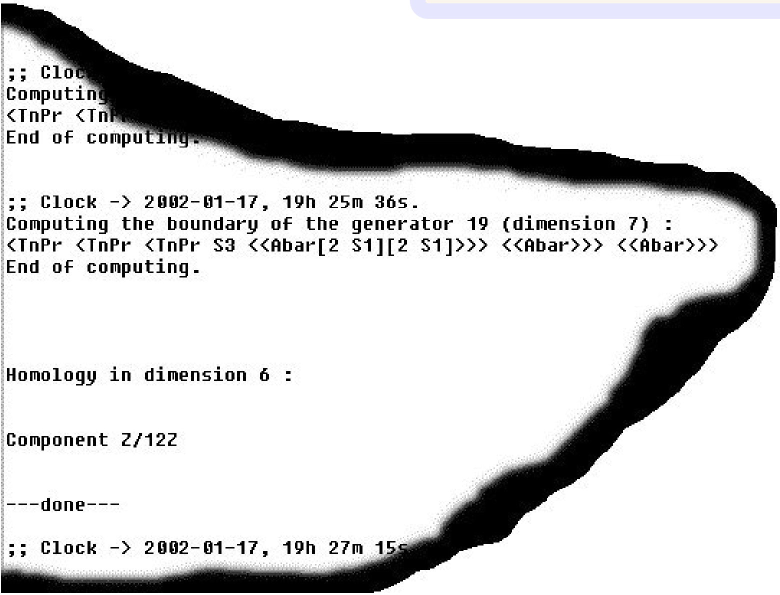
# From a constructiveness problem

# to

# Discrete Vector Fields





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*Mulhouse, May 24, 2012*

**Semantics of colours:**

| | | |
|---|---|---|
| **Blue** | = | **“Standard” Mathematics** |
| **Red** | = | **Constructive, effective, algorithm, machine object, ...** |
| **Violet** | = | **Problem, difficulty, obstacle, disadvantage, ...** |
| **Green** | = | **Solution, essential point, mathematicians, ...** |
| **Dark Orange** | = | **Fuzzy objects.** |
| **Pale grey** | = | **Hyper-Fuzzy objects.** |

**Plan.**

1. **Constructiveness.**
2. **Algebraic Topology and Constructiveness.**
3. **Homological Reduction.**
4. **Homotopy and Vector Fields.**
5. **Discrete Vector Fields.**
6. **Eilenberg-Zilber and Vector Fields.**
7. **Twisted Eilenberg-Zilber.**
8. **Results.**

## 1/8. Constructiveness ? Standard example:

Problem: Determine real numbers $\alpha$ and $\beta$ satisfying:

1. $\alpha, \beta \in \mathbb{R} - \mathbb{Q}$,
2. $\alpha^{\beta} \in \mathbb{Q}$.

"Solution": Consider $\gamma = \sqrt{2}^{\sqrt{2}}$.

Then: If $\gamma \in \mathbb{Q}$, $\alpha = \beta = \sqrt{2}$ = solution. .

Else $\alpha = \gamma$ and $\beta = \sqrt{2}$ = solution. $[(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2]$

But which one of $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$

finally is a solution ???

# 2/8. Standard Algebraic Topology is not constructive.

**Standard Algebraic Topology:**

$$\text{Topological object} \longmapsto \text{Algebraic object}$$

$$\text{Topological Object} \longmapsto \text{Chain Complex} \longmapsto \text{Homology group}$$

**Chain complex:**

$$\cdots \xleftarrow{d_{p-2}} C_{p-2} \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \xleftarrow{d_{p+2}} \cdots$$

$$d_{p-2}d_{p-1} = 0 \qquad d_{p-1}d_p = 0 \qquad d_pd_{p+1} = 0 \qquad d_{p+2}d_{p+1} = 0$$

$$dd = 0 \quad \Leftrightarrow \quad \ker d_p \supset \operatorname{im} d_{p+1} \quad \Rightarrow \quad H_p := \frac{\ker d_p}{\operatorname{im} d_{p+1}}$$

**Standard work in Algebraic Topology:**

1. **Some topological space $X$.**
2. **$\Rightarrow$ Chain complex $(C_*(X), d_*)$.**
3. **$\Rightarrow$ Homology groups $\{H_p(X)\}_{p \in \mathbb{N}}$.**

