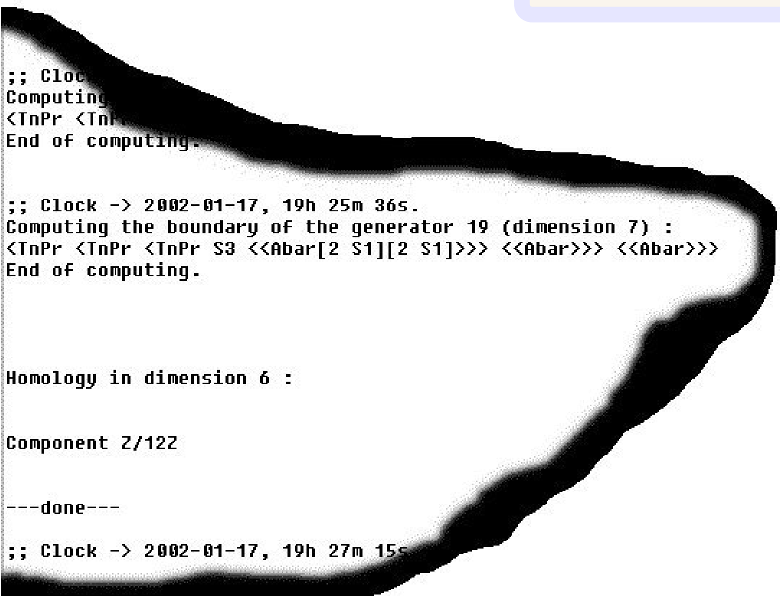
Effective

Moore-Postnikov

Factorization



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Semantics of colours:

Blue	=	**"Standard" Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

<u>Plan</u>.

1. **Eilenberg-MacLane spaces.**

2. **Map $\Rightarrow$ Mapping Cone.**

3. **Postnikov's fundamental Theorem.**

4. **Postnikov construction**
   for spaces with effective homology.

5. **Postnikov construction for square diagrams.**

1/5. $\boxed{\text{Eilenberg-MacLane spaces}}$.

Given: π = Abelian group ; $2 \leq n \in \mathbb{N}$.

$K(\pi, n)$:= CW-complex **well-defined** up to homotopy **by:**

$$\pi_n(K(\pi, n)) = \pi \quad \text{and} \quad \pi_k(K(\pi, n)) = 0 \text{ for } k \neq n$$

$K(\pi, n)$ represents **the** functor:

$$\underline{\text{Top}} \ni X \longmapsto H^n(X; \pi) \in \underline{\text{Ab}}$$

Characteristic class: $\chi_{\pi,n} \in H^n(K(\pi, n), \pi)$.

Canonical bijection**:**

$$[X, K(\pi, n)] \ni f \xleftrightarrow{\cong} f^*(\chi_{\pi,n}) \in H^n(X; \pi)$$

Canonical simplicial model of $K(\pi, n)$

by Eilenberg and MacLane.

$$E(\pi, n)_k \ := \ C^n(\Delta^k, \pi)$$

$$K(\pi, n)_k \ := \ Z^n(\Delta^k, \pi)$$

If $\alpha : [0...k] \to [0...\ell] = \Delta^k \to \Delta^\ell$ and $s \in E(\pi, n)_\ell$, then:

$$\alpha^*(s) := \alpha^*(s)$$

$$\alpha : 0...k \to 0...\ell \ \Rightarrow \ \alpha^* : E(\pi, n)_\ell \to E(\pi, n)_k$$

$$\alpha : \Delta^k \to \Delta^\ell \ \Rightarrow \ \alpha^* : C^n(\Delta^\ell, \pi) \to C^n(\Delta^k, \pi)$$

$\boxed{\text{Normalized}}$ **cochains (and cocycles):**

$$C^n(\Delta^k, \pi) := \{c : \Delta^k_n \to \pi\}$$

$$\Delta^k_n := \{0 \le i_0 \le \cdots \le i_n \le k\}$$

$$\Delta^k_{n,ND} := \{0 \le i_0 < \cdots < i_n \le k\} \quad (= \emptyset \text{ if } k < n)$$

$$\Delta^k_{n,D} := \Delta^k_n - \Delta^k_{n,ND}$$

Cochain c normalized $:= \; c|\Delta^k_{n,D} \equiv 0$

$$\Rightarrow \quad [(k < n) \;\Rightarrow\; C^n(\Delta^k, \pi) = 0 = Z^n(\Delta_k, \pi)]$$

$\Rightarrow$ **Eilenberg-MacLane's $K(\pi, n)$ begins only in dimension n.**

$$K(\pi, n)_k = \{*_k\} = \{0 \in Z^n(\Delta^k, \pi)\} \text{ for } k < n$$

$$K(\pi, n)_n = \pi$$

Canonical identifications:

Given: $2 \leq n \in \mathbb{N}$, $\pi \in \underline{\mathrm{Ab}}$, $X =$ simplicial set.

$\underline{\mathrm{SSet}}$:= Category of simplicial sets.

$\underline{\mathrm{SSet}}(\mathrm{X,Y})$:= $\{$Simplicial maps : $X \to Y\}$.

Then:

$$C^n(X, \pi) \;\cong\; \underline{\mathrm{SSet}}(X, E^n(\pi, n))$$

$$Z^n(X, \pi) \;\cong\; \underline{\mathrm{SSet}}(X, Z^n(\pi, n))$$

Proof: abstract nonsense.

Universal characteristic objects:

$$\underline{\mathbf{SSet}}(X, E^n(\pi, n)) \leftrightarrow C^n(X, \pi)$$

$$\underline{\mathbf{SSet}}(E(\pi, n-1), E(\pi, n-1)) \ni \mathrm{id} \leftrightarrow$$
$$\leftrightarrow \chi_{E,\pi,n-1} \in C^{n-1}(E(\pi, n-1), \pi)$$

$$\Rightarrow \chi_{E,\pi,n-1} = \text{Universal characteristic cochain.}$$

$$\underline{\mathbf{SSet}}(X, Z^n(\pi, n)) \leftrightarrow Z^n(X, \pi)$$

$$\underline{\mathbf{SSet}}(K(\pi, n), K(\pi, n)) \ni \mathrm{id} \leftrightarrow \chi_{K,\pi,n} \in Z^n(K(\pi, n), \pi)$$

$$\chi_{K,\pi,n} = \text{Universal characteristic cocycle.}$$

Canonical Eilenberg-MacLane Kan-fibration:

$$K(\pi, n-1) \hookrightarrow E(\pi, n-1) \xrightarrow{p} K(\pi, n)$$

$$p : E(\pi, n-1)_k = C^{n-1}(\Delta^k, \pi) \ni c \mapsto dc \in Z^n(\Delta^k, n) = K(\pi, n)_k$$

Δ^k contractible

$\Rightarrow$ **every cocycle is a coboundary**

$\Rightarrow$ **short exact sequence:**

$$0 \to Z^{n-1}(\Delta^k, \pi) \xrightarrow{i} C^{n-1}(\Delta^k, \pi) \xrightarrow{d} Z^n(\Delta^k, \pi) \to 0$$

