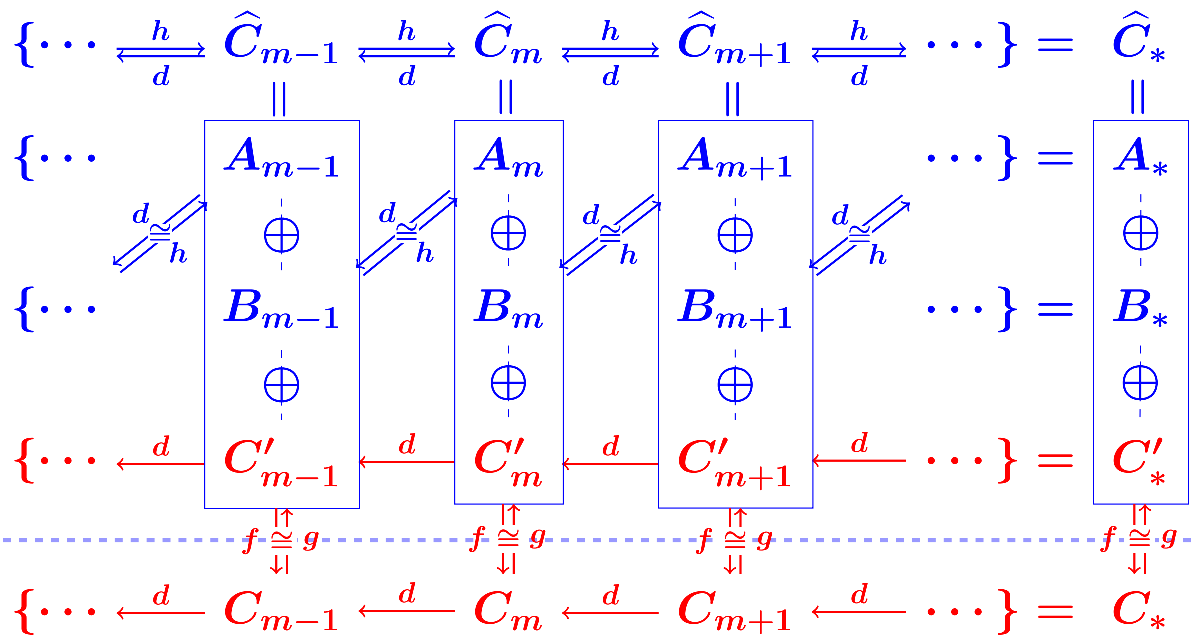
Definition: A (homological) reduction is a diagram:

$$\rho = (f, g, h) = \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$$

with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree $+1$).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.

Meaning = Reduction Diagram:



Homological Perturbation Theorem (HPT)

<u>Definition</u>: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d{+}\delta).$$

Let $\rho : \boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$ be a given reduction

and $\widehat{\delta}$ a perturbation of $\widehat{d}$

satisfying $h\widehat{\delta}$ $\boxed{\text{pointwise nilpotent}}$.

<u>Theorem</u>: The HPT determines a new reduction:

$$\rho' : \boxed{h + \delta_h \circlearrowleft (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \underset{f + \delta_f}{\overset{g + \delta_g}{\leftrightarrows}} (C_*, d_* + \delta_{d*})}$$

4/6. Hexagonal Lemma.

R = Unitary ring

$\varepsilon, \varphi, \psi, \beta \in R$ with ε invertible.

Gauss discussion of (1) + (2):

$$(1) \qquad \varepsilon x + \varphi y = a$$

$$(2) \qquad \psi x + \beta y = b$$

$(2) - \psi\varepsilon^{-1}(1) \;\Rightarrow$

$$(2') \qquad (\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a)$$

$\Rightarrow$ (1) + (2) has a solution $\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; y = \cdots$$

$$\Rightarrow\; x = \varepsilon^{-1}a - \varepsilon^{-1}\varphi y$$

Matrix translation:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Leftrightarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} x + \varepsilon^{-1}\varphi y \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}}_{\begin{pmatrix} a \\ -\psi\varepsilon^{-1}a + b \end{pmatrix}}$$

$\Leftrightarrow$

$$\varepsilon(x + \varepsilon^{-1}\varphi\, y) = a$$

$$(\beta - \psi\varepsilon^{-1}\varphi)\, y = (b - \psi\varepsilon^{-1}a)$$

$\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; \ldots$$

Diagram translation:

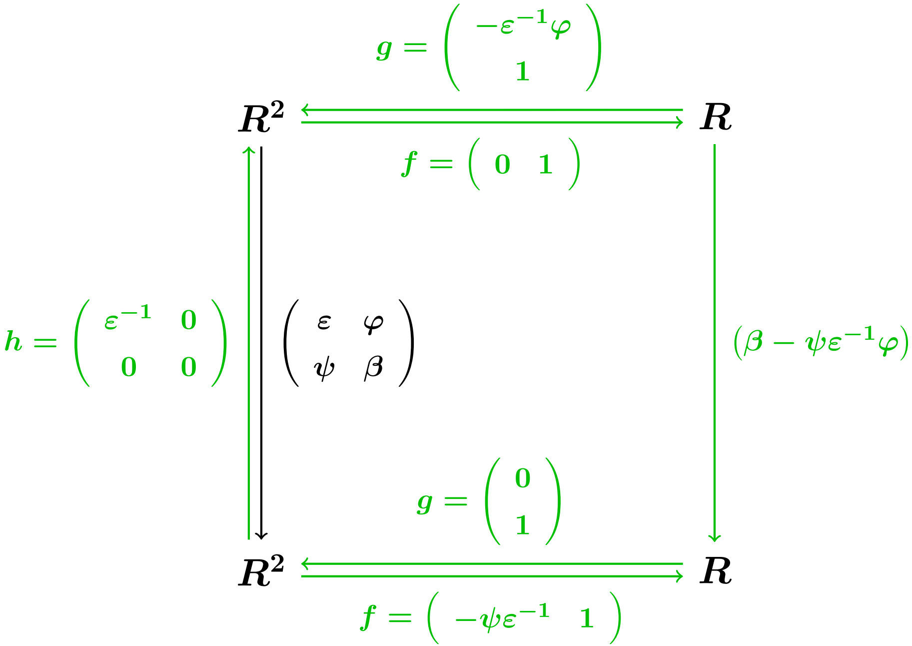
$$
\begin{array}{ccc}
\boldsymbol{R}^2 & \overset{\begin{pmatrix}1 & -\varepsilon^{-1}\varphi \\ 0 & 1\end{pmatrix}}{\underset{\begin{pmatrix}1 & \varepsilon^{-1}\varphi \\ 0 & 1\end{pmatrix}}{\rightleftarrows}} & \boldsymbol{R}^2 \\
\begin{pmatrix}\varepsilon & \varphi \\ \psi & \beta\end{pmatrix} \Big\downarrow & & \Big\downarrow \begin{pmatrix}\varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi\end{pmatrix} \\
\boldsymbol{R}^2 & \overset{\begin{pmatrix}1 & -0 \\ \psi\varepsilon^{-1} & 1\end{pmatrix}}{\underset{\begin{pmatrix}1 & -0 \\ -\psi\varepsilon^{-1} & 1\end{pmatrix}}{\rightleftarrows}} & \boldsymbol{R}^2
\end{array}
$$

