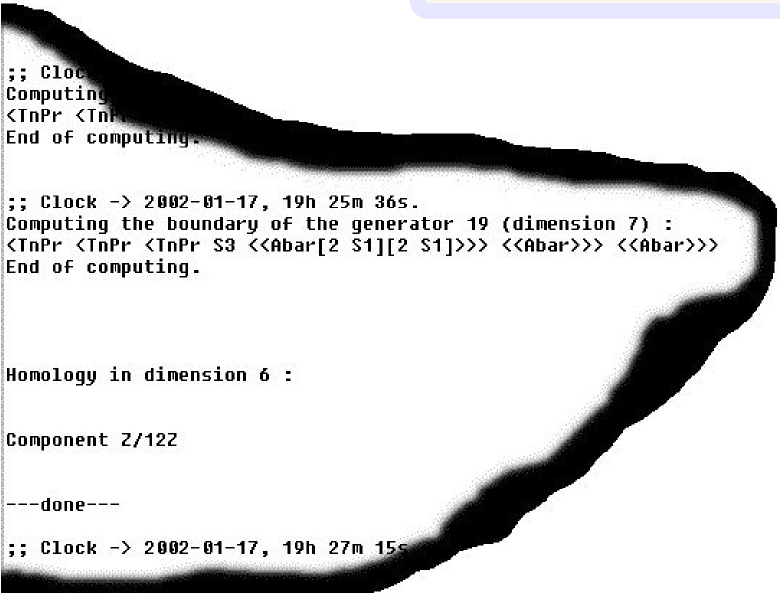
# Morphisms between

# Discrete Vector Fields

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*Oberwolfach, May-2013*

**Semantics of colours:**

| | | |
|---|---|---|
| **Blue** | = | **“Standard” Mathematics** |
| **Red** | = | **Constructive, effective,** |
| | | **algorithm, machine object, . . .** |
| **Violet** | = | **Problem, difficulty,** |
| | | **obstacle, disadvantage, . . .** |
| **Green** | = | **Solution, essential point,** |
| | | **mathematicians, . . .** |

**Plan.**

- $\rightarrow$ **Introduction.**
- **Basics of Algebraic Discrete Vector Fields.**
- **Discrete Vector Field $\Rightarrow$ Canonical Reduction.**
- **Guessing the reduction formulas.**
- **Vector Fields and Morphisms.**
- **Seeing complicated products.**
- **The Eilenberg-Zilber vector field.**
- **The Eilenberg-Zilber vector field is natural.**
- **EZ vector field $\Rightarrow$ EZ reduction.**

# Introduction.

**Eilenberg-MacLane conjecture (1953):**

$G$ = simplicial group $\Rightarrow$

$BG$ = Classifying space of $G$

$\mathrm{Bar}(C_*G)$ = Algebraic classifying object of $C_*G$

Then $\exists$ a reduction $\rho : C_*(BG) \Rrightarrow \mathrm{Bar}(C_*G)$

Essential for effective methods computing $\pi_n X$.

**First proof = Pedro Real (1993). Never implemented.**

**Second "proof" through discrete vector fields.**

**Easily implemented** $\Rightarrow \left\{ \begin{array}{l} \textbf{Program code divided by} \sim 3 \\ \textbf{Computing times divided by} \sim 20 \end{array} \right.$

**But proof quite complex not yet finished.**

**Main problem =**

**Compatibility: Vector fields** $\leftrightarrow$ **Algebraic structures**

**Typical problem = Naturality of reductions**

**coming from discrete vector fields.**

**Basic Algebraic Topology:**

**Serre + Eilenberg-Moore spectral sequences**

**Non-constructive**

**Making basic algebraic topology constructive:**

1. **Eilenberg-Zilber theorem.**
2. **Twisted Eilenberg-Zilber theorem.**
3. **Eilenberg-MacLane correspondence:**

**Topological Classifying space**

$$\Updownarrow$$

**Algebraic Classifying space**

**But constructive requires: with explicit homotopies.**

**Example:** Rubio-Morace homotopy for Eilenberg-Zilber:

$$RM : C_*(X \times Y) \to C_{*+1}(X \times Y)$$

$$RM(x_p\ ,\ y_p) \ = \sum_{\substack{0\leq r\leq p-1\\ 0\leq s\leq p-r-1\\ (\eta,\eta')\in \mathrm{Sh}(s+1,r)}} (-1)^{p-r-s}\,\varepsilon(\eta,\eta')\ldots$$

$$\ldots(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p\ ,\ \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p)$$

with $\mathrm{Sh}(p,q) = \{(p,q)\text{-shuffles}\} = \{(\eta_{i_{p-1}}\cdots\eta_{i_0}, \eta_{j_{q-1}}\cdots\eta_{j_0})\}$

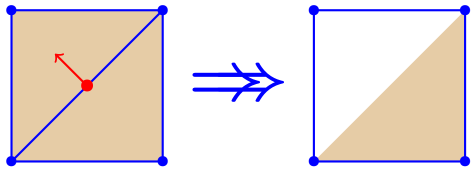
for $0 \leq i_0 < \cdots < i_{p-1} \leq p+q-1$

and $0 \leq j_0 < \cdots < j_{q-1} \leq p+q-1$

and $\{i_0, \ldots, i_{p-1}\} \cap \{j_0, \ldots, j_{q-1}\} = \emptyset$.

and $\uparrow^k (\eta_\alpha\eta_\beta\cdots) = \eta_{\alpha+k}\eta_{\beta+k}\cdots$ $\quad$ ($\uparrow^k$ = $k$-shift operator.)

**Simpler:**



**once the notion of discrete vector field is understood.**

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## Basics of Algebraic Discrete Vector Fields.

**Definition**: $R$ = unitary commutative ring.

A cellular $R$-chain complex is an indexed triple:

$$(C_p, \beta_p, d_p)_{p\in\mathbb{Z}}$$

with:

- $C_p$ = free $R$-module (non necessarily of finite type);
- $\beta_p$ = distinguished $R$-basis of $C_p$;
- $d_p : C_p \to C_{p-1}$ = differential.

The elements of $\beta_p$ are the $p$-cells of the cellular complex.

# Main examples:

- **Geometrical cellular complexes** ($\boldsymbol{R} = \mathbb{Z}$):
  - **Cubical complexes (digital images)**;
  - **Simplicial complexes** ;
  - **Simplicial sets** ;
  - **CW-complexes** ;
- **Algebraic cellular complexes** ($\boldsymbol{R} = \mathfrak{k} =$ **field**):
  - **Free resolutions**;
  - **Koszul complexes**;
  - ... ...

**Natural distinguished basis** $\Rightarrow$ **Cellular complex.**

$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = Cellular chain complex.

Definition: A $p$-cell is an element of $\beta_p$.

Definition: If $\tau \in \beta_p$ and $\sigma \in \beta_{p-1}$,

then $\varepsilon(\sigma,\tau) :=$ coefficient of $\sigma$ in $d\tau$

is called the incidence number between $\sigma$ and $\tau$.

Definition: $\sigma$ is a face of $\tau$ if $\varepsilon(\sigma,\tau) \neq 0$.

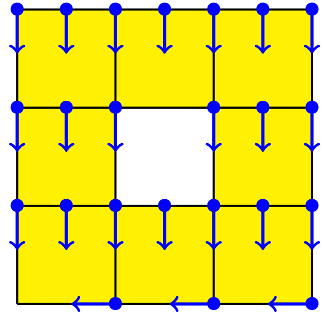
Definition: $\sigma$ is a regular face of $\tau$ if $\varepsilon(\sigma,\tau)$ is $R$-invertible.

