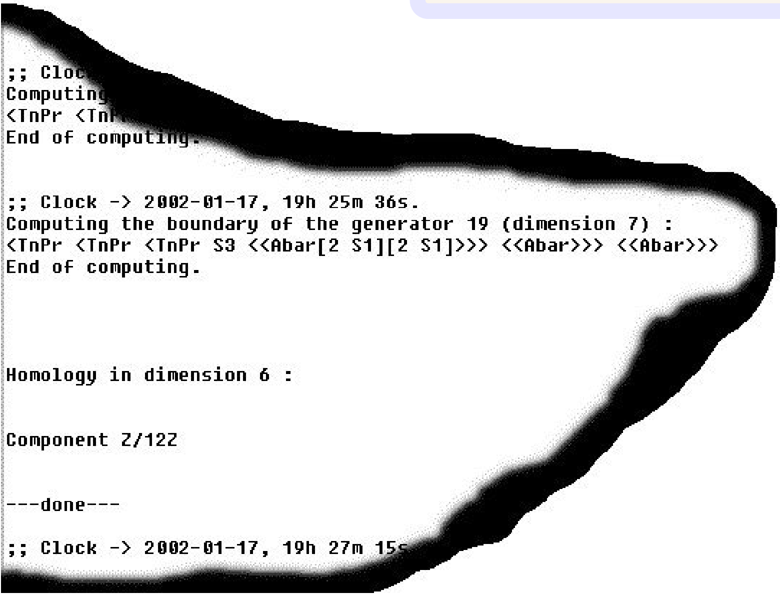
The Eilenberg-Zilber

Combinatorial Reduction



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Noon Seminar, ETH Zurich, June 14, 2012

Plan.

1. **Homological Reduction.**
2. **Homotopy and Vector Fields.**
3. **Discrete Vector Fields.**
4. **Eilenberg-Zilber and Vector Fields.**

1/4. Homological Reduction.

Standard Algebraic Topology:

$$\text{Topological object} \longmapsto \text{Algebraic object}$$

$$\text{Topological Object} \longmapsto \text{Chain Complex} \longmapsto \text{Homology group}$$

Chain complex:

$$\cdots \xleftarrow{d_{p-2}} C_{p-2} \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \xleftarrow{d_{p+2}} \cdots$$

$$d_{p-2}d_{p-1} = 0 \quad d_{p-1}d_p = 0 \quad d_p d_{p+1} = 0 \quad d_{p+2}d_{p+1} = 0$$

$$dd = 0 \quad \Leftrightarrow \quad \ker d_p \supset \operatorname{im} d_{p+1} \quad \Rightarrow \quad H_p := \frac{\ker d_p}{\operatorname{im} d_{p+1}}$$