$\Rightarrow$ $p := d : E(\pi, n-1) \to K(\pi, n)$ **surjective.**

In a diagram:

$$\begin{tikzcd} & E(\pi, n-1) \arrow[d, "p"] \\ X \arrow[ur, "F"] \arrow[r, "f"'] & K(\pi, n) \end{tikzcd}$$

$$\chi_E \in C^{n-1}(E(\pi, n-1), \pi)$$

$$\chi_K \in Z^n(K(\pi, n), \pi)$$

$$\underline{\mathrm{SSet}}(X, K(\pi, n)) \ni f \xleftrightarrow{\cong} f^*(\chi_K) \in Z^n(X, \pi)$$

$$\underline{\mathrm{SSet}}(E(\pi, n-1), E(\pi, n-1)) \ni \mathrm{id} \longleftrightarrow \chi_E \in C^{n-1}(E(\pi, n-1))$$

$$\underline{\mathrm{SSet}}(E(\pi, n-1), K(\pi, n)) \ni p \longleftrightarrow p^*(\chi_K) \in Z^n(E(\pi, n-1), \pi)$$

$$\boxed{p^*(\chi_K) = d(\chi_E)}$$

$$f^*(\chi_K) = F^* p^*(\chi_K) = F^* d(\chi_E) = d(F^*(\chi_E))$$

Theorem: In the diagram:

$$\begin{tikzcd} & E(\pi, n-1) \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "F\ ??"] & K(\pi, n) \end{tikzcd}$$

$$\chi_E \in C^{n-1}(E(\pi, n-1), \pi)$$

$$\chi_K \in Z^n(K(\pi, n), \pi)$$

a lifting F of f along p can be defined

if and only if $f^*(\chi_K) \in Z^n(X, \pi)$ is a coboundary:

$f^*(\chi_K) = dc$ with $c \in C^{n-1}(X, \pi)$.

$$C^{n-1}(X, \pi) \ni c \leftrightarrow F \in \underline{\mathrm{SSet}}(X, E(n-1, \pi))$$

Cocycle $\Rightarrow$ Principal Fibration:

Given: $X \in \underline{\text{SSet}}$, $z \in Z^n(X;\pi)$.

$\Rightarrow$ **Principal Fibration:**

$$K(\pi, n-1) \hookrightarrow X \;{}_z\widetilde{\times}_p\; E(\pi, n-1) \xrightarrow{p'} X$$

$$\begin{tikzcd}
X \;{}_z\widetilde{\times}_p\; E(\pi, n-1) \arrow[r, "c'"] \arrow[d, "p'"'] & E(\pi, n-1) \arrow[d, "p"] \\
X \arrow[r, "z"'] & K(\pi, n)
\end{tikzcd}$$

with $[X \;{}_z\times_p E(\pi, n-1)]_k :=$

$$\{(a, b) \in X_k \times E(\pi, n-1)_k \ \underline{\text{st}}\ z(a) = p(b)\}$$

<u>Proposition</u>: Given: A = simplicial set;

π = abelian group;

$z, z' \in Z^{n+1}(A, \pi)$.

Then the $K(\pi, n)$-principal fibrations defined by z and z' are isomorphic if and only if z and z' are cohomologous.

A cochain c satisfying $dc = z' - z$ canonically defines such an isomorphism.

$$\begin{tikzcd}
 & (\sigma, \tau + c(\sigma)) \in A \ {}_{z'}\times_p \ E(\pi, n) \arrow[dr] \arrow[dd] & \\
(\sigma, \tau) \in A \ {}_{z}\times_p \ E(\pi, n) \arrow[ur, "\varphi"] \arrow[ur, shift right, "\varphi = ???"'] \arrow[rr] \arrow[dd] & & E(\pi, n) \ni \tau \arrow[dd, "p"] \\
 & \sigma \in A \arrow[dr, "z'"'] & \\
\sigma \in A \arrow[rr, "z"'] & & K(\pi, n+1)
\end{tikzcd}$$

<u>**Solution**</u>: $\varphi(\sigma, \tau) = (\sigma, \tau + c(\sigma))$.

$$(\sigma, \tau) \in \ {}_{z}\times_p \Leftrightarrow z(\sigma) = p(\tau)$$

$$(\sigma, \tau + c(\sigma)) \overset{?}{\in} \ {}_{z'}\times_p \Leftrightarrow z'(\sigma) \overset{?}{=} p(\tau + c(\sigma))$$

But $z'(\sigma) = z(\sigma) + dc(\sigma) = p(\tau) + p(c(\sigma)) \Rightarrow$ **OK.** **QED**

Generalized lifting problem.

$$
\begin{tikzcd}
 & X \, {}_z\widetilde{\times}_p \, E(\pi, n-1) \arrow[r, "c'"] \arrow[d, "p'"'] & E(\pi, n-1) \arrow[d, "p"] \\
A \arrow[r, "f"'] \arrow[ur, dashed, "F\ ??"] \arrow[urr, dashed, bend right=10, "c\ ??"] & X \arrow[r, "z"'] & K(\pi, n) \ni \chi_K
\end{tikzcd}
$$

Theorem: A lifting F of f along p' can be defined

if and only if $f^* z^* \chi_K$ is a coboundary $f^* z^* \chi_K = dc$.

Proof: $c \in C^{n-1}(A, \pi) = \underline{\mathrm{SSet}}(A, E(\pi, n-1))$.

Take:

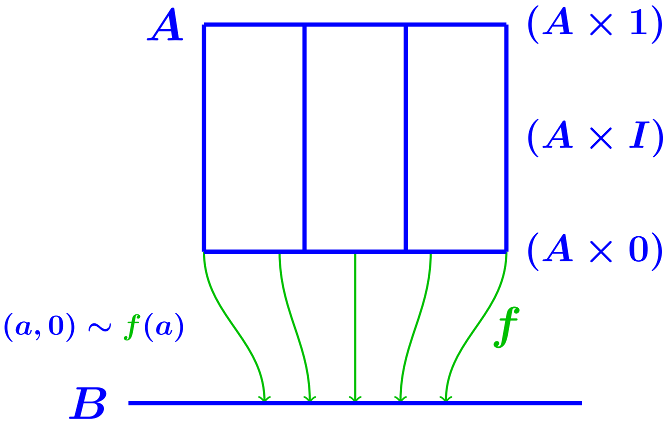
$$F := (f, c)$$

QED

2/5. $\boxed{\text{Map} \Rightarrow \text{Mapping Cone.}}$

Mapping cylinder of $f : A \longrightarrow B$:

$$\mathrm{Cyl}(f) := [(A \times I) \coprod B] \;/\; [(A, 0) \sim f(a)].$$

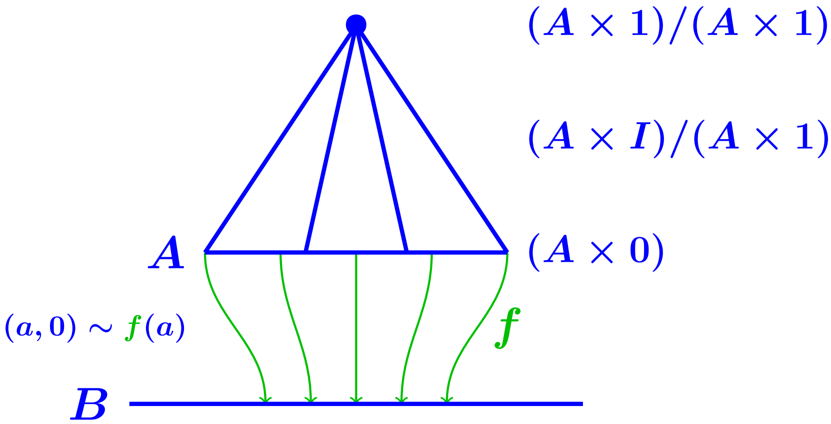


The canonical projection $\mathrm{Cyl}(f) \longrightarrow B$

is a homotopy equivalence.

Mapping cone of $f : A \longrightarrow B$:

$$\mathrm{Cone}(f) = [B \coprod (A \times I)] \;/\; [(A \times 1) \;\&\; (a,0) \sim f(a)].$$



$$\mathrm{Cone}(f) := \mathrm{Cyl}(f)/(A \times 1)$$

Algebraic translation.