($\Leftrightarrow \varepsilon(\sigma,\tau) = \pm 1$ if $R = \mathbb{Z}$)

**Definition:**

$(C_*, \beta_*, d_*)$ = Cellular complex

A Discrete Vector Field is a pairing:



$$V = \{(\sigma_i, \tau_i)\}_{i \in I}$$

satisfying:

- $\forall i \in I, \quad \tau_i$ = some $k_i$-cell and $\sigma_i$ = some $(k_i - 1)$-cell.
- $\forall i \in I, \quad \sigma_i$ is a regular face of $\tau_i$.
- $\forall i \neq j \in I, \quad \{\sigma_i, \tau_i\} \cap \{\sigma_j, \tau_j\} = \emptyset$.

The vector field $V$ is admissible or not.

$C_* = \{C_p, \beta_p, d_p\}_{p \in \mathbb{Z}}$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)\}_{i \in I}$ = Vector field.

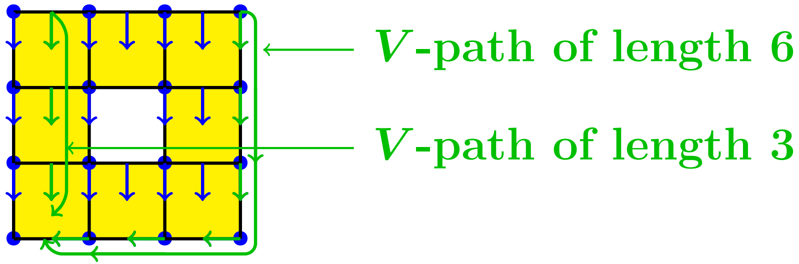
**Definition**: $V$-path = sequence $(\sigma_{i_1}, \tau_{i_1}, \sigma_{i_2}, \tau_{i_2}, \ldots, \sigma_{i_n}, \tau_{i_n})$

satisfying:

1. $(\sigma_{i_j}, \tau_{i_j}) \in V$.
2. $\sigma_{i_j}$ face of $\tau_{i_{j-1}}$.
3. $\sigma_{i_j} \neq \sigma_{i_{j-1}}$.

**Remark:** $\sigma_{i_j}$ not necessarily regular face of $\tau_{i_{j-1}}$.

**Examples:**



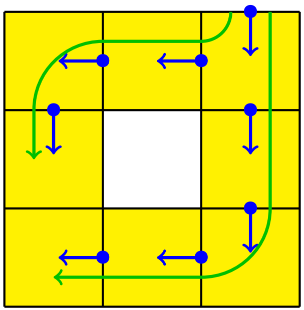
**Definition**: A vector field is admissible if

for every source cell $\sigma$,

the length of any path starting from $\sigma$

is bounded by a fixed integer $\lambda(\sigma)$.

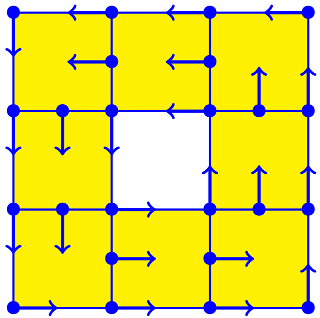
**Example of two different paths with the same starting cell.**



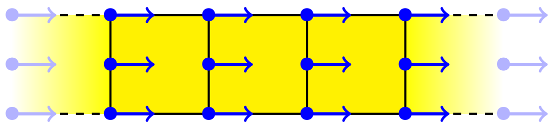
**Remark**: The paths from a starting cell

are not necessarily organized as a tree.

# Typical examples of non-admissible vector fields.



???!!! $\sim$ $\emptyset$



???!!! $\sim$ $\emptyset$

$\mathbb{R} \times \boldsymbol{I}$

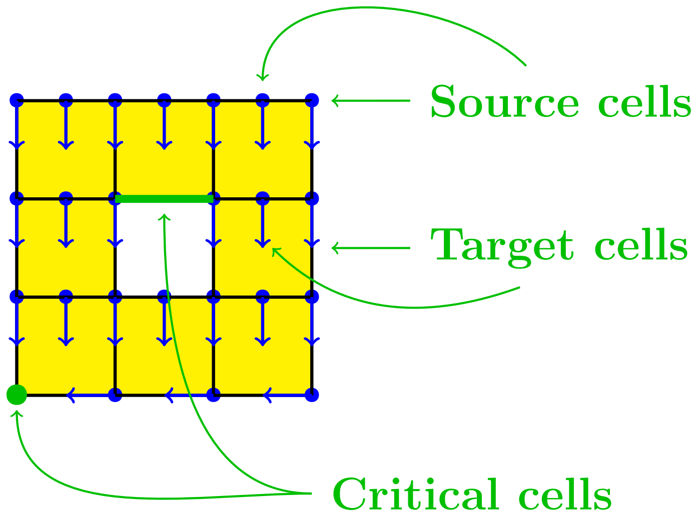
$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = **Cellular chain complex.**

$V = \{(\sigma_i, \tau_i)\}_{i\in I}$ = **Vector field.**

**Definition:** A critical $p$-cell is an element of $\beta_p$ which does not occur in $V$.

Other cells divided in source cells and target cells.

**Example:**



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# Discrete Vector Field ⇒ Canonical Reduction.

**Fundamental Theorem:**

**Given:** $C_* = (C_p, \beta_p, d_p)_{p\in\mathbb{Z}}$ = Cellular chain complex.

$V = (\sigma_i, \tau_i)_{i\in I}$ = Admissible Discrete Vector Field.

⇒

A canonical process constructs: $d_*^c + f + g + h$

defining a canonical reduction:

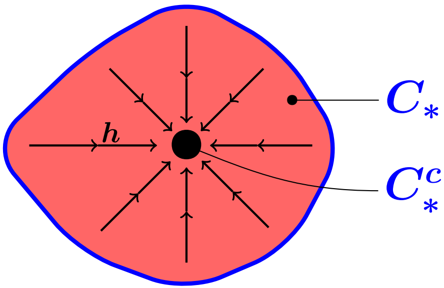
$$\rho_V = \boxed{h \circlearrowright (C_p, \beta_p, d_p)_{p\in\mathbb{Z}} \underset{f}{\overset{g}{\rightleftarrows}} (C_p^c, \beta_p^c, d_p^c)_{p\in\mathbb{Z}}}$$

Initial Complex $\overset{\rho_V}{\Rrightarrow}$ Critical complex

**In** $\rho = (f, g, h) = \boxed{h \circlearrowleft (C_*, \beta_*, d_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*^c, \beta_*^c, d_*^c)}$,

**the most important component is** $h$.

**To be understood as a contraction** $C_* \Rrightarrow C_*^c$.



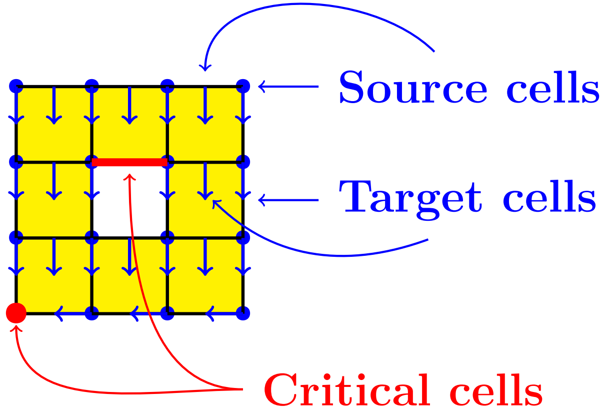

**The homotopy** $h$ **reduces the "cherry"** $C_*$

**onto the "stone"** $C_*^c$.

$\Rightarrow h =$ **the "flow" generated by the vector field.**

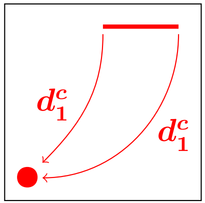
# Toy Example.