**But most often $C_*(X)$ not of finite type.**

**$\Rightarrow$ The groups $H_p(X)$ not directly calculable.**

**$\Rightarrow$ How to determine these $H_p(X)$ ???**

**Standard "solution":**

**Use the numerous exact sequences and spectral sequences of the relevant textbooks.**

**Not enough !!**

**Typical example: Serre computing $\pi_6(S^3) = ???$ in 1950:**

**Serre constructs a complicated space $Y$ and his famous spectral sequence produces the exact sequence:**

$$0 \longleftarrow \mathbb{Z}/6 \longleftarrow H_6(Y) \longleftarrow \mathbb{Z}/2 \longleftarrow 0$$

$$\Rightarrow H_6(Y) = \mathbb{Z}/12 \text{ or } \mathbb{Z}/2 \oplus \mathbb{Z}/6 \ ???$$

**Analysis of the problem:**

**Standard Algebraic Topology is non-constructive.**

**Typical example:**

**Exact translation of $H_6(Z)$ "$=$" $\mathbb{Z}/6$:**

**There exists an isomorphism $H_6(C_*(Z)) \cong \mathbb{Z}/6$,**

**but producing such an isomorphism**

**remains unsolved**

**in standard Algebraic Topology.**

**Elementary $\mathbb{Z}$-linear algebra $\Rightarrow$**

**such an isomorphism $\varphi : H_p(C_*) \cong A$**

**for $A$ some $\mathbb{Z}$-module of finite type**

**is elementary when $C_*$ is of finite type.**

**What about the usual case with $C_*(X)$ not of finite type ???**

**3/8. Homological Reduction.**

**Notion of Reduction**

**between big chain complexes and small chain complexes.**

**In particular between chain complexes not of finite type**

**and chain complexes of finite type.**

**<u>Definition</u>: A (homological) reduction is a diagram:**

$$\rho = (f, g, h) = \boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$$

**with:**

1. $\widehat{C}_*$ **and** $C_*$ = **chain complexes.**

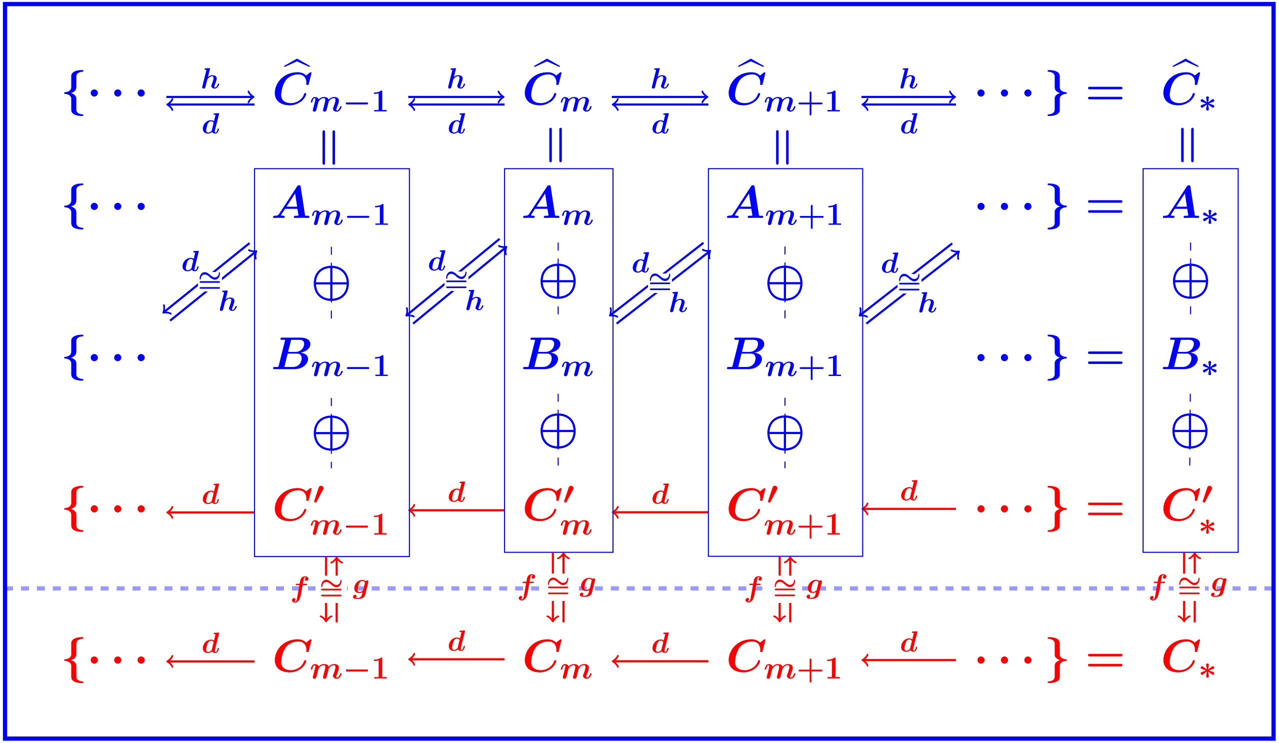
2. $f$ **and** $g$ = **chain complex morphisms.**

3. $h$ = **homotopy operator (degree +1).**

4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.