<u>Definition</u>: Given:

$$f : C'_* \longrightarrow C_*$$

= chain complex morphism.

Then Cone(f) is

the chain complex T_*

defined by:

$$\begin{tikzcd}
{} \arrow[d, dashed] & & \\
C_{n+1} \arrow[d, "d_{n+1}"'] & {} \arrow[l, dashed] \arrow[d, dashed] & \\
C_n \arrow[d, "d_n"'] & C'_n \arrow[l, "f_n"] \arrow[d, "-d'_n"] & {} \arrow[d, dashed] \\
C_{n-1} \arrow[d, dashed] & C'_{n-1} \arrow[l, "f_{n-1}"] \arrow[d, "-d'_{n-1}"] & T_{n+1} \arrow[d, "d^T_{n+1}"] \\
{} & C'_{n-2} \arrow[l, dashed] \arrow[d, dashed] & T_n \arrow[d, "d^T_n"] \\
& {} & T_{n-1}
\end{tikzcd}$$

$$T_n := C_n \oplus C'_{n-1}$$

$$d^T_n : T_n \longrightarrow T_{n-1} := \begin{bmatrix} d_n & f_{n-1} \\ 0 & -d'_{n-1} \end{bmatrix}$$

Notation: $\mathrm{Cone}(f) = C_* \oplus_f {C'_*}^{[1]}$

Remarks:

C_* subcomplex of

$$\mathrm{Cone}(f) = C_* \oplus_f {C'_*}^{[1]}$$

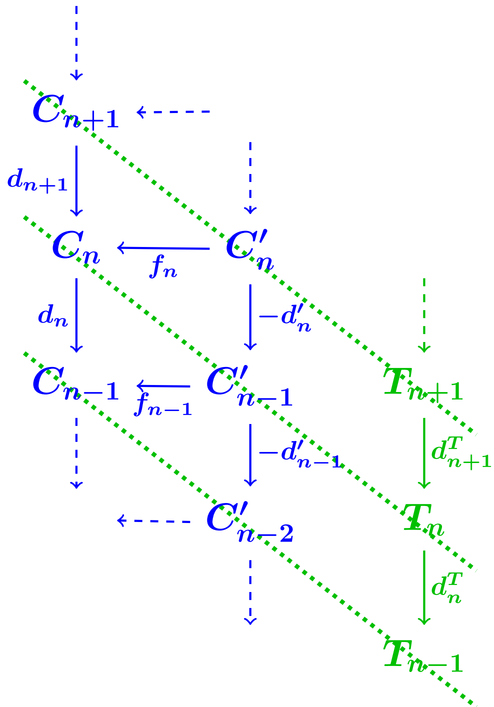
${C'_*}^{[1]}$ $\boxed{\text{not}}$ subcomplex of

$$\mathrm{Cone}(f) = C_* \oplus_f {C'_*}^{[1]}$$

$${C'_*}^{[1]} = \{C_* \oplus_f {C'_*}^{[1]}\} \,/\, C_*$$

$\Leftrightarrow$ short exact sequence:

$$0 \to C_* \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} {C'_*}^{[1]} \to 0$$



$\Rightarrow$ Long homology exact sequence:

$$\dashrightarrow H_{n+1}(C'^{[1]}_*) \longrightarrow H_n(C_*) \longrightarrow H_n(\mathrm{Cone}(f)) \longrightarrow H_n(C'^{[1]}_*) \longrightarrow H_{n-1}(C_*) \dashrightarrow$$

Better viewed as:

$$\dashrightarrow H_{n+1}(\mathrm{Cone}(f)) \xrightarrow{p} H_n(C'_*) \xrightarrow{f_*} H_n(C_*) \xrightarrow{i} H_n(\mathrm{Cone}(f)) \xrightarrow{p} H_{n-1}(C'_*) \dashrightarrow$$

Analogous long homotopy exact sequence.

$$\dashrightarrow \pi_{n+1}(\mathrm{Cyl}(f), A) \xrightarrow{p} \pi_n(A) \xrightarrow{f_*} \pi_n(C_*) \xrightarrow{i} \pi_n(\mathrm{Cyl}(f), A) \xrightarrow{p} \pi_{n-1}(A) \dashrightarrow$$

No excision in homotopy: $\pi_n(\mathrm{Cyl}(f), A) \not\cong \pi_n(\mathrm{Cone}(f))$.

<u>Definition</u>: The map f is n-connected

if $\pi_i(\mathrm{Cyl}(f), A) = 0$ for $0 \leq i \leq n$.

$\Longleftrightarrow$

$$f_i : \pi_i(C'_*) \to \pi_i(C_*) = \text{isomorphism for } 0 \leq i < n,$$

$$+ \;\; f_n : \pi_n(C'_*) \to \pi_n(C_*) = \text{epimorphism.}$$

Also $\mathrm{Cone}(f) := \mathrm{Cyl}(f)/(A \times 1 = A)$

and $\widetilde{H}_*(\mathrm{Cone}(f)) = H_*(\mathrm{Cyl}(f), A)$

Hurewicz Theorem $\Rightarrow$

$$
\begin{aligned}
f \text{ is } n\text{-connected} := \quad & \pi_i(\mathrm{Cyl}(f), A) = 0 \ \text{ for } 0 \le i \le n \\
\Leftrightarrow\ & H_i(\mathrm{Cyl}(f), A) = 0 \ \text{ for } 0 \le i \le n \\
\Leftrightarrow\ & \widetilde{H}_i(\mathrm{Cone}(f)) = 0 \ \text{ for } 0 \le i \le n \\
\Leftrightarrow\ & \pi_i(\mathrm{Cone}(f)) = 0 \ \text{ for } 0 \le i \le n.
\end{aligned}
$$

3/5. Postnikov's fundamental theorem.

$***$ Simplicial Set Category $***$

$***$ All objects assumed 1-connected $***$

Postnikov Theorem: Given: $f : A \to B =$ an n-connected map.

Then $\exists$ a factorization:

$$\begin{tikzcd} & B' = B \times_\tau K(\pi, n) \arrow[d, "p"] \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & B \end{tikzcd}$$

with: f' $(n+1)$-connected;

p a principal fibration:

$$K(\pi, n) \to [B \times_\tau K(\pi, n) = B'] \xrightarrow{p} B \quad ;$$

$$\pi = H_{n+1}(\mathrm{Cone}(f)).$$

Proof.

$$
\begin{tikzcd}
C_n(\mathrm{Cone}(f)) & C_{n+1}(\mathrm{Cone}(f)) \arrow[l, "d"'] \arrow[ddddd, bend right, "k_n"'] & C_{n+2}(\mathrm{Cone}(f)) \arrow[l, "d"'] \arrow[ddddl, "d"] \\
 & \cup & \\
0 \arrow[uu] & Z_{n+1} \arrow[l, "d"'] \arrow[ddd, bend right, "\mathrm{pr}"'] & \\
 & \cup & \\
 & B_{n+1} & \\
 & H_{n+1}(\mathrm{Cone}(f)) = Z_{n+1}/B_{n+1} =: \pi &
\end{tikzcd}
$$

$$C_{n+1}(\mathrm{Cone}(f)) = Z_{n+1} \oplus \text{(some) } Z_{n+1}^{\perp}$$

$\Rightarrow k_n : C_{n+1} \longrightarrow H_{n+1} = \pi =$ non-canonical extension of pr.

$$d(k_n) = k_n \circ d = 0$$

$\Rightarrow H^{n+1}(\mathrm{Cone}(f), \pi) \ni k_n =$ Postnikov $\boxed{\text{class}}$ $k_n(f)$.