$C_* =$




Fundamental Reduction Theorem $\Rightarrow$

$\rho : C_* \Rrightarrow C_*^c =$




$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} =$ Circle

| |
|---|
| $\mathbf{Rank}(C_*) = (16, 24, 8)$ vs $\mathbf{Rank}(C_*^c) = (1, 1, 0)$ |

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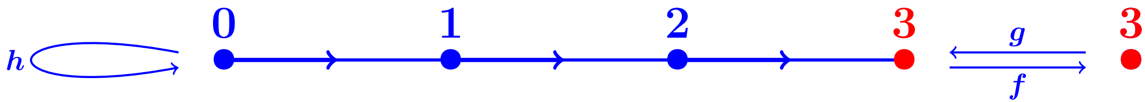
## Guessing the reduction formulas.

**Transcription for discrete vector fields:**

**Elementary example of :**




$$f(0) = f(1) = f(2) = f(3) = 3$$

$$f(01) = f(12) = f(23) = 0$$

$$g(3) = 3$$

$$\begin{aligned} h(0) &= -01 - 12 - 23 \\ h(1) &= -12 - 23 \\ h(2) &= -23 \\ h(3) &= 0 \end{aligned}$$

$$\mathrm{id} = fg$$

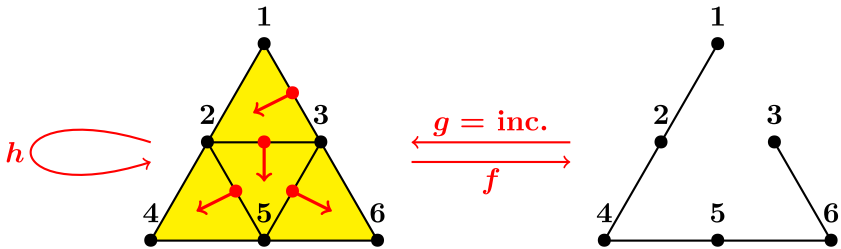
$$\mathrm{id} = gf + dh + hd$$

$$fh = 0$$

$$hg = 0$$

$$hh = 0$$

# Just a little more complicated:



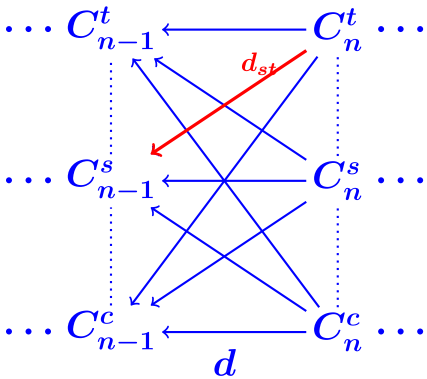

$$\begin{aligned}
f(25) &= 24 + 45 = -dh(25) + 25 \\
h(25) &= -245 = -\tau(25) \qquad \text{(25 = negative face of 245)} \\
f(23) &= 24 + 45 + 56 - 36 \\
h(23) &= -245 + 235 - 356 = \tau(23) + h(25) - h(35) \\
f(13) &= 12 + 24 + 45 + 56 - 36 \\
h(13) &= -123 - 245 + 235 - 356 = -\tau(13) + h(23)
\end{aligned}$$

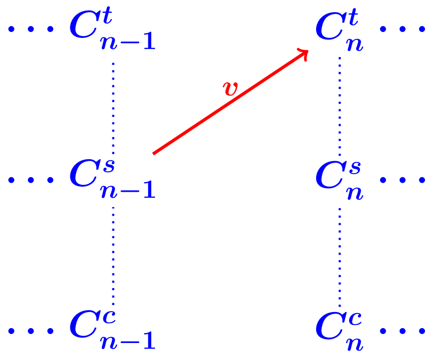
$\Rightarrow$ General formula for $h$(source cells):

$$h(\sigma) = v(\sigma) - h(d_{st}v(\sigma) - \sigma)$$

to be explained.

## $(C_*, \beta_*, d_*)$ + $V = \{(\sigma_i, \tau_i)\}$ = vector field defines a splitting:




$V$ defines a codifferential: $v(\sigma) = \varepsilon(\sigma, \tau)\tau$ if $(\sigma, \tau) \in V$

$$d = \begin{bmatrix} d_{tt} & d_{ts} & d_{tc} \\ d_{st} & d_{ss} & d_{sc} \\ d_{ct} & d_{cs} & d_{cc} \end{bmatrix} \qquad v = \begin{bmatrix} 0 & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h(\sigma) = v(\sigma) - h(d_{st}v(\sigma) - \sigma) = [v - h(dv - 1)](\sigma)$$

Deciding also $h(\sigma) = 0$ if $\sigma$ = target or critical cell

+ using some elementary facts, gives the general formula:

$$h(\sigma) = v(\sigma) - h(dv(\sigma) - \sigma)$$

valid even for $\sigma$ target or critical cell.

In particular: $\boxed{\boxed{v - hdv = 0}}$

Then:

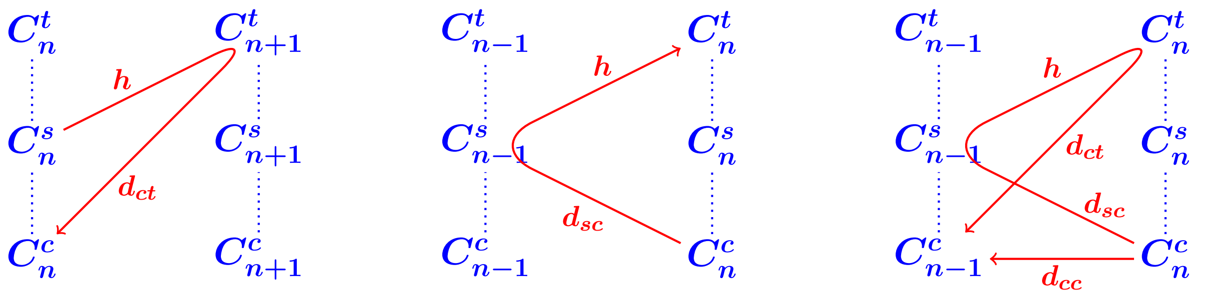
$$f = \mathrm{pr}_c(1 - dh) \qquad d^c_* = d_{cc} - d_{ct} h d_{sc} \qquad g = (1 - hd)$$

defines the searched reduction:

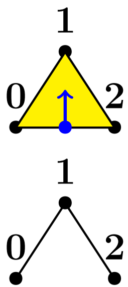
$$\rho = (f, g, h) = \boxed{h \circlearrowright (C_*, \beta_*, d_*) \underset{f}{\overset{g}{\leftrightarrows}} (C^c_*, \beta^c_*, d^c_*)}$$

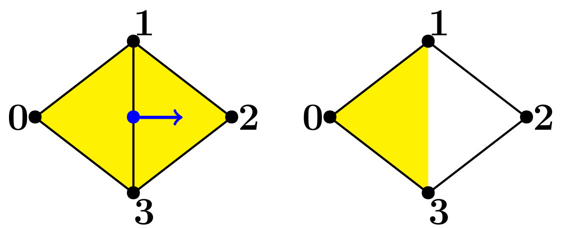
# Understanding: $h \Rightarrow f$ and $g$ and $d^c$:

$$h = v - h(dv - 1) : C^s_* \to C^t_{*+1}$$




$f = \mathrm{pr}_c(1 - dh)$ $\quad$ $g = (1 - hd)$ $\quad$ $d^c = d_{cc} - d_{ct} h d_{sc}$




$f(01) = 01$

$f(02) = 01 + 12$

$f(012) = \emptyset$

$f(013) = 013$

$f(123) = \emptyset$

$g(013) = 013 + 123$

$d^c(013) = 01 + 12 + 23 - 03$

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## Vector Fields and Morphisms.