5. $fh = 0$, $hg = 0$ **and** $hh = 0$.

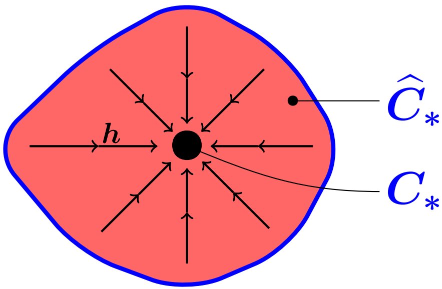
# Meaning = Reduction Diagram:



**In** $\rho = (f, g, h) = \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}_*) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d_*)}$,

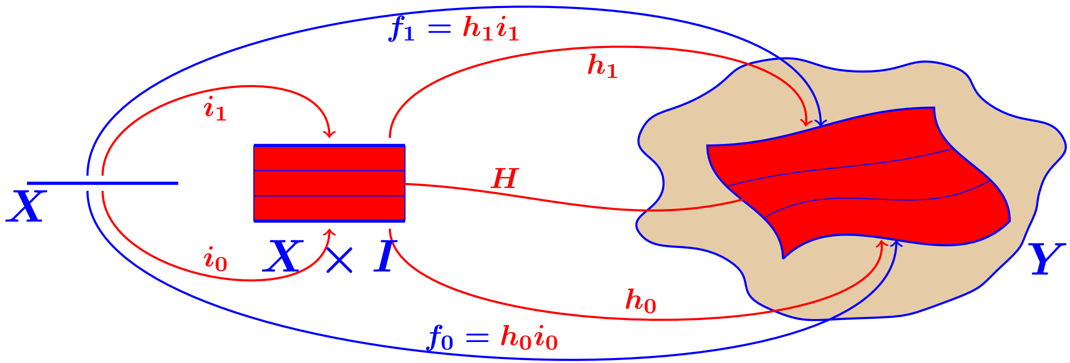
**the most important component is** $h$.

**To be understood as a contraction** $\widehat{C}_* \Rrightarrow C_*$.




**The homotopy** $h$ **contracts the "cherry"** $\widehat{C}_*$

**onto the "stone"** $C_*$.




**General notion of homotopy.**

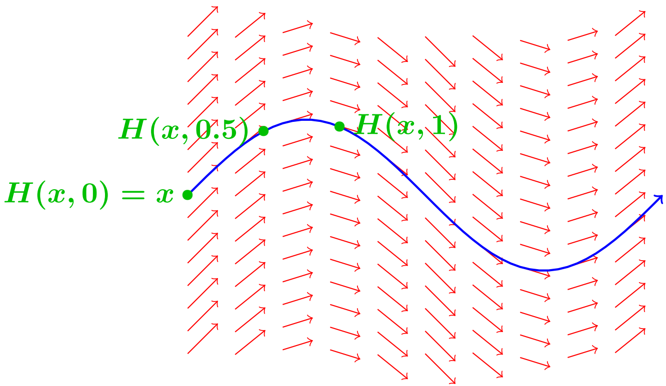
**Given:** $f_0, f_1 : X \to Y$.

**Definition:** $f_0$ **homotope to** $f_1$ $\Leftrightarrow$

$\exists H : X \times I \to Y$ **satisfying:**

$$f_0(x) = H(x, 0)$$
$$f_1(x) = H(x, 1)$$

# 4/8. Vector field ⇒ Flow ⇒ Homotopy.



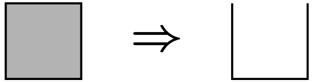

**Vector field** $= V : x \mapsto V(x) \in T_x X$.

⇒ **Corresponding flow** $\Phi(x,t)$:

$$\text{solution of } \left\{ \frac{\partial \Phi(x,t)}{\partial t} = V(\Phi(x,t)) \quad + \quad \Phi(x,0) = x \right\}.$$

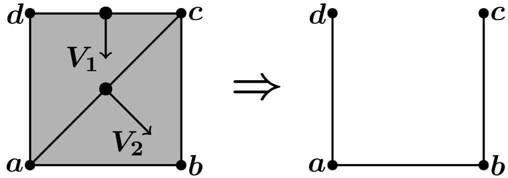
⇒ **Auto-Homotopy** $H(x,t) := \Phi(x,t)$ **for** $t \in [0,1]$.