Combined with an obvious reduction:

$$
\begin{array}{ccccc}
\boldsymbol{R}^2 & \begin{array}{c} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \\ \longleftarrow \\ \longrightarrow \\ \begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \end{array} & \boldsymbol{R}^2 & \begin{array}{c} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \longrightarrow \\ \longleftarrow \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array} & \boldsymbol{R} \\
\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \Big\downarrow & h = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 0 \end{pmatrix} & \Big\uparrow \Big\downarrow \begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} & & \begin{pmatrix} \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} \Big\downarrow \\
\boldsymbol{R}^2 & \begin{array}{c} \begin{pmatrix} 1 & 0 \\ \psi\varepsilon^{-1} & 1 \end{pmatrix} \\ \longleftarrow \\ \longrightarrow \\ \begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \end{array} & \boldsymbol{R}^2 & \begin{array}{c} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \longrightarrow \\ \longleftarrow \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array} & \boldsymbol{R}
\end{array}
$$

$\Rightarrow$

$\Rightarrow$ Canonical reduction induced by ε invertible



The same is valid with

$$\boldsymbol{R}^2 = \boldsymbol{R} \oplus \boldsymbol{R} \text{ replaced by } \boldsymbol{A}_n \oplus \boldsymbol{B}_n = \boldsymbol{C}_n$$

$$\text{or by } \boldsymbol{A}_{n-1} \oplus \boldsymbol{B}_{n-1} = \boldsymbol{C}_{n-1}$$

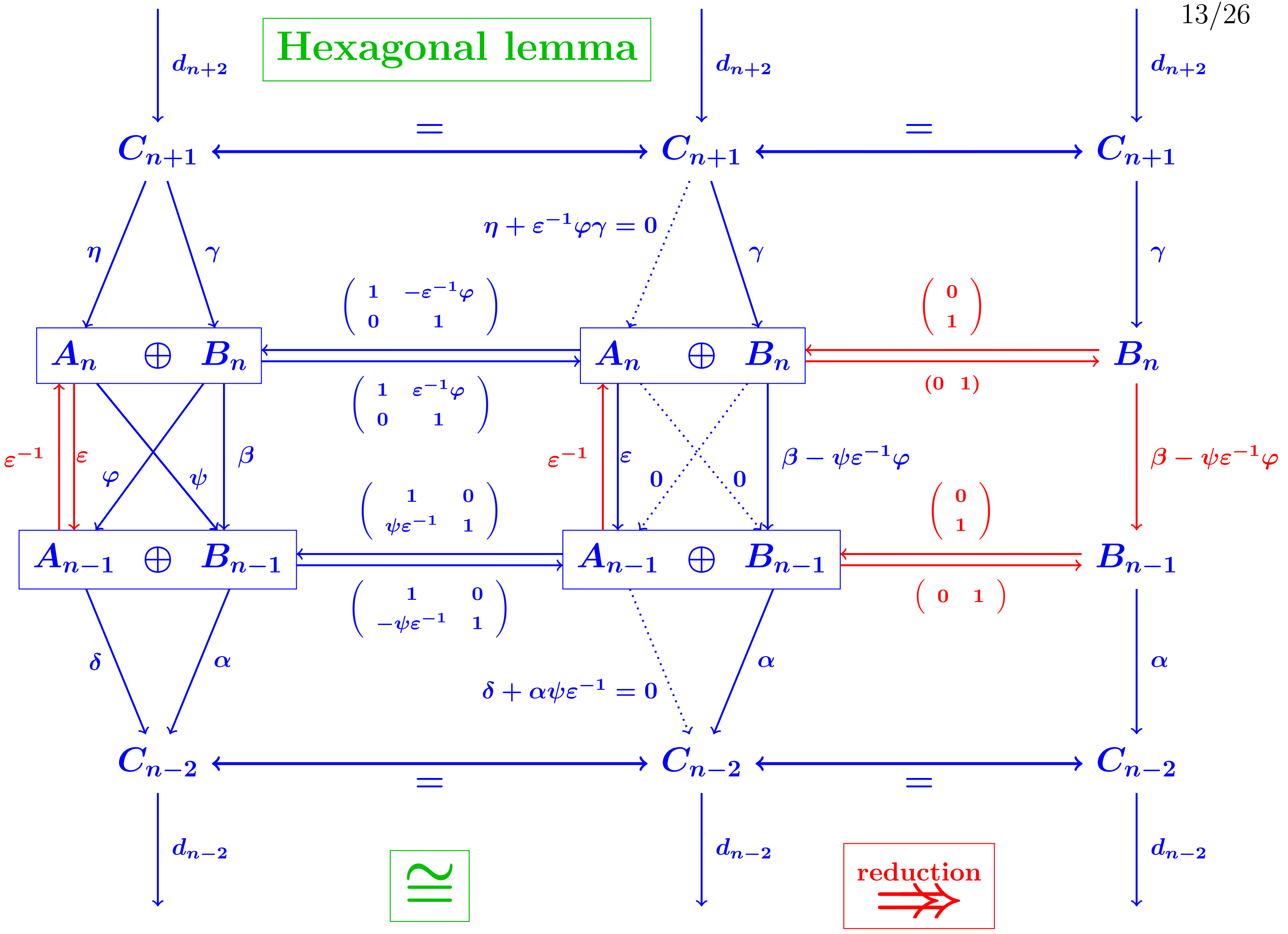
and:

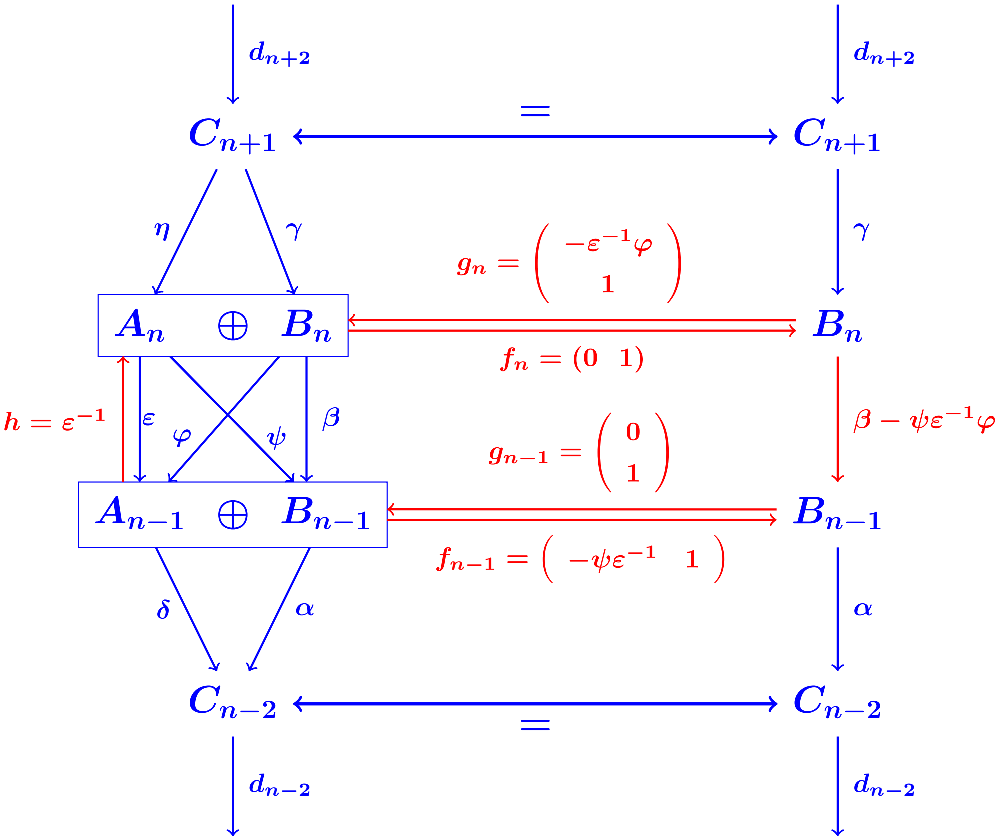
$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} : \boldsymbol{A}_n \oplus \boldsymbol{B}_n \to \boldsymbol{A}_{n-1} \oplus \boldsymbol{B}_{n-1}$$