**Problem**: Let $C_*$ and $C'_*$ be two cellular chain complexes respectively provided with vector fields $V$ and $V'$.

**Question**: Right notion of morphism $\phi : (C_*, V) \to (C'_*, V')$ ???

1. Not trivial.
2. Essential to master the Eilenberg-Zilber vector fields.
3. Quite amazing !!

<u>**Definition**</u>**: A cellular morphism:**

$$\phi : (C_*, d_*, \beta_*) \to (C'_*, d'_*, \beta'_*)$$

**is a chain complex morphism** $\phi : (C_*, d_*) \to (C'_*, d'_*)$

**satisfying the extra condition:**

**For every** $p$**-cell** $\sigma \in \beta_p$,

$\phi(\sigma)$ **is null or** $\in \beta'_p$.

$(C_*, d, \beta, V)$ **and** $(C'_*, d', \beta', V')$

$=$ **cellular chain complexes**

**with respective admissible discrete vector fields** $V$ **and** $V'$**.**

**Definition: A vectorious morphism:**

$$\phi : (C_*, d, \beta, V) \to (C'_*, d', \beta', V')$$

**is a cellular morphism** $\phi := (C_*, d, \beta) \to (C'_*, d', \beta')$
**satisfying the extra conditions:**

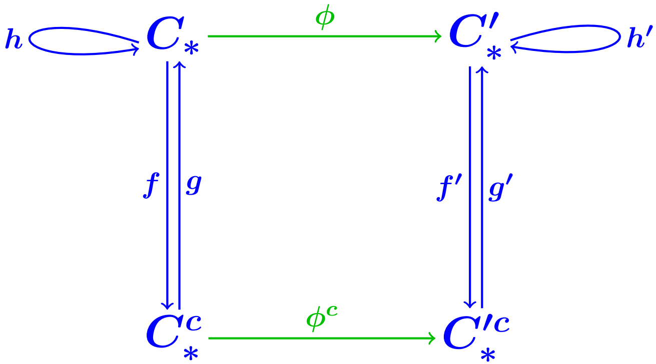
1. **For every critical cell** $\chi \in \beta^c_p$, $\quad \phi(\chi)$ **is null or** $\in \beta'^c_p$.
2. **For every target cell** $\tau \in \beta^t_p$, $\quad \phi(\tau)$ **is null or** $\in \beta'^t_p$.
3. **No condition at all for the source cells !!**

<u>Theorem</u>: $\phi : (C_*, d, \beta, V) \to (C'_*, d', \beta', V')$

$=$ vectorious morphism.

Then $\phi$ defines a morphism $(\phi, \phi^c)$

between the corresponding reductions:




with:

$$d'^c\phi^c = \phi^c d^c$$
$$f'\phi = \phi^c f$$
$$g'\phi^c = \phi g$$
$$h'\phi = \phi h$$

Note: $\phi^c := \phi | C^c_*$

**Proof:**

**Definition:**

$$\lambda_\sigma = \begin{cases} 0 & \text{for target and critical cells,} \\ \text{maximal length of a } V\text{-path} \\ & \text{starting from the source cell } \sigma. \end{cases}$$

**Remember: Recursive formula:**

$$h(\sigma) = v(\sigma) - h(dv(\sigma) - \sigma)$$

$$\Rightarrow hdv(\sigma) = v(\sigma)$$

$$\Rightarrow hd\tau = \tau \text{ for every target cell } \tau$$

1. $h'\phi\sigma = \phi h\sigma$ ??

Obvious for $\sigma$ target or critical cell.

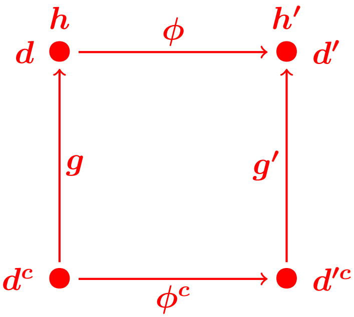
Assumed known for $\lambda_\sigma < k$.

Let $\sigma$ be a source cell with $\lambda_\sigma = k$.

$$\begin{array}{rrl} & \phi h\sigma & = \phi v\sigma - \phi h(dv\sigma - \sigma) \\ \text{OK for } (dv\sigma - \sigma) \Rightarrow & & = \phi v\sigma - h'\phi(dv\sigma - \sigma) \\ \phi d = d'\phi \Rightarrow & & = \phi v\sigma - h'd'\phi v\sigma + h'\phi\sigma \\ \phi v\sigma = \text{target cell} \Rightarrow & & = h'\phi\sigma \end{array}$$

QED

**2.** $g'\phi^c = \phi g$ ??



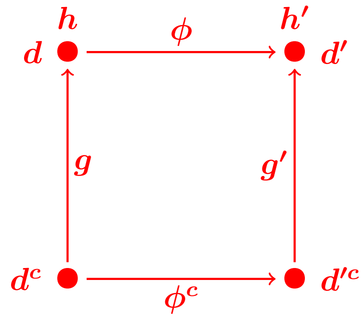

**For a critical cell $\chi$:** $g\chi = \chi - hd\chi = (1 - hd)\chi$

$$\Rightarrow \quad \phi g\chi = \phi(1 - hd)\chi$$

$$\phi hd = h'd'\phi \quad \Rightarrow \quad = (1 - h'd')\phi\chi$$

$$\phi\chi = \phi^c\chi \quad \Rightarrow \quad = g'\phi^c\chi$$

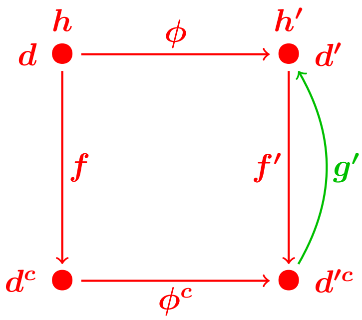
**QED**




**3.** $\phi^c d^c = d'^c \phi^c$ ??

$$
\begin{aligned}
g'\phi^c = \phi g &\quad\Rightarrow\quad & g'\phi^c d^c &= \phi g d^c \\
g d^c = d g &\quad\Rightarrow\quad & &= \phi d g \\
\phi d = d'\phi &\quad\Rightarrow\quad & &= d'\phi g \\
\phi g = g'\phi^c &\quad\Rightarrow\quad & &= d' g' \phi^c \\
d' g' = g' d'^c &\quad\Rightarrow\quad & &= g' d'^c \phi^c \\
g' \text{ injective} &\quad\Rightarrow\quad & \phi^c d^c &= d'^c \phi^c
\end{aligned}
$$

**QED**




**4.** $f'\phi = \phi^c f$ ??

$g'$ **injective** $\Rightarrow [(f'\phi = \phi^c f) \quad \Leftrightarrow \quad (g'f'\phi = g'\phi^c f)]$

$$
\begin{aligned}
& & g'f'\phi &= (1 - d'h' - h'd')\phi \\
(d'\phi = \phi d) + (h'\phi = \phi h) & \quad \Rightarrow & &= \phi(1 - dh - hd) \\
& & &= \phi g f \\
& & &= g'\phi^c f
\end{aligned}
$$

**QED**

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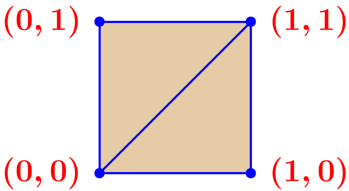
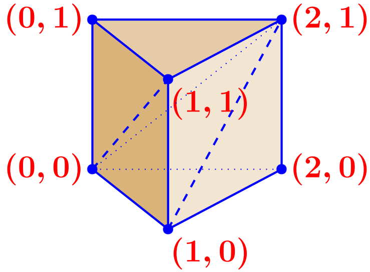
# Seeing complicated products.