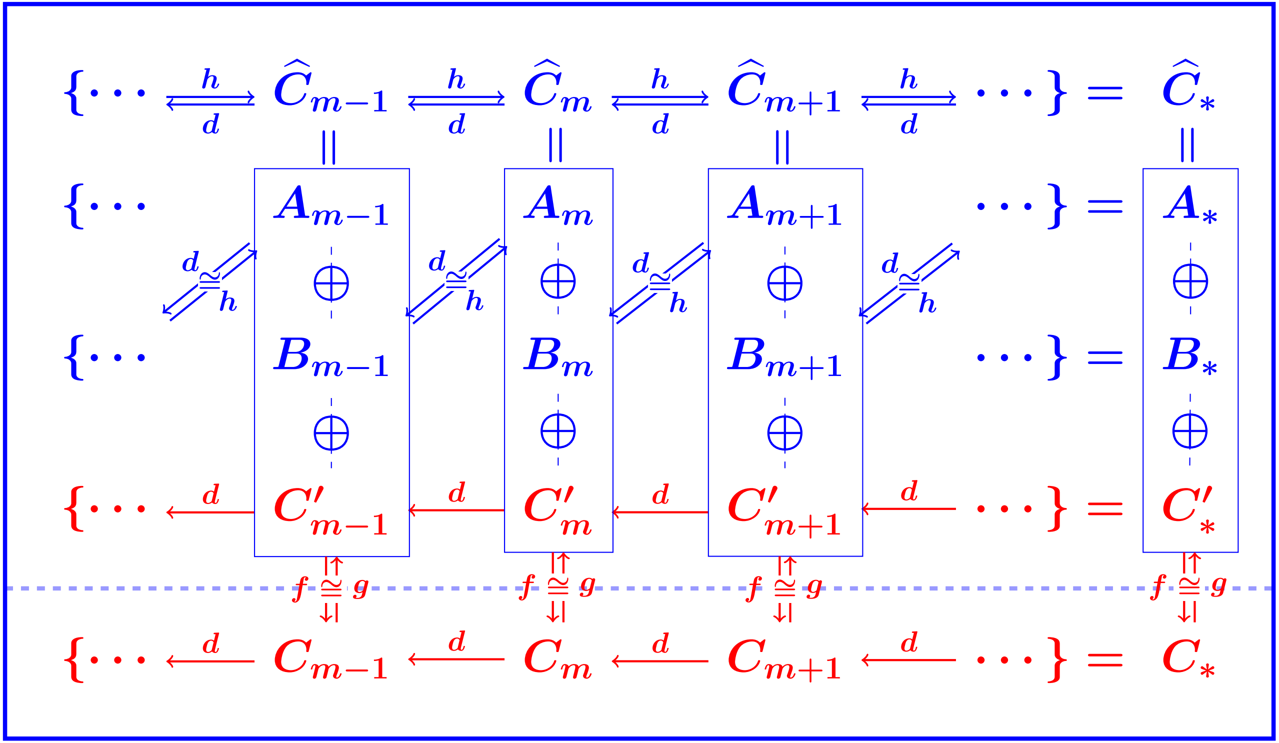
Definition: A (homological) reduction is a diagram:

$$\rho = (f, g, h) = \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$$

with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree $+1$).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.

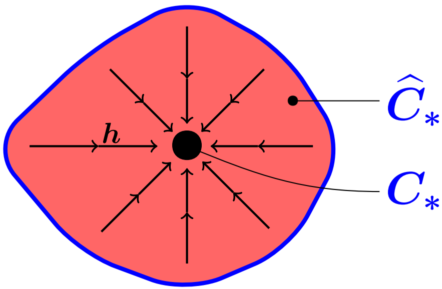
Meaning = Reduction Diagram:



In $\rho = (f, g, h) = \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$,

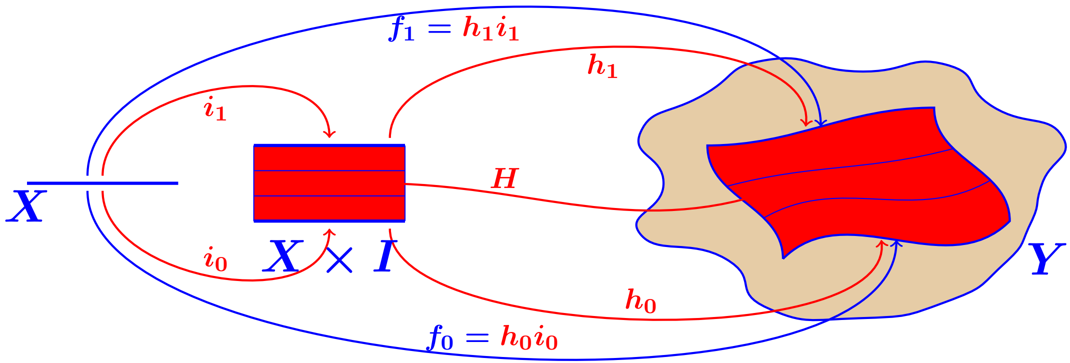
the most important component is h**.**

To be understood as a reduction $\widehat{C}_* \Rrightarrow C_*$**.**



The homotopy h **reduces the "cherry"** $\widehat{C}_*$

onto the "stone" C_***.**



General notion of homotopy.

