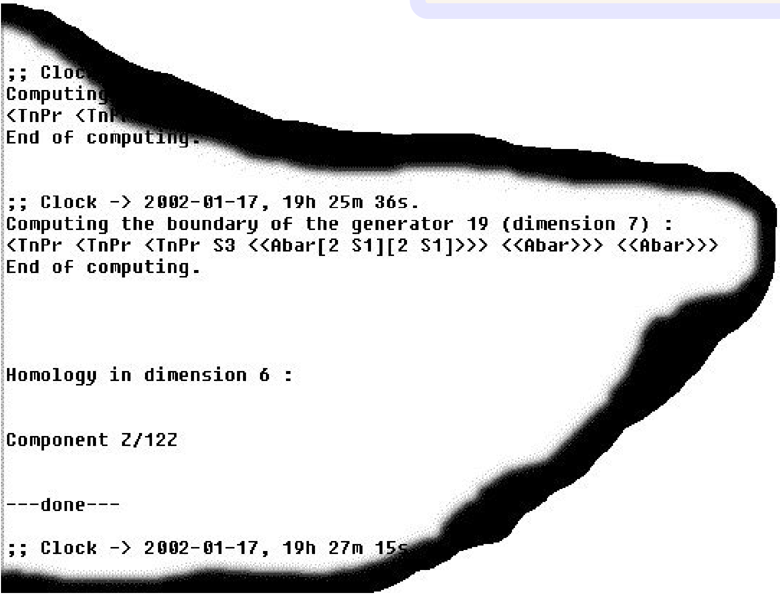
Effectiveness vs Gödel



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Semantics of colours:

Blue	=	**"Standard" Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, ...
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, ...
Green	=	**Solution, essential point,**
		mathematicians, ...

First significant computability result in Algebraic Topology:

Annals of Mathematics
Vol. 65, No. 1, January, 1957
Printed in U.S.A.

FINITE COMPUTABILITY OF POSTNIKOV COMPLEXES[1]

By Edgar H. Brown, Jr.

(Received March 3, 1956)

In [4] Postnikov associates with each arcwise connected space X a sequence of algebraically defined complexes $P_1(X)$, $P_2(X)$, $\cdots$, and in [5] he asserts

. . . .

have a finite number of non-degenerate simplexes.)

It must be emphasized that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.

In the first section of this paper we give some preliminary definitions, con-

Problem: Tractable algorithms ???

Main obstacle: Infinite objects !!!

Didactic example: $\pi_3(S^2) := \pi_0(\mathcal{C}(S^3, S^2)) =$???

Hurewicz Theorem $\Rightarrow \pi_2(S^2) = H_2(S^2) = \mathbb{Z}$. (easy)

$X_3 = K(\mathbb{Z}, 1) \times_\tau S^2$ with appropriate $\tau \Rightarrow$

$$\pi_3(S^2) \stackrel{\tau}{=} \pi_3(X_3) \stackrel{\text{Hurewicz}}{=} H_3(X_3)$$

But $K(\mathbb{Z}, 1) =$ infinite simplicial set

$\Rightarrow$ Implementation of $K(\mathbb{Z}, 1)$ on a machine ???

Functional trick:

The functional trick often allows one

to implement infinite objects.

Example: $\boldsymbol{K(\mathbb{Z},1)}$:

$\{$**n-simplices**$\}$:= $\{$**lists of n integers**$\}$

$$\boldsymbol{\sigma = (12\ 23\ 34\ 45\ 56) = \text{5-simplex of } K(\mathbb{Z},1).}$$

Example of geometric structure: 3-face of $\boldsymbol{\sigma}$ = ???

$$\boldsymbol{\partial_3\sigma := (12\ 23\ 79\ 56) \qquad (34+45=79)}$$

$H_*(K(\mathbb{Z},1)) = ???$

Theorem (**Gödel**, **Church**, **Turing**, **Post**):

$\nexists$ **general algorithm**:

$$[\text{Simplicial set functionnally coded } X] \longmapsto H_*(X)$$

E. Brown solution: **Approximation of infinite objects by sufficiently large finite approximations.**

Effective Homology solution: **Defining pairs:**

$$\text{Functional Infinite objects} \overset{\varepsilon}{\Longleftrightarrow} \text{Finite objects}$$

connected by functional objects.

Definition: An object with effective homology X is a 4-tuple:

$$X = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = "the" chain complex "traditionally" associated with X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \stackrel{\varepsilon}{\Longleftarrow\!\!\Longrightarrow} EC_*$.