## Triangulating prisms $\Delta^p \times \Delta^q$:




$\Delta^1 \times \Delta^1$




$\Delta^2 \times \Delta^1$

Two $\Delta^2$ in $\Delta^1 \times \Delta^1$: $(0,0) < (0,1) < (1,1)$

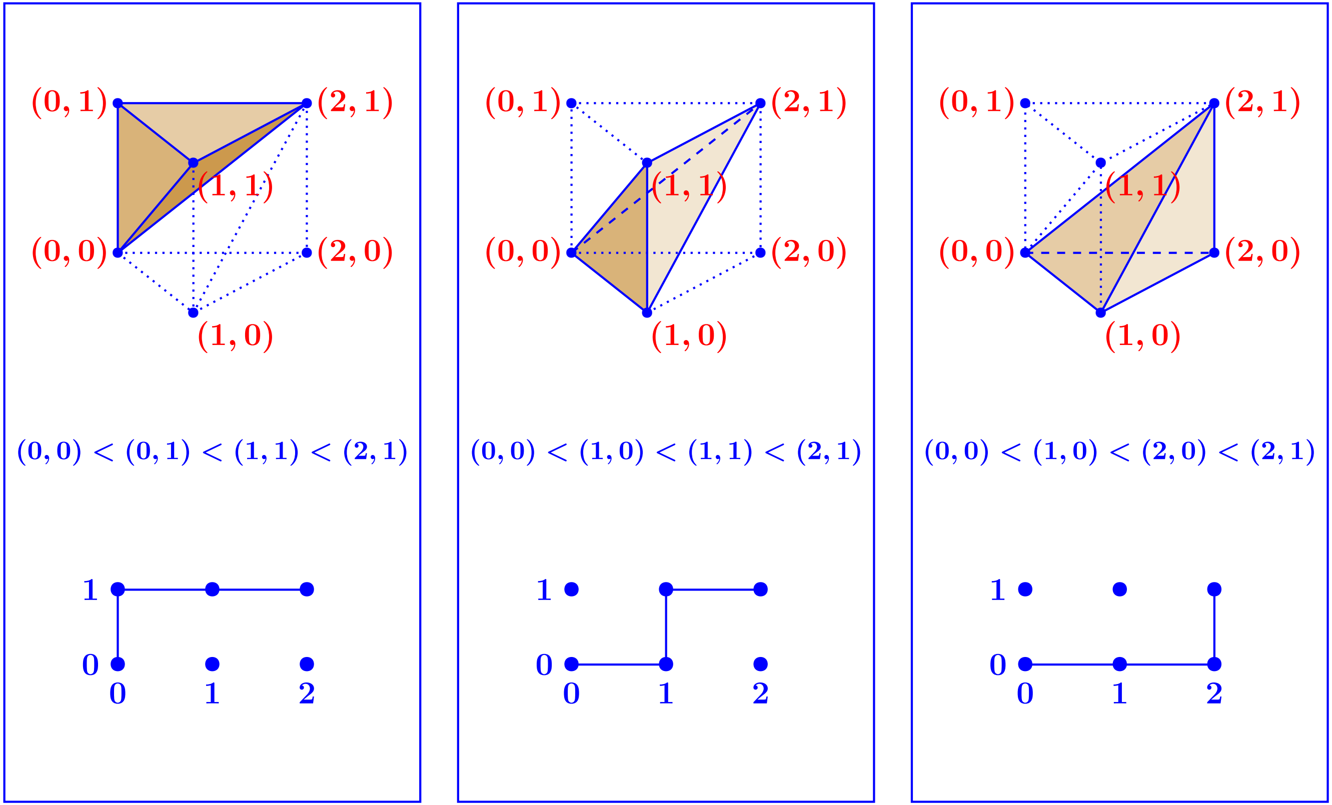
$(0,0) < (1,0) < (1,1)$

Three $\Delta^3$ in $\Delta^2 \times \Delta^1$: $(0,0) < (0,1) < (1,1) < (2,1)$

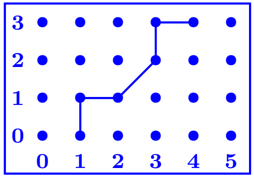
$(0,0) < (1,0) < (1,1) < (2,1)$

$(0,0) < (1,0) < (2,0) < (2,1)$

# Rewriting the triangulation of $\Delta^2 \times \Delta^1$.

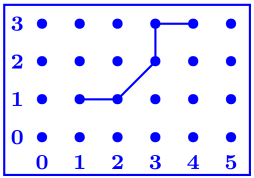


# Planar representations of simplices of $\Delta^p \times \Delta^q$:

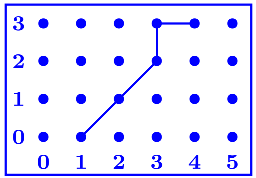
Example of 5-simplex : 

$= \sigma \in (\Delta^5 \times \Delta^3)_5$

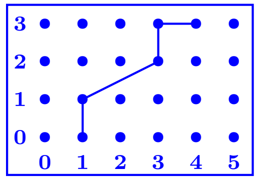
$\Rightarrow$ 6 faces:



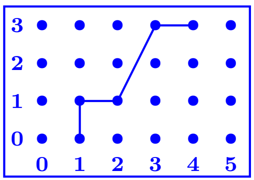

$\partial_0\sigma$



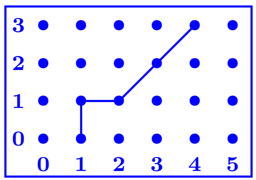

$\partial_1\sigma$



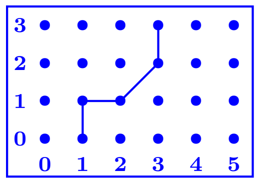

$\partial_2\sigma$




$\partial_3\sigma$




$\partial_4\sigma$




$\partial_5\sigma$

**Plan.**

- ✓ • **Introduction.**
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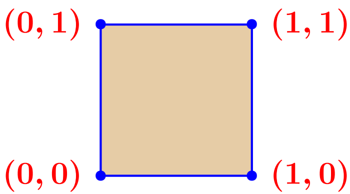
- ✓ • **Seeing complicated products.**
- $\rightarrow$ • **The Eilenberg-Zilber vector field.**
- • **The Eilenberg-Zilber vector field is natural.**
- • **EZ vector field $\Rightarrow$ EZ reduction.**

# The Eilenberg-Zilber vector field.

**Eilenberg-Zilber problem** for $\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$:

**Cubical** version of $\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$:



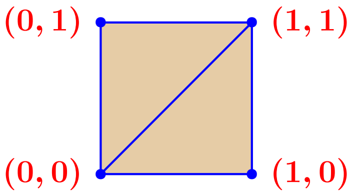

$$C_0 \longleftarrow C_1 \longleftarrow C_2$$

$$\mathbb{Z}^4 \longleftarrow \mathbb{Z}^4 \longleftarrow \mathbb{Z}$$

$$C_*\boldsymbol{\Delta}^1 \otimes C_*\boldsymbol{\Delta}^1$$

$$C_0\boldsymbol{\Delta}^1 \otimes C_0\boldsymbol{\Delta}^1 \longleftarrow (C_0\boldsymbol{\Delta}^1 \otimes C_1\boldsymbol{\Delta}^1) \oplus (C_1\boldsymbol{\Delta}^1 \otimes C_0\boldsymbol{\Delta}^1) \longleftarrow C_1\boldsymbol{\Delta}^1 \otimes C_1\boldsymbol{\Delta}_1$$

