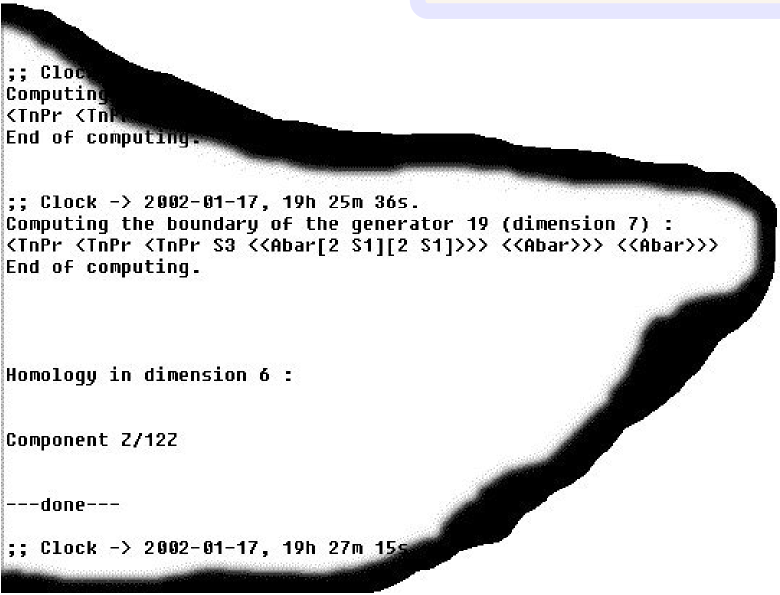
Homological Perturbation Theorem

and

Eilenberg-Zilber vector field



Ana Romero, Universidad de La Rioja
Francis Sergeraert, Institut Fourier, Grenoble

ETH Zurich, June 2012

R = Unitary ring

$\varepsilon, \varphi, \psi, \beta \in R$ with ε invertible.

Gauss discussion of (1) + (2):

$$(1) \qquad \varepsilon x + \varphi y = a$$

$$(2) \qquad \psi x + \beta y = b$$

$(2) - \psi\varepsilon^{-1}(1) \;\Rightarrow$

$$(2') \qquad (\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a)$$

$\Rightarrow$ (1) + (2) has a solution $\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; y = \cdots$$

$$\Rightarrow\; x = \varepsilon^{-1}a - \varepsilon^{-1}\varphi y$$

Matrix translation:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Leftrightarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} x + \varepsilon^{-1}\varphi y \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}}_{\begin{pmatrix} a \\ -\psi\varepsilon^{-1}a + b \end{pmatrix}}$$

$\Leftrightarrow$

$$\varepsilon(x + \varepsilon^{-1}\varphi\, y) = a$$

$$(\beta - \psi\varepsilon^{-1}\varphi)\, y = (b - \psi\varepsilon^{-1}a)$$

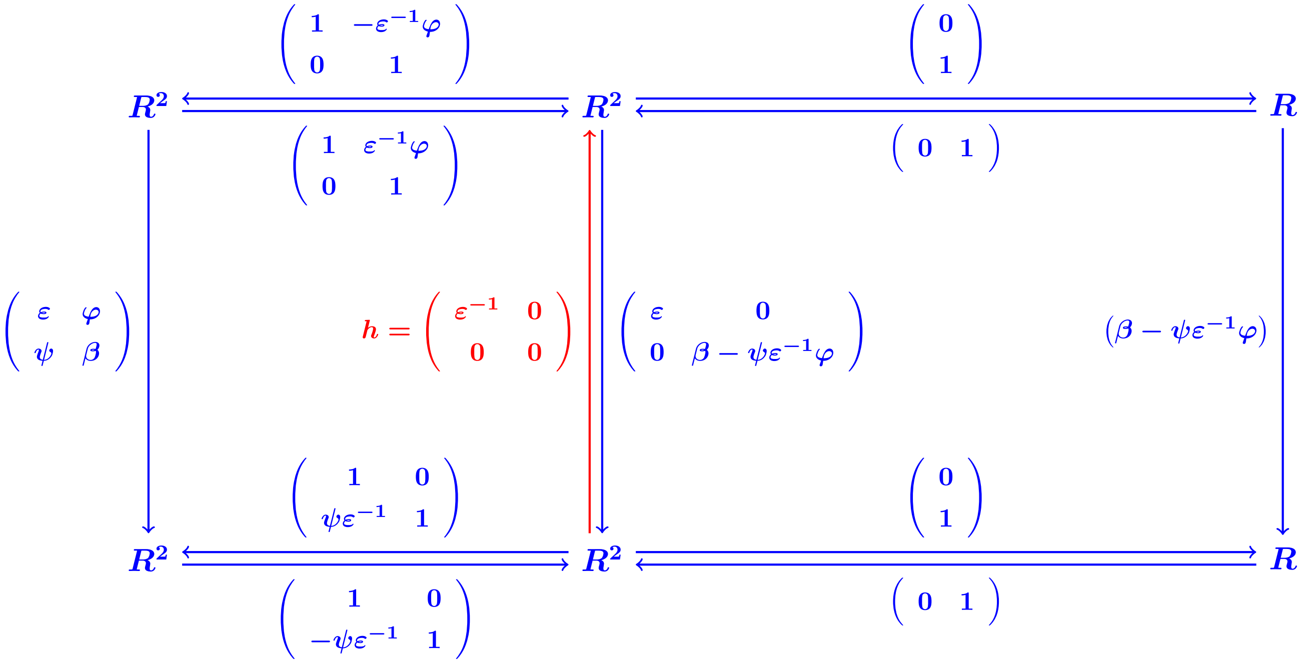
$\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \;\Rightarrow\; \ldots$$

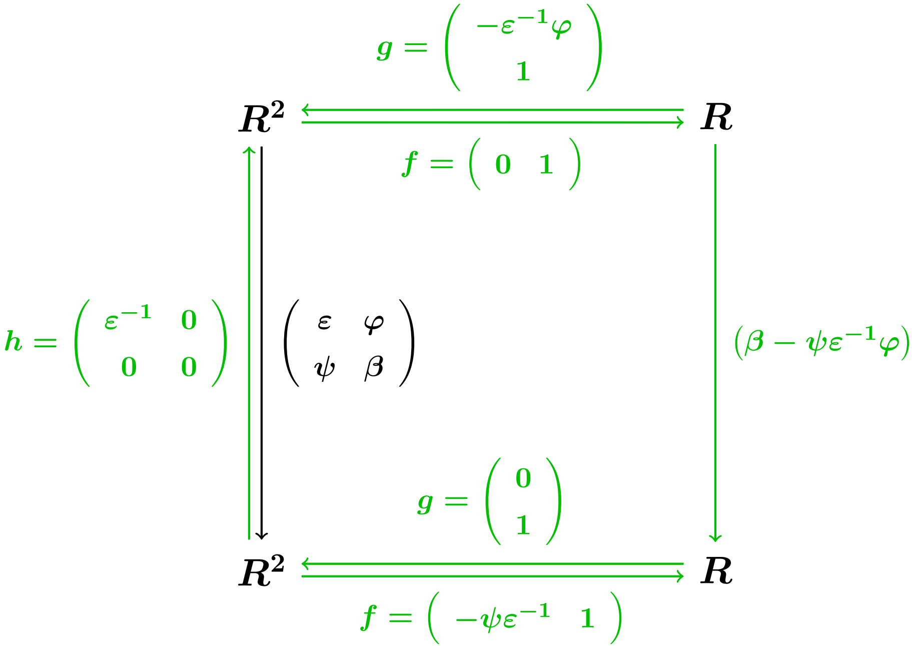
Diagram translation:

$$
\begin{array}{ccc}
\boldsymbol{R}^2 & \underset{\begin{pmatrix}1 & \varepsilon^{-1}\varphi\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & -\varepsilon^{-1}\varphi\\ 0 & 1\end{pmatrix}}{\xleftrightarrows{\qquad\qquad\qquad\qquad}}} & \boldsymbol{R}^2 \\
\begin{pmatrix}\varepsilon & \varphi\\ \psi & \beta\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix}\varepsilon & 0\\ 0 & \beta-\psi\varepsilon^{-1}\varphi\end{pmatrix} \\
\boldsymbol{R}^2 & \underset{\begin{pmatrix}1 & -0\\ -\psi\varepsilon^{-1} & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & -0\\ \psi\varepsilon^{-1} & 1\end{pmatrix}}{\xleftrightarrows{\qquad\qquad\qquad\qquad}}} & \boldsymbol{R}^2
\end{array}
$$

Combined with an obvious reduction:



⇒ Canonical reduction induced by ε invertible



The same is valid with

$R^2 = R \oplus R$ **replaced by** $A_n \oplus B_n = C_n$

or by $A_{n-1} \oplus B_{n-1} = C_{n-1}$

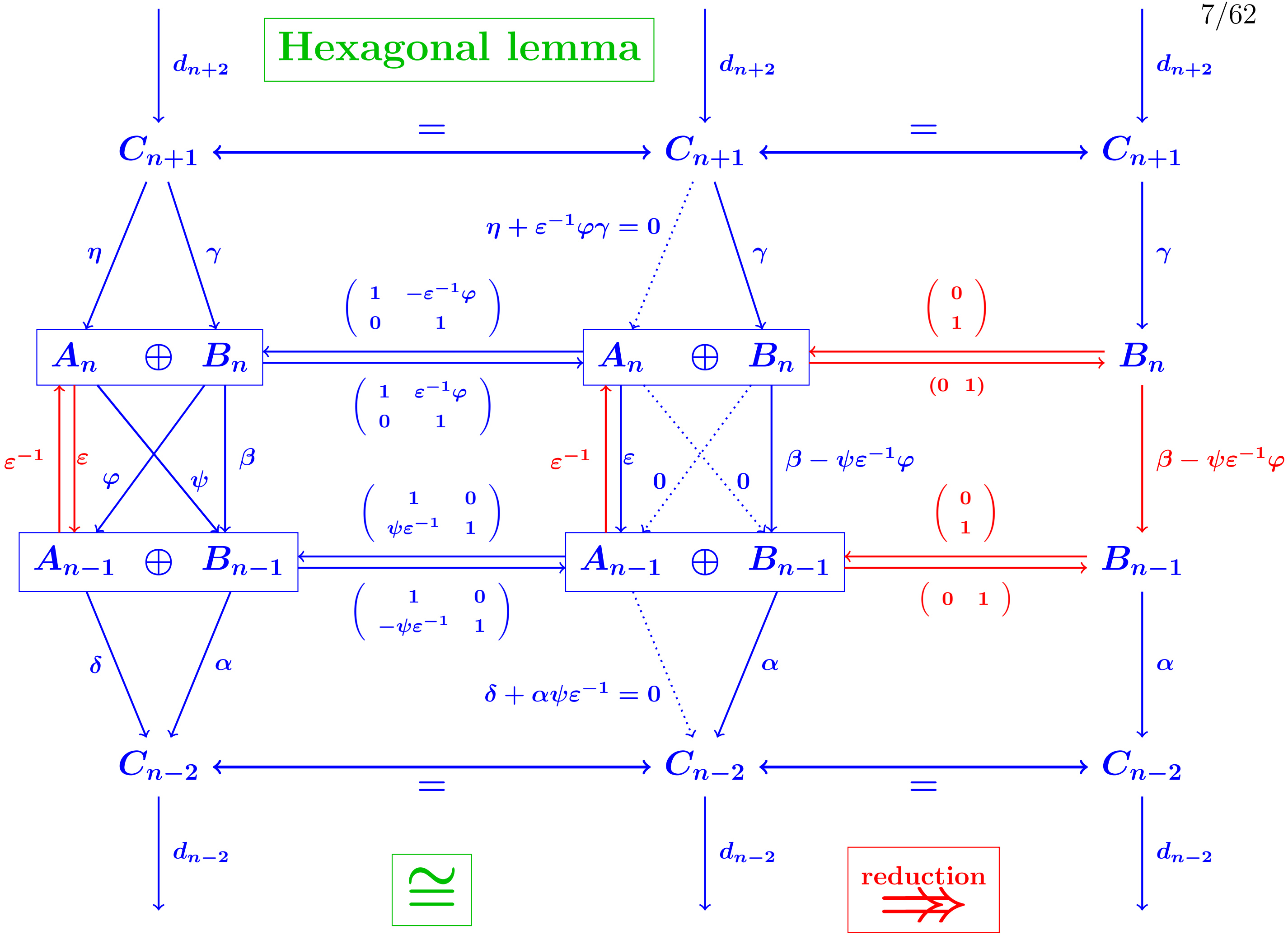
and:

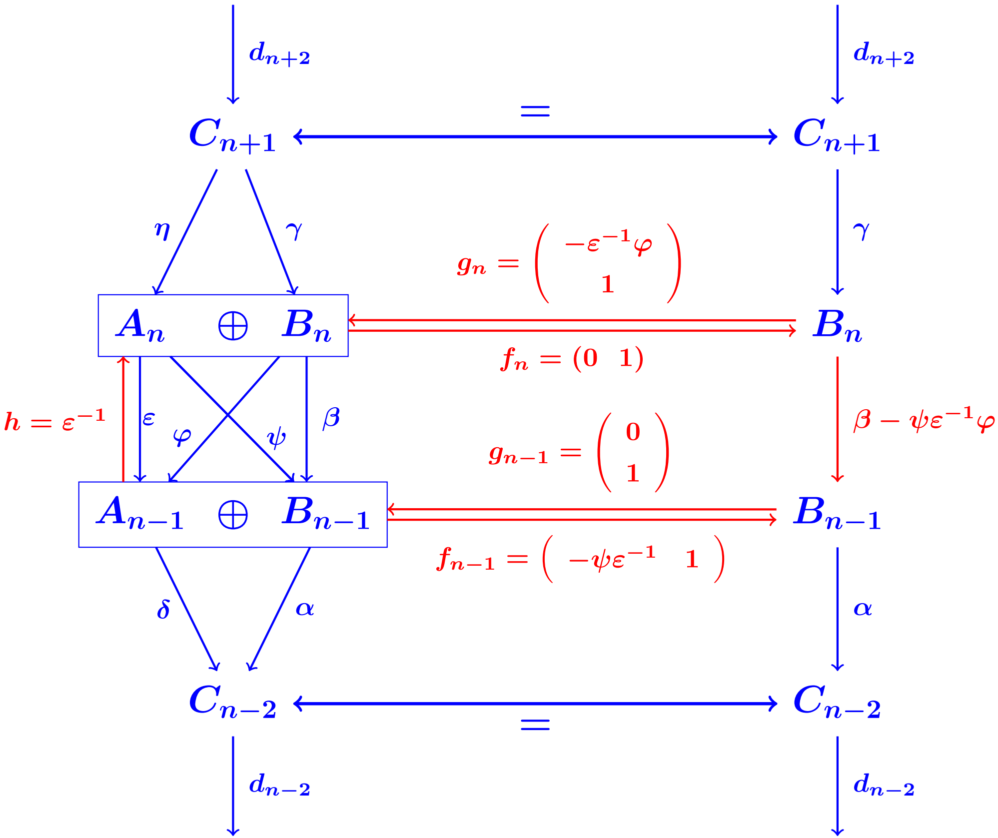
$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} : A_n \oplus B_n \to A_{n-1} \oplus B_{n-1}$$

with $\varepsilon : A_n \to A_{n-1}$ **isomorphism.**

$\Rightarrow$ **Hexagonal lemma.**

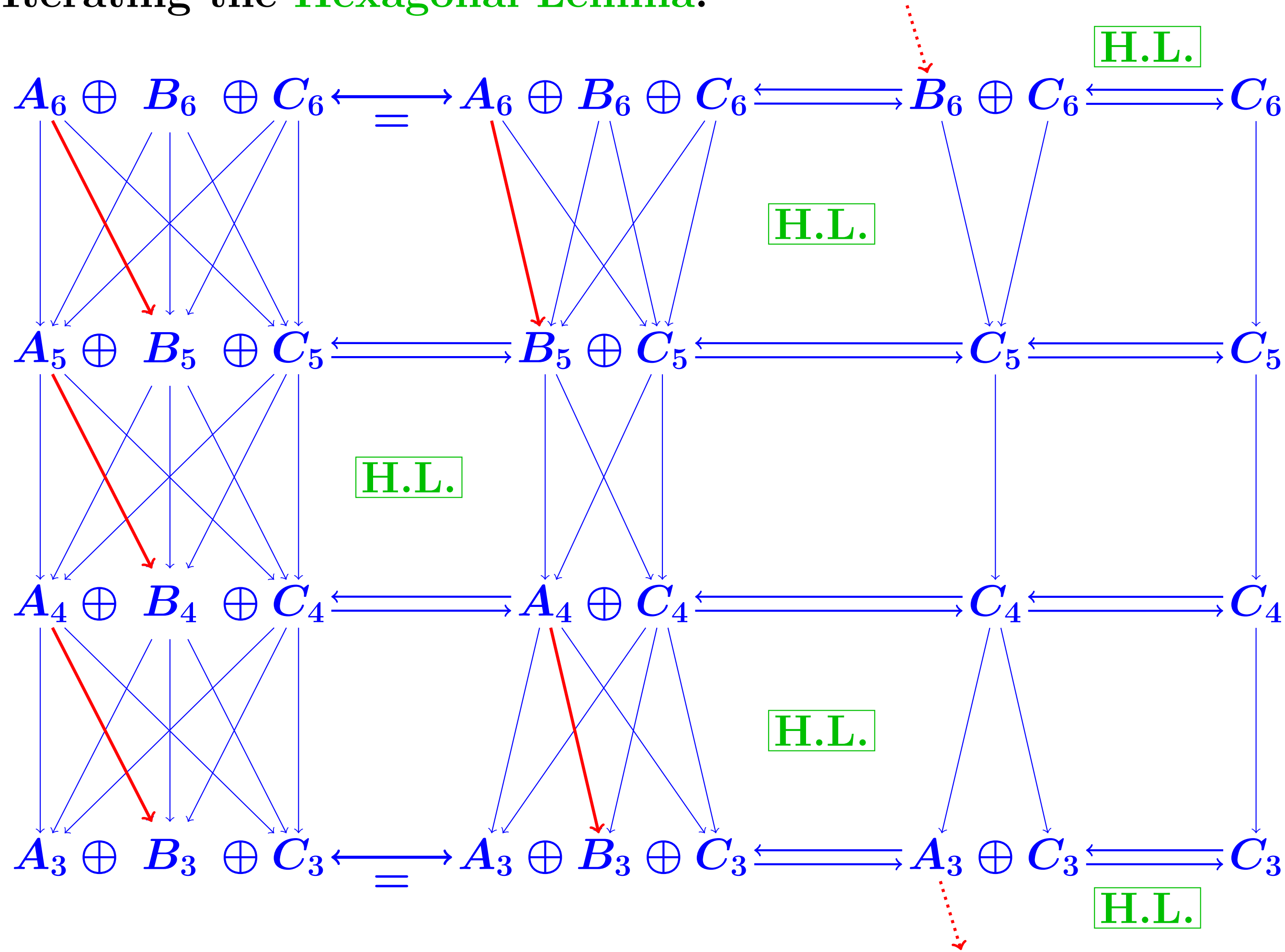
Hexagonal lemma



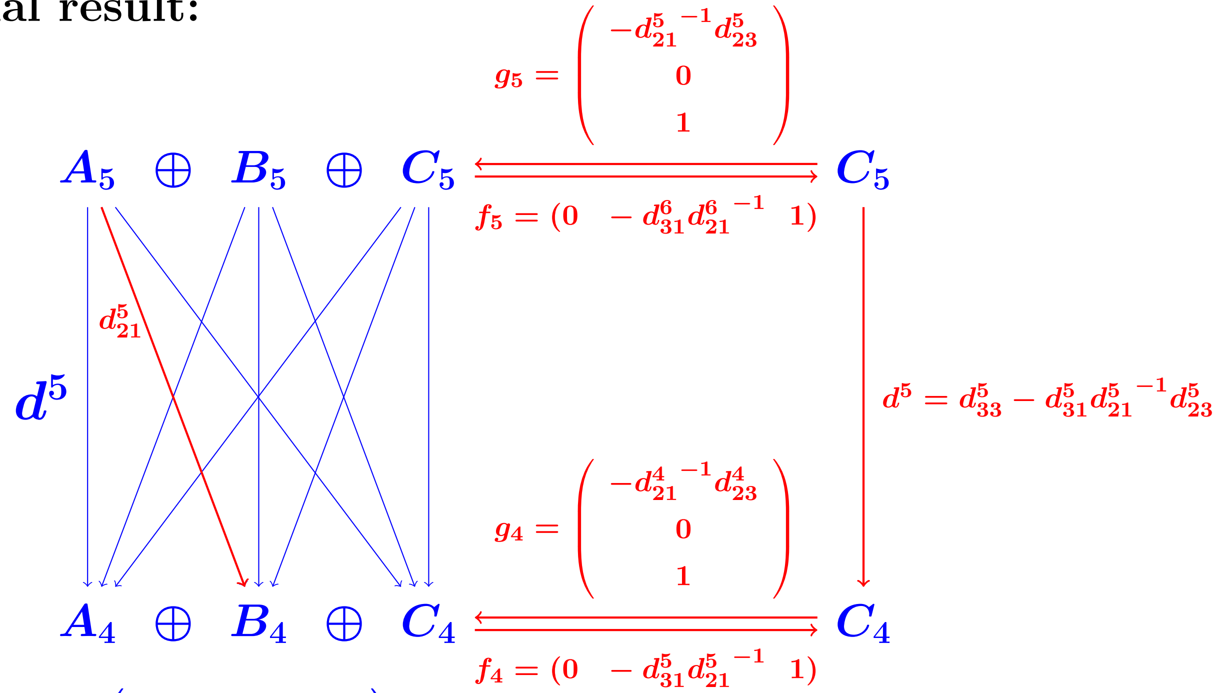


Hexagonal lemma

Iterating the Hexagonal Lemma:

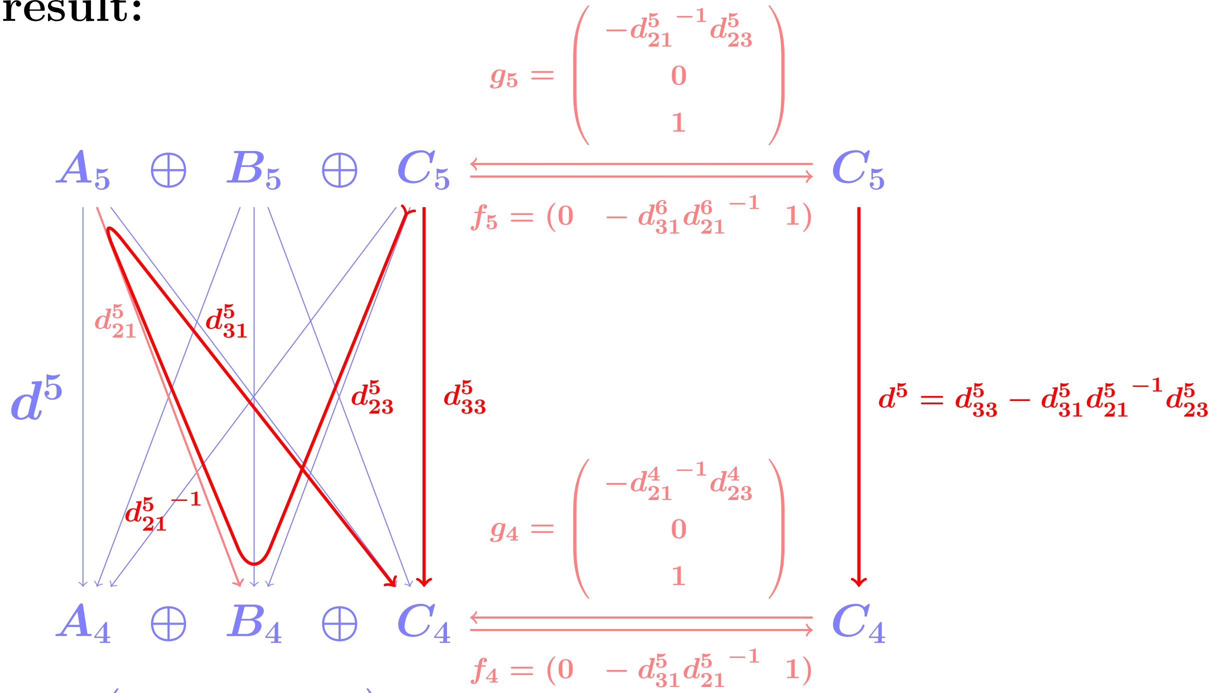


Final result:



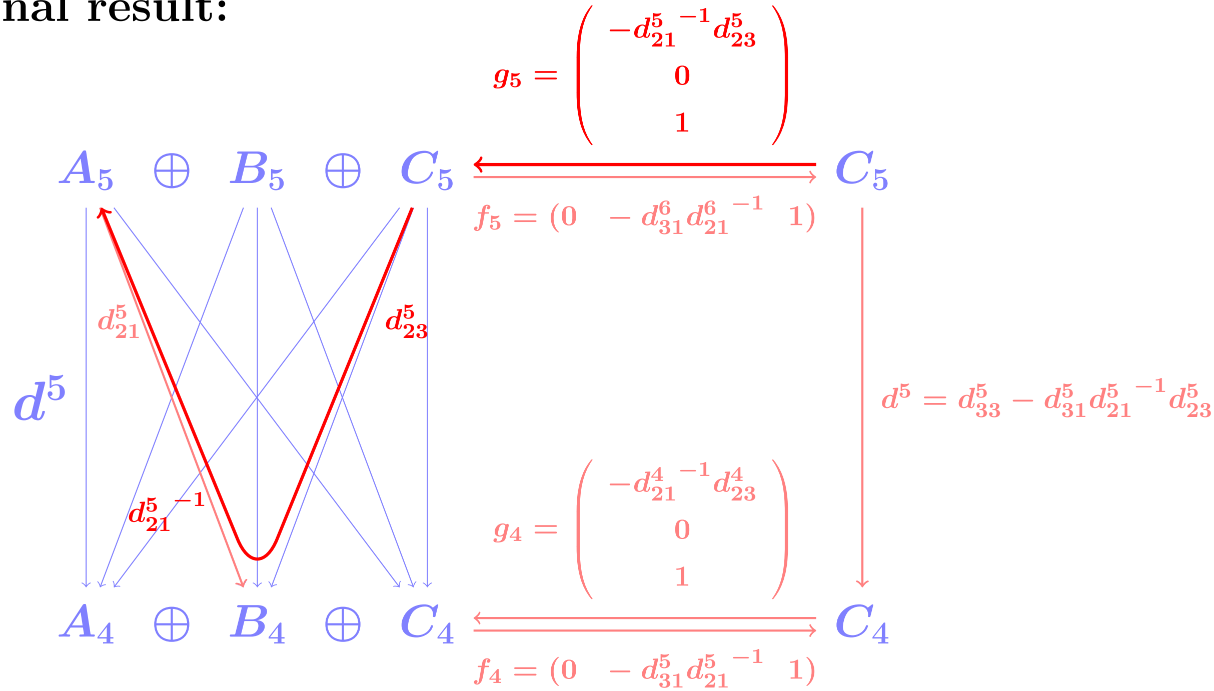
$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Final result:



$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Final result:

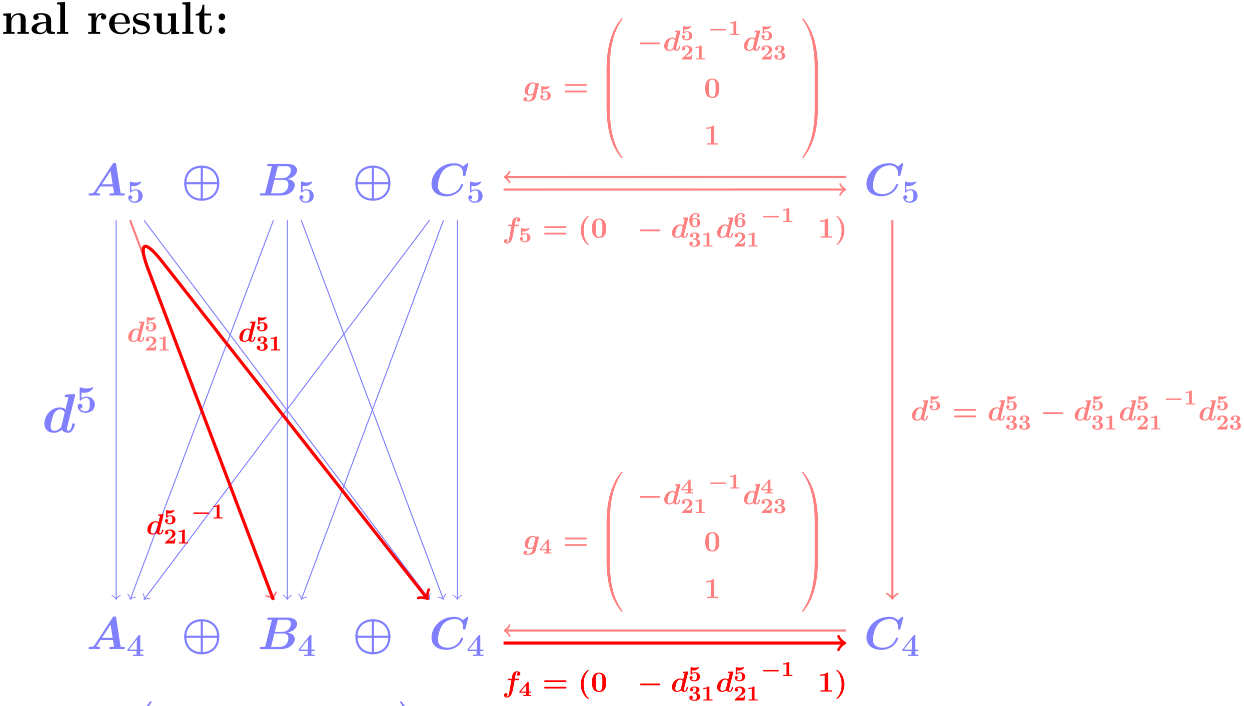


$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Note : $\mathrm{im}(g) \subset A_* \oplus C_*$

$= \text{graph } {d^5_{21}}^{-1} d^5_{23} : C_* \to A_*$

Final result:



$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Global Hexagonal Theorem:

Input: A chain complex (C_*, d_*)

with for every $n \in \mathbb{Z}$ a decomposition:

$$C_n = C_n^1 \oplus C_n^2 \oplus C_n^3 \qquad d_n = \begin{pmatrix} d_{n,11} & d_{n,12} & d_{n,13} \\ d_{n,21} & d_{n,22} & d_{n,23} \\ d_{n,31} & d_{n,32} & d_{n,33} \end{pmatrix}$$

with $d_{n,21} : C_n^1 \to C_{n-1}^2$ isomorphism $\forall n$.

Output: A canonical reduction:

$$(C_*, d_*) = (C_*^1 \oplus C_*^2 \oplus C_*^3, d_*) \Rrightarrow (C_*^3, d'_*)$$

Application: Basic Perturbation Lemma (BPL)

Definition: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

Let $\rho : \boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$ be a given reduction

and $\widehat{\delta}$ a perturbation of $\widehat{d}$

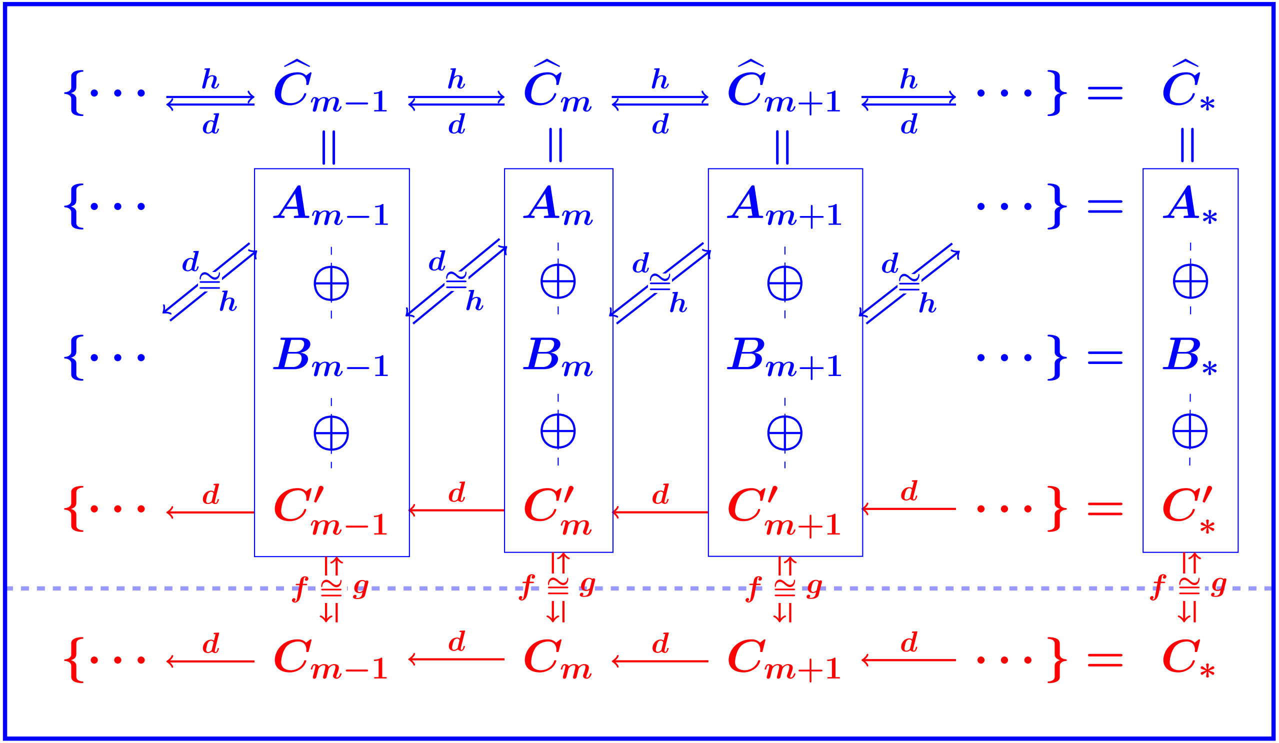
satisfying $h\widehat{\delta}$ pointwise nilpotent.

Theorem: The BPL determines a new reduction:

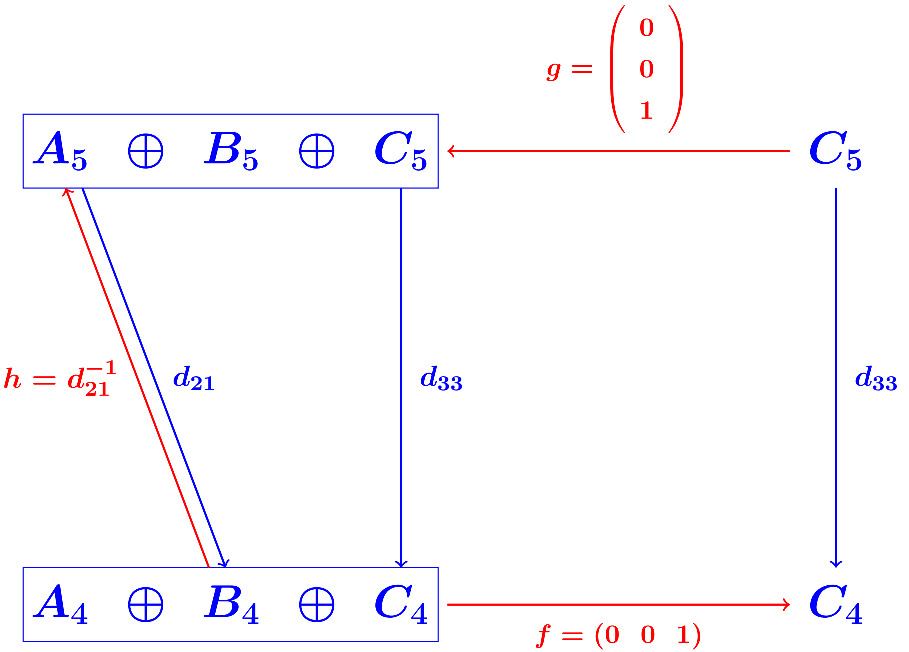
$$\rho' : \boxed{h + \delta_h \circlearrowleft (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \underset{f + \delta_f}{\overset{g + \delta_g}{\leftrightarrows}} (C_*, d_* + \delta_{d*})}$$