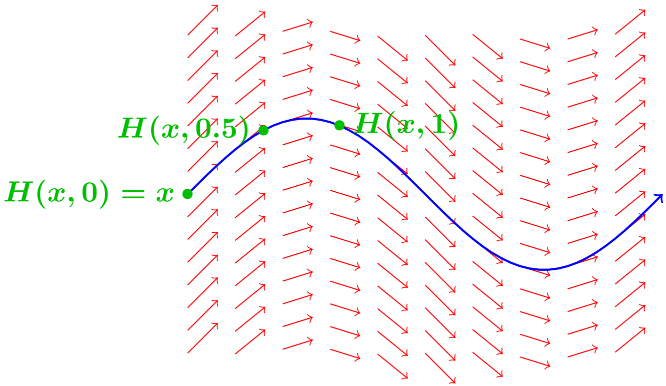
Given: $f_0, f_1 : X \to Y$.

Definition: f_0 **homotope to** f_1 $\Leftrightarrow$

$\exists H : X \times I \to Y$ **satisfying:**

$$f_0(x) = H(x, 0)$$
$$f_1(x) = H(x, 1)$$

2/4. Vector field ⇒ Flow ⇒ Homotopy.



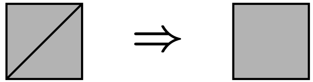
Vector field $= V : x \mapsto V(x) \in T_x X$.

⇒ Corresponding flow $\Phi(x,t)$:

$$\text{solution of } \left\{ \frac{\partial \Phi(x,t)}{\partial t} = V(\Phi(x,t)) \quad + \quad \Phi(x,0) = x \right\}.$$

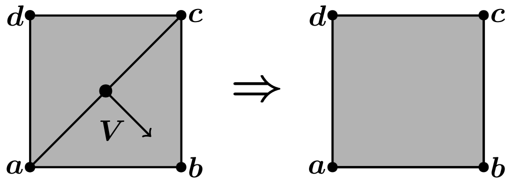
⇒ Auto-Homotopy $H(x,t) := \Phi(x,t)$ for $t \in [0,1]$.

3/4. Corresponding combinatorial notion ??

Example of reduction: 

Appropriate discrete vector field

on a triangulation of the square:



V **contracts** $abc \Rightarrow ab||bc$;

V **inflates the triangle** dac **to the square** $abcd$.

C = cellular complex = collection of cells satisfying...

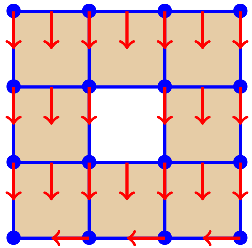
Definition: A discrete vector field

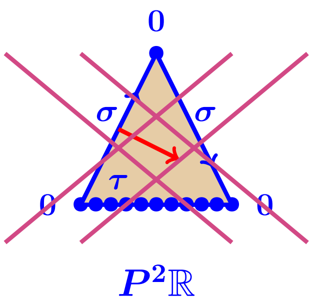
is a set of pairs of cells $V = \{(\sigma_i, \tau_i)_{i \in v}\}$

satisfying:

- $(\forall i)$ σ_i is regular face of τ_i.
- $(\forall i \neq j)$ $\sigma_i \neq \sigma_j \neq \tau_i \neq \tau_j$.

Examples:





σ_i regular face of $\tau_i \Leftrightarrow$ incidence number $\varepsilon(\sigma_i, \tau_i) = \pm 1$.

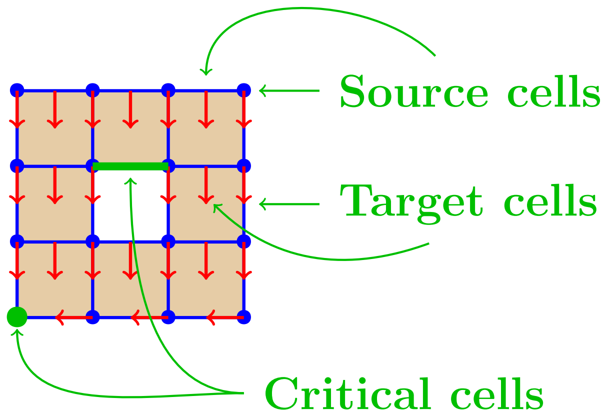
C = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)_{i \in v}\}$ = Vector field.