with $\varepsilon : \boldsymbol{A}_n \to \boldsymbol{A}_{n-1}$ **isomorphism.**

$\Rightarrow$ **Hexagonal lemma.**

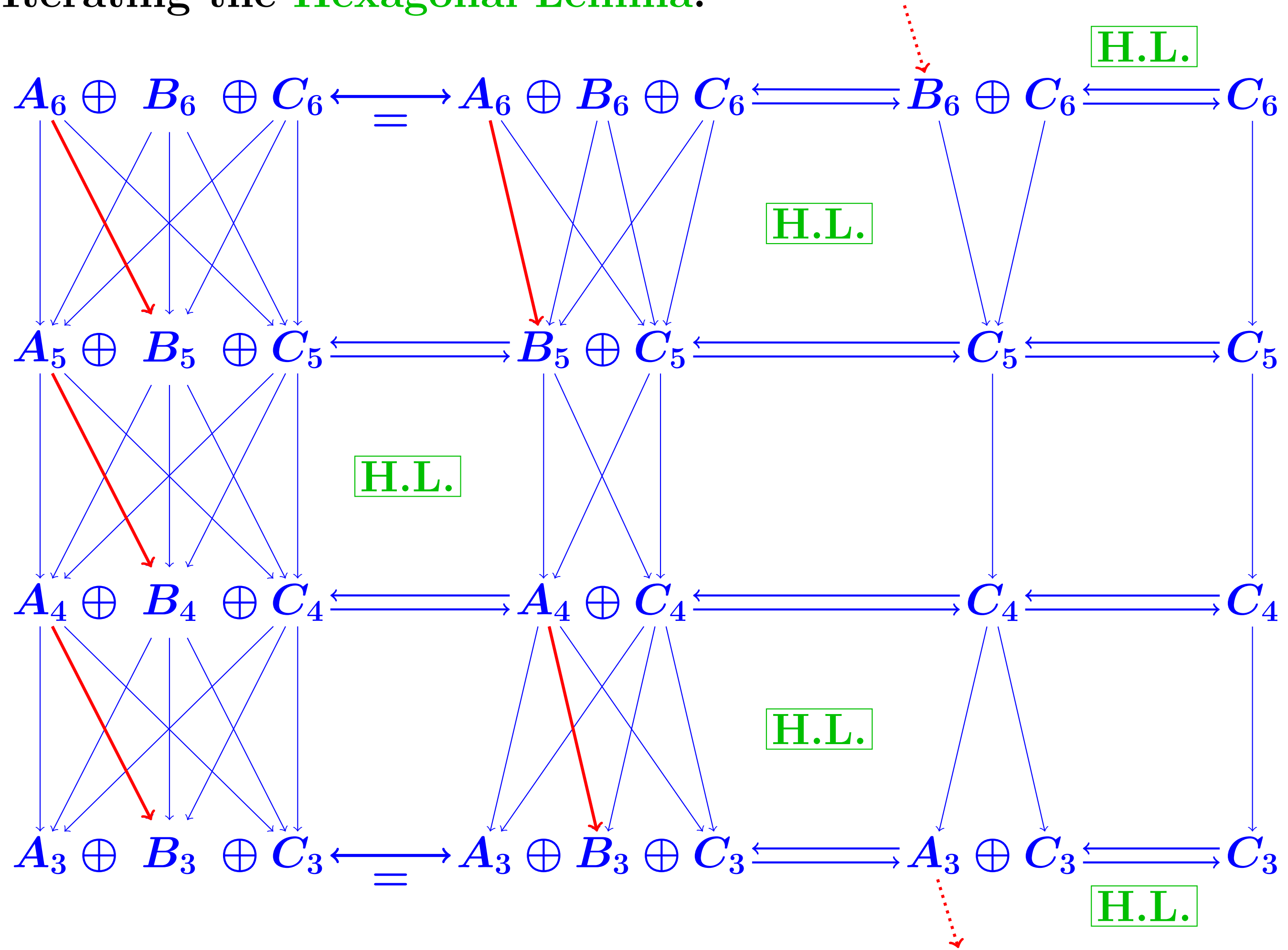
Hexagonal lemma



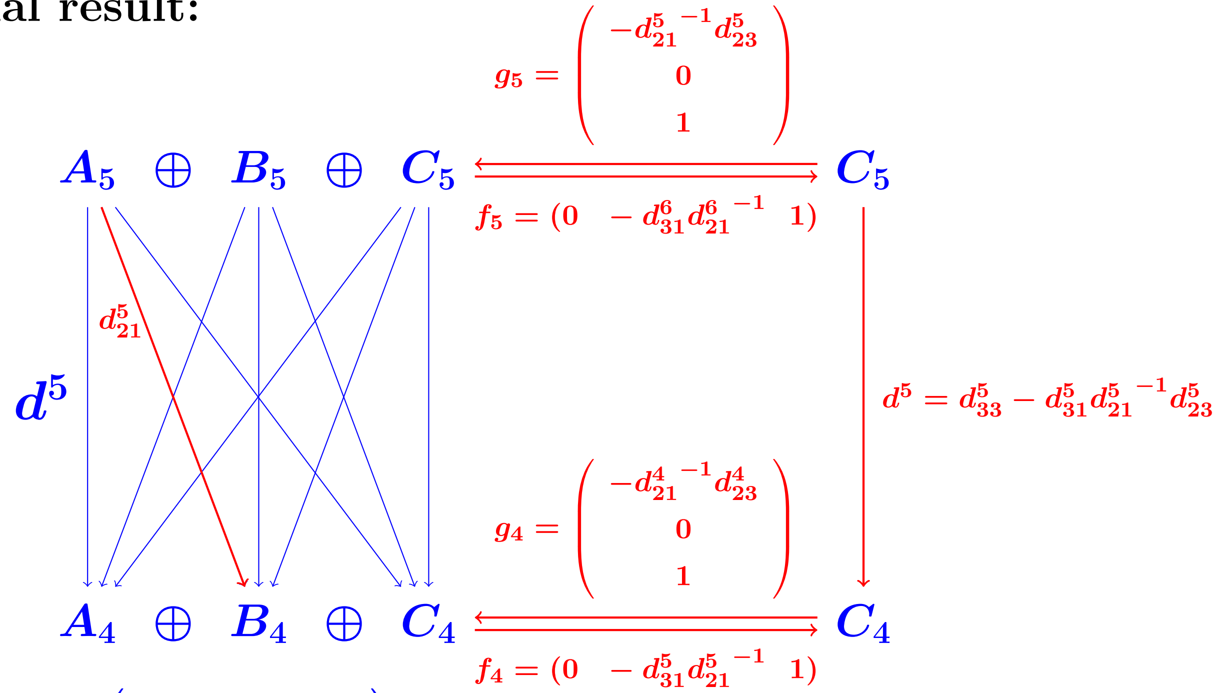


Hexagonal lemma

Iterating the Hexagonal Lemma:

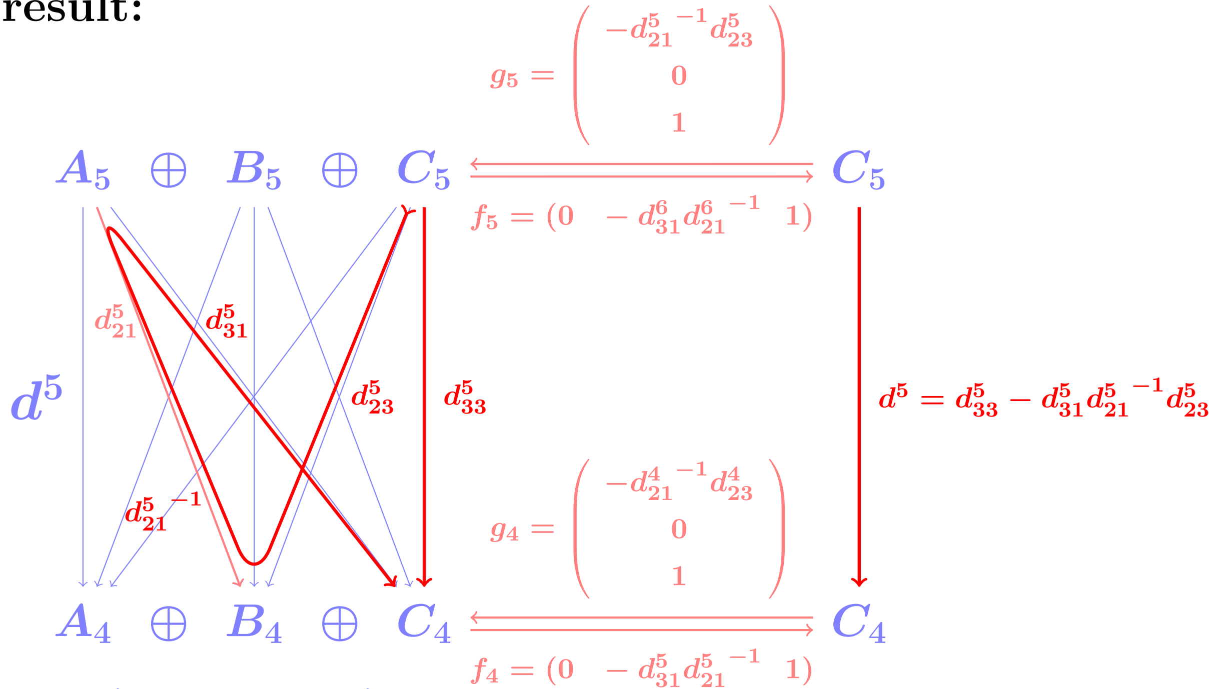


Final result:



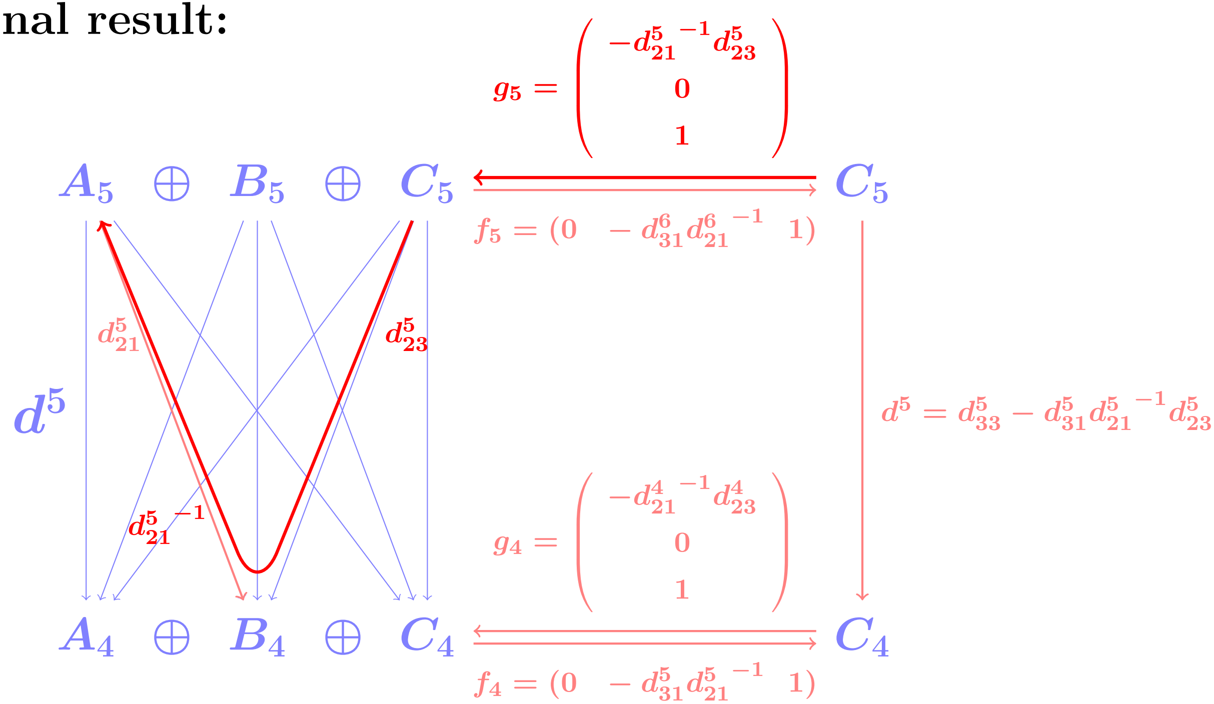
$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Final result:



$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Final result:

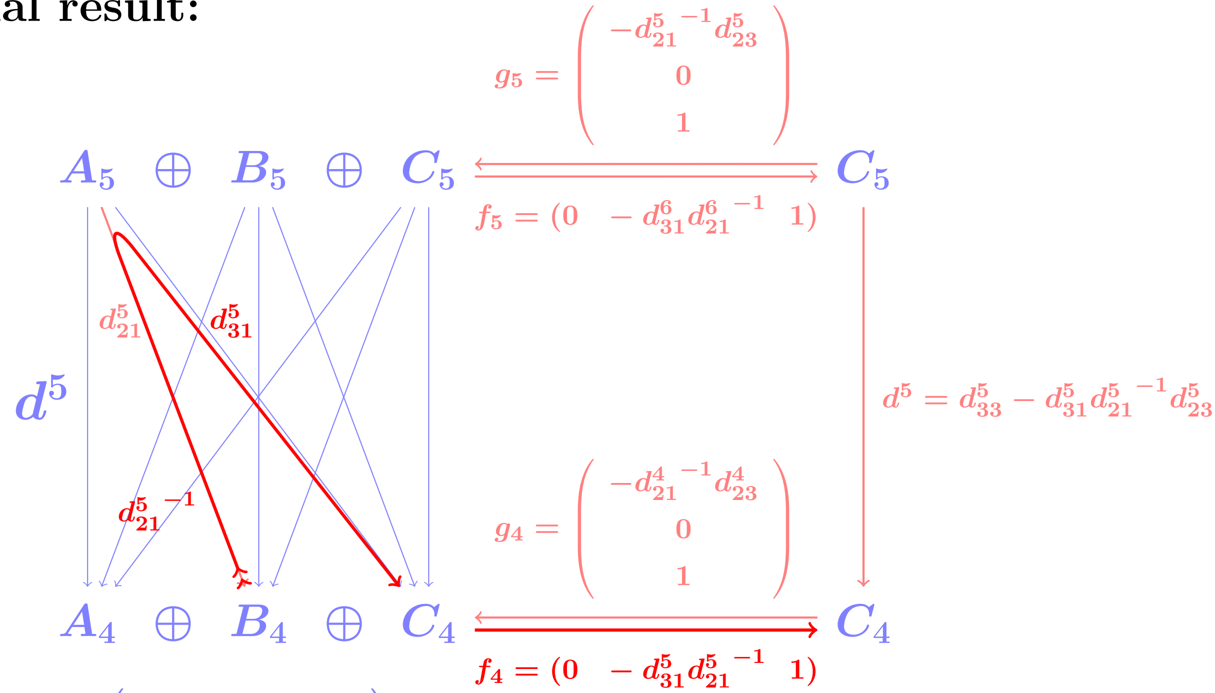


$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Note : $\mathrm{im}(g) \subset A_* \oplus C_*$

$= \text{graph } {d^5_{21}}^{-1} d^5_{23} : C_* \to A_*$

Final result:



$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Global Hexagonal Theorem:

Input: A chain complex (C_*, d_*)

with for every $n \in \mathbb{Z}$ a decomposition:

$$C_n = C_n^1 \oplus C_n^2 \oplus C_n^3 \qquad d_n = \begin{pmatrix} d_{n,11} & d_{n,12} & d_{n,13} \\ d_{n,21} & d_{n,22} & d_{n,23} \\ d_{n,31} & d_{n,32} & d_{n,33} \end{pmatrix}$$

with $d_{n,21} : C_n^1 \to C_{n-1}^2$ isomorphism $\forall n$.