Proof:

Reduction Diagram:

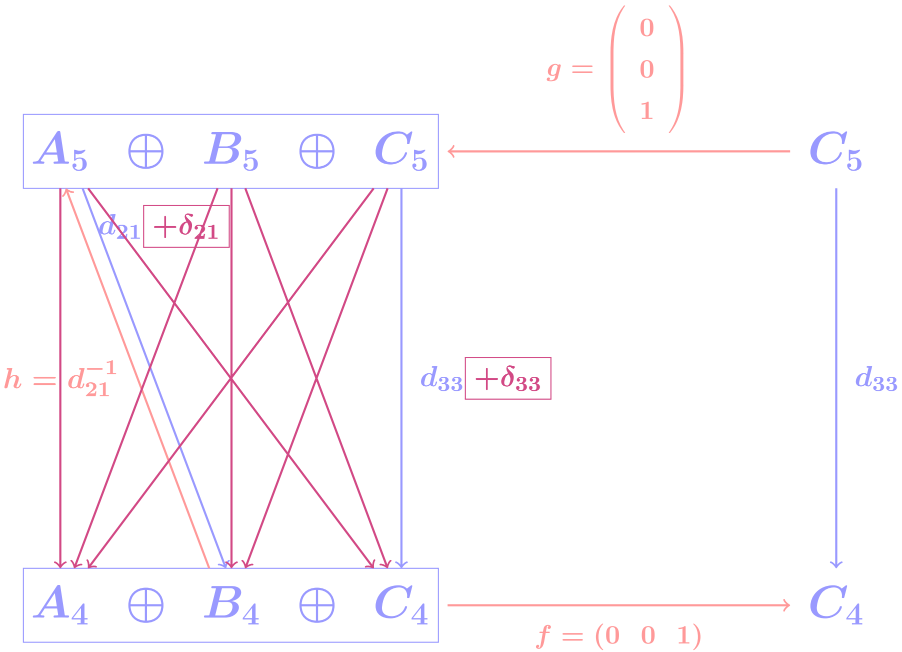


Main part:



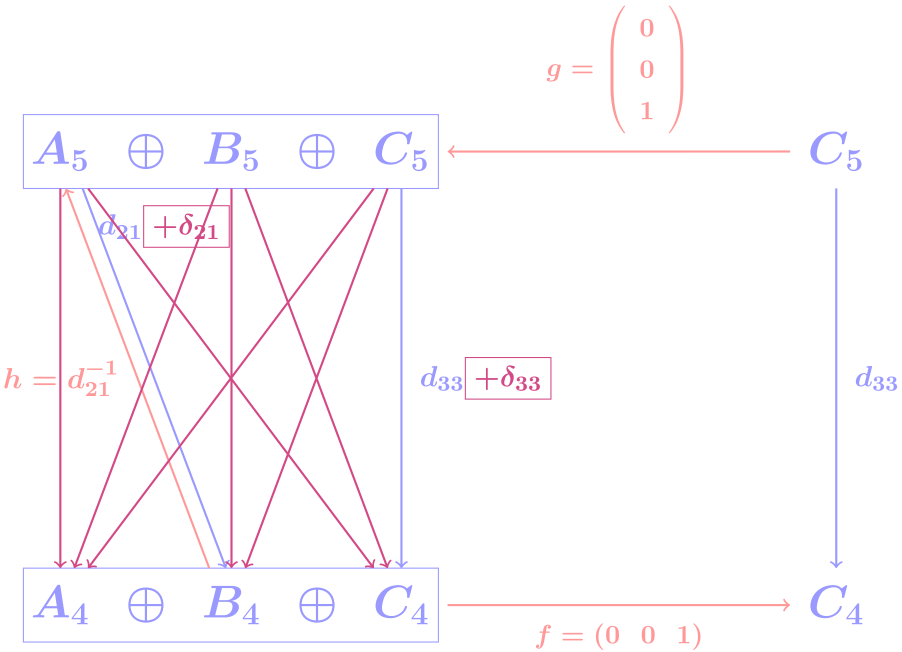
with d_{21} = isomorphism.

Perturbation $= \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$:



Question: $(d_{21} + \delta_{21})$ again isomorphism?

(applying the Global Hexagonal Theorem possible ?)



But d_{21} invertible with $d_{21}h = 1 \quad \Rightarrow$

$$d_{21} + \delta_{21} = d_{21} + d_{21}h\delta_{21} = d_{21}(1 + h\delta_{21})$$

$\Rightarrow \quad d_{21} + \delta_{21}$ **invertible** $\Leftrightarrow (1 + h\delta_{21})$ **invertible.**

A sufficient condition is $h\delta_{21}$ nilpotent, in which case:

$$(1 + h\delta_{21})^{-1} = \sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i$$

Then:

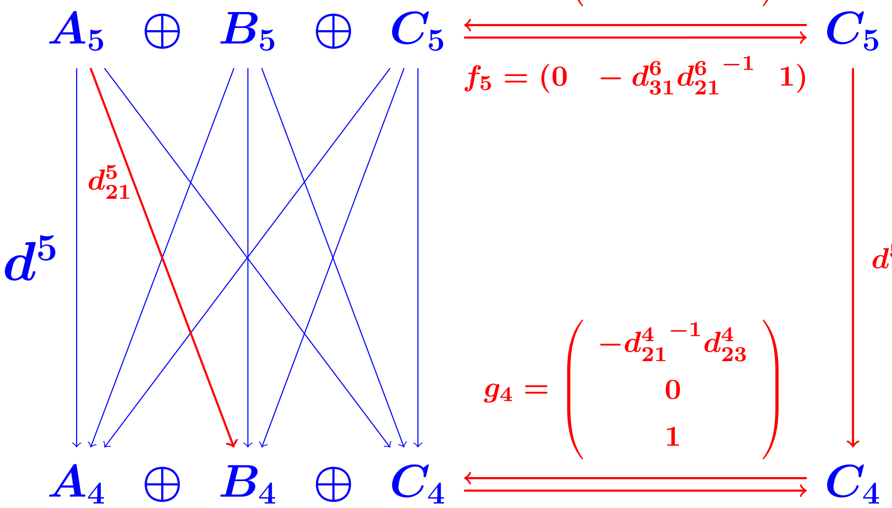
$$(d_{21} + \delta_{21})^{-1} =: h' := \left(\sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h$$

Remark:

$$\left(\sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h = \left(\sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

Global Hexagonal Theorem:

$$g_5 = \begin{pmatrix} -{d^5_{21}}^{-1} d^5_{23} \\ 0 \\ 1 \end{pmatrix}$$



$$f_5 = \begin{pmatrix} 0 & -d^6_{31} {d^6_{21}}^{-1} & 1 \end{pmatrix}$$

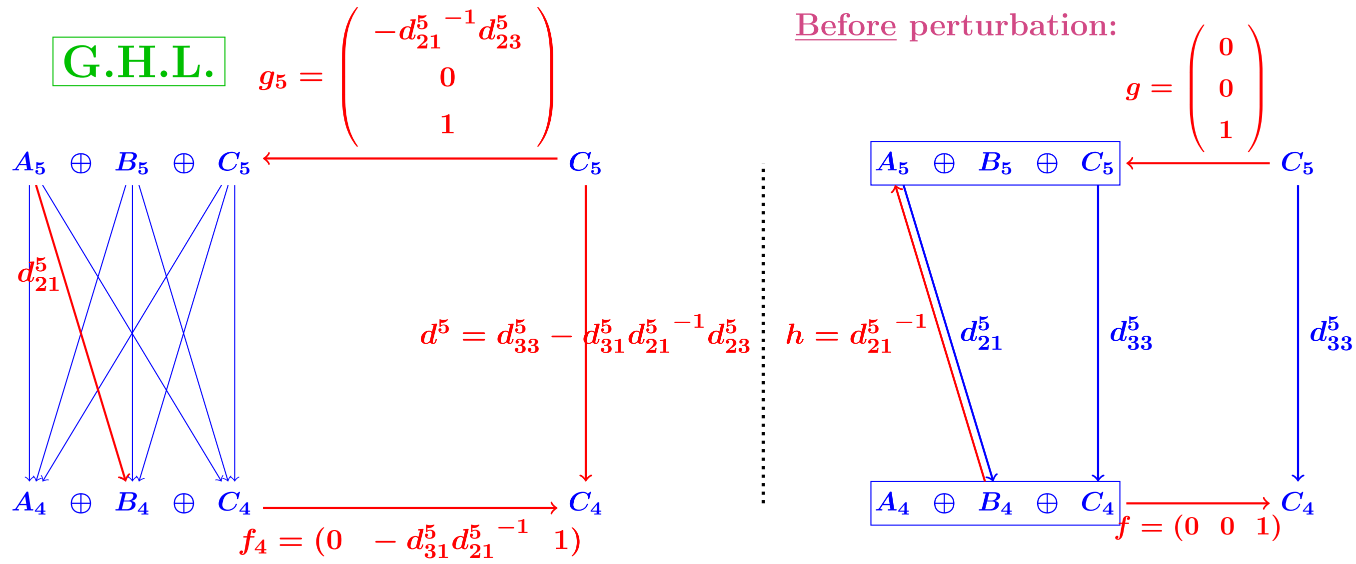
$$d^5 = d^5_{33} - d^5_{31} {d^5_{21}}^{-1} d^5_{23}$$

$$g_4 = \begin{pmatrix} -{d^4_{21}}^{-1} d^4_{23} \\ 0 \\ 1 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} 0 & -d^5_{31} {d^5_{21}}^{-1} & 1 \end{pmatrix}$$

$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Applying to our situation:



$$d^5_{21} \mapsto d^5_{21} + \delta^5_{21}$$

$$g_5 \mapsto (1 - h'\delta)g$$

$${d^5_{21}}^{-1} \mapsto h' = \left(\sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

$$f_4 \mapsto f(1 - \delta h')$$

$$d^5 \mapsto (d^5_{33} + \delta^5_{33}) - f\delta h' \delta g$$

$$= d^5_{33} + f\delta g - f\delta h' \delta g$$

= Basic Perturbation Lemma

QED

Application to Vector Fields.

(C_*, d_*, β_*) = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)\}_{i \in I}$ = Admissible discrete vector field.

$\Rightarrow$ Canonical reduction:

$$(C_*, d_*, \beta_*) \Rrightarrow (C^c{}_*, d^c{}_*, \beta^c{}_*)$$

on the critical chain complex $C^c{}_*$:

$$\rho : \boxed{h \circlearrowright (C_*, d_*) \underset{f}{\overset{g}{\leftrightarrows}} (C^c{}_*, d^c{}_*)}$$

Forman's Morse subcomplex $= g(C^c{}_*) \subset C_*$

Definition: Incidence number $\varepsilon(\sigma,\tau)$:= coefficient of σ in $d\tau$.

Remark: V vector field $\Rightarrow$ $\varepsilon(\sigma,\tau)$ invertible if $(\sigma,\tau)\in V$.