<u>Definition</u>: A critical p-cell is a p-cell which does not occur in V.

Other cells divided in source cells and target cells.

<u>Example</u>:



C = Cellular complex.

$V = \{(\sigma_i, \tau_i)_{i \in v}\}$ = vector field.

<u>Theorem</u>: A "critical" chain complex $C_*^c = ((\beta_p^c)_{p \in \mathbb{Z}}, d^c)$

can be constructed:

- β_p^c = the set of critical p-cells of V.
- $d_p^c : \mathbb{Z}[\beta_p^c] \to \mathbb{Z}[\beta_{p-1}^c]$

  an appropriate "critical" differential

  deduced from the initial differential d

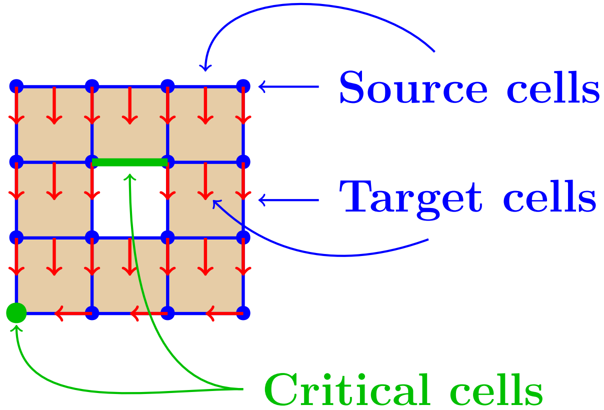
  and the vector field V.

Also a canonical reduction $\rho : C_* \Rightarrow\!\!\!\!\!\!\Rightarrow C_*^c$ is provided.

$\Rightarrow$ Any homological problem in C_* can be solved in C_*^c.

Simple example.

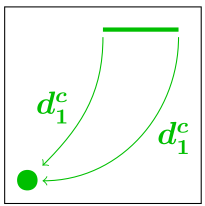
$C =$



$$C_* = \{0 \leftarrow \mathbb{Z}^{16} \leftarrow \mathbb{Z}^{24} \leftarrow \mathbb{Z}^{8} \leftarrow 0\}$$

Theorem $\Rightarrow$

$$\rho : C_* \Rrightarrow C^c_* =$$



$$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \text{Circle}$$

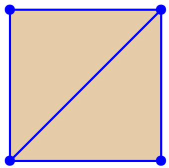
4/4. Eilenberg-Zilber Vector Field.

Vector Fields ⇒ Eilenberg-Zilber

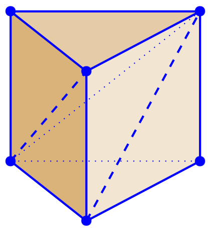
⇒ Twisted Eilenberg-Zilber ⇒ Serre spectral sequence

⇒ Eilenberg-Moore spectral sequence.

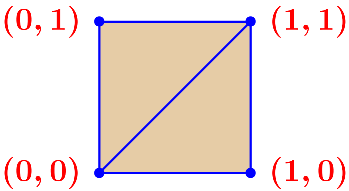
Main problem: Triangulation of $\Delta^p \times \Delta^q$???