Output: A canonical reduction:

$$(C_*, d_*) = (C_*^1 \oplus C_*^2 \oplus C_*^3, d_*) \Rrightarrow (C_*^3, d'_*)$$

5/6. Algebraic HPT Proof.

Definition: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d{+}\delta).$$

Let $\rho : \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \overset{g}{\underset{f}{\leftrightarrows}} (C_*, d_*)}$ be a given reduction

and $\widehat{\delta}$ a perturbation of $\widehat{d}$

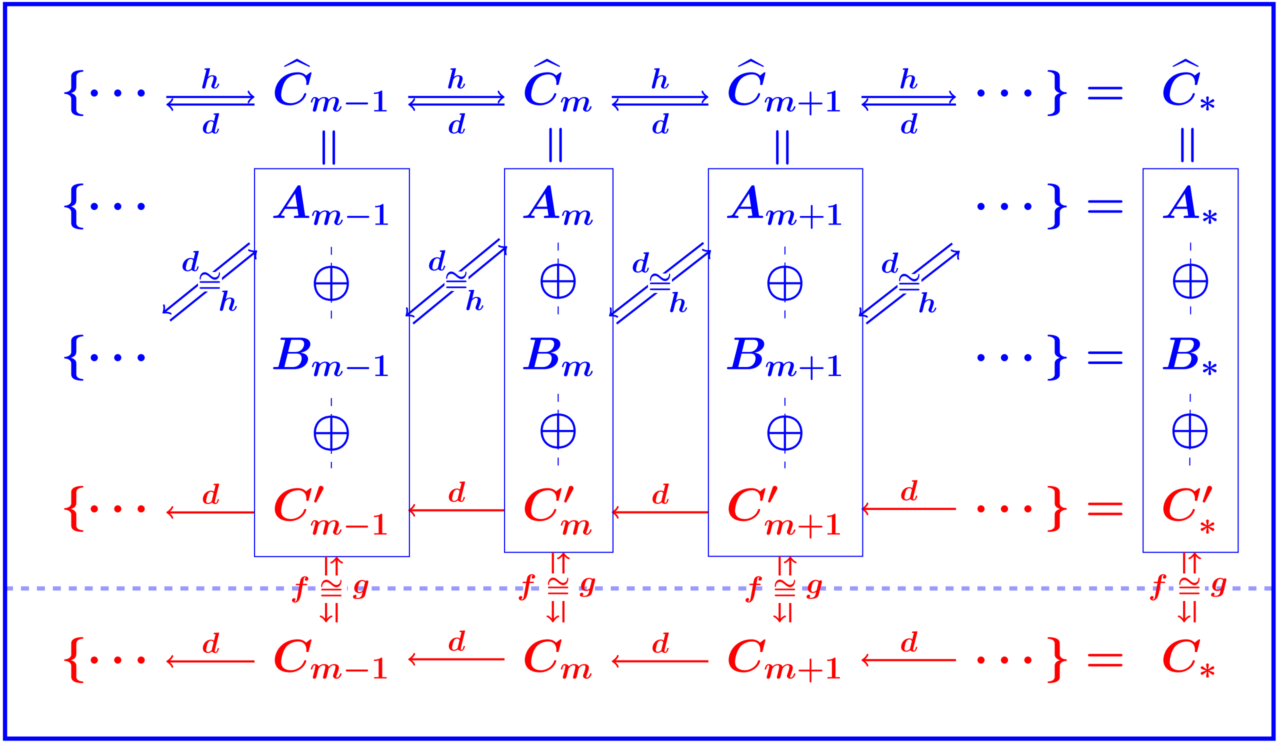
satisfying $h\widehat{\delta}$ pointwise nilpotent.

Theorem: The BPL determines a new reduction:

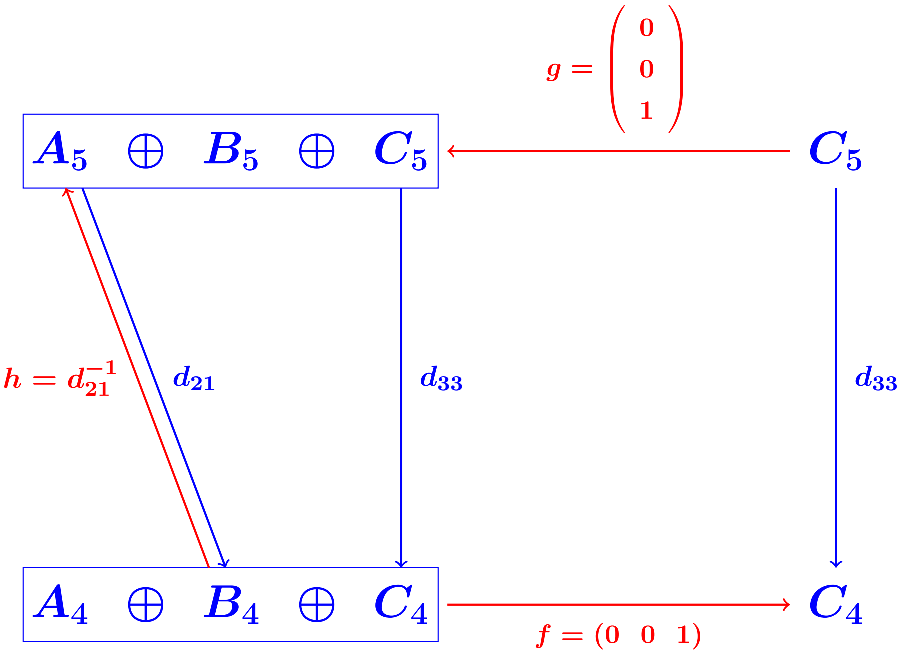
$$\rho' : \boxed{h + \delta_h \circlearrowright (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \overset{g + \delta_g}{\underset{f + \delta_f}{\leftrightarrows}} (C_*, d_* + \delta_{d*})}$$

Proof:

Reduction Diagram:

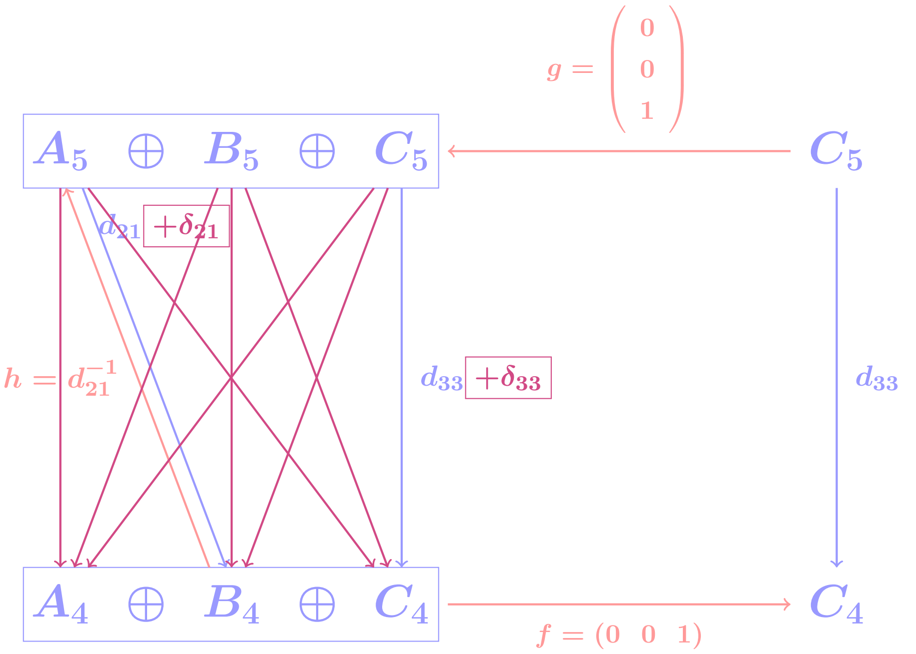


Main part:



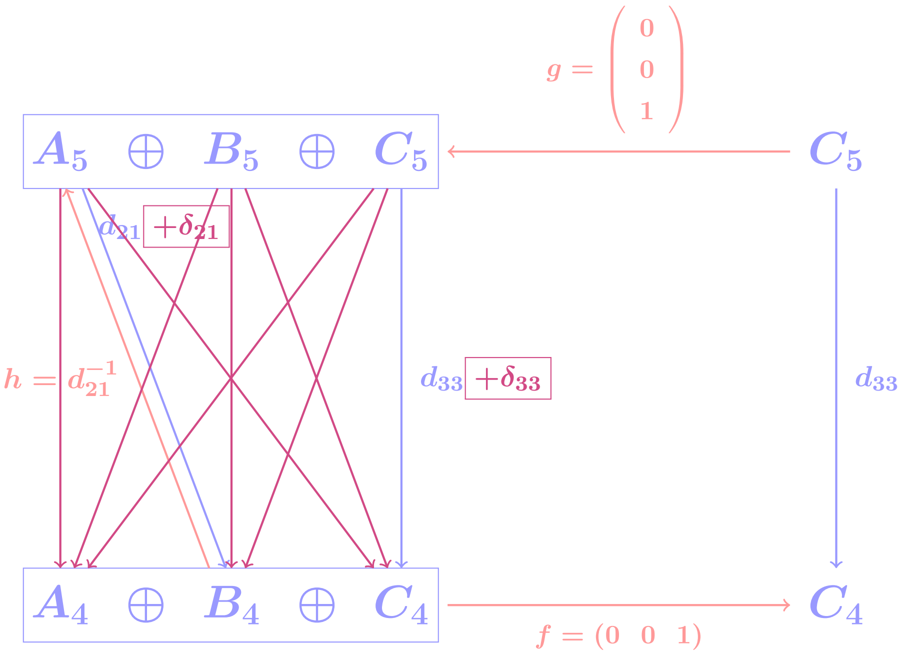
with d_{21} **= isomorphism.**

$$\textbf{Perturbation} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}:$$



Question: $(d_{21} + \delta_{21})$ again isomorphism?

(applying the Global Hexagonal Theorem possible ?)



But d_{21} invertible with $d_{21}h = 1 \quad \Rightarrow$

$$d_{21} + \delta_{21} = d_{21} + d_{21}h\delta_{21} = d_{21}(1 + h\delta_{21})$$

$\Rightarrow \quad d_{21} + \delta_{21}$ **invertible** $\Leftrightarrow (1 + h\delta_{21})$ **invertible.**

A sufficient condition is $h\delta_{21}$ nilpotent, in which case:

$$(1 + h\delta_{21})^{-1} = \sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i$$

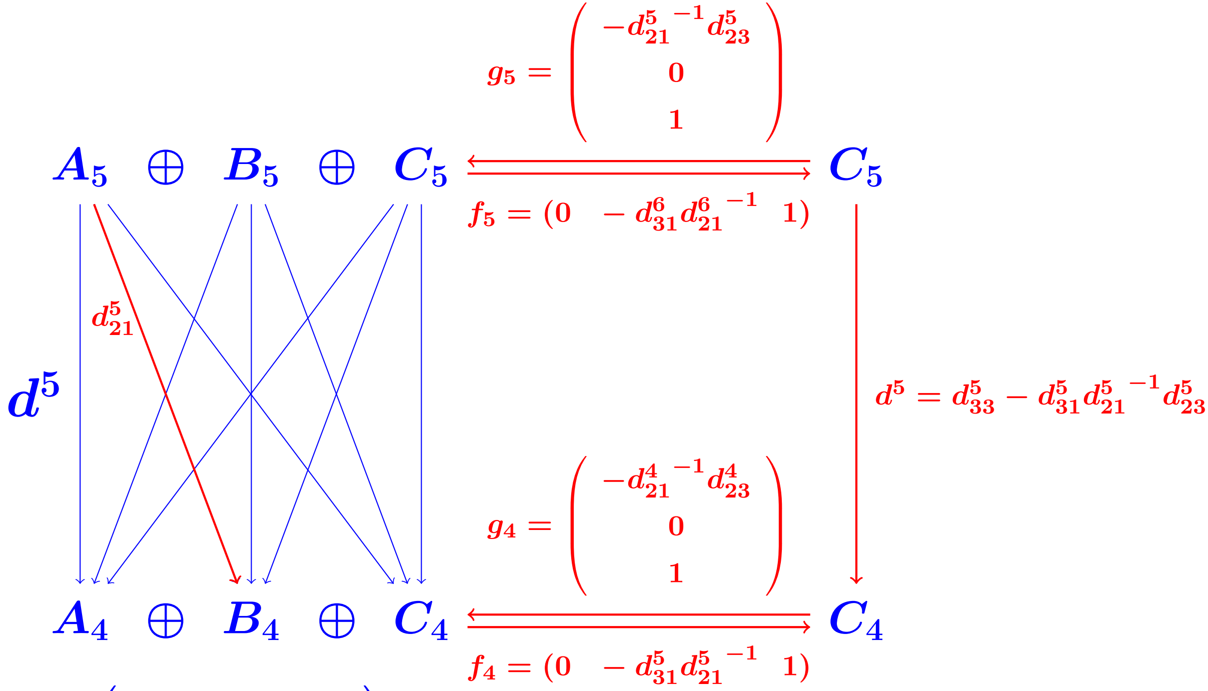
Then:

$$(d_{21} + \delta_{21})^{-1} =: h' := \left(\sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h$$

Remark:

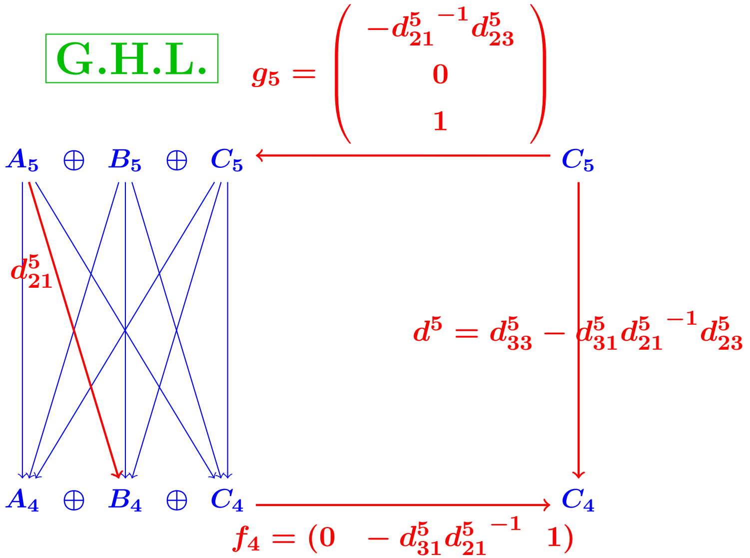
$$\left(\sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h = \left(\sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

Global Hexagonal Theorem:

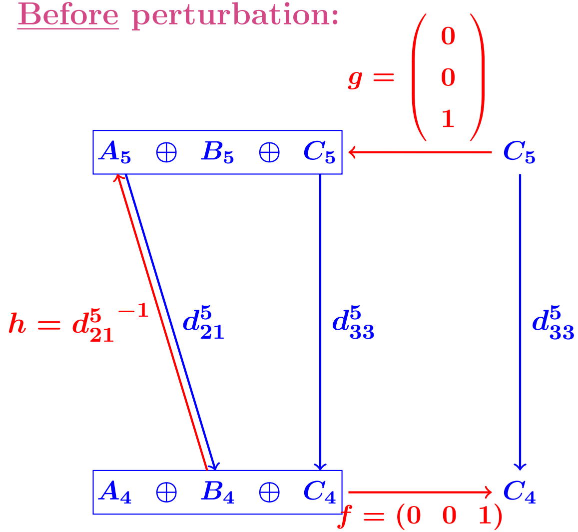


$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Applying to our situation:



Before perturbation:



$$d^5_{21} \mapsto d^5_{21} + \delta^5_{21}$$

$$d^5_{21}{}^{-1} \mapsto h' = \left(\sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

$$g_5 \mapsto (1 - h'\delta) g$$

$$f_4 \mapsto f(1 - \delta h')$$

$$\begin{aligned} d^5 &\mapsto (d^5_{33} + \delta^5_{33}) - f\delta h'\delta g \\ &= d^5_{33} + f\delta g - f\delta h'\delta g \end{aligned}$$

= Homological Perturbation Theorem **QED**

6/6. The topological case.

Corollary: The HPT can easily be extended to topological situations.

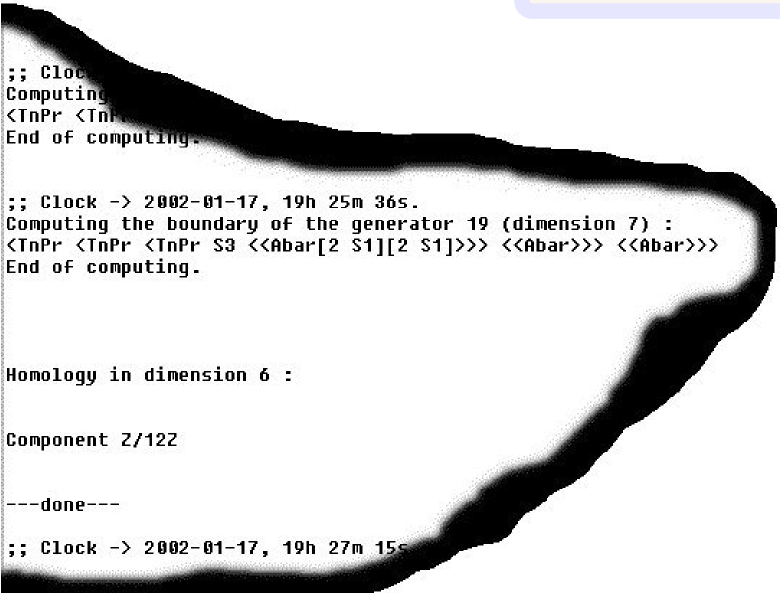
Example 1: Banach situations:

$$\|h\delta_{21}\| < 1 \Rightarrow (1 + h\delta_{21}) \text{ invertible } \Rightarrow \text{OK.}$$

Example 2: Frechetic situations:

The Nash-Moser-Schwartz technology often allows to prove $(1 - h\delta_{21})$ is invertible $\Rightarrow$ OK.

The END



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