<u>Lemma</u>: $k_n(f)$ as a cohomology class unambiguously defined.

$$
\begin{tikzcd}
C_n \arrow[dddd, bend left, "\mathrm{pr}''"] & & C_{n+1} \arrow[ll, "d"'] \arrow[dd, bend left, "\mathrm{pr}'"] \arrow[dddddd, bend right, "k_n(f)"'] \\
\| & & \| \\
Z_n^{\perp} & & Z_{n+1}^{\perp} \arrow[ddll, "d", "\cong"'] \arrow[dddd, bend left, "\varepsilon"] \\
\oplus & & \oplus \\
Z_n = B_n & & Z_{n+1} \arrow[dd, "\mathrm{pr}"'] \\
\updownarrow & & \\
H_n = 0 & & H_{n+1}
\end{tikzcd}
$$

$\Leftrightarrow$ ε arbitrary

$\Rightarrow$ $\varepsilon.\mathrm{pr}' \stackrel{??}{=}$ coboundary ??

But $\varepsilon.\mathrm{pr}' = \varepsilon d^{-1}\mathrm{pr}'' d$

$= d(\varepsilon d^{-1}\mathrm{pr}'')$.

QED

Remark: $H_n = 0 \quad \Rightarrow \quad \mathrm{pr}''$ defined.

Postnikov Construction = Diagram of $K(\pi, n)$-fibrations:

$$
\begin{tikzcd}
E_A \arrow[r] \arrow[d] & E_B \arrow[r, "\sim"] \arrow[d] & E_{\mathrm{Cyl}} \arrow[r] \arrow[d] & E_{\mathrm{Cone}} \arrow[r] \arrow[d] & E(\pi, n) \arrow[d] \\
A \arrow[r, "f"] \arrow[ur, "f'"] \arrow[u, bend left] & B \arrow[r, "\sim"] & \mathrm{Cyl} \arrow[r, "\mathrm{pr}"] & \mathrm{Cone} \arrow[r, "z"] & K(\pi, n+1)
\end{tikzcd}
$$

Claim: f n-connected $+$ $z = k_n(f) \Rightarrow f'$ $(n+1)$-connected.

Postnikov Construction = Functor:

$$\{[f : A \to B], z \in Z^{n+1}(\mathrm{Cone}(f), \pi)\} \longmapsto$$

$$\longmapsto \{\text{Postnikov Diagram}\}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & \mathrm{Cyl}(f) \arrow[r] \arrow[d] & \mathrm{Cone}(f) \arrow[dr, "z"] \arrow[d] & \\
 & & & & K(\pi, n+1) \\
A' \arrow[r, "f'"] & B' \arrow[r] & \mathrm{Cyl}(f') \arrow[r] & \mathrm{Cone}(f') \arrow[ur, "z'"'] &
\end{tikzcd}$$

+ the $K(\pi, n)$-fibrations induced by $\{\cdots \to K(\pi, n+1)\}$.

Postnikov Construction $\Rightarrow$ Diagram of exact sequences:

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & \pi_{q+1}(A) \arrow[r] \arrow[d, "f'"'] & \pi_{q+1}(B) \arrow[r] \arrow[d, "="'] & \pi_{q+1}(f) \arrow[r] \arrow[d, "\lambda = ??"] & \pi_q(A) \arrow[r, dashed] \arrow[d, "f'"'] & \cdots \\
\cdots \arrow[r, dashed] & \pi_{q+1}(E_B) \arrow[r] & \pi_{q+1}(B) \arrow[r] & \pi_q(K(\pi, n)) \arrow[r] & \pi_q(E_B) \arrow[r, dashed] & \cdots
\end{tikzcd}
$$

Top sequence = Exact sequence of pair (Cyl, A)

with $\mathrm{Cyl} \sim B$.

Bottom sequence =

Exact sequence of fibration $K(\pi, n) \hookrightarrow E_{\mathrm{Cyl}} \rightarrow \mathrm{Cyl}$.

Vertical arrow $\pi_{q+1}(f) \xrightarrow{\lambda = ??} \pi_q(K(\pi, n))$???

Vertical arrow $\pi_{q+1}(f) \xrightarrow{??} \pi_q(K(\pi,n))$ **???**

Fibration $E_A \to A$ **trivial** $\Rightarrow$ $E_A = A \times K(\pi,n)$ $\Rightarrow$

$$0 \longrightarrow \pi_q(K(\pi,n)) \underset{i}{\overset{\rho}{\leftrightarrows}} \pi_q(E_A) \underset{p}{\overset{\sigma}{\leftrightarrows}} \pi_q(A) \longrightarrow 0$$

with $\mathrm{id}_{\pi_q(E_A)} = \sigma p + i\rho$.

$$\Rightarrow \quad \begin{tikzcd} \pi_{q+1}(E_{\mathrm{Cyl}}, E_A) \arrow[r, "\partial"] \arrow[d, "p", "\cong"'] & \pi_q(E_A) \arrow[r, "\rho"] & \pi_q(K(\pi,n)) \\ \pi_{q+1}(f) := \pi_{q+1}(\mathrm{Cyl}, A) \arrow[urr, "\lambda = \rho\partial p^{-1}"'] & & \end{tikzcd}$$

Three squares to be proved commutative.

Square 1 =

$$\begin{tikzcd}
\pi_{q+1}(A) \arrow[r, "f"] \arrow[d, "f'"] \arrow[dr, phantom, "??"] & \pi_{q+1}(B) \arrow[d, "="] \\
\pi_{q+1}(E_B) \arrow[r, "p"'] & \pi_{q+1}(B)
\end{tikzcd}$$

Variant of:

$$\begin{tikzcd}
E_A \arrow[r, "f"] \arrow[d, "p"] & E_B \arrow[d, "p"] \\
A \arrow[u, bend left, "\sigma"] \arrow[ur, "f' := f\sigma"] \arrow[r, "f"'] & B
\end{tikzcd}$$

Proof: $$pf' = pf\sigma = fp\sigma = f$$ **QED**

$$
\textbf{Square 2} =
\begin{tikzcd}
\pi_{q+1}(B) \arrow[r, "j"] \arrow[d, "="] \arrow[dr, phantom, "??"] & \pi_{q+1}(f) \arrow[d, "\lambda"] \\
\pi_{q+1}(B) \arrow[r, "\partial_{\mathrm{fib}}"'] & \pi_q(K(\pi, n))
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \pi_{q+1}(B) \arrow[ddl, "j"'] \arrow[d, "\cong"'] \arrow[rr, bend left, "\partial_{\mathrm{fib}}"] & \pi_{q+1}(E_B, K(\pi, n)) \arrow[l, "\cong"', "p"] \arrow[r, "\partial_{\mathrm{pair}}"] \arrow[d, "\cong"'] & \pi_q(K(\pi, n)) \arrow[d, "="'] \\
 & \pi_{q+1}(\mathrm{Cyl}) \arrow[d, "j"'] & \pi_{q+1}(E_{\mathrm{Cyl}}, K(\pi, n)) \arrow[l, "\cong"', "p"] \arrow[r, "\partial_{\mathrm{pair}}"] \arrow[d, "j"'] & \pi_q(K(\pi, n)) \arrow[d, shift right, "j"'] \\
\pi_{q+1}(f) \arrow[r, phantom, "="] & \pi_{q+1}(\mathrm{Cyl}, A) \arrow[urr, bend left, "\lambda"] & \pi_{q+1}(E_{\mathrm{Cyl}}, E_A) \arrow[l, "\cong"', "p"] \arrow[r, "\partial_{\mathrm{pair}}"] & \pi_q(E_A) \arrow[u, shift right, "\rho"']
\end{tikzcd}
$$

$$\lambda j = \rho \partial_{\mathrm{pair}} p^{-1} j = \rho \partial_{\mathrm{pair}} j p^{-1} = \rho j \partial_{\mathrm{pair}} p^{-1} = \partial_{\mathrm{pair}} p^{-1} = \partial_{\mathrm{fib}}$$

