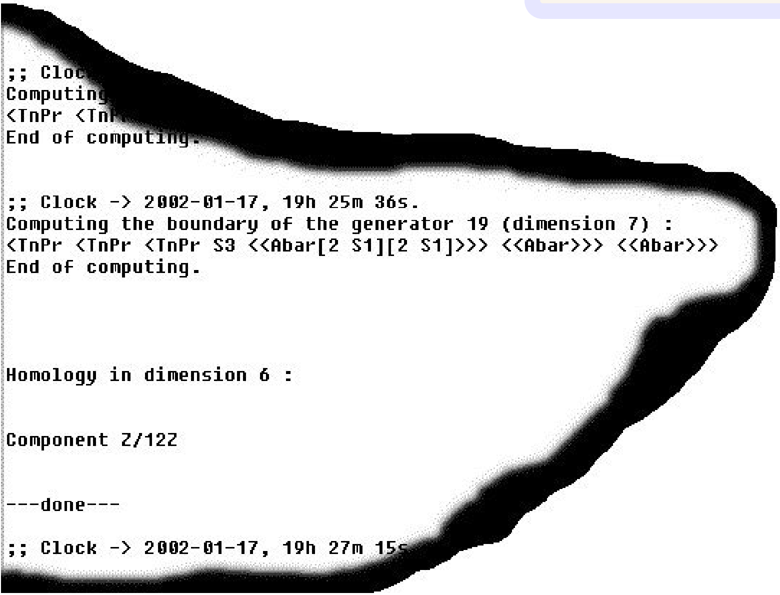
Constructive Logic

and

Constructive Algebraic Topology



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Leçons de Mathématiques d'Aujourd'hui
Bordeaux, March 3, 2011

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, . . .
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, . . .
Green	=	**Solution, essential point,**
		mathematicians, . . .
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

Plan:

1. **The Computability Problem in Algebraic Topology.**
2. **A harder problem can be easier.**
3. **Basic Homological Algebra and questionable ∃'s.**
4. **Mathematical structures and Functional Programming.**
5. **Effective vs locally effective objects.**
6. **Homological Reductions.**
7. **Basic Perturbation Lemma.**
8. **⇒ Constructive Algebraic Topology OK !!!**

1/8. The computability problem in Algebraic Topology.

Typical example.

Serre (1951): $\pi_5(S^2) = \mathbb{Z}/2$ ($\Rightarrow$ Fields Medal).

Definition: $\pi_5(S^2) = \pi_0(\mathrm{Cont}(S^5, S^2))$

with π_0 := set of connected components.

Function $\pi : (n_{\geq 2},$ Topological Space$) \mapsto$ Abelian Group

$$(5, S^2) \mapsto \mathbb{Z}/2$$

Observation: The argument $(5, S^2)$ can be

an input of a computer program.

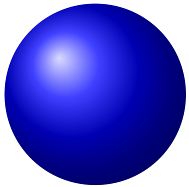
The value $\mathbb{Z}/2$ could be

an output of a computer program.

Lisp coding of $S^2 = \partial D^3 = \partial\Delta^3$:

$$((1\ 2\ 3)\ (0\ 2\ 3)\ (0\ 1\ 3)\ (0\ 1\ 2))$$

$S^2 =$



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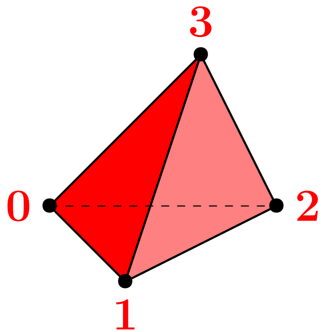


Coding of an abelian group of finite type:

$$\mathbb{Z}/6 \oplus \mathbb{Z}/30 \oplus \mathbb{Z}^2 = (6\ 30\ 0\ 0)$$

$$\mathbb{Z}/2 = (2)$$

Jean-Pierre Serre (1953):

Theorem: For every "reasonable" space X,

the homology groups $H_n(X)$ and

the homotopy groups $\pi_n(X)$ have finite type.

Example of reasonable space:

$$\mathrm{Cont}(S^1, \quad D^3 \cup_2 \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R}))$$

Example of computability problem:

$$H_4(\mathrm{Cont}(S^1, \quad D^3 \cup_2 \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R}))) = ???$$

Natural problem:

Does there exist an algorithm:

Input: (n, X)

n = **natural number**

X = **topological space comb. coded**

Output: $(d_1\ d_2\ \cdots d_k)$

= **integer list**

coding an abelian group of finite type

satisfying $\pi_n(X) = \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k$

First positive answer:

Edgar Brown (1956): X = simply connected finite simplicial complex.