## 5/8. Corresponding combinatorial notion ??

**Example of contraction:** 

**Appropriate discrete vector field**

**on a triangulation of the square:**




$V_1$ **contracts** $acd \Rightarrow da||ac$ = **cancels the pair** $(cd, dac)$.

$V_2$ **contracts** $abc \Rightarrow ab||bc$ = **cancels the pair** $(ac, abc)$.

$\Rightarrow$ **There remains only** $da||ab||bc$.

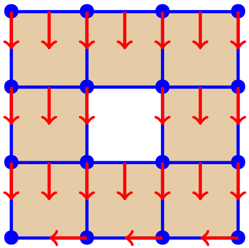
$C$ = cellular complex = collection of cells satisfying...

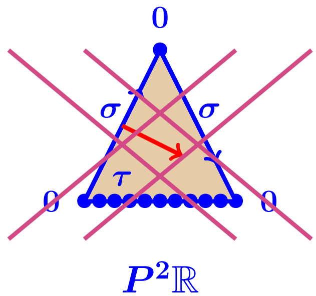
**Definition**: A vector field

is a set of cells $V = \{(\sigma_i, \tau_i)_{i\in v}\}$ satisfying:

- $(\forall i)$ $\sigma_i$ is regular face of $\tau_i$.
- $(\forall i \neq j)$ $\sigma_i \neq \sigma_j \neq \tau_i \neq \tau_j$.

**Examples**:






$\sigma_i$ regular face of $\tau_i \Leftrightarrow$ incidence number $\varepsilon(\sigma_i, \tau_i) = \pm 1$.

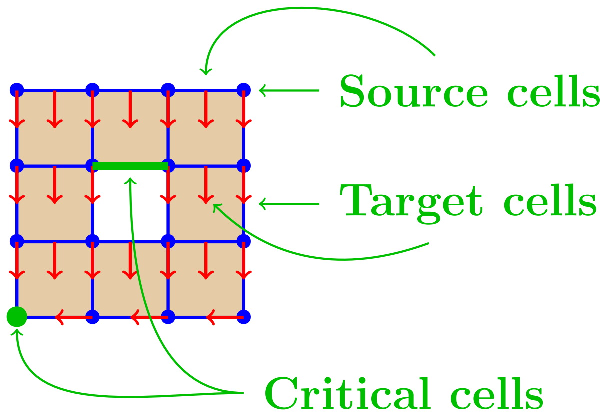
$C$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)_{i \in v}\}$ = Vector field.

**Definition**: A critical $p$-cell is a $p$-cell which does not occur in $V$.

Other cells divided in source cells and target cells.

**Example**:



$\boldsymbol{C}$ = **Cellular complex.**

$\boldsymbol{V} = \{(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i)_{i \in v}\}$ = **vector field.**

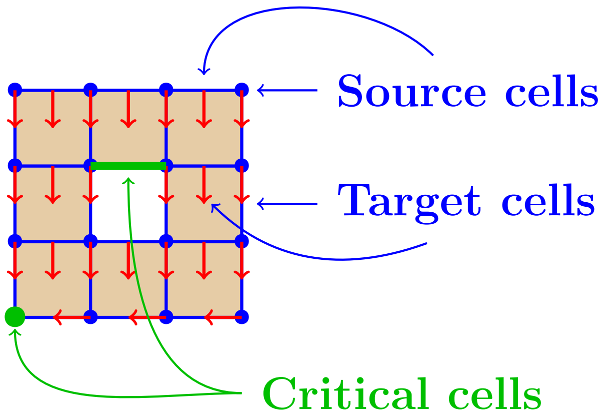
**<u>Theorem</u>: A "critical" chain complex $C_*^c = ((\beta_p^c)_{p \in \mathbb{Z}}, d^c)$ can be constructed:**

- $\beta_p^c$ = **the set of critical $p$-cells of $V$.**
- $d_p^c : \mathbb{Z}[\beta_p^c] \to \mathbb{Z}[\beta_{p-1}^c]$ **an appropriate "critical" differential deduced from the initial differential $d$ and the vector field $V$.**