Definition: The *gradient* v

associated to the vector field $V=\{(\sigma_i,\tau_i)\}_{i\in I}$

is the *codifferential* $v_* : C_* \to C_{*+1}$ defined by:

$$v(\sigma)=\begin{cases}\varepsilon(\sigma,\tau)^{-1}\tau & \text{if } (\sigma,\tau)\in V,\\ 0 & \text{otherwise.}\end{cases}$$

Definition: The *cogradient* v^{-1}

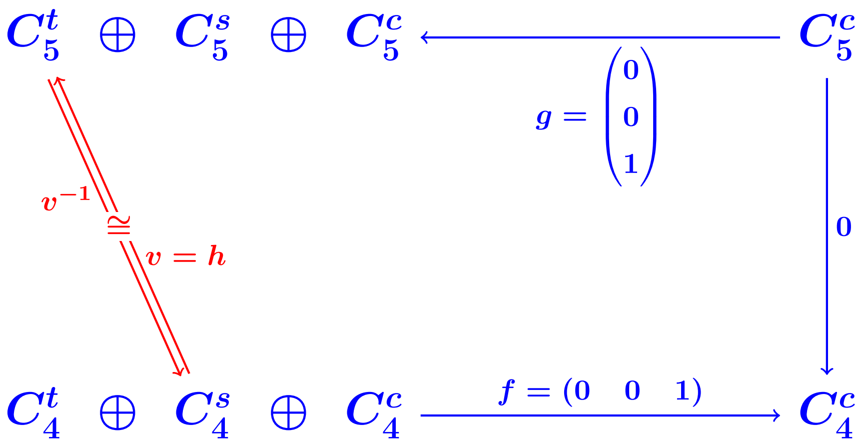
associated to the vector field $V=\{(\sigma_i,\tau_i)\}_{i\in I}$

is the *differential* $v_*^{-1} : C_* \to C_{*-1}$ defined by:

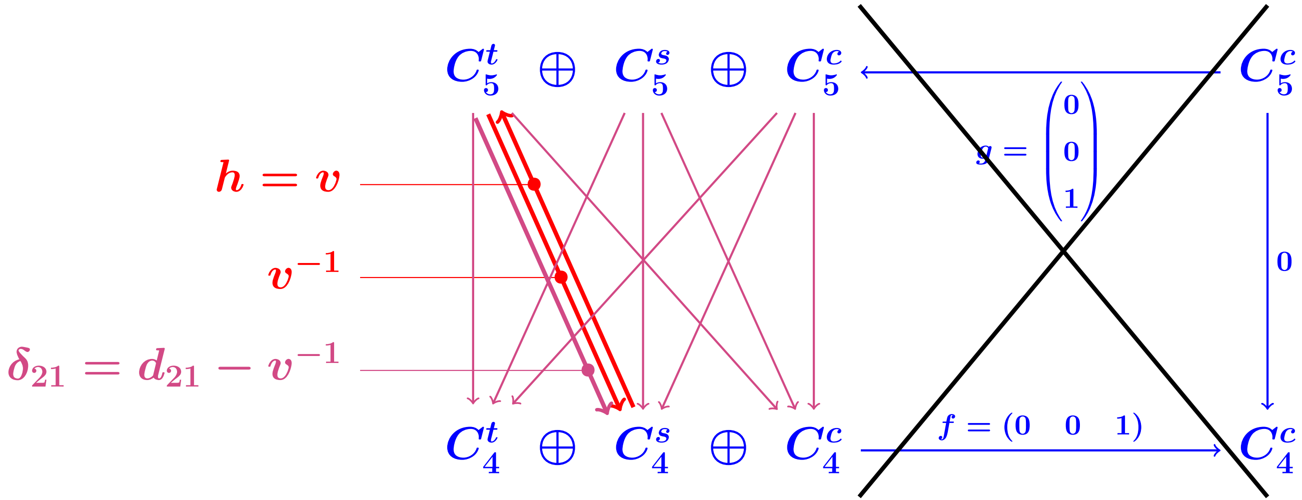
$$v^{-1}(\tau)=\begin{cases}\varepsilon(\sigma,\tau)\sigma & \text{if } (\sigma,\tau)\in V,\\ 0 & \text{otherwise.}\end{cases}$$

Applying the BPL.

Initial situation:



Restoring the differential d of C_*:



$$d_{21} = v^{-1} + (d_{21} - v^{-1}) = v^{-1} + \delta_{21}$$

But $h\ \delta_{21} = v(d_{21} - v^{-1})$ nilpotent ???

Examining: $\xrightarrow{v} \xrightarrow{d_{21}-v^{-1}} \xrightarrow{v} \xrightarrow{d_{21}-v^{-1}} \xrightarrow{v} \xrightarrow{d_{21}-v^{-1}} \xrightarrow{v} \cdots$

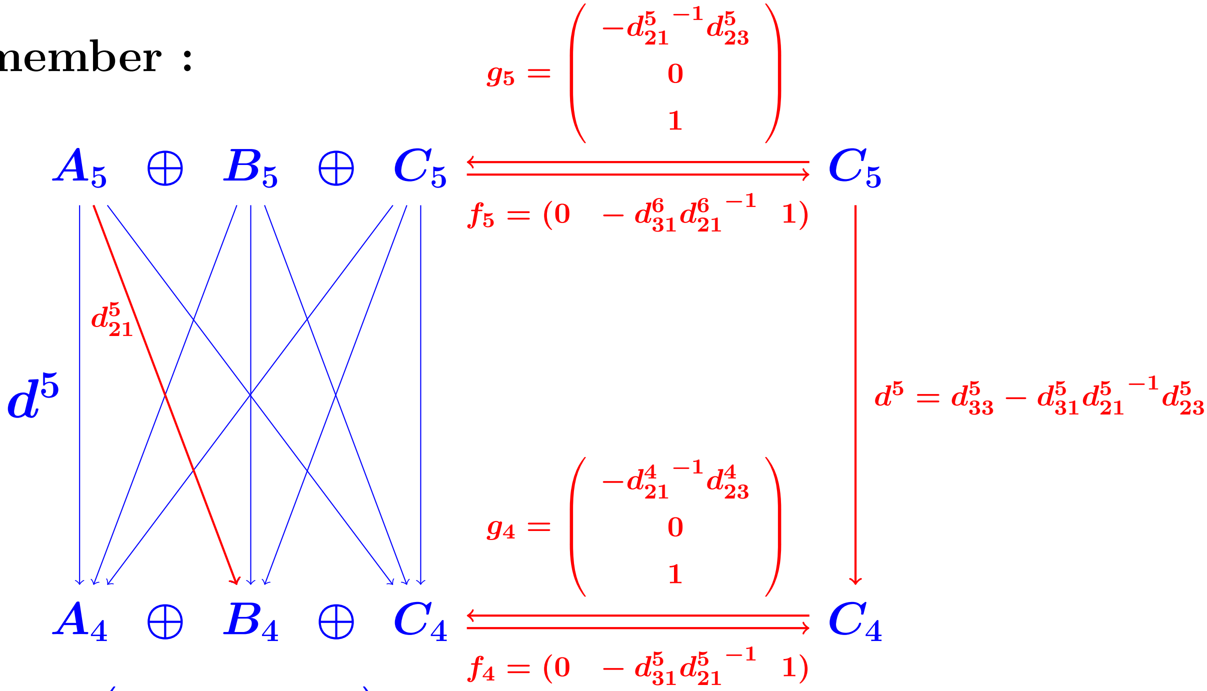
1. **Starting cell** $= \sigma \in C_4^s$.
2. $v(\sigma) = \varepsilon(\sigma, \tau)^{-1}\tau \sim$ **target cell associated by** v.
3. $v^{-1}(\varepsilon(\sigma, \tau)^{-1}\tau) = \sigma$

   $\Rightarrow (d_{21} - v^{-1})(\tau) =$ **Comb. of {sources faces of** $\tau \neq \sigma$**}**

   $\Rightarrow$ **beginning all the** V**-paths starting from** σ.
4. $v \Rightarrow$ **trying to extend** V**-paths.**
5. $(d_{21} - v^{-1})(\tau) \Rightarrow$ **trying to extend** V**-paths.**
6. **. . . . .**

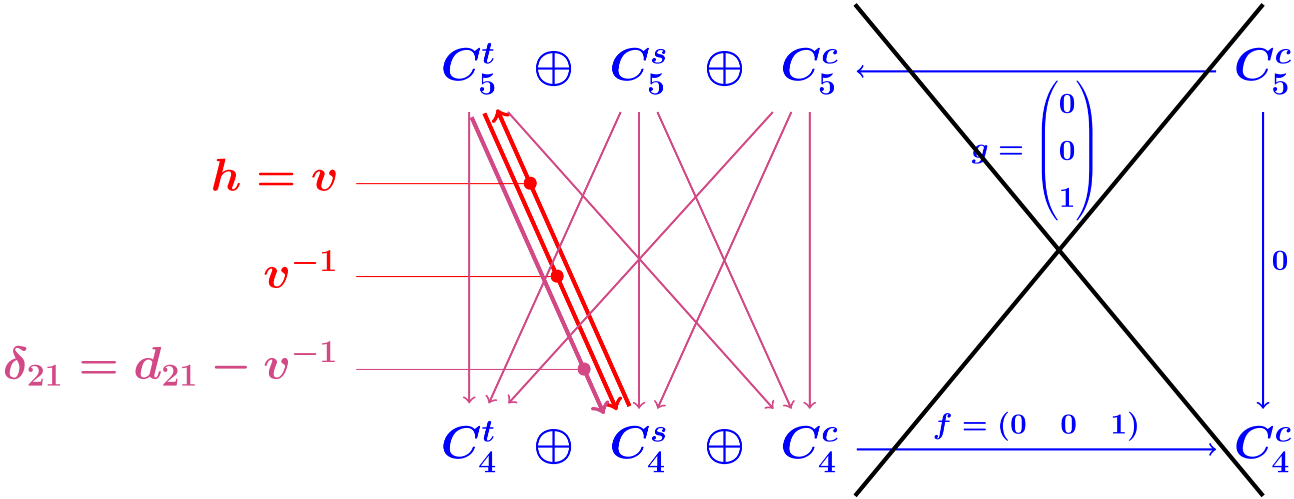
$\Rightarrow$ $\delta_{21}h$ nilpotent $\Leftrightarrow$ V admissible.

Remember :



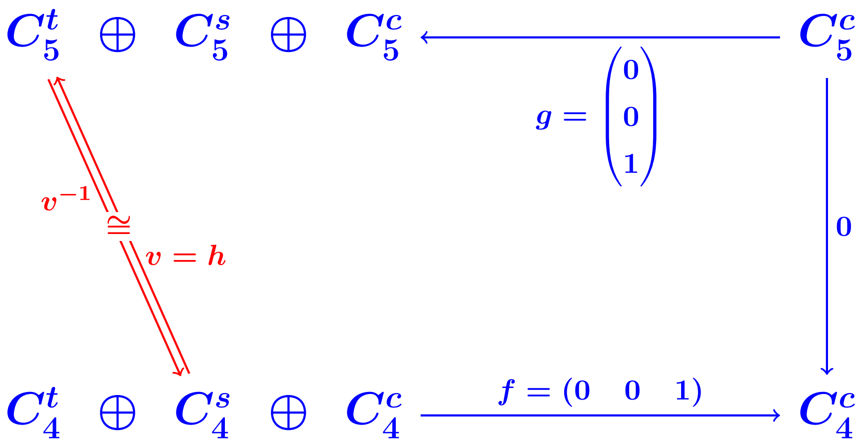
$$d^5 = \begin{pmatrix} d^5_{11} & d^5_{12} & d^5_{13} \\ d^5_{21} & d^5_{22} & d^5_{23} \\ d^5_{31} & d^5_{32} & d^5_{33} \end{pmatrix}$$

Applying in our situation ⇒

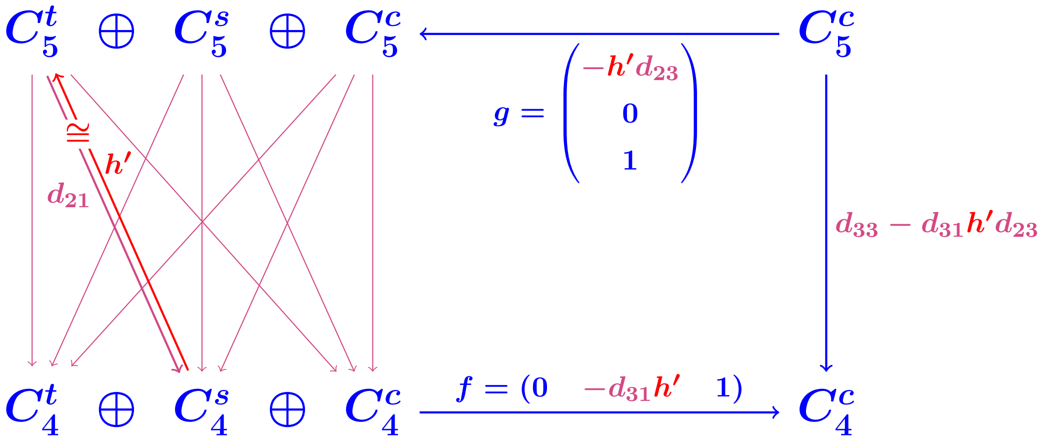


$$h \quad \mapsto \quad h' = \left(\sum_{i=0}^{\infty} (-1)^i (v\delta_{21})^i \right) v \qquad \textbf{and}$$

Initial situation:



Final situation:



Forman's Morse Subcomplex.