$\Rightarrow \pi_n(X)$ is computable.

But Edgar Brown also warned his method is:

"much too complicated to be considered practical".

ANNALS OF MATHEMATICS
Vol. 65, No. 1, January, 1957
Printed in U.S.A.

FINITE COMPUTABILITY OF POSTNIKOV COMPLEXES[1]

BY EDGAR H. BROWN, JR.

(Received March 3, 1956)

In [4] Postnikov associates with each arcwise connected space X a sequence

. . . .

simply connected simplicial complex. From these results we are then able to prove:

(i) If X is a simply connected simplicial complex, then $\pi_n(X)$ is finitely computable for each $n > 0$.

(ii) If X and Y are simply connected simplicial complexes with finite ho-

. . . .

have a finite number of non-degenerate simplexes.)

It must be emphasized that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.

In the first section of this paper we give some preliminary definitions, con-

50 years later ⇒ same appreciation:

"Much too complicated to be considered practical" ???

Yes, because of hyper- ⋯ -hyper-exponential complexity.

⇒

Second natural problem:

Does there exist an algorithm $(n, X) \mapsto \pi_n(X)$

concretely usable ???

Two potential current solutions (1-2)

+ one effective solution (3):

1. Approximating infinite objects

by inductive limits of finite objects.

(Rolf Schön + Alain Clément)

2. Operadic solution.

(Peter May + Michael Mandell + Benoit Fresse)

3. Constructive homological algebra.

(FS + Julio Rubio + Ana Romero)

2/8. A harder problem can be easier.

GCD between all these methods:

Design a more ambitious problem

+ Functional programming.

Didactic analogous problem: Zeros of $f \in \mathcal{F}_{\infty}^{\infty}$??

Definition: $f \in \mathcal{F}_{\infty}^{\infty}$ **(= Functions with infinite limit)**

is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$

satisfying $\lim_{n\to\infty} f(n) = \infty$.

Theorem: $f \in \mathcal{F}_{\infty}^{\infty} \Rightarrow$

$$Z(f) := \#\{n \in \mathbb{N} \text{ st } f(n) = 0\} \text{ is finite.}$$

Problem 1: **Algorithm** $\mathcal{F}_{\infty}^{\infty} \xrightarrow{???} \mathbb{N} : f \longmapsto Z(f)$???

Theorem 1: **Such an algorithm does not exist.**

Theorem: $f, g \in \mathcal{F}_{\infty}^{\infty} \Rightarrow g \circ f \in \mathcal{F}_{\infty}^{\infty}$.

Problem 2: **Algorithm**:

$$\begin{array}{ccccccccc} (\mathcal{F}_{\infty}^{\infty} & \times & \mathbb{N} & \times & \mathcal{F}_{\infty}^{\infty} & \times & \mathbb{N}) & \xrightarrow{???} & \mathbb{N} \\ (f & , & Z(f) & , & g & , & Z(g)) & \longmapsto & Z(g \circ f) \text{ ???} \end{array}$$

Theorem 2: **Such an algorithm does not exist.**

Analysis of the problem:

Translation of $\lim_{n\to\infty} f(n) = \infty$**:**

$$(\forall m \in \mathbb{N})\ (\exists N \in \mathbb{N})\ (\forall n \geq N)\ (f(n) \geq m)$$

Analysis of the problem:

Translation of $\lim_{n\to\infty} f(n) = \infty$**:**

$$(\forall m \in \mathbb{N})\ \boxed{(\exists N \in \mathbb{N})}\ (\forall n \geq N)\ (f(n) \geq m)$$

The key point is in the quantifier $(\exists N \in \mathbb{N})$**:**

if $\boxed{\text{non-constructive}}$**, the penalty is certain :**

no algorithms to process the interesting questions.

The constructive existence of N

consists in having a process (algorithm)

producing N **when** m **is given.**

Constructive version of $\mathcal{F}_\infty^\infty$:

Definition: $C\mathcal{F}_\infty^\infty = \{(f, \overline{f})\}$ **satisfying:**

$$f = \text{algorithm } \mathbb{N} \to \mathbb{N};$$

$$\overline{f} = \text{algorithm } \mathbb{N} \to \mathbb{N} \text{ st}:$$

$$(\overline{f}(m) = N) \Rightarrow [(n \geq N) \Rightarrow (f(n) \geq m)]$$

$$\overline{f} = \text{constructive version of } \lim_{n \to \infty} = \infty$$

In this constructive context,

Theorems 1 and 2 have positive answers.