QED

Square 3 =

$$\begin{tikzcd}
\pi_{q+1}(f) \arrow[r, "\partial_{\text{pair}}"] \arrow[d, "\lambda"] \arrow[dr, phantom, "??"] & \pi_q(A) \arrow[d, "f'"] \\
\pi_q(K(\pi,n)) \arrow[r, "i"'] & \pi_q(E_B)
\end{tikzcd}$$

$$\begin{tikzcd}
& \pi_{q+1}(\mathrm{Cyl}) \arrow[d, "i"] & \pi_{q+1}(E_{\mathrm{Cyl}}, K(\pi,n)) \arrow[l, "\cong", "p"'] \arrow[r, "\partial_{\text{pair}}"] \arrow[d, "i"] & \pi_q(K(\pi,n)) \arrow[d, shift right, "i"'] \arrow[dr, "i"] & \\
\pi_{q+1}(f) = & \pi_{q+1}(\mathrm{Cyl}, A) \arrow[d, "\partial_{\text{pair}}"'] \arrow[urr, bend left, "\lambda"] & \pi_{q+1}(E_{\mathrm{Cyl}}, E_A) \arrow[l, "\cong", "p"'] \arrow[r, "\partial_{\text{pair}}"] \arrow[ddrr, "0"] & \pi_q(E_A) \arrow[u, shift right, "\rho"'] \arrow[r, "f"] \arrow[dll, shift right, "p"'] \arrow[ddr, "f"] & \pi_q(E_B) \arrow[dd, "i", "\cong"'] \\
& \pi_q(A) \arrow[urr, shift right, "\sigma"'] \arrow[urrr, bend right, "f'"] & & & \\
& & & & \pi_q(E_{\mathrm{Cyl}})
\end{tikzcd}$$

$$0 = f\partial_{\text{pair}}p^{-1} = f(i\rho + \sigma p)\partial_{\text{pair}}p^{-1} = fi\rho\partial_{\text{pair}}p^{-1} + f\sigma p\partial_{\text{pair}}p^{-1} =$$

$$= i\rho\partial_{\text{pair}}p^{-1} + f'\partial_{\text{pair}}pp^{-1} = i\lambda + f'\partial_{\text{pair}} = 0$$

QED up to sign

Remark: The Postnikov construction is a functor.

Canonical morphism :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow k_n'(f) \\ * & \longrightarrow & K(\pi, n+1) \end{array}$$

$k_n(f) \in Z^n(\text{Cone})$

$\text{id} \in Z^n(\text{Cone})$

$\Rightarrow$ Diagram:

$$\begin{tikzcd}
\pi_{n+1}(f) = \pi_{n+1}(\mathrm{Cyl}, A) \arrow[r, "k'", "\cong"'] \arrow[ddd, "\lambda"'] & \pi_{n+1}(K(\pi, n+1), *) \\
\pi_{n+1}(E_{\mathrm{Cyl}}, E_A) \arrow[u, "\cong", "p^{-1}"'] \arrow[r, "k'"] \arrow[d, "\partial_{\mathrm{pair}}"] & \pi_{n+1}(E(\pi, n), K(\pi, n)) \arrow[u, "\cong", "p^{-1}"'] \arrow[d, "\cong"', "\partial_{\mathrm{pair}}"] \\
\pi_n(E_A) \arrow[r, "k'"] \arrow[d, "\rho"] & \pi_n(K(\pi, n)) \arrow[d, "\cong"', "\rho"] \\
\pi_n(K(\pi, n)) \arrow[r, "="] & \pi_n(K(\pi, n))
\end{tikzcd}$$

$\Rightarrow$ $\lambda : \pi_{n+1}(f) \to \pi_n(K(\pi, n)) =$ isomorphism.

Using the morphism of exact sequence – I.

$$\begin{tikzcd}
\pi_{n+1}(B) \arrow[r] \arrow[d, "="] & \pi_{n+1}(f) \arrow[r] \arrow[d, "\lambda"', "\cong"] & \pi_n(A) \arrow[r] \arrow[d, "f'"'] & \pi_n(B) \arrow[r] \arrow[d, "="] & \cancel{\pi_n(f)} \arrow[d, "\lambda"'] \\
\pi_{n+1}(B) \arrow[r] & \pi_n(K(\pi,n)) \arrow[r] & \pi_n(E_B) \arrow[r] & \pi_{n+1}(B) \arrow[r] & \cancel{\pi_{n-1}(K(\pi,n))}
\end{tikzcd}$$

$\Rightarrow f' : \pi_n(A) \to \pi_n(E_B) =$ **isomorphism**

and the same for $f' : \pi_i(A) \to \pi_i(E_B)$ **for** $i < n$.

Using the morphism of exact sequence – II.

$$\begin{tikzcd}
& \pi_{n+1}(A) \arrow[r] \arrow[d, "f'"'] & \pi_{n+1}(B) \arrow[r] \arrow[d, "="] & \pi_{n+1}(f) \arrow[d, "\lambda"', "\cong"] \\
\cancel{\pi_{n+1}(K(\pi, n))} \arrow[r] & \pi_{n+1}(E_B) \arrow[r] & \pi_{n+1}(B) \arrow[r] & \pi_n(K(\pi, n))
\end{tikzcd}$$

Standard diagram chasing $\Rightarrow$

$f' : \pi_{n+1}(A) \to \pi_{n+1}(E_B)$ **is epimorphism.**

Proved: $\boxed{f' : A \to E_B \text{ is } (n+1)\text{-connected}}$.

QED

Corollary: Given: A, B simply connected and $f : A \to B$.

Then:

Moore-Postnikov decomposition

of f with:

- f_i is i-connected.
- p_i is the fibration:

$$\begin{tikzcd} K(\pi_{i+1}(f_i), i) \arrow[r, hook] & B_{i+1} \arrow[d, "p_i"] \\ & B_i \end{tikzcd}$$

$$\begin{tikzcd} & \varprojlim B_i = B_\infty \arrow[d, dashed] \\ & B_3 \arrow[d, "p_2"] \\ & B_2 \arrow[d, "p_1"] \\ A \arrow[r, "f =: f_1"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"'] \arrow[uuur, "f_\infty"] & B =: B_1 \end{tikzcd}$$

defined by the Postnikov class $k_i(f_i) \in H^{i+1}(B_i, \pi_{i+1}(f_i))$.

In particular f_∞ is a homotopy equivalence.

Particular case 1. $f : A \to * \Rightarrow$ **Postnikov tower of** A.

$$\begin{tikzcd}
 & \varprojlim A_i = A_\infty \arrow[d, dashed] \\
 & A_3 = K(\pi_3, 3) \times_{k_3} A_2 \arrow[d, "p_2"] \\
 & A_2 = K(\pi_2, 2) \arrow[d, "p_1"] \\
A \arrow[ruuu, "f_\infty"] \arrow[ruu, "f_3"] \arrow[ru, "f_2"] \arrow[r, "f_1 := *"'] & *
\end{tikzcd}$$

$A_{i-1} = (i-1)$-th stage of the Postnikov tower.

$\pi_i(A) = H_{i+1}(\mathrm{Cone}(f_{i-1}), \mathbb{Z})$.