**Also a canonical reduction $\rho : C_* \Rrightarrow C_*^c$ is provided.**

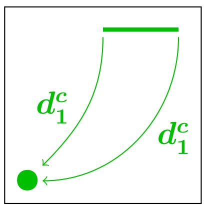
$\Rightarrow$ **Any homological problem in $C_*$ can be solved in $C_*^c$.**

## Simple example.

$C =$ 


$$C_* = \{0 \leftarrow \mathbb{Z}^{16} \leftarrow \mathbb{Z}^{24} \leftarrow \mathbb{Z}^{8} \leftarrow 0\}$$

**Theorem** $\Rightarrow$

$\rho : C_* \Rrightarrow C^c_* =$  $= \mathbb{Z} \xleftarrow{d^c_1 = 0} \mathbb{Z} =$ **Circle**

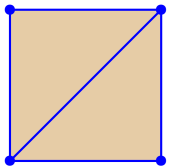
**6/8. Eilenberg-Zilber Vector Field.**

**Vector Fields ⇒ Eilenberg-Zilber**

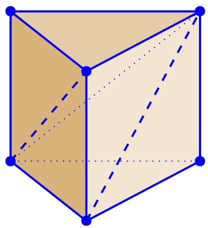
**⇒ Twisted Eilenberg-Zilber ⇒ Serre spectral sequence**

**⇒ Eilenberg-Moore spectral sequence.**

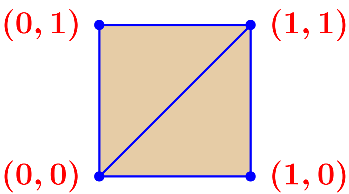
**Main problem: Triangulation of $\boldsymbol{\Delta}^p \times \boldsymbol{\Delta}^q$ ???**



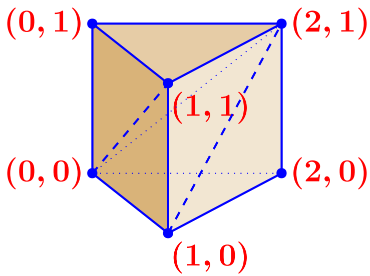
$\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$