Theorem 1': $\exists$ **algorithm**:

$$Z : C\mathcal{F}^{\infty}_{\infty} \longrightarrow \mathbb{N} : (f, \overline{f}) \longmapsto Z(f)$$

Solution: **Examine** $\{f(n)\}_{0 \leq n < \overline{f}(1)}$.

Theorem 2': $\exists$ **algorithm**:

$$\mathbf{Cmp} : C\mathcal{F}^{\infty}_{\infty} \times C\mathcal{F}^{\infty}_{\infty} \to C\mathcal{F}^{\infty}_{\infty} : [(f, \overline{f}), (g, \overline{g})] \mapsto (g \circ f, \overline{g \circ f})$$

Proof:

$$(g \circ f)(n) \geq m \Leftarrow f(n) \geq \overline{g}(m) \Leftarrow n \geq \overline{f}(\overline{g}(m))$$

$\Rightarrow$ **Take** $\overline{g \circ f} := \overline{f} \circ \overline{g}$. **QED**

3/8. Basic Homological Algebra and questionable ∃'s.

1. **Locate the ∃'s in the definitions of Homological Algebra.**
2. **Examine whether these ∃'s are constructive.**
3. **If not, improve the definition to have only constructive ∃'s.**
4. **The computability problems can then have natural solutions.**

Requires a high level of functional programming.

Locating non-constructive ∃'s in standard homological algebra.

Definition: Chain complex C_* :

$$C_* = (C_*, d) = [\cdots \leftarrow C_{m-1} \xleftarrow{d_m} C_m \xleftarrow{d_{m+1}} C_{m+1} \leftarrow \cdots]$$

with $d_m \circ d_{m+1} = 0$.

$\Leftrightarrow \quad \ker d_m \supset \operatorname{im} d_{m+1} \quad \Rightarrow$

Definition:

$$H_m(C_*) := \frac{\ker d_m}{\operatorname{im} d_{m+1}}$$

Typical statement in Algebraic Topology:

$$H_5(\Omega^2 S^3) = H_5(\mathrm{Cont}(S^2, S^3)) = \mathbb{Z}/6$$

Implicit translation:

$$\exists f : H_5(\Omega^2 S^3) \xrightarrow{\cong} \mathbb{Z}/6$$

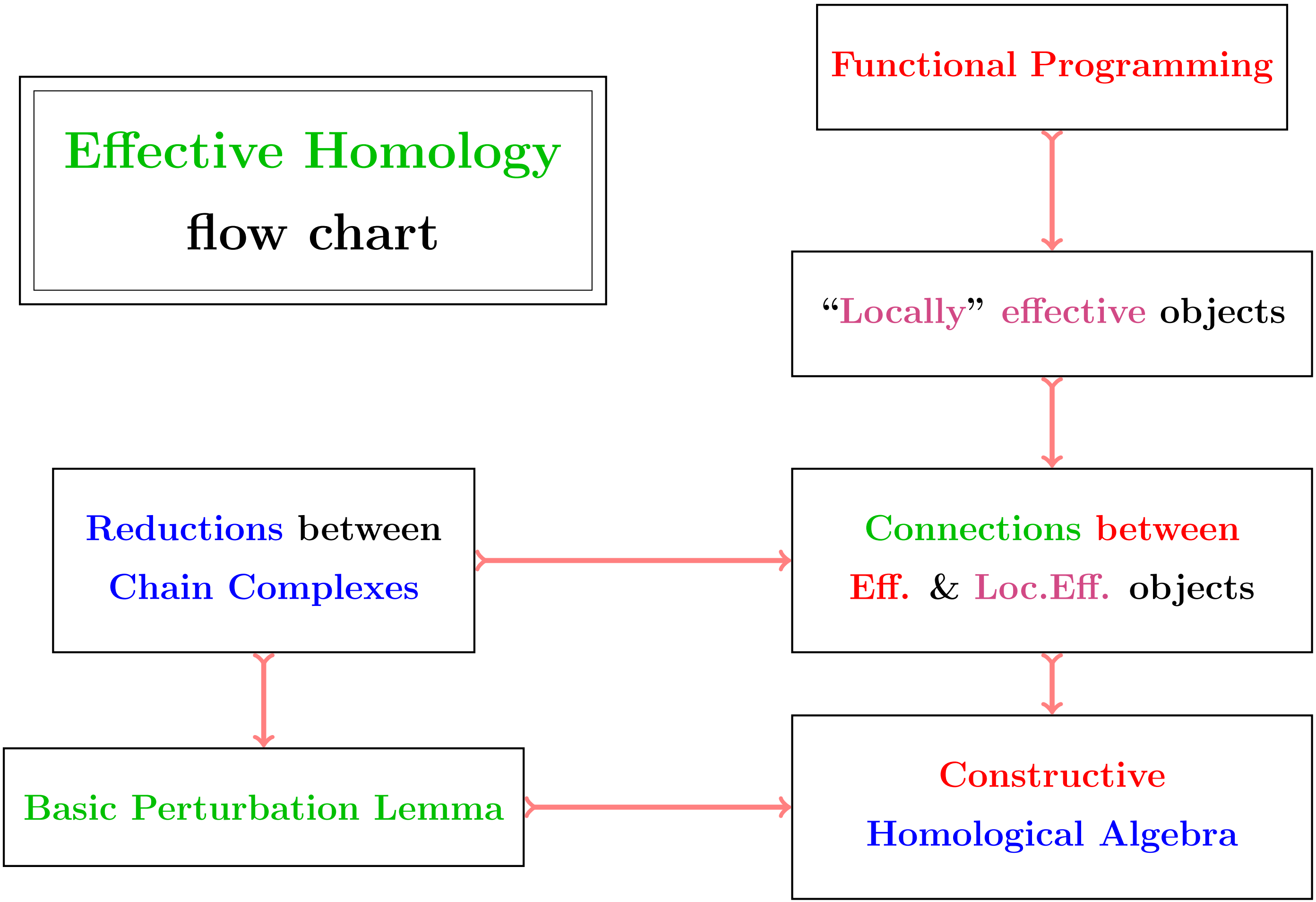
But most often the initial $\exists$ is non-constructive.