**Simplicial version of $\Delta^1 \times \Delta^1$:**




$$C'_0 \longleftarrow C'_1 \longleftarrow C'_2$$

$$\mathbb{Z}^4 \longleftarrow \mathbb{Z}^5 \longleftarrow \mathbb{Z}^2$$

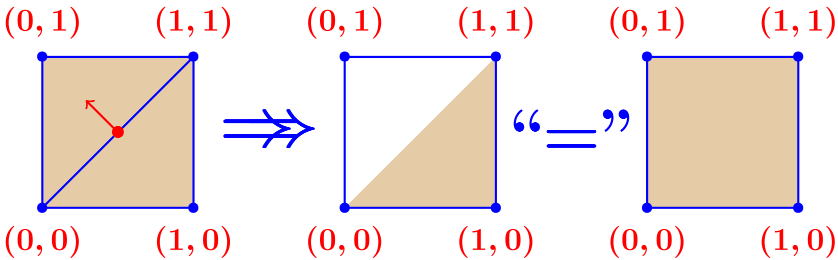
**To be compared with:**

$$C_0 \longleftarrow C_1 \longleftarrow C_2$$

$$\mathbb{Z}^4 \longleftarrow \mathbb{Z}^4 \longleftarrow \mathbb{Z}^1$$

**Difference = a vector field with a unique "vector"**

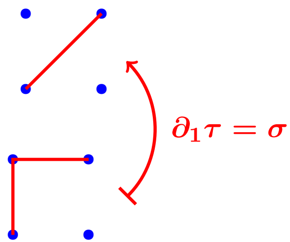
## Eilenberg-Zilber reduction as induced by a vector field:




## Translation in planar diagrams: $V = \{(\sigma, \tau)\}$

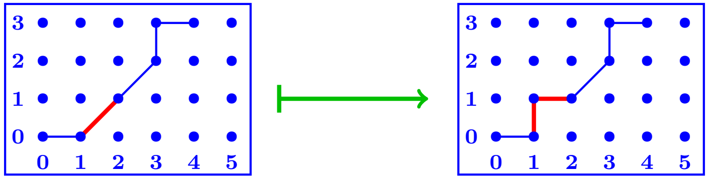
$\sigma$ = source cell = edge$[(0,0),(1,1)]$ =

$\tau$ = target cell = triangle$[(0,0),(0,1),(1,1)]$ =

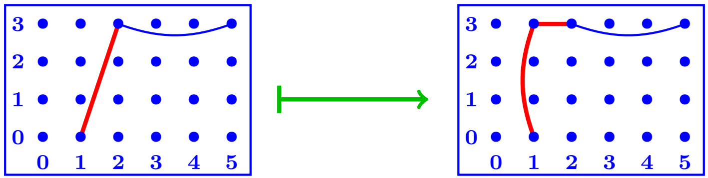


# Generalizing the idea $\Rightarrow$

## Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

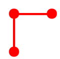



Source $= \partial_2($ Target$)$




Source $= \partial_1($ Target$)$

Recipe: First "event" = Diagonal step =  $\Rightarrow$ Source cell.

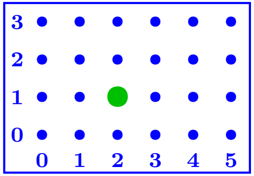
= (-90°)-bend =  $\Rightarrow$ Target cell.

# Critical cells ??

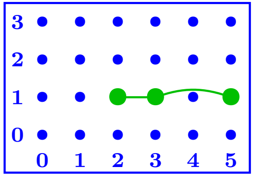
**Critical cell = cell without any "event"**

**= without any diagonal or −90°-bend.**

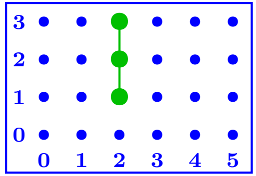
## Examples.



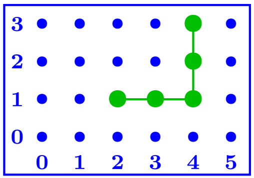

$\Delta^0_2 \otimes \Delta^0_1$



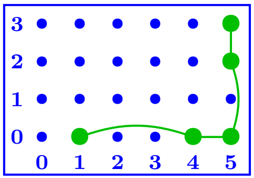

$\Delta^2_{2,3,5} \otimes \Delta^0_1$



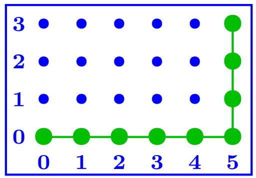

$\Delta^0_2 \otimes \Delta^2_{1,2,3}$




$\Delta^2_{2,3,4} \otimes \Delta^2_{1,2,3}$




$\Delta^2_{1,4,5} \otimes \Delta^2_{0,2,3}$




$\Delta^5_{0,1,2,3,4,5} \otimes \Delta^3_{0,1,2,3}$

**Conclusion:**

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

**Fundamental theorem of vector fields** $\Rightarrow$

**Canonical Homological Reductions:**

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

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# The Eilenberg-Zilber vector field is natural.

Naturality of the Eilenberg-Zilber reduction

with respect to product morphisms.

Standard methods $\Rightarrow$ $(\Delta^p \rightarrow \Delta^{p'}) \times (\Delta^q \rightarrow \Delta^{q'})$ is enough.

**Given:** $\begin{array}{l} \phi : \Delta^p \rightarrow \Delta^{p'} \\ \psi : \Delta^q \rightarrow \Delta^{q'} \end{array}$ simplicial.

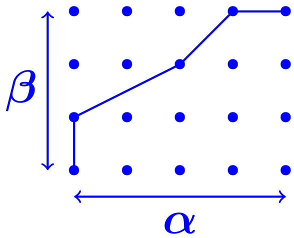
**Prove:**

$$\begin{array}{ccccc} h \circlearrowright & C_*(\Delta^p \times \Delta^q) & \xrightarrow{\phi \times \psi} & C_*(\Delta^{p'} \times \Delta^{q'}) & \circlearrowleft h' \\ & f \downarrow\uparrow g & \boxed{\text{com ?}} & f' \downarrow\uparrow g' & \\ & C_*\Delta^p \otimes C_*\Delta^q & \xrightarrow{\phi \otimes \psi} & C_*\Delta^{p'} \otimes C_*\Delta^{q'} & \end{array}$$

is commutative ??

**Proof:**

**Representation of a simplex of $\Delta^p \times \Delta^q$ via an s-path.**




**= subsimplex of $\alpha \times \beta \subset (\Delta^p \times \Delta^q)_4$**

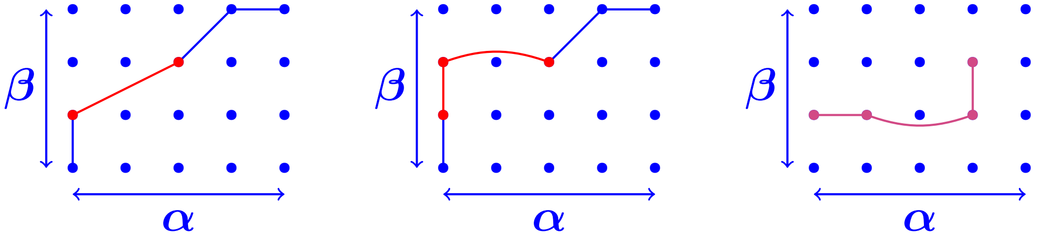
**spanned by the vertices (0,0) – (0,1)– (2,2) – (3,3) – (4,3).**

**The game first event "diagonal**  **"**

**or "right-angle bend**  **"**

**determines the nature source, target or critical.**

**Examples:**




source $= \sigma$ target $= \tau$ critical $= \chi$

**Here:**

$$\partial_2(\tau) = \sigma$$

$$\Rightarrow \quad v(\sigma) = \tau$$

**Two** maps $\left| \begin{array}{l} \phi : \Delta^p \to \Delta^{p'} \\ \psi : \Delta^q \to \Delta^{q'} \end{array} \right|$ = simplicial morphisms.