Theorem: (C_*, d, β_*) = cellular complex.

$V = \{(\sigma_i, \tau_i)\}_{i \in I}$ =

admissible discrete vector field on C_*.

$\rho = (f, g, h) : C_* \Rrightarrow C_*^c$ the induced reduction.

Then $\ker(dh + hd) = \ker(dv + vd) = \mathrm{im}(g)$

is Forman's Morse subcomplex C_*^{inv}.

Forman's definition: $C_*^{\mathrm{inv}} := \{\sigma \ \underline{\mathrm{st}}\ (1 - dv - vd)(\sigma) = \sigma\}$.

Proposition:

$$\begin{aligned} h(\sigma) &= v(\sigma) - h\delta_{21}v(\sigma) \\ &= v(\sigma) - h(d_{21} - v^{-1})v(\sigma) \\ &= v(\sigma) - h(d - v^{-1})v(\sigma) \\ hdv(\sigma) &= v(\sigma) \end{aligned}$$

Proof: **For a source cell σ:**

$$\begin{aligned} h(\sigma) &= v(\sigma) - v\delta_{21}v(\sigma) + v\delta_{21}v\delta_{21}v(\sigma) - \cdots \\ &= v(\sigma) - (v - v\delta_{21}v + \cdots)\delta_{21}v(\sigma) \\ &= v(\sigma) - h\delta_{21}v(\sigma) \end{aligned}$$

+ Everything null for σ target or critical cell $\Rightarrow$ QED

Proof of Theorem:

$$
\begin{aligned}
& & \sigma &\in \ker(dv + vd) \\
&\Rightarrow & dv\sigma + vd\sigma &= 0 \\
&\Rightarrow & hdv\sigma + hvd\sigma &= 0 \\
hv = 0 &\Rightarrow & hdv\sigma &= 0 \\
hdv = v &\Rightarrow & v\sigma &= 0 \\
&\Rightarrow & vd\sigma &= 0 \\
&\Rightarrow & dh\sigma + hd\sigma &= 0 \\
&\Rightarrow & \sigma &\in \ker(dh + hd) \\
& & &= \mathrm{im}(g)
\end{aligned}
$$

Conversely:

$$\begin{array}{rrcl} & \boldsymbol{\sigma} & \in & \ker(dh + hd) \\ \Rightarrow & \boldsymbol{\sigma} & = & g(\chi) \\ \Rightarrow & \multicolumn{3}{l}{\boldsymbol{\sigma} \text{ made of target and critical cells}} \\ dg = gd^c \Rightarrow & d\boldsymbol{\sigma} & = & g(\chi) \\ \Rightarrow & \multicolumn{3}{l}{d\boldsymbol{\sigma} \text{ made of target and critical cells}} \\ \Rightarrow & dv\boldsymbol{\sigma} + vd\boldsymbol{\sigma} & = & 0 \\ \Rightarrow & \boldsymbol{\sigma} & \in & \ker(dv + vd) \end{array}$$

Finally: $\ker(dv + vd) = \ker(dh + hd)$ **QED**

Vector Fields and Morphisms

Problem: Let C_* and C'_* be two cellular chain complexes respectively provided with vector fields V and V'.

Question: Right notion of morphism $\varphi : (C_*, V) \to (C'_*, V')$???

1. Not trivial.
2. Essential to master the Eilenberg-Zilber vector fields.
3. Quite amazing !!

<u>Definition</u>: A cellular morphism:

$$\varphi : (C_*, d_*, \beta_*) \to (C'_*, d'_*, \beta'_*)$$

is a chain complex morphism $\varphi : (C_*, d_*) \to (C'_*, d'_*)$ satisfying the extra condition:

For every p-cell $\sigma \in \beta_p$,

$\varphi(\sigma)$ is null or $\in \beta'_p$.

(C_*, d, β, V) **and** (C'_*, d', β', V')

= cellular chain complexes

with respective admissible discrete vector fields V **and** V'**.**

<u>**Definition**</u>**: A vectorious morphism:**

$$\varphi : (C_*, d, \beta, V) \rightarrow (C'_*, d', \beta', V')$$

is a cellular morphism $\varphi := (C_*, d, \beta) \rightarrow (C'_*, d', \beta')$
satisfying the extra conditions:

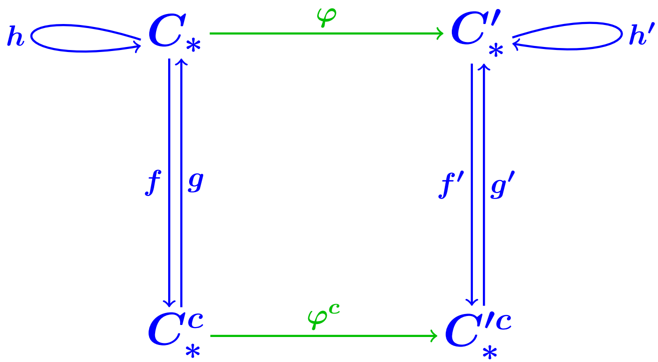
1. **For every critical cell** $\chi \in \beta_p^c$, $\varphi(\chi)$ **is null or** $\in \beta_p'^c$**.**
2. **For every target cell** $\tau \in \beta_p^t$, $\varphi(\tau)$ **is null or** $\in \beta_p'^t$**.**
3. **No condition at all for the source cells !!**

<u>Theorem</u>: $\varphi : (C_*, d, \beta, V) \to (C'_*, d', \beta', V')$

= **vectorious morphism.**

Then φ defines a morphism (φ, φ^c)

between the corresponding reductions:



with:

$$d'^c \varphi^c = \varphi^c d^c$$
$$f' \varphi = \varphi^c f$$
$$g' \varphi^c = \varphi g$$
$$h' \varphi = \varphi h$$

Proof:

Definition:

$$\lambda_\sigma = \begin{cases} 0 \quad \text{for target and critical cells,} \\ \text{maximal length of a } V\text{-path} \\ \qquad \text{starting form the source cell } \sigma. \end{cases}$$

Remember: Recursive formula:

$$h(\sigma) = v(\sigma) - h(d - v^{-1})v(\sigma)$$

$$\Rightarrow hdv(\sigma) = v(\sigma)$$

$$\Rightarrow hd\tau = \tau \text{ for every target cell } \tau$$

1. $h'\varphi\sigma = \varphi h\sigma$??

Obvious for σ target or critical cell.

Assumed known for $\lambda_\sigma < k$.

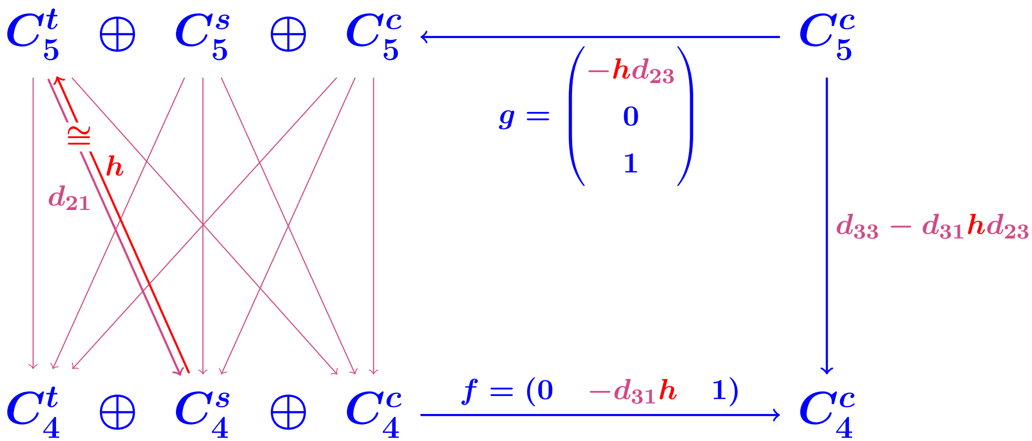
Let σ be a source cell with $\lambda_\sigma = k$.

$$\begin{aligned}
& & \varphi h\sigma &= \varphi v\sigma - \varphi h(d - v^{-1})v\sigma \\
\textbf{OK for } (d - v^{-1})v\sigma &\Rightarrow & &= \varphi v\sigma - h'\varphi(d - v^{-1})v\sigma \\
\varphi d = d'\varphi &\Rightarrow & &= \varphi v\sigma - h'd'\varphi v\sigma + h'\varphi\sigma \\
\varphi v\sigma = \textbf{target cell} &\Rightarrow & &= h'\varphi\sigma
\end{aligned}$$

QED

2. $g'\varphi^c = \varphi g$??

Remember:



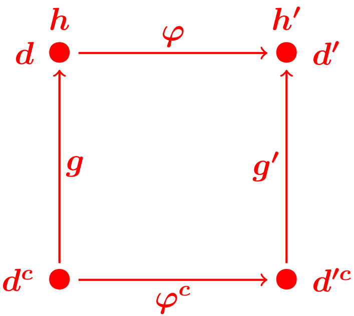
For a critical cell χ: $g\chi = \chi - hd\chi = (1 - hd)\chi$ $\Rightarrow$

$$
\begin{aligned}
& \Rightarrow & \varphi g \chi &= \varphi(1 - hd)\chi \\
\varphi h d = h' d' \varphi & \Rightarrow & &= (1 - h'd')\varphi\chi \\
\varphi\chi = \varphi^c \chi & \Rightarrow & &= g'\varphi^c\chi
\end{aligned}
$$

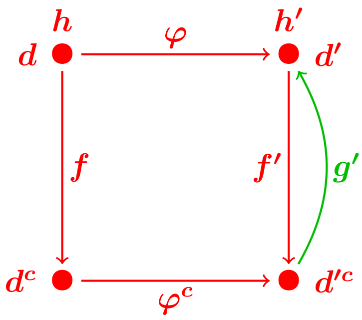
QED

$\varphi^c d^c = d'^c \varphi^c$??

$$
\begin{aligned}
g'\varphi^c = \varphi g & \Rightarrow & g'\varphi^c d^c &= \varphi g d^c \\
g d^c = d g & \Rightarrow & &= \varphi d g \\
\varphi d = d' \varphi & \Rightarrow & &= d' \varphi g \\
\varphi g = g' \varphi^c & \Rightarrow & &= d' g' \varphi^c \\
d' g' = g' d'^c & \Rightarrow & &= g' d'^c \varphi^c \\
g' \text{ injective} & \Rightarrow & \varphi^c d^c &= d'^c \varphi^c
\end{aligned}
$$



QED



$f'\varphi = \varphi^c f \quad ??$

g' **injective** $\Rightarrow [(f'\varphi = \varphi^c f) \quad \Leftrightarrow \quad (g'f'\varphi = g'\varphi^c f)]$

$$(d'\varphi = \varphi d) + (h'\varphi = \varphi h) \quad \Rightarrow \quad \begin{aligned} g'f'\varphi &= (1 - d'h' - h'd')\varphi \\ &= \varphi(1 - dh - hd) \\ &= \varphi g f \\ &= g'\varphi^c f \end{aligned}$$

QED

Application:

Theorem: Given: $\left| \begin{array}{l} \varphi : X \to X' \\ \varphi' : Y \to Y' \end{array} \right|$ = simplicial morphisms.