$H_5(\Omega_2 S^3) := \ker d_5 / \operatorname{im} d_6$ **generates another problem.**

$$(z \in \ker d_5) \wedge (f(\overline{z}) = 0) \Leftrightarrow \exists c \in C_6(\Omega^2 S^3) \ \underline{\text{st}} \ d_6 c = z$$

But the $\exists$ again is rarely constructive.

Effective Homology flow chart



4/8. Mathematical Structures and Functional Programming.

The art of handling and creating functional objects.

Examples of functional objects:

$$(\mathbb{Z}, +, -, \times) \quad (\mathbb{Z}[X], +, -, \times)$$

Other example:

Kan model for the loop space $\Omega S^3 := \mathrm{Cont}(S^1, S^3)$:

$$(\mathcal{S}_{\Omega S^3}, \{\partial_i^n\}_{n \geq 1, 0 \leq i \leq n}, \{\eta_i^n\}_{n \geq 0, 0 \leq i \leq n})$$

with $\mathcal{S}_{\Omega S^3}$ = the simplex set of the Kan model.

= "Locally" effective objects.

Main problem:

Designing programs $(f_1, \ldots, f_n) \mapsto f$.

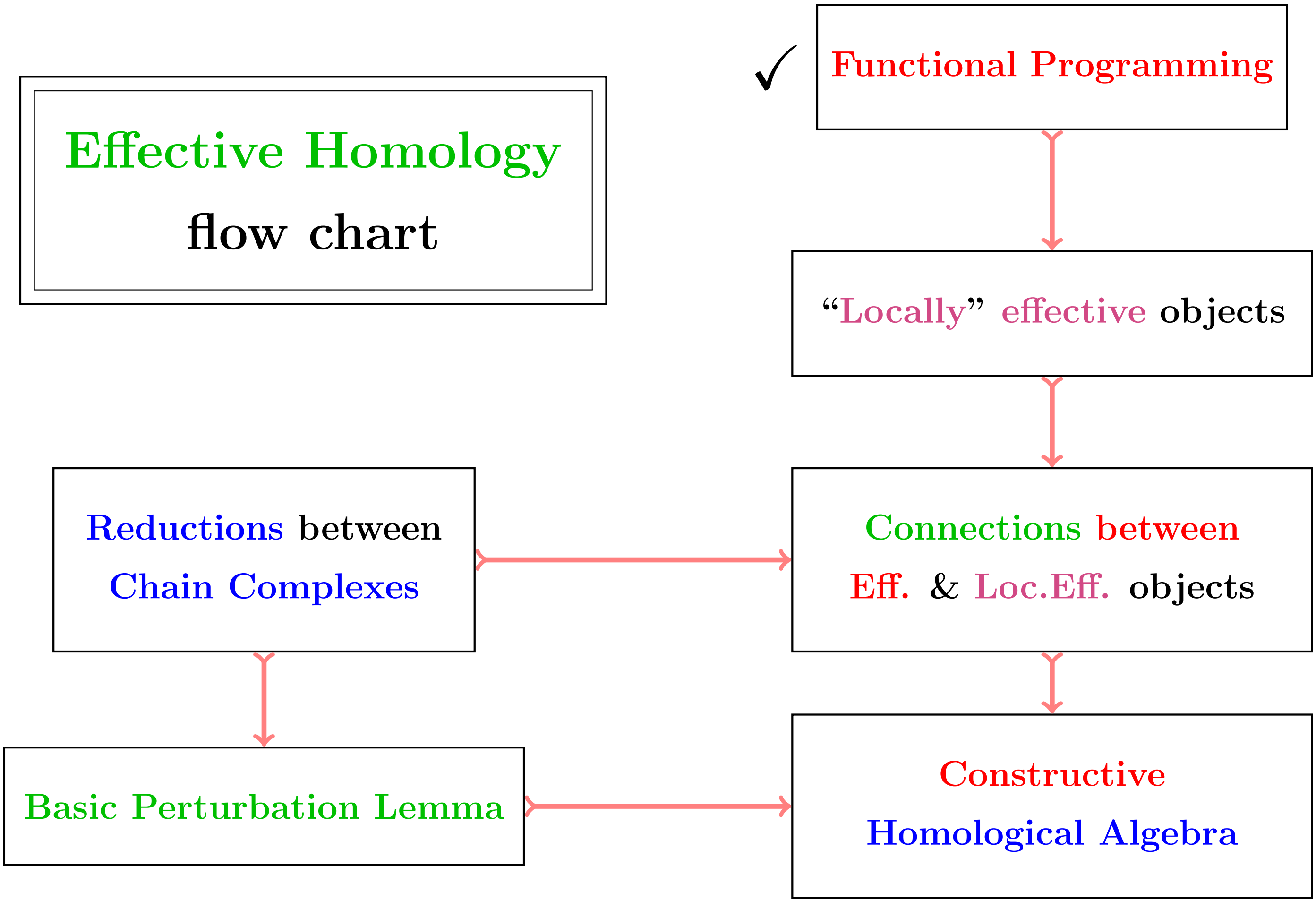
Example:

$$(\mathfrak{R}, +_{\mathfrak{R}}, -_{\mathfrak{R}}, \times_{\mathfrak{R}}) \mapsto (\mathfrak{R}[X], +_{\mathfrak{R}[X]}, -_{\mathfrak{R}[X]}, \times_{\mathfrak{R}[X]})$$

Topological example. X = **topological space.**

$$(\mathcal{S}_X, \{\partial(X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(X)_i^n\}_{n\geq 0, 0\leq i\leq n}) \mapsto (\mathcal{S}_{\Omega X}, \{\partial(\Omega X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(\Omega X)_i^n\}_{n\geq 0, 0\leq i\leq n})$$

Solution = λ-calculus, Lisp, ML, Axiom, Haskell...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Eff. & Loc.Eff. objects