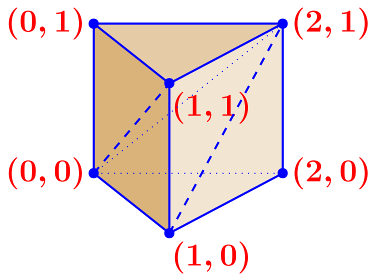
$\Delta^1 \times \Delta^1$



$\Delta^2 \times \Delta^1$



$\Delta^1 \times \Delta^1$



$\Delta^2 \times \Delta^1$

Two Δ^2 in $\Delta^1 \times \Delta^1$: $(0,0) < (0,1) < (1,1)$

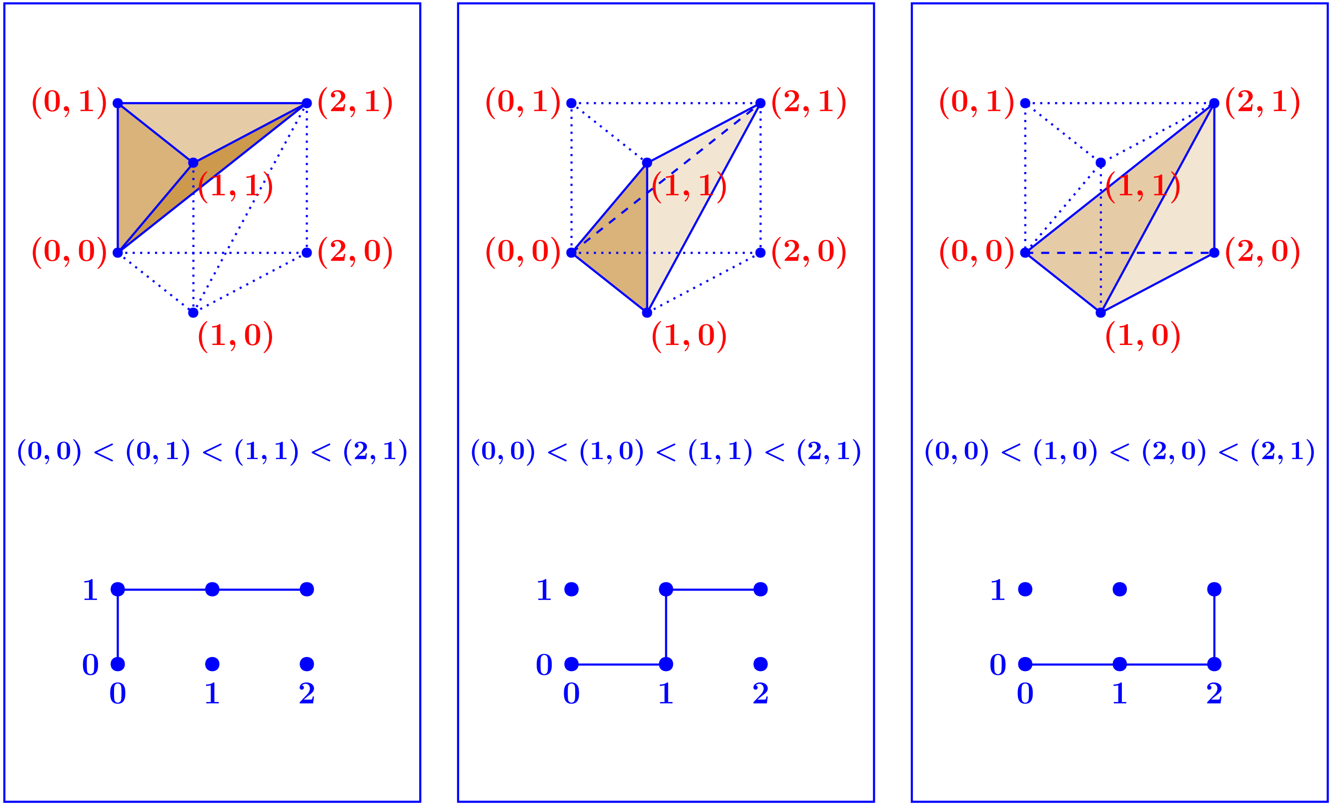
$(0,0) < (1,0) < (1,1)$

Three Δ^3 in $\Delta^2 \times \Delta^1$: $(0,0) < (0,1) < (1,1) < (2,1)$

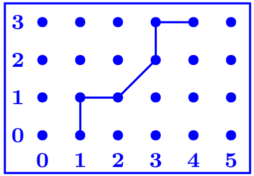
$(0,0) < (1,0) < (1,1) < (2,1)$

$(0,0) < (1,0) < (2,0) < (2,1)$

Rewriting the triangulation of $\Delta^2 \times \Delta^1$.