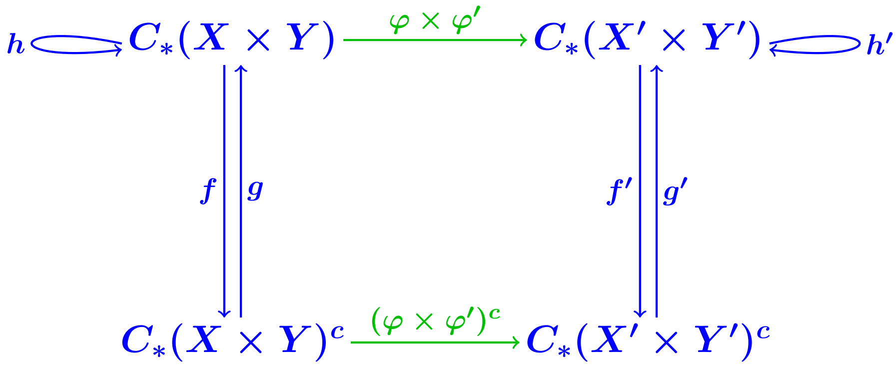
Then: φ and φ' induce a morphism between the reductions defined by the Eilenberg-Zilber vector fields:

$$\begin{array}{ccccc} h \circlearrowright & C_*(X \times Y) & \xrightarrow{\varphi \times \varphi'} & C_*(X' \times Y') & \circlearrowleft h' \\ & f \downarrow \uparrow g & & f' \downarrow \uparrow g' & \\ & C_*(X \times Y)^c & \xrightarrow{(\varphi \times \varphi')^c} & C_*(X' \times Y')^c & \end{array}$$

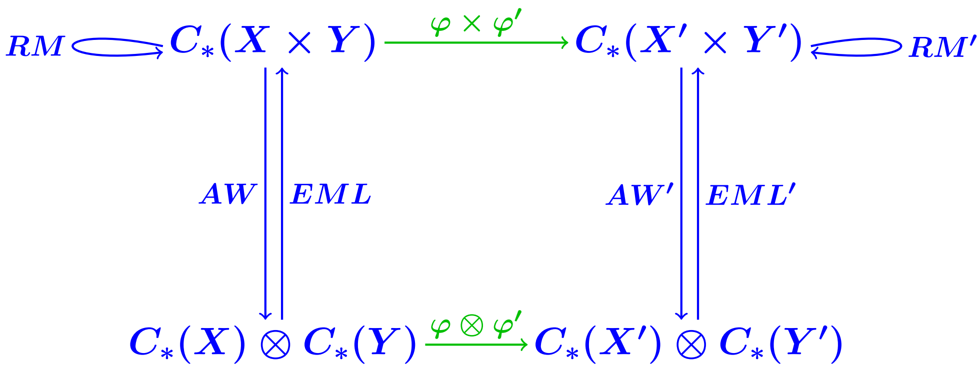
with:

$$\begin{array}{rcl} d'(\varphi \times \varphi')^c & = & (\varphi \times \varphi')^c d \\ f'(\varphi \times \varphi') & = & (\varphi \times \varphi')^c f \\ g'(\varphi \times \varphi')^c & = & (\varphi \times \varphi') g \\ h'(\varphi \times \varphi') & = & (\varphi \times \varphi') h \end{array}$$

Important Note:



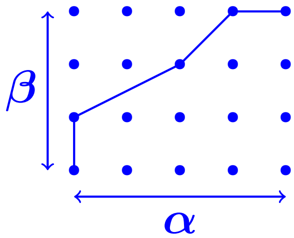
= same diagram as:



But not yet known!

<u>Proof</u>:

Representation of a simplex of $\boldsymbol{X \times Y}$ via an s-path.



= subsimplex of $\boldsymbol{\alpha \times \beta \subset (X \times Y)_7}$

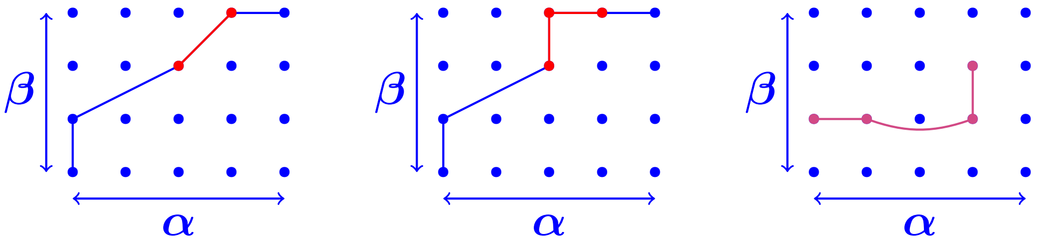
spanned by the vertices (0,0) – (0,1)– (2,2) – (3,3) – (4,3).

The game first event "diagonal  "

or "right-angle bend  "

determines the nature source, target or critical.

Examples:



source $= \boldsymbol{\sigma}$ **target** $= \boldsymbol{\tau}$ **critical** $= \chi$

Here:

$$\partial_3(\boldsymbol{\tau}) = \boldsymbol{\sigma}$$

$$\Rightarrow \quad v(\boldsymbol{\sigma}) = -\boldsymbol{\tau} \qquad (\textbf{gradient})$$

$$v^{-1}(\boldsymbol{\tau}) = -\boldsymbol{\sigma} \qquad (\textbf{cogradient})$$

Two maps $\left|\begin{array}{l}\varphi : X \to X' \\ \varphi' : Y \to Y'\end{array}\right|$ = simplicial morphisms.

Claim:

τ target cell in $X \times Y \Rightarrow$
$(\varphi \times \varphi')(\tau)$ target or degenerate cell in $X' \times Y'$

χ critical cell in $X \times Y \Rightarrow$
$(\varphi \times \varphi')(\chi)$ critical or degenerate cell in $X' \times Y'$

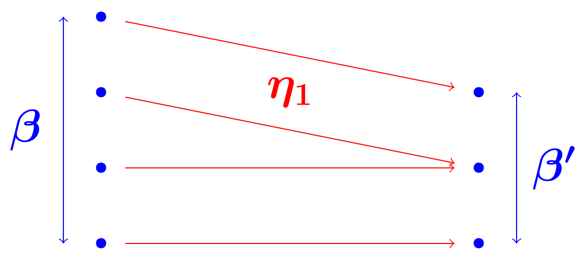
$\alpha \in X^{ND} \Rightarrow \varphi(\alpha) = \eta\alpha'$

for some multi-degeneracy η and some $\alpha' \in X'^{ND}$.

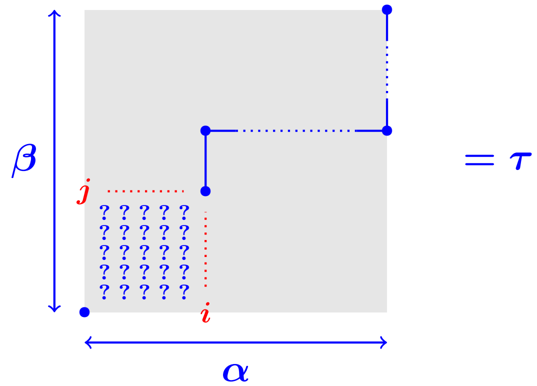
$\beta \in Y^{ND} \Rightarrow \psi(\beta) = \theta\beta'$

for some multi-degeneracy θ and some $\beta' \in Y'^{ND}$.

Example: $Y_3 \ni \beta \mapsto \theta\beta' = \eta_1 \beta' \in Y_3'$**:**



General shape of an Eilenberg-Zilber target cell:



$$(\varphi \times \psi)(\alpha \times \beta) = (\eta\alpha' \times \theta\beta')$$

If no index of η is $\geq i$ and no index of θ is $\geq j$,

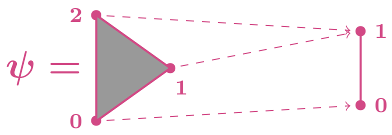
then $(\varphi \times \psi)(\tau)$ has the same shape and therefore is a target cell.

Otherwise $(\varphi \times \psi)(\tau)$ is degenerate.

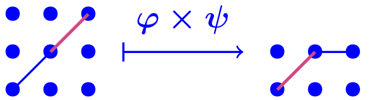
Same argument for critical simplices. $\Rightarrow$ **QED**

Typical accidents with source cells.

$\boldsymbol{\alpha = \mathrm{id} : \Delta^2 \to \Delta^2}$ **and** $\boldsymbol{\psi : \Delta^2 \to \Delta^1}$ **as below:**



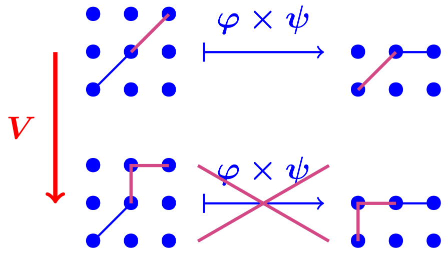
1)



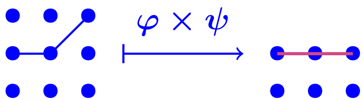
Then $\boldsymbol{(\varphi \times \psi)(\mathrm{source}) = \mathrm{source}}$

but for reasons which do not match!

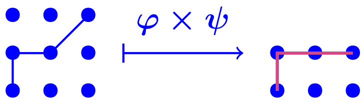
Compare corresponding target cells.



2) The image of a source cell can be a critical cell:



or target:



But we don't care about source cells!

Eilenberg-Zilber formulas

<u>Theorem</u>: The Eilenberg-Zilber vector field previously described gives the standard Eilenberg-Zilber reduction.

Standard Eilenberg-Zilber reduction:

$$\boldsymbol{EZ} : \boxed{{}_{RM} \circlearrowright \boldsymbol{C_*(X \times Y)} \underset{AW}{\overset{EML}{\leftrightarrows}} \boldsymbol{C_*(X) \otimes C_*(Y)}}$$

$\boldsymbol{AW}$ = **Alexander-Whitney**

$\boldsymbol{EML}$ = **Eilenberg-MacLane**

$\boldsymbol{RM}$ = **Rubio-Morace**

$\boldsymbol{EZ} = \boldsymbol{AW} + \boldsymbol{EML} + \boldsymbol{RM}$:

$$\boldsymbol{AW}(x_p \times y_p) = \sum_{i=0}^{p} \partial_{p-i+1} \cdots \partial_p x_p \otimes \partial_0 \cdots \partial_{p-i-1} y_p$$

$$\boldsymbol{EML}(x_p \otimes y_q) = \sum_{(\eta,\eta')\in \mathrm{Sh}(p,q)} \varepsilon(\eta,\eta')\,(\eta' x_p \times \eta y_q)$$

$$\boldsymbol{RM}(x_p \times y_p) = \sum_{\substack{0\le r\le p-1\\ 0\le s\le p-r-1\\ (\eta,\eta')\in \mathrm{Sh}(s+1,r)}} (-1)^{p-r-s}\,\varepsilon(\eta,\eta')\ldots$$

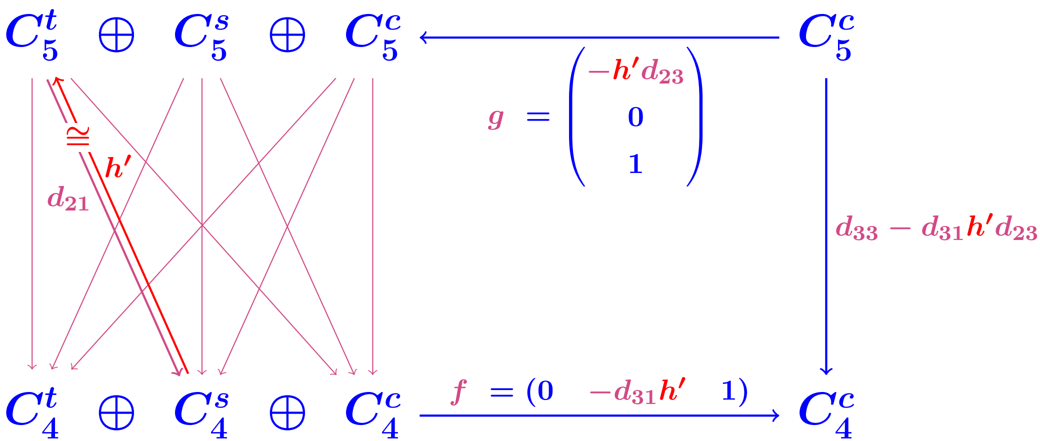
$$\ldots(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \ldots$$

$$\ldots\uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1} y_p)$$

Plan of proof:

1) Prove the RM-formula for the homotopy induced by the Eilenberg-Zilber vector field.