$k_i \in H^{i+1}(A_{i-1}, \pi_i(A)) =$ Postnikov $\boxed{\text{class}}$ $=$ intrinsic.

$k_i \in H^{i+1}(A_{i-1}, \pi_i)$ $\boxed{\text{depending}}$ on $\pi_i \cong \pi_i(A)$.

$A_i = K(\pi_i(A), i) \times_{k_i} A_{i-1}$.

Particular case 2. $f : * \to B \Rightarrow$ **Whitehead tower of** B.

$$
\begin{tikzcd}
 & \varprojlim B_i = B_\infty \sim * \arrow[d, dashed] \\
 & B_3 = K(\pi_3(B), 2) \times_{k_3} B_2 \arrow[d, "p_2"] \\
 & B_2 = K(\pi_2(B), 1) \times_{k_2} B \arrow[d, "p_1"] \\
* \arrow[r, "f_1 := *"'] \arrow[ur, "f_2"] \arrow[uur, "f_3"] \arrow[uuur, "f_\infty"] & B =: B_1
\end{tikzcd}
$$

B_i = i**-th stage of the Whitehead tower.**

$= B$ **with** $\{\pi_j\}_{j \leq i}$ **killed.**

$=$ i**-th generalized covering space of** B.

General case:

$$
\begin{tikzcd}
 & \varprojlim B_i =: B_\infty \arrow[d, dashed] & F_\infty := \varprojlim F_i \arrow[l, hook'] \arrow[d, dashed] \\
 & K(\pi_3(\mathrm{Cone}(f_2)), 2) \times_{k_3} B_2 =: B_3 \arrow[d, "p_2"] & F_2 = K(\pi_3(\mathrm{Cone}(f_2)), 2) \times_{k_3} F_1 \arrow[l, hook'] \arrow[d, "p_2"] \\
 & K(\pi_2(\mathrm{Cone}(f_1)), 1) \times_{k_2} B_1 =: B_2 \arrow[d, "p_1"] & F_1 = K(\pi_2(\mathrm{Cone}(f_1)), 1) \arrow[l, hook'] \arrow[d, "p_1"] \\
A \arrow[r, "f"'] \arrow[ur, "f_2"] \arrow[uur, "f_3"] \arrow[uuur, "f_\infty"] & B_1 := B & * \arrow[l, hook']
\end{tikzcd}
$$

The restriction of the Postnikov tower of f above B

to the base point $*$

is the Postnikov tower of the homotopy fiber F of $f : A \to B$.

Particular case of the general case:

$$\begin{tikzcd}
 & \varprojlim B_i =: B_\infty \arrow[d, dashed] & F_\infty := \varprojlim F_i \arrow[l, hook'] \arrow[d, dashed] \\
 & K(\pi_3(B),2) \times_{k_3} B_2 =: B_3 \arrow[d, "p_2"] & F_2 = K(\pi_3(B),2) \times_{k_3} F_1 \arrow[l, hook'] \arrow[d, "p_2"] \\
 & K(\pi_2(B),1) \times_{k_2} B_1 =: B_2 \arrow[d, "p_1"] & F_1 = K(\pi_2(B),1) \arrow[l, hook'] \arrow[d, "p_1"] \\
* \arrow[r, "f"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"'] \arrow[uuur, "f_\infty"] & B_1 := B & * \arrow[l, hook']
\end{tikzcd}$$

F_∞ = Homotopy fiber of $[* \to B] =: \Omega B$ = loop space of B.

<u>Corollary</u>: The Postnikov classes of ΩB

are the Whitehead classes of B.

4/5. Postnikov construction for spaces with effective homology

<u>Given</u>: $f : A \to B$ an n-connected simplicial map ($n \geq 1$) between 1-reduced simplicial sets with effective homology.

<u>Then</u>: A general algorithm produces the diagram:

$$\begin{tikzcd} & B' \arrow[d, "p"] \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & B \end{tikzcd}$$

with:

- f' is $(n+1)$-connected;
- p is a fibration $K(\pi, n) \hookrightarrow B' \xrightarrow{p} B$;
- B' is a simplicial set with effective homology.

Let C_* be a free $\mathbb{Z}$-chain complex,

n-connected ($n \geq 1$), with effective homology.

$\Rightarrow$ Equivalence $C_* \lll\!\Longleftrightarrow\!\ggg EC_*$

with EC_* = free Z-chain complex of finite type.

$\Rightarrow$ $\pi := H_{n+1}(C_*) = H_{n+1}(EC_*)$ is computable.

$$
\begin{tikzcd}
EC_n & EC_{n+1} \arrow[l, "d"'] \arrow[ddddd, bend right, "E\mathfrak{h}"'] & EC_{n+2} \arrow[l, "d"'] \arrow[ddddl, "d"] \\
\cup & \cup & \\
0 & Z_{n+1} \arrow[l, "d"'] \arrow[ddd, bend right, "\mathrm{pr}"'] & \\
 & \cup & \\
 & B_{n+1} & \\
 & \pi &
\end{tikzcd}
$$

$\Rightarrow$ $E\mathfrak{h}$ = fund. cohomology class

of EC_* computable.

Effective homology of C_*:

$$
\begin{tikzcd}
C_* \arrow[loop left, "\ell h"] \arrow[r, shift right, "\ell g"'] \arrow[drr, "\mathfrak{h} = E\mathfrak{h} \circ rf \circ \ell g"'] & \widehat{C}_* \arrow[l, shift right, "\ell f"'] \arrow[r, shift left, leftarrow, "rg"] \arrow[r, shift right, "rf"'] & EC_* \arrow[loop right, "rh"] \arrow[d, "E\mathfrak{h}"] \\
 & & \pi
\end{tikzcd}
$$

$\Rightarrow$ **The fundamental cohomology class of C_*:**

$$\mathfrak{h} := E\mathfrak{h} \circ rf \circ \ell g$$

is computable.

Cone Theorem: **Given**: $f : A_* \to B_*$

a chain complex morphism

between two chain complexes

with effective homology.

Then: **A general algorithm computes**

a version with effective homology of Cone(f).

Proof: **Particular case of the SES$_2$ Theorem.**

Given $f : A \to B$ **n-connected +**

$$C_*(A) \xrightarrow{f_*} C_*(B) \xrightarrow{i} \mathrm{Cone}(f_*)$$

+ A and B with effective homology

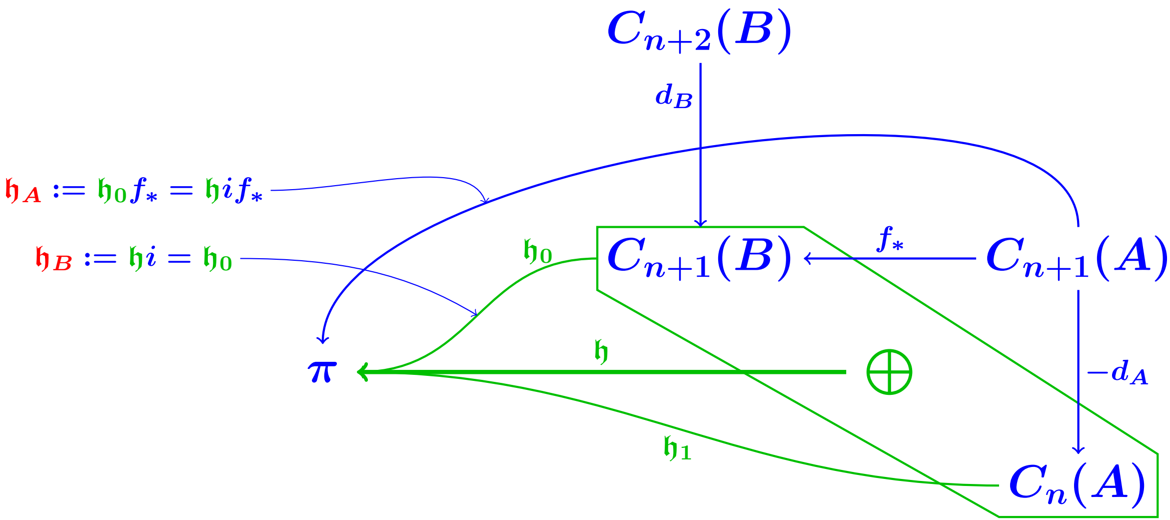
$\Rightarrow$

$$C_*(A) \xrightarrow{f_*} C_*(B) \xrightarrow{i} \mathrm{Cone}(f_*) \xrightarrow{\mathfrak{h}} \pi$$