$\boldsymbol{\Delta}^2 \times \boldsymbol{\Delta}^1$




$\Delta^1 \times \Delta^1$




$\Delta^2 \times \Delta^1$

**Two $\Delta^2$ in $\Delta^1 \times \Delta^1$:** $(0,0) < (0,1) < (1,1)$

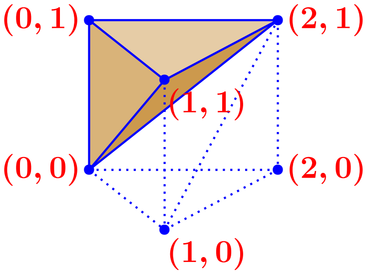
$(0,0) < (1,0) < (1,1)$

**Three $\Delta^3$ in $\Delta^2 \times \Delta^1$:** $(0,0) < (0,1) < (1,1) < (2,1)$

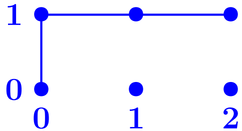
$(0,0) < (1,0) < (1,1) < (2,1)$

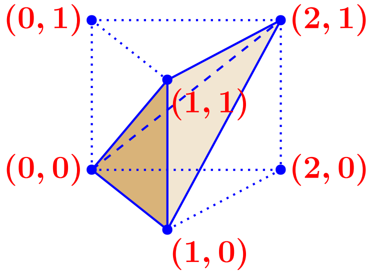
$(0,0) < (1,0) < (2,0) < (2,1)$

# Rewriting the triangulation of $\Delta^2 \times \Delta^1$.

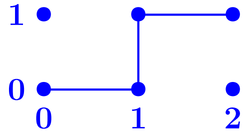


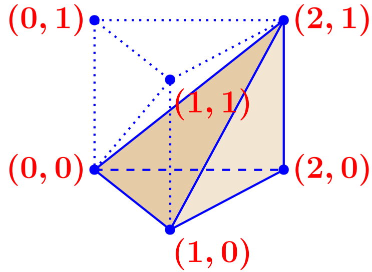

$(0,0) < (0,1) < (1,1) < (2,1)$



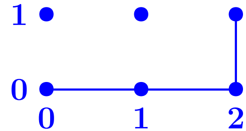




$(0,0) < (1,0) < (1,1) < (2,1)$

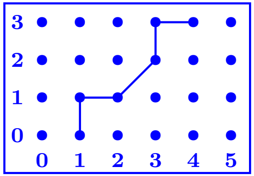



$(0,0) < (1,0) < (2,0) < (2,1)$

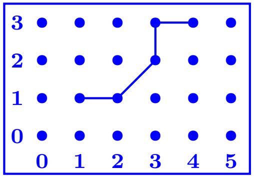


# “Seeing” the triangulation of $\Delta^5 \times \Delta^3$.

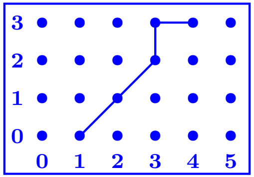
**Example of 5-simplex :** 

$= \sigma \in (\Delta^5 \times \Delta^3)_5$

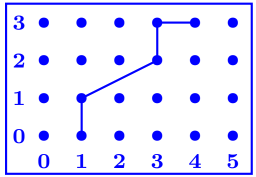
**$\Rightarrow$ 6 faces:**



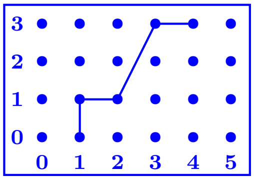

$\partial_0\sigma$



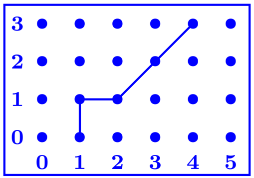

$\partial_1\sigma$



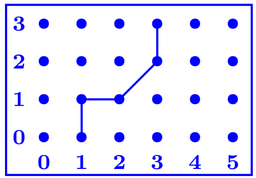

$\partial_2\sigma$




$\partial_3\sigma$

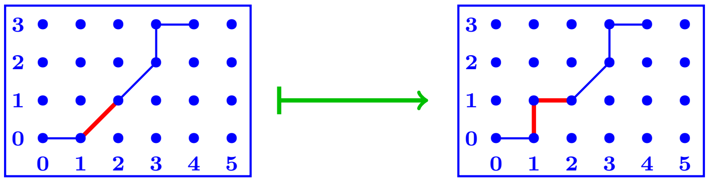



$\partial_4\sigma$

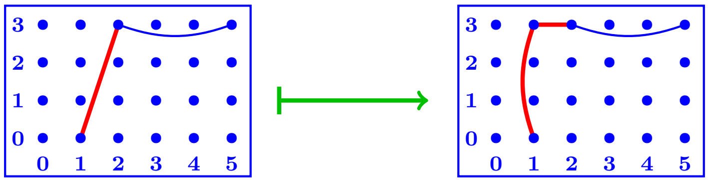



$\partial_5\sigma$

# ⇒ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

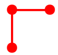



Source $= \partial_2($ Target$)$




Source $= \partial_1($ Target$)$

**Recipe:** First "event" = Diagonal step =  ⇒ Source cell.

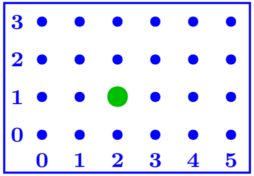
= (-90°)-bend =  ⇒ Target cell.

# Critical cells ??

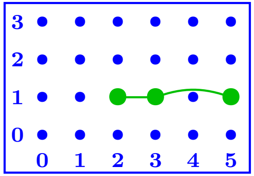
**Critical cell = cell without any "event"**

**= without any diagonal or −90°-bend.**

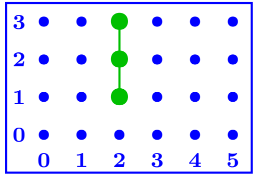
**Examples.**



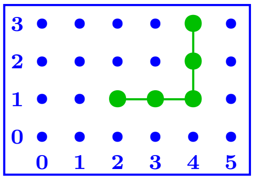

$\Delta^0_2 \times \Delta^0_1$



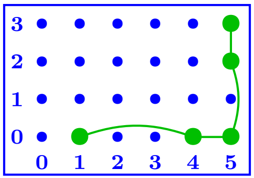

$\Delta^2_{2,3,5} \times \Delta^0_1$



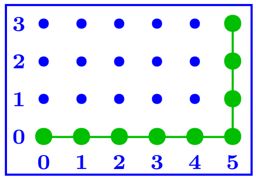

$\Delta^0_2 \times \Delta^2_{1,2,3}$




$\Delta^2_{2,3,4} \times \Delta^2_{1,2,3}$




$\Delta^2_{1,4,5} \times \Delta^2_{0,2,3}$




$\Delta^5_{0,1,2,3,4,5} \times \Delta^3_{0,1,2,3}$

**Conclusion:**

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

**Fundamental theorem of vector fields** $\Rightarrow$

**Canonical Homological Reductions:**

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

More generally: $X$ and $Y$ = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$.