**<u>Claim</u>:**

$\tau$ target cell in $\Delta^p \times \Delta^q \Rightarrow$

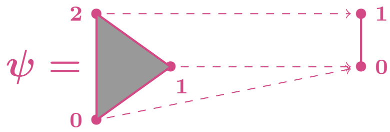
$(\phi \times \psi)(\tau)$ target or degenerate cell in $\Delta^{p'} \times \Delta^{q'}$

$\chi$ critical cell in $\Delta^p \times \Delta^q \Rightarrow$

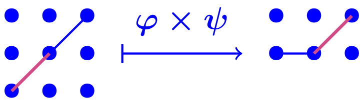
$(\phi \times \psi)(\chi)$ critical or degenerate cell in $\Delta^{p'} \times \Delta^{q'}$

**Typical accidents with source cells.**

$\alpha = \mathrm{id} : \Delta^2 \to \Delta^2$ **and** $\psi : \Delta^2 \to \Delta^1$ **as below:**



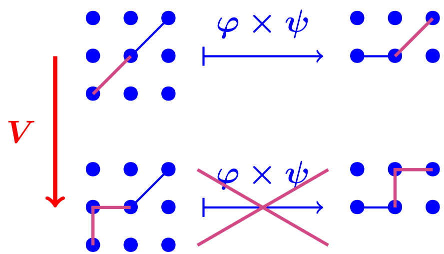

**1)**



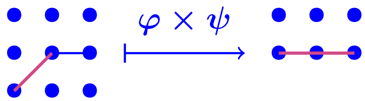

**Then** $(\phi \times \psi)(\text{source}) = \text{source}$

**but for reasons which do not match!**

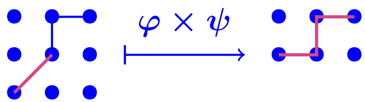
**Compare corresponding target cells.**



**2) The image of a source cell can be a critical cell:**

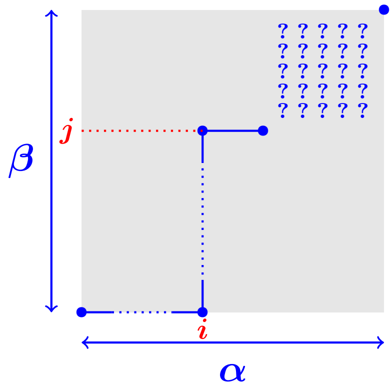



**or a target cell:**




**But we don't care about source cells!**

**General shape of an Eilenberg-Zilber target cell:**




$= \tau$

$$(\phi \times \psi)(\alpha \times \beta) = (\eta\alpha' \times \theta\beta')$$

**If no index of $\eta$ is $< i + 1$ and no index of $\theta$ is $< j$,**

**then $(\phi \times \psi)(\tau)$ has the same shape and therefore is a target cell**

**(or can be degenerate),**

**otherwise $(\phi \times \psi)(\tau)$ is degenerate.**

**Same study for critical cells (easier) $\Rightarrow$ OK.**

**Finally: $(\phi \times \psi)$(target cell) = target cell or 0.**

**$(\phi \times \psi)$(critical cell) = critical cell or 0.**

**$\Rightarrow \phi \times \psi$ is a vectorious morphism.**

$$\Rightarrow \qquad \begin{array}{ccccc} h \circlearrowright & C_*(\Delta^p \times \Delta^q) & \xrightarrow{\phi \times \psi} & C_*(\Delta^{p'} \times \Delta^{q'}) & \circlearrowleft h' \\ & f \downarrow \uparrow g & \boxed{\text{com. OK!}} & f' \downarrow \uparrow g' & \\ & C_*\Delta^p \otimes C_*\Delta^q & \xrightarrow{\phi \otimes \psi} & C_*\Delta^{p'} \otimes C_*\Delta^{q'} & \end{array}$$

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- → • **EZ vector field ⇒ EZ reduction.**

## EZ vector field $\Rightarrow$ EZ reduction.

**Theorem**: The Eilenberg-Zilber vector field

previously described

gives the standard Eilenberg-Zilber reduction.

**Standard Eilenberg-Zilber reduction:**

$$EZ : \boxed{{\scriptstyle RM} \circlearrowleft C_*(X \times Y) \underset{AW}{\overset{EML}{\leftrightarrows}} C_*(X) \otimes C_*(Y)}$$

$AW$ = Alexander-Whitney

$EML$ = Eilenberg-MacLane

$RM$ = Rubio-Morace

$\boldsymbol{EZ} = \boldsymbol{AW} + \boldsymbol{EML} + \boldsymbol{RM}$:

$$\boldsymbol{AW}(x_p \times y_p) = \sum_{i=0}^{p} \partial_{p-i+1} \cdots \partial_p x_p \otimes \partial_0 \cdots \partial_{p-i-1} y_p$$

$$\boldsymbol{EML}(x_p \otimes y_q) = \sum_{(\eta,\eta') \in \mathrm{Sh}(p,q)} \varepsilon(\eta, \eta') \, (\eta' x_p \times \eta y_q)$$

$$\boldsymbol{RM}(x_p \times y_p) = \sum_{\substack{0 \le r \le p-1 \\ 0 \le s \le p-r-1 \\ (\eta,\eta') \in \mathrm{Sh}(s+1,r)}} (-1)^{p-r-s} \varepsilon(\eta, \eta') \ldots$$

$$\ldots (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_p x_p \times \ldots$$

$$\ldots \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

**Plan of Proof:**

1. Prove $h_V = RM$.

2. Use the dependency $h \mapsto (f, g, d^c)$ to prove:

$$\begin{aligned} f_V &= AW \\ g_V &= EML \\ (C^c, d^c) &= (C_*X \otimes C_*Y, d^{\otimes}) \end{aligned}$$

**Proof of $h_V = RM$:**

1. Prove $RM = 0 = h_V$ for target and critical cells.

2. Prove $RM$ satisfies the same recursive formula as $h_V$:

$$RM(\sigma) = v(\sigma) - RM(dv(\sigma) - \sigma) \qquad .$$

**Reminders:**

**1. Commutation relations face $\leftrightarrow$ degeneracy operators:**

$$\begin{aligned} \partial_i \eta_j &= \eta_{j-1}\partial_i \quad \text{for } i < j \\ &= \text{id} \quad \text{for } i = j \text{ or } j+1 \\ &= \eta_j \partial_{i-1} \quad \text{for } i \geq j+2 \end{aligned}$$

**2. Canonical form of a degenerate simplex:**

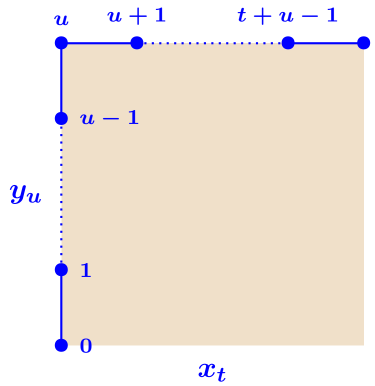
$$\sigma = \eta_{i_{k-1}} \cdots \eta_{i_0} \sigma' \quad \text{with } i_{k-1} > \cdots > i_0.$$

**3. A product simplex in canonical form:**

$$\sigma = (\eta_{i_{k-1}} \cdots \eta_{i_0} x \ , \ \eta_{j_{\ell-1}} \cdots \eta_{j_0} y)$$