with $\pi := H_{n+1}(\mathrm{Cone}(f_*), \mathbb{Z})$

$\Rightarrow$ **π-cochains:**

$$\begin{aligned} \mathfrak{h} &\in Z^{n+1}(\mathrm{Cone}(f_*), \pi) \\ \mathfrak{h}_B := \mathfrak{h} i &\in Z^{n+1}(B, \pi) \\ \mathfrak{h}_A := \mathfrak{h} i f_* &\in Z^{n+1}(A, \pi) \end{aligned}$$



$\mathfrak{h} = [\mathfrak{h}_0 \quad \mathfrak{h}_1]$ **cocycle** $\Leftrightarrow$

$$[\mathfrak{h}_0 \quad \mathfrak{h}_1] \begin{bmatrix} d_B & f_* \\ 0 & -d_A \end{bmatrix} = [\mathfrak{h}_0 d_B \quad \mathfrak{h}_0 f_* - \mathfrak{h}_1 d_A] = 0$$

$\Rightarrow \mathfrak{h}_A = \mathfrak{h}_0 f_* = \mathfrak{h}_1 d_A = d_A(\mathfrak{h}_1)$

$\Rightarrow \mathfrak{h}_A$ **is a cohomology class** $\boxed{\textbf{constructively}}$ **null.**

$\Rightarrow (K_n := K(\pi, n))$

$$
\begin{tikzcd}
 & B' := K_n \times_{\mathfrak{h}_B} B \arrow[r, "i_*"'] \arrow[d] & K_n \times_{\mathfrak{h}} \mathrm{Cone}(f) \arrow[r, "\mathfrak{h}_*"'] \arrow[d] & E(\pi, n) \arrow[d, "p = d"] \\
A \arrow[ur, "f' := \mathfrak{h}_1 \times f"] \arrow[urrr, bend left, "\mathfrak{h}_1"] \arrow[r, "f"'] \arrow[rrr, bend right, "\mathfrak{h}_A"'] & B \arrow[r, "i"'] \arrow[rr, bend right, "\mathfrak{h}_B"'] & \mathrm{Cone}(f) \arrow[r, "\mathfrak{h}"'] & K(\pi, n+1)
\end{tikzcd}
$$

Constructive solution of Postnikov process $= f' : A \rightarrow B'$.

QED

5/5. Postnikov construction for morphisms.

Given: A commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

with A, B, C, D 1-connected, and f, g n-connected.

Then: A morphism between Postnikov constructions

is induced:

$$\begin{tikzcd} & B' \arrow[rr, "\beta'"] \arrow[d, "p"] & & D' \arrow[d, "q"] \\ A \arrow[ur, "f'"] \arrow[r, "f"] \arrow[rr, bend right, "\alpha"'] & B \arrow[rr, bend right, "\beta"'] & C \arrow[ur, "g'"] \arrow[r, "g"] & D \end{tikzcd}$$

Remember the functor:

$$\{[f : A \to B], z \in Z^{n+1}(\mathrm{Cone}(f), \pi)\} \longmapsto$$
$$\longmapsto \{\text{Postnikov Diagram}\}$$

$$\begin{tikzcd}
E_A \arrow[r] \arrow[d] & E_B \arrow[r, "\sim"] \arrow[d] & E_{\mathrm{Cyl}} \arrow[r] \arrow[d] & E_{\mathrm{Cone}} \arrow[r] \arrow[d] & E(\pi, n) \arrow[d] \\
A \arrow[r, "f"] \arrow[u, bend left] \arrow[ur, "f'"] & B \arrow[r, "\sim"] & \mathrm{Cyl} \arrow[r, "\mathrm{pr}"] & \mathrm{Cone} \arrow[r, "z"] & K(\pi, n+1)
\end{tikzcd}$$

Serious general coherence problem

with cocycle coherence.

Problem visible in this simple case:

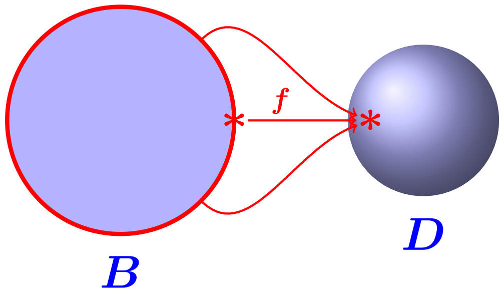
$A = C = * \Rightarrow$ **Cone** $= B$ **or** D.

$B =$ **Disk bounded by a circle.**

$$\Rightarrow C_*(B) = [0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \leftarrow 0]$$

$D =$ **2-sphere** $\Rightarrow C_*(D) = [0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0]$

$f =$ **the map [circle of** B**]** $\mapsto$ **[base point of** D**].**



B, D **and** f **are 1-connected.**

$\Rightarrow \pi = H_2(B,\mathbb{Z}) = 0$ **and** $\pi' = H_2(D,\mathbb{Z}) = \mathbb{Z}$.

$\Rightarrow z = 0 \in Z^2(B,0)$ **and** $z' = \mathrm{id} \in Z^2(D,\mathbb{Z})$ **[no choice !]**.

$\Rightarrow$

$$\begin{tikzcd} B \times_0 K(0,1) \arrow[r, "???"'] \arrow[d, "p"] & D \times_1 K(\mathbb{Z},1) \arrow[d, "p"] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

Only one natural intermediate object:

$$\begin{tikzcd} B \times_0 K(0,1) \arrow[r, "???"'] \arrow[d, "p"] & B \times_{?z''?} K(\mathbb{Z},1) \arrow[r, "???"'] \arrow[d, "p"] & D \times_1 K(\mathbb{Z},1) \arrow[d, "p"] \\ B \arrow[r, "\mathrm{id}"] & B \arrow[r, "f"] & D \end{tikzcd}$$

with in particular $?z''?$ to be determined.

$$\begin{tikzcd} B \times_0 K(0,1) \arrow[r, "???"'] \arrow[d, "p"] & B \times_{?z''?} K(\mathbb{Z},1) \arrow[r, "???"'] \arrow[d, "p"] & D \times_1 K(\mathbb{Z},1) \arrow[d, "p"] \\ B \arrow[r, "\mathrm{id}"] & B \arrow[r, "f"] & D \end{tikzcd}$$

Left-coherence $\Rightarrow z'' = f_*(z) = f_*(0) = 0$

with $f_* : Z^2(B, \pi) \to Z^2(B, \pi')$.

Right-coherence $\Rightarrow z'' = f^*(z') = f^*(1) = 1$

with $f^* : Z^2(D, \pi') \to Z^2(B, \pi')$.