Eilenberg-Zilber Theorem: Canon. homological reduction:

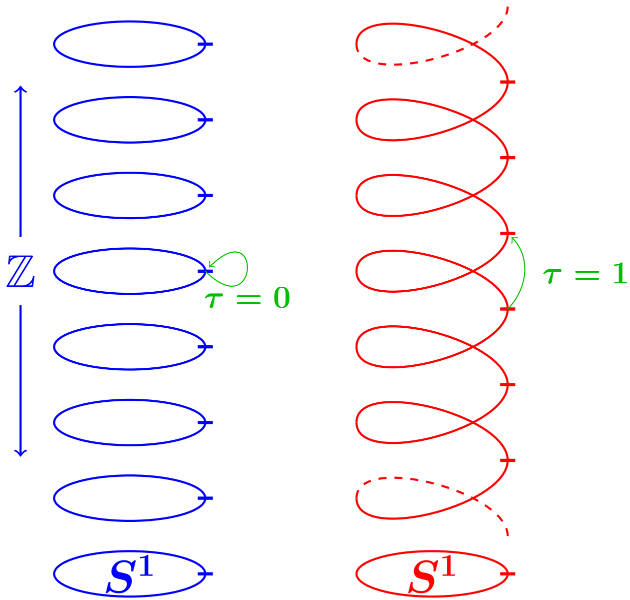
$$\rho_{EZ} : C_*(X \times Y) \Rrightarrow C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

# 7/8. Twisted Eilenberg-Zilber.

**Notion of twisted product.**

**Simplest example:** $\mathbb{Z} \times \boldsymbol{S}^1$ **vs** $\mathbb{Z} \times_{\boldsymbol{\tau}} \boldsymbol{S}^1 = \mathbb{R}$**:**



**General notion of twisted product:** $B$ = base space.

$F$ = fibre space.

$G$ = structural group.

Action $G \times F \to F$.

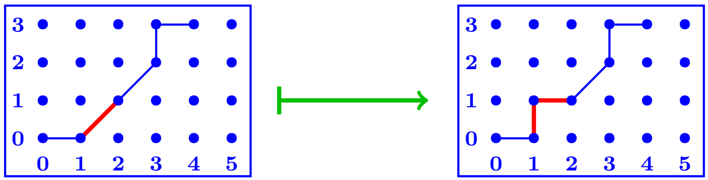
$\tau : B \to G$ = Twisting function.

**Structure of $F \times_\tau B$:**

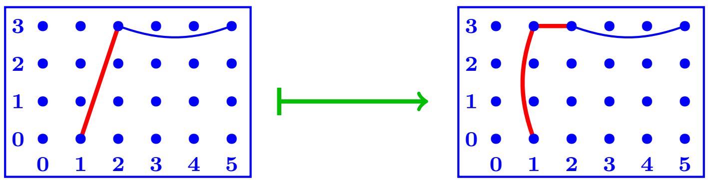
$$\partial_i(\sigma_f, \sigma_b) = (\partial_i\sigma_f, \partial_i\sigma_b) \quad \text{for } i > 0$$
$$\partial_0(\sigma_f\ ,\ \sigma_b) = (\tau(\sigma_b).\partial_0\sigma_f\ ,\ \partial_0\sigma_b)$$

$\Rightarrow$ **Only the 0-face is modified in the twisted product.**

# Reminder about the EZ-vector field of $\Delta^5 \times \Delta^3$.




Source $= \partial_2($ Target$)$




Source $= \partial_1($ Target$)$

The vector field is concerned by faces $\partial_i$ only if $i > 0$.

1. The twisting function $\tau$ modifies only $\boxed{0}$-faces.

2. The EZ-vector field $V_{EZ}$ of $X \times Y$

uses only $\boxed{i}$-faces with $i \geq 1$.

$\Rightarrow$ $V_{EZ}$ is defined and admissible as well on $X \times_{\boxed{\tau}} Y$.

Fundamental theorem of admissible vector fields $\Rightarrow$

$$V_{EZ} \Rightarrow \begin{array}{c} C_*(X \times Y) \\ \Downarrow \\ C_*(X) \otimes C_*(Y) \end{array} \qquad\qquad V_{EZ} \Rightarrow \begin{array}{c} C_*(X \times_{\boxed{\tau}} Y) \\ \Downarrow \\ C_*(X) \otimes_{\boxed{t}} C_*(Y) \end{array}$$

Known as the twisted Eilenberg-Zilber Theorem.