**is non-degenerate iff** $\{i_0, \cdots, i_{k-1}\} \cap \{j_0, \ldots, j_{\ell-1}\} = \emptyset$.

## Example 1: Case of the target cell:




$$\sigma = (\eta_{u-1} \cdots \eta_0\ x_t\ ,\ \eta_{t+u-1} \cdots \eta_u\ y_u)$$

Generic $RM$-term (sign omitted):

$$(\uparrow^{p-r-s}(\eta')\ \eta_{p-r-s-1}\ \partial_{p-r+1} \cdots \partial_p\ \eta_{u-1} \cdots \eta_0\ x_t\ , \ldots$$
$$\ldots, \uparrow^{p-r-s}(\eta)\ \partial_{p-r-s} \cdots \partial_{p-r-1}\ \eta_{t+u-1} \cdots \eta_u\ y_u)$$

$(\eta, \eta') \in \mathrm{Sh}(s+1, r) \ \Rightarrow$

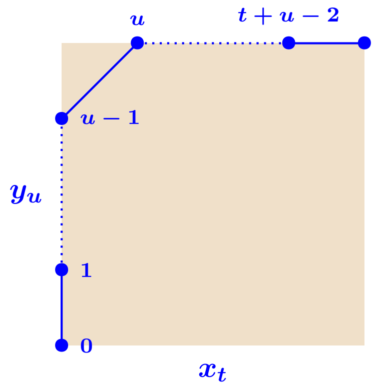
$$\#(\eta) \geq r + 1 + (u - r) + s + 1 + (t - s) = t + u + 2$$

$\Rightarrow$ too many degeneracy operators $\Rightarrow$ collision $\Rightarrow$ $RM(\sigma) = 0$

$\Rightarrow$ OK !

(General case of a target cell combinatorially more complicated but analogous)

## Example 2: Case of the source cell:




$$\sigma = (\eta_{u-2} \cdots \eta_0 x_t \;,\; \eta_{t+u-2} \cdots \eta_u y_u)$$

($\eta_{u-1}$ absent)

Generic *RM*-term (sign omitted):

$$(\uparrow^{p-r-s}(\eta')\ \eta_{p-r-s-1}\ \partial_{p-r+1} \cdots \partial_p\ \eta_{u-2} \cdots \eta_0\ x_t\ , \ldots$$
$$\ldots,\ \uparrow^{p-r-s}(\eta)\ \partial_{p-r-s} \cdots \boxed{\partial_{p-r-1}\ \eta_{t+u-2}} \cdots \eta_u\ y_u)$$

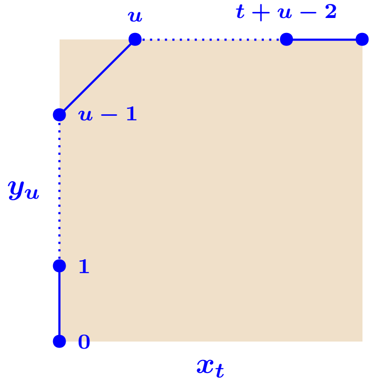
Same computation about $\#(\eta) \Rightarrow \#(\eta) \geq t+u$

$\Rightarrow$ no collision only if perfect simplifications between $\partial$'s and $\eta$'s.

$\Rightarrow$ in particular $\partial_{p-r-1}\eta_{t+u-2}$ must simplify +

$p =$dimension $= t+u-1 \;\Rightarrow r = 0$.

## Case of the source cell:




$$\sigma = (\eta_{u-2} \cdots \eta_0 x_t \ , \ \eta_{t+u-2} \cdots \eta_u y_u)$$

($\eta_{u-1}$ absent)

$r = 0$ + simplifications in the second factor ⇒ Generic $RM$-term:

$$(\boxed{\eta_{p-s-1}}\ \eta_{u-2} \cdots \eta_0\ x_t \ , \ \uparrow^{p-s}(\eta)\ \boxed{\eta_{t+u-s-2}} \cdots \eta_u\ y_u)$$

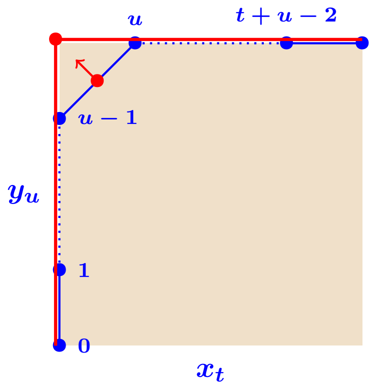
$p = t + u - 1 \quad \Rightarrow \quad p - s - 1 = t + u - s - 2 \quad \Rightarrow$ collision

except if $\quad t + u - s - 2 = u - 1 \quad \Leftrightarrow \quad s = t - 1$.

Finally only one remaining term:

$$(\boxed{\eta_{u-1}}\eta_{u-1} \cdots \eta_0\ x_t \ , \ \eta_{t+u-1} \cdots \eta_u\ y_u)$$

## Case of the source cell:




$$\sigma = (\eta_{u-2} \cdots \eta_0 x_t \; , \; \eta_{t+u-2} \cdots \eta_u y_u)$$

($\eta_{u-1}$ absent)

$r = 0$ + simplifications in the second factor ⇒ Generic *RM*-term:

$$(\boxed{\eta_{p-s-1}}\, \eta_{u-2} \cdots \eta_0 \; x_t \; , \; \uparrow^{p-s}(\eta)\; \boxed{\eta_{t+u-s-2}} \cdots \eta_u \; y_u)$$

$p = t + u - 1 \quad \Rightarrow \quad p - s - 1 = t + u - s - 2 \quad \Rightarrow$ collision

except if $\quad t + u - s - 2 = u - 1 \quad \Leftrightarrow \quad s = t - 1$.

Finally only one remaining term:

$$(\boxed{\eta_{u-1}}\, \eta_{u-1} \cdots \eta_0 \; x_t \; , \; \eta_{t+u-1} \cdots \eta_u \; y_u)$$

= previous target cell.

**More complex but analogous calculations $\Rightarrow$**

**$\boldsymbol{RM}$ satisfies the same recursive formula**

**as the Eilenberg-Zilber vector field.**

**+ Dependency $\boldsymbol{h \mapsto (f, g, d^c)}$ $\Rightarrow$ QED.**

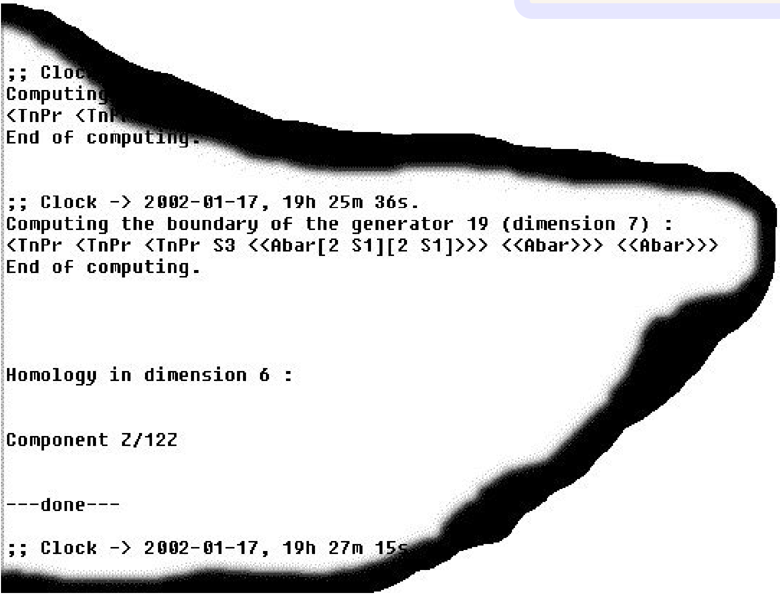
**The Eilenberg-Zilber vector field is the key point**

**to obtain a very efficient algorithm**

**computing the effective homology**

**of the Eilenberg-MacLane spaces.**

# The END

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