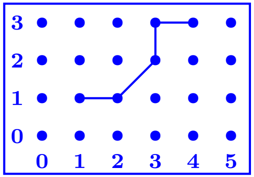


“Seeing” the triangulation of $\Delta^5 \times \Delta^3$.

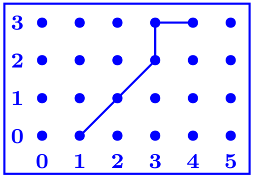
Example of 5-simplex : 

$= \sigma \in (\Delta^5 \times \Delta^3)_5$

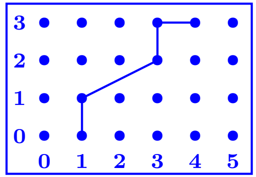
$\Rightarrow$ 6 faces:



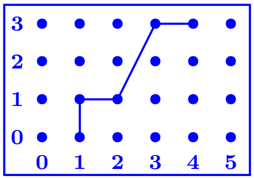
$\partial_0\sigma$



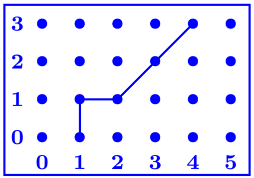
$\partial_1\sigma$



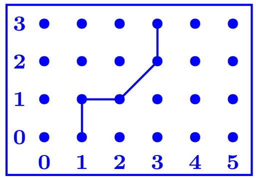
$\partial_2\sigma$



$\partial_3\sigma$

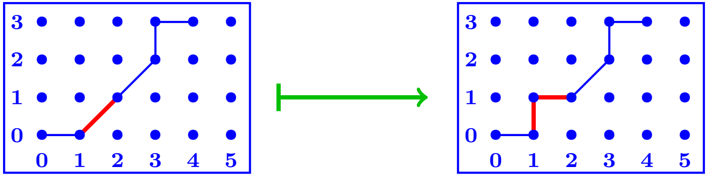


$\partial_4\sigma$

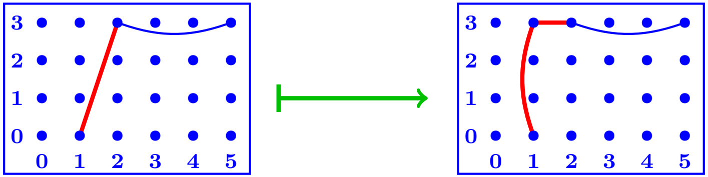


$\partial_5\sigma$

⇒ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

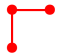


Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

Recipe: First "event" = Diagonal step =  ⇒ Source cell.

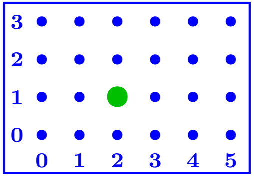
= (-90°)-bend =  ⇒ Target cell.

Critical cells ??

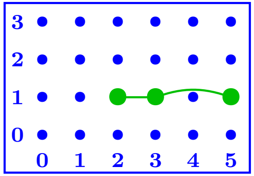
Critical cell = cell without any "event"

= without any diagonal or −90°-bend.

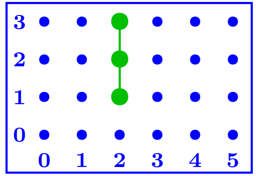
Examples.



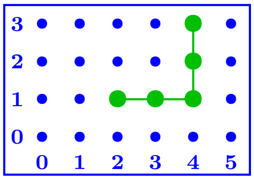
$\Delta^0_2 \times \Delta^0_1$



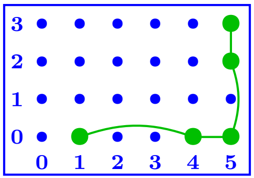
$\Delta^2_{2,3,5} \times \Delta^0_1$



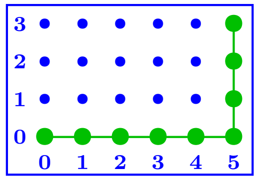
$\Delta^0_2 \times \Delta^2_{1,2,3}$



$\Delta^2_{2,3,4} \times \Delta^2_{1,2,3}$



$\Delta^2_{1,4,5} \times \Delta^2_{0,2,3}$



$\Delta^5_{0,1,2,3,4,5} \times \Delta^3_{0,1,2,3}$

Conclusion:

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

Fundamental theorem of vector fields $\Rightarrow$

Canonical Homological Reductions:

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

More generally: X and Y = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$.