Main result of effective homology:

<u>Meta-theorem</u>: Let $X_1, \ldots, X_n$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_1, \ldots, X_n) \mapsto X.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}\colon \left(\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}\right)$$

$$\mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is perfectly stable

and can be again used with X for further calculations.

Example:

Julio Rubio's solution of Adams' problem:

$\Omega X := \mathcal{C}(S^1, X)$. **Adams: Computation of** $H_*(\Omega^n X)$ **???**

$$\boxed{X = (X,\ C_*(X),\ EC_*^X,\ \varepsilon^X)}$$

$$\Downarrow\Downarrow\Downarrow \qquad \Omega_{EH}$$

$$\boxed{\Omega X = (\Omega X,\ C_*(\Omega X),\ EC_*^{\Omega X},\ \varepsilon^{\Omega X})}$$

$\Longrightarrow$ **Trivial iteration now available.**

$\Rightarrow$ **Very simple solution of Adam's problem :**

Indefinite iteration of the Cobar construction ???

$$\begin{aligned}
\boldsymbol{X} &= (\boldsymbol{X}, C_*(\boldsymbol{X}), \boldsymbol{EC_*^X}, \varepsilon^{\boldsymbol{X}}) \\
&\Downarrow \ \boldsymbol{\Omega_{EH}} \\
\boldsymbol{\Omega X} &= (\boldsymbol{\Omega X}, C_*(\boldsymbol{\Omega X}), \boldsymbol{EC_*^{\Omega X}}, \varepsilon^{\boldsymbol{\Omega X}}) \\
&\Downarrow \ \boldsymbol{\Omega_{EH}} \\
\boldsymbol{\Omega^2 X} &= (\boldsymbol{\Omega^2 X}, C_*(\boldsymbol{\Omega^2 X}), \boldsymbol{EC_*^{\Omega^2 X}}, \varepsilon^{\boldsymbol{\Omega^2 X}}) \\
&\Downarrow \ \boldsymbol{\Omega_{EH}} \\
\boldsymbol{\Omega^3 X} &= (\boldsymbol{\Omega^3 X}, C_*(\boldsymbol{\Omega^3 X}), \boldsymbol{EC_*^{\Omega^3 X}}, \varepsilon^{\boldsymbol{\Omega^3 X}}) \\
&\Downarrow \ \boldsymbol{\Omega_{EH}} \\
\boldsymbol{\Omega^4 X} &= \ldots
\end{aligned}$$

"Cobar" **3** $(\boldsymbol{EC_*^X})$

Simple example of calculation now possible.