**Corollary:** Base $B$ 1-reduced $\Rightarrow$ Algorithm:

$$[(F, C_*(F), EC_*^F, \varepsilon_F) + (B, C_*(B), EC_*^B, \varepsilon_B) + G + \tau]$$
$$\longmapsto (F \times_\tau B, C_*(F \times_\tau B), EC_*^{F \times_\tau B}, \varepsilon_{F \times_\tau B}).$$

Version of $F$ with effective homology

+ Version of $B$ with effective homology

+ $G + \tau$ describing the fibration $F \hookrightarrow F \times_\tau B \to B$

$\Rightarrow$ Version with effective homology of the total space $F \times_\tau B$.

= Version with effective homology

of the Serre Spectral Sequence

**Analogous result for the Eilenberg-Moore spectral sequence.**

**Key results:**

$G$ = Simplicial group $\Rightarrow$ $BG$ = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

$X$ = Simplicial set $\Rightarrow$ $KX$ = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

**Analogous process $\Rightarrow$ Algorithms:**

$$(G, C_*G, EC_*^G, \varepsilon_G) \mapsto (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(G, C_*X, EC_*^X, \varepsilon_X) \mapsto (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

## 8/8. Main result.

**"Standard" Algebraic Topology becomes constructive.**

**The most basic standard spectral sequences**

**(Serre, Eilenberg-Moore) become effective.**

**Example:** $\pi_5(\Omega S^3 \cup_2 D^3) = (\mathbb{Z}/2)^4$.

$$\#[S^5, \mathrm{Cont}(S^1, S^3) \cup_2 D^3] = 16$$

**Computing plan: Five fibrations (twisted products):**

$$X_2 = \Omega S^3 \cup_2 D^3$$

$$\begin{array}{lcl} K(\mathbb{Z}/2,1) \to X_3 \to X_2 & & \pi_2(X_2) = \mathbb{Z}/2 \\ K(\mathbb{Z}/2,2) \to X_4 \to X_3 & & \pi_3(X_3) = \mathbb{Z}/2 \\ K(\mathbb{Z},3) \to X_4' \to X_4 & & \pi_4(X_4) = \mathbb{Z}/4 \oplus \mathbb{Z} \\ K(\mathbb{Z}/2,3) \to X_4'' \to X_4' & & \pi_4(X_4') = \mathbb{Z}/4 \\ K(\mathbb{Z}/2,3) \to X_5 \to X_4'' & & \pi_4(X_4'') = \mathbb{Z}/2 \end{array}$$

**Applying five times our effective version**

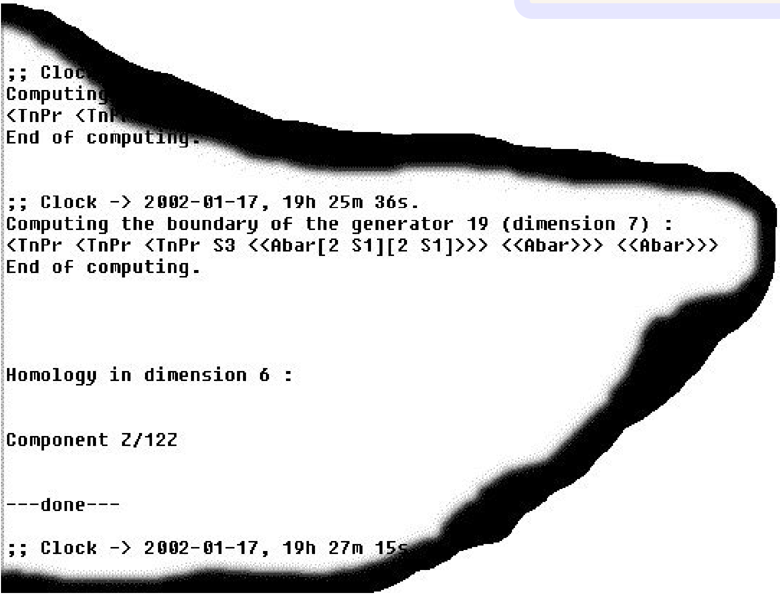
**of the Serre spectral sequence and**

**five times the effective version**

**of the Eilenberg-Moore spectral sequence**

$$\Rightarrow \quad \pi_5(X_2) = \pi_5(X_5) = H_5(X_5) = (\mathbb{Z}/2)^4.$$

# The END





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