Eilenberg-Zilber Theorem: Canon. homological reduction:

$$\rho_{EZ} : C_*(X \times Y) \Rrightarrow C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

Analogous result for the Eilenberg-Moore spectral sequence.

Key results:

G = Simplicial group $\Rightarrow$ BG = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

X = Simplicial set $\Rightarrow$ KX = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

Analogous process $\Rightarrow$ Algorithms:

$$(G, C_*G, EC_*^G, \varepsilon_G) \;\mapsto\; (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(G, C_*X, EC_*^X, \varepsilon_X) \;\mapsto\; (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

Example: $\boldsymbol{\pi_5(\Omega S^3 \cup_2 D^3) = (\mathbb{Z}/2)^4}$.

$$\#[S^5, \mathrm{Cont}(S^1, S^3) \cup_2 D^3] = 16$$

Computing plan: Five fibrations (twisted products):

$$X_2 \;=\; \Omega S^3 \cup_2 D^3$$

$$\begin{array}{rll} K(\mathbb{Z}/2,1) \to X_3 \to X_2 & \qquad & \pi_2(X_2) = \mathbb{Z}/2 \\ K(\mathbb{Z}/2,2) \to X_4 \to X_3 & & \pi_3(X_3) = \mathbb{Z}/2 \\ K(\mathbb{Z},3) \to X_4' \to X_4 & & \pi_4(X_4) = \mathbb{Z}/4 \oplus \mathbb{Z} \\ K(\mathbb{Z}/2,3) \to X_4'' \to X_4' & & \pi_4(X_4') = \mathbb{Z}/4 \\ K(\mathbb{Z}/2,3) \to X_5 \to X_4'' & & \pi_4(X_4'') = \mathbb{Z}/2 \end{array}$$

Applying five times our effective version

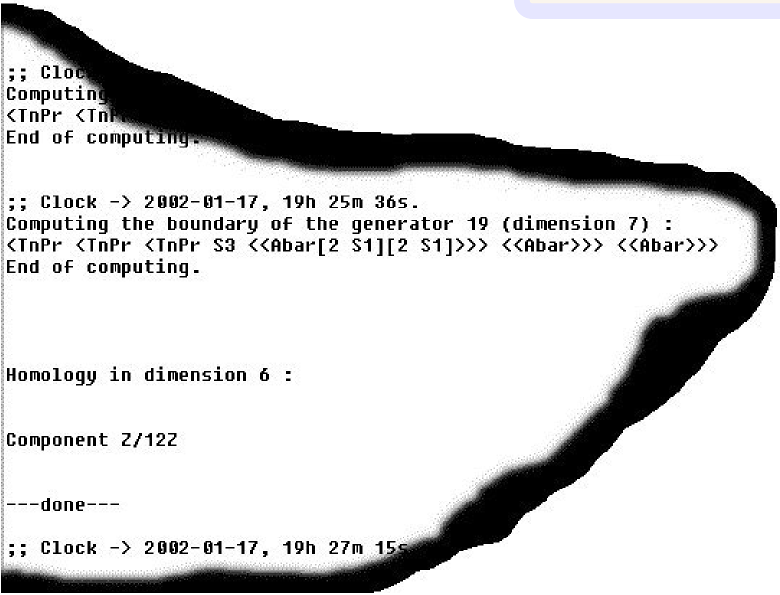
of the Serre spectral sequence and

five times the effective version

of the Eilenberg-Moore spectral sequence

$$\Rightarrow \quad \pi_5(X_2) = \pi_5(X_5) = H_5(X_5) = (\mathbb{Z}/2)^4.$$

The END



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