2) Use the diagram ($h' = RM$, $f = ?AW$, $g = ?EML$):



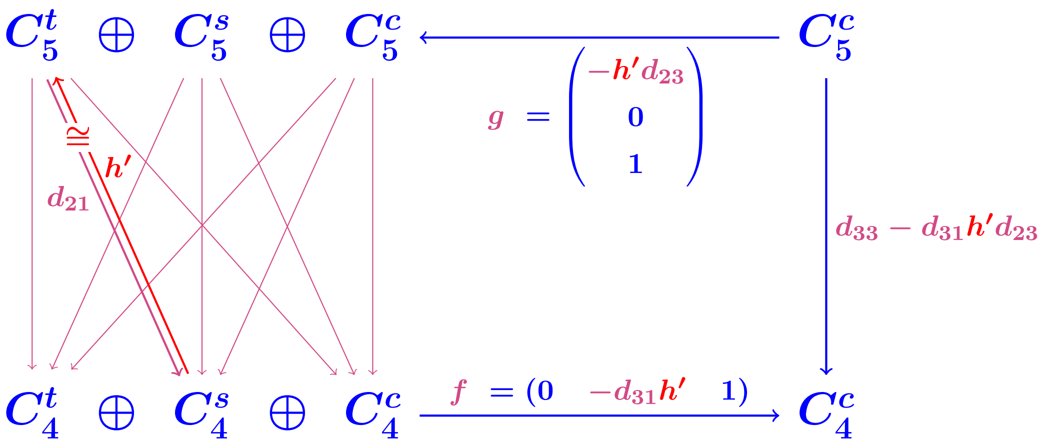
which produces: $h' \Rightarrow f$ and g.

Plan for the *RM*-formula:

1) True for target and critical simplices.

2) Use the recursive formula:

$$h'(\sigma) = v(\sigma) - h'\delta_{21}v(\sigma)$$



with $\delta_{21} = d_{21} - v^{-1}$.

Study of the generic term of the RM-formula:

$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \ldots$$

$$\ldots \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1} y_p)$$

$\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}$ **and** $\uparrow^{p-r-s}(\eta)$ = **known degeneracy part.**

$\partial_{p-r+1}\cdots\partial_p x_p$ **and** $\partial_{p-r-s}\cdots\partial_{p-r-1} y_p$ = **problematic part.**

$$(\eta, \eta') \in \mathrm{Sh}(s+1, r)$$

$$\Rightarrow \quad \mathrm{Ind}_{\eta} \cup \mathrm{Ind}_{\eta'} = \{0, \ldots, r+s\}$$

$$\Rightarrow \quad \mathrm{Ind}_{\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}} \cup \mathrm{Ind}_{\uparrow^{p-r-s}(\eta)} = \{p-r-s-1, \ldots p\}$$

Collision lemma: $(\eta x, \eta' y) \in (X \times Y)_p$

$+ \ \mathrm{Ind}_\eta \cup \mathrm{Ind}_{\eta'} = \{k, \ldots, p-1\}$

$+ \ \eta_i$ present in x or y with $i \geq k$

$\Rightarrow \ (\eta x, \eta' y)$ degenerate.

Proof by examples: $p = 10$, $k = 6$, $i = 6$,

$(\eta_9 \eta_7 x, \eta_8 \eta_6 y)$ degenerate?

$$\eta_6 \in x \Rightarrow (\eta_9\eta_7 x, \eta_8\eta_6 y) = (\eta_9\eta_7\eta_6 x', \eta_8\eta_6 y) = \eta_6(\eta_8\eta_6 x', \eta_7 y)$$

$$\eta_6 \in y \Rightarrow (\eta_9\eta_7 x, \eta_8\eta_6 y) = (\eta_9\eta_7 x, \eta_8\eta_6\eta_6 y') = \eta_7(\eta_8 x, \eta_7\eta_6 y)$$

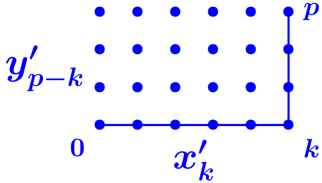
Putting both components in canonical form

$\Rightarrow$ a common factor.

QED

Application: $RM(x_p \times y_p) = 0$ if $(x_p \times y_p) =$ critical cell.

Proof: $(x_p, y_p) = (\eta_{p-1} \cdots \eta_k x'_k \times \eta_{k-1} \cdots \eta_0 y'_{p-k})$



1) Examine $\underbrace{\partial_{p-r+1} \cdots \partial_p}_{r} \underbrace{\eta_{p-1} \cdots \eta_k}_{p-k} x'_k$.

$r < p-k \ \Rightarrow$ there remains $\eta_{p-r-1} \cdots x'_k$

but $\{p-r-1 \ \geq p-r-s-1 \Rightarrow \text{collision}\} \Rightarrow \ r \geq p-k$.

???

2) Examine $\underbrace{\partial_{p-r-s} \cdots \partial_{p-r-1}}_{s} \underbrace{\eta_{k-1} \cdots \eta_0}_{k} y'_{p-k}$

$s < k \ \Rightarrow$ there remains $\eta_{k-s-1} \cdots y'_{p-k}$

$\Rightarrow \ k-s-1 < p-r-s-1 \ \Rightarrow \ r < p-k$.

QED

Same sort of argument $\Rightarrow$

$$RM(x_p \times y_p) = 0 \text{ for } (x_p \times y_p) = \text{target cell.}$$

$$\Rightarrow h(x_p \times y_p) = RM(x_p \times y_p) = 0$$

$$\text{for } (x_p \times y_p) = \text{critical or target cells.}$$

If $(x_p \times y_p) = \text{source cell}$, $\boxed{h(x_p \times y_p) \overset{???}{=} RM(x_p \times y_p)}$.

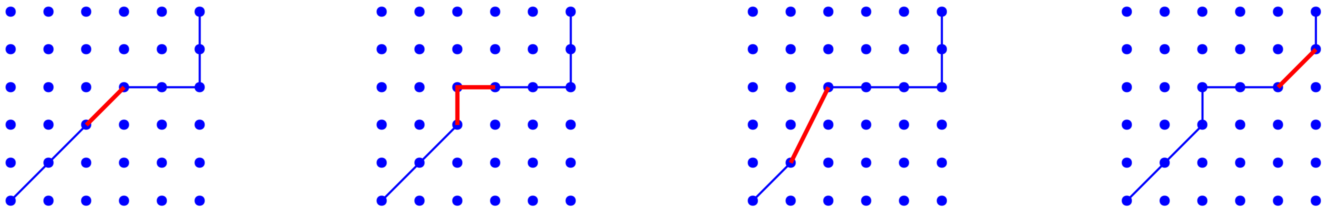
Remember:

$$\lambda_\sigma = \begin{cases} 0 & \text{for target and critical cells,} \\ \text{maximal length of a } V\text{-path} & \\ \quad\quad \text{starting form the source cell } \sigma. & \end{cases}$$

Remember: Recursive formula for source cells:

$$h(\sigma) = v(\sigma) - h(d_{21} - v^{-1})v(\sigma)$$

Example:



$\boldsymbol{\sigma}$ $\lambda = 13$ | $\boldsymbol{\tau}$ | $\boldsymbol{\sigma_1}$ $\lambda = 6$ | $\boldsymbol{\sigma_2}$ $\lambda = 12$

$v(\sigma) = -\tau$ $\quad$ $d_{21}(\tau) = -\sigma + \sigma_1 + \sigma_2$ $\quad$ $v^{-1}(\tau) = -\sigma$

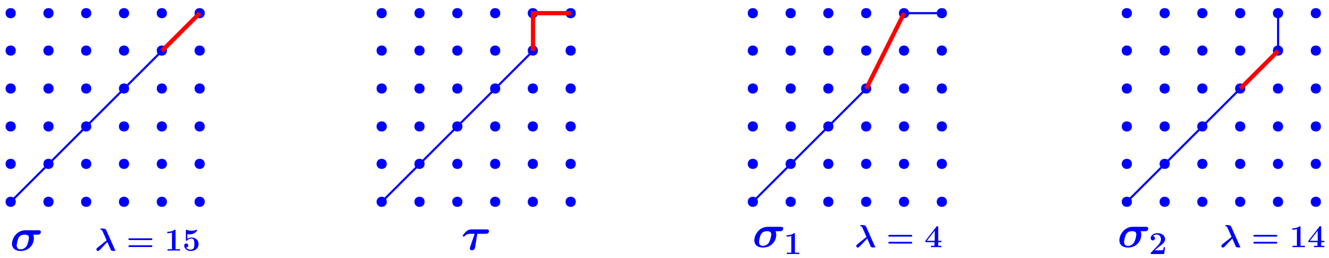
$$\Rightarrow \qquad h(\sigma) = -\tau + h(\sigma_1) + h(\sigma_2)$$

Then we prove the summands of $RM(\sigma_1)$

$$\text{are exactly divided into} \quad \begin{cases} -\tau \\ \text{the summands of } h(\sigma_1) \\ \text{the summands of } h(\sigma_2) \end{cases}$$

$\Rightarrow$ Recursive proof

Easiest and last case:



$$x_p = y_p = \Delta^p$$

$$\sigma = (x_p \times y_p) \qquad \tau = (\eta_{p-1} x_p \times \eta_p y_p)$$

$$\sigma_1 = (x_p \times \eta_{p-1} \partial_{p-1} y_p) \qquad \sigma_2 = (\eta_{p-1} \partial_p x_p \times y_p)$$

We recursively assume

$h(\sigma_1) = RM(\sigma_1)$ and $h(\sigma_2) = RM(\sigma_2)$

Then we prove $RM(\sigma) = \pm \tau \pm h(\sigma_1) \pm h(\sigma_2) =: h(\sigma)$ QED

***RM*-formula:**

Indices =

$$\{0 \le r \le p-1,\ 0 \le s \le p-r-1,\ (\eta|\eta') \in \mathrm{Sh}(s+1, r)\}$$

Generic term of the *RM*-formula:

$$\pm(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \ldots$$

$$\ldots\uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p)$$

We must prove $RM(\sigma) = \pm\tau \pm RM(\sigma_1) \pm RM(\sigma_2)$

$\sigma = (x_p \times y_p)$ $\qquad$ $\tau = (\eta_{p-1}x_p \times \eta_p y_p)$

$\sigma_1 = (x_p \times \eta_{p-1}\partial_{p-1}y_p)$ $\qquad$ $\sigma_2 = (\eta_{p-1}\partial_p x_p \times y_p)$

Elementary applications of Collision Lemma $\Rightarrow$

$$\tau = RM_{r=s=0}(\sigma)$$

$RM(\sigma_1) = RM_{r=0,s>0}(\sigma)$ $\qquad$ $RM(\sigma_2) = RM_{r>0}(\sigma)$

$RM = RM_{r=s=0} + RM_{r=0,s>0} + RM_{r>0}(\sigma) \quad \Rightarrow \quad QED$

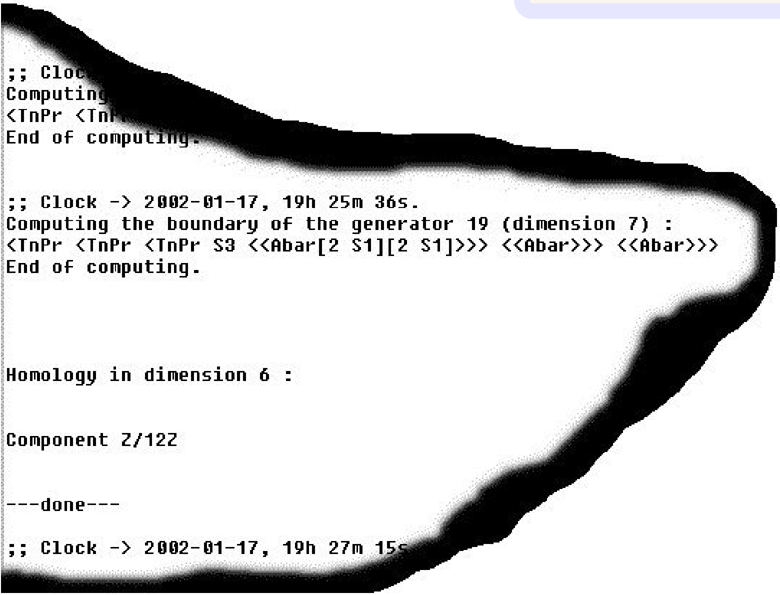
Analogous proof for the general case $\Rightarrow$

OK for $RM = h_{\text{EZ-vector-field}}$

$h_{\text{EZ-vector-field}} \Rightarrow f_{\text{EZ-vector-field}}$ **and** $g_{\text{EZ-vector-field}}$

the EZ-vector-field defines the standard EZ-reduction.

QED



```
;; Cloc
Computing
<TnPr <TnP
End of computing.

;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.

Homology in dimension 6 :

Component Z/12Z

---done---

;; Clock -> 2002-01-17, 19h 27m 15s
```

The END

Ana Romero, Universidad de La Rioja
Francis Sergeraert, Institut Fourier, Grenoble
ETH Zurich, June 2012