$$X := P^\infty\mathbb{R}/P^3\mathbb{R}$$

X **is** not a suspension $\Rightarrow$ **computation of** $H_*\Omega^3 X = ???$

X **has finite type** $\Rightarrow$

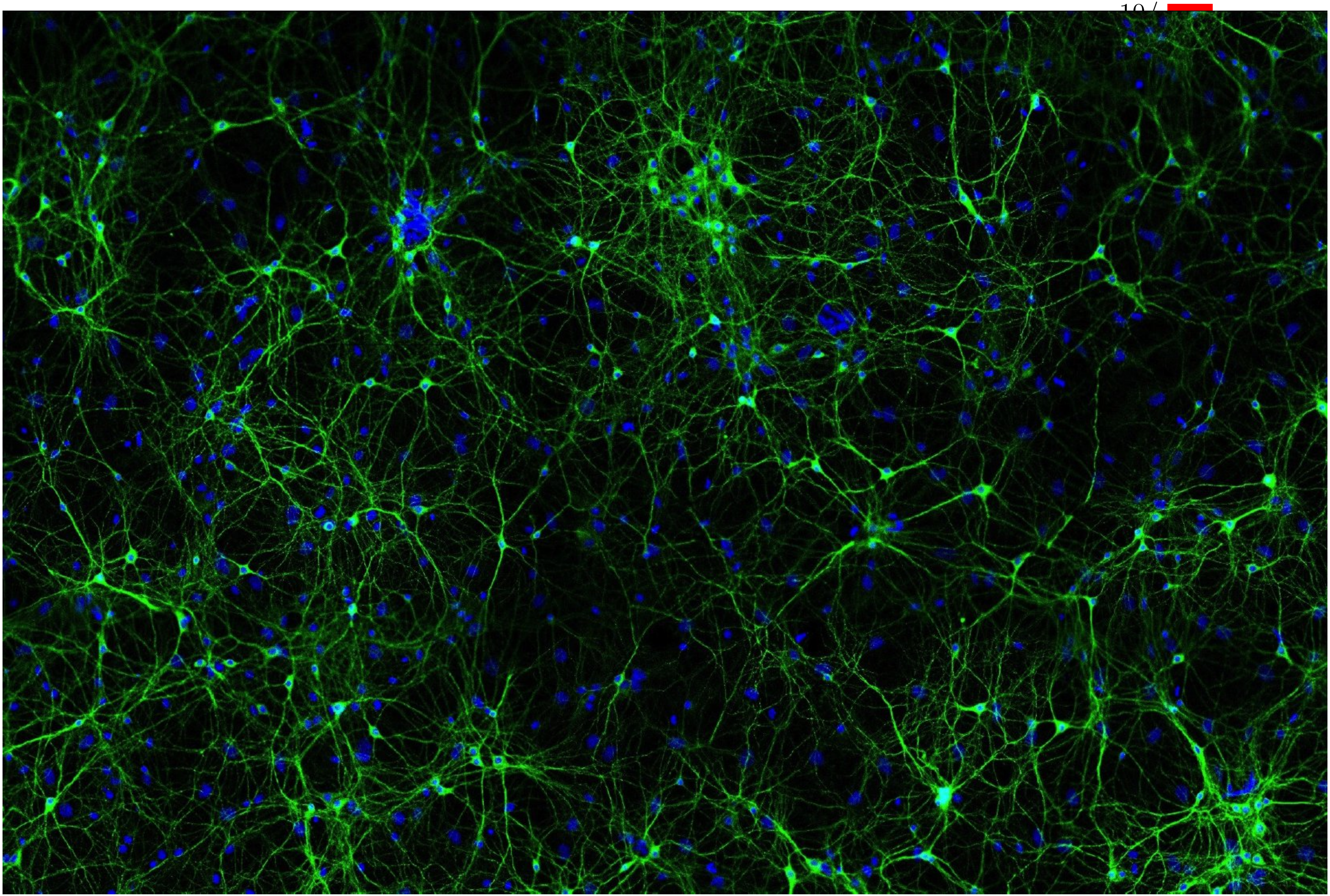
trivial object with eff. hom.: $(X, C_*X, C_*X, \varepsilon_0 = \mathrm{id})$.

Julio Rubio $\Rightarrow$ $(\Omega^3 X, C_*\Omega^3 X, EC_*\Omega^3 X, \varepsilon_3)$

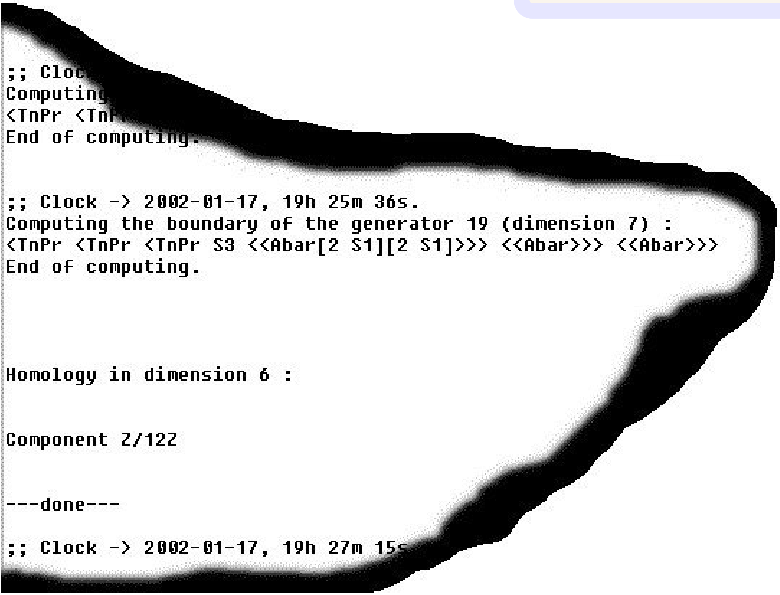
$EC_*\Omega^3 X$ **has finite type** $\Rightarrow$

Computing $H_*(EC_*\Omega^3 X)$ **is elementary.**

Example: $H_5\Omega^3 X = H_5(EC_*\Omega^3 X) = (\mathbb{Z}/3) \oplus (\mathbb{Z}/2)^5 \oplus \mathbb{Z}$



The END



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