$0 \neq 1 \Rightarrow$ **Problem!**

At the level of chain complexes:

$$
\begin{tikzcd}
\pi = H_2 = 0 \arrow[rrr, bend left, "f_*"] & C_2 = \mathbb{Z} \arrow[l, "z = 0"'] \arrow[r, "1", "f"'] \arrow[d, "1"'] \arrow[r, phantom, bend left, "\boxed{\text{Non-comm!}}"] & C'_2 = \mathbb{Z} \arrow[r, "z' = 1"] \arrow[d, "0"'] & \mathbb{Z} = H'_2 = \pi' \\
 & C_1 = \mathbb{Z} \arrow[r, "0", "f"'] \arrow[d, "0"'] & C'_1 = 0 \arrow[d, "0"'] & \\
 & C_0 = \mathbb{Z} \arrow[r, "1", "f"'] & C'_0 = \mathbb{Z} &
\end{tikzcd}
$$

C_* and C'_* chain complexes, $f : C_* \to C'_*$ morphism.

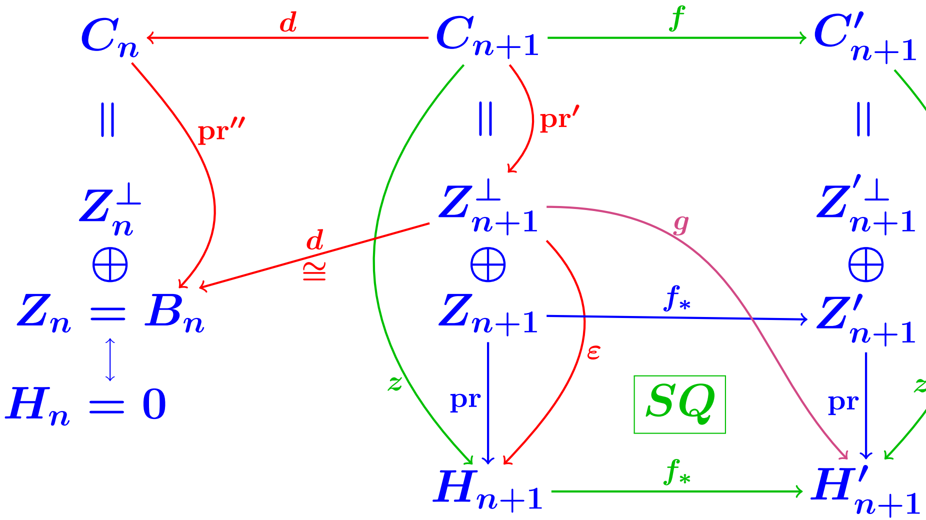
$z \in Z^2(C_*, H_2)$, $z' \in Z^2(C'_*, H'_2)$ = characteristic cocycles.

$f_* : Z^2(C_*, H_2) \to Z^2(C_*, H'_2) : z = 0 \mapsto 0 = f_*(z)$.

$f^* : Z^2(C'_*, H'_2) \to Z^2(C_*, H'_2) : z' = 1 \mapsto 1 = f^*(z')$.

Bug: $0 = f_*(z) \neq f^*(z') = 1$!!!

<u>Theorem</u>: In this context, $f_*(z)$ and $f^*(z')$ are cohomologous.



$$z'f \overset{??}{\sim} f_* z$$

1. The square SQ is commutative.
2. For g arbitrary:

$$g.\mathrm{pr}' = g.d^{-1}.\mathrm{pr}''.d = d(g.d^{-1}.\mathrm{pr}'')$$

QED

$\Rightarrow$ Diagram-1:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\text{Com.}"] & B \arrow[r, "j"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\text{Com.}"] & \mathrm{Cyl}(f) \arrow[r, "p"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\text{Com.}"] & \mathrm{Cone}(f) \arrow[r, "z"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "z'\Phi \sim \Phi_* z"] & K(\pi, n+1) \arrow[d, "\Phi_*"'] \\
C \arrow[r, "g"] & D \arrow[r, "j"] & \mathrm{Cyl}(g) \arrow[r, "p"] & \mathrm{Cone}(g) \arrow[r, "z'"] & K(\pi', n+1)
\end{tikzcd}$$

$\Rightarrow$ Diagram-2:

$$
\begin{tikzcd}
 & \mathrm{Cone}(f) \times_{z} K(\pi, n) \arrow[dl, "p"'] \arrow[d, "\Phi_*"] \arrow[ddd, bend right=60, "\overline{\Phi}"'] \\
\mathrm{Cone}(f) \arrow[dd, "\Phi"] & \mathrm{Cone}(f) \times_{\Phi_* z} K(\pi', n) \arrow[l, "p"'] \arrow[d, "\Phi_* z \sim \Phi^* z'"] \\
 & \mathrm{Cone}(f) \times_{\Phi^* z'} K(\pi', n) \arrow[ul, "p"'] \arrow[d, "\Phi_*"] \\
\mathrm{Cone}(f') & \mathrm{Cone}(f') \times_{z'} K(\pi', n) \arrow[l, "p"']
\end{tikzcd}
$$

$\Rightarrow$ Diagram-3:

$$
\begin{tikzcd}
 & B' \arrow[r] \arrow[d, "p"] \arrow[dd, bend left, "\beta'"] & \mathrm{Cone}(f) \times_z K(\pi, n) \arrow[d, "p"] \arrow[dd, "\overline{\Phi}"] \\
A \arrow[ur, "f'"] \arrow[r, "f"] \arrow[dd, "\alpha"] & B \arrow[r, "j"] \arrow[dd, bend right, "\beta"'] & \mathrm{Cone}(f) \arrow[dd, bend right, "\Phi"'] \\
 & D' \arrow[r] \arrow[d, "p"] & \mathrm{Cone}(g) \times_{z'} K(\pi', n) \arrow[d, "p"] \\
C \arrow[ur, "g'"] \arrow[r, "g"] & D \arrow[r, "j"] & \mathrm{Cone}(g)
\end{tikzcd}
$$

QED

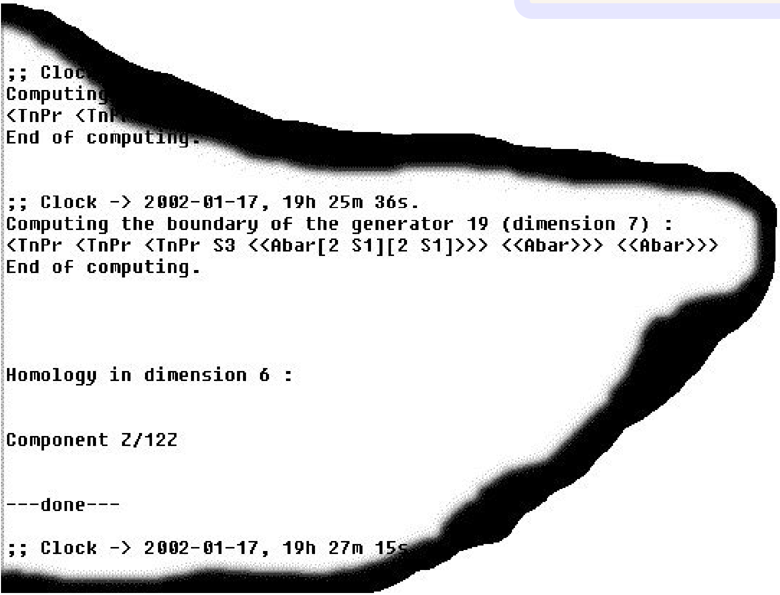
Corollary: All the Postnikov constructions:

- **Moore-Postnikov factorization;**
- **Postnikov tower;**
- **Whitehead tower;**
- **Postnikov tower of the homotopy fiber;**

can be functorially organized.

In an effective way in context of effective homology.

The END



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