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5/8. Effective vs Locally Effective Objects.

An effective object is an object

which is essentially entirely known.

In particular the standard global information

concerning this object is reachable (= *computable*).

A locally effective object is most often a quite infinite object.

For any "local" ingredient of this object,

any necessary information is reachable.

But in general *no global information*

for the underlying object is reachable.

Notion of effective chain complex :

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \stackrel{d_n}{\leftarrow} C_n \stackrel{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\beta, d)$$

where:

1. β: $\mathbb{Z} \rightarrow \mathcal{L}\text{ist}$: $n \mapsto [g_1^n, \ldots, g_{k_n}^n]$ = distinguished basis of C_n.
2. d: $\mathbb{Z} \tilde{\times} \mathbb{N}_* \rightarrow \mathcal{U}$: $(n, i) \mapsto d_n(g_i^n) \in C_{n-1}$ when g_i^n makes sense.

In particular every C_n is a free $\mathbb{Z}$-module with a finite distinguished basis.

$\Rightarrow$ Every $d_n : C_n \rightarrow C_{n-1}$ is computable.

$\Rightarrow$ Every homology group $H_n(C_*)$ is computable

(every global information is reachable).

Notion of locally effective chain complex:

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \stackrel{d_n}{\leftarrow} C_n \stackrel{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\chi, d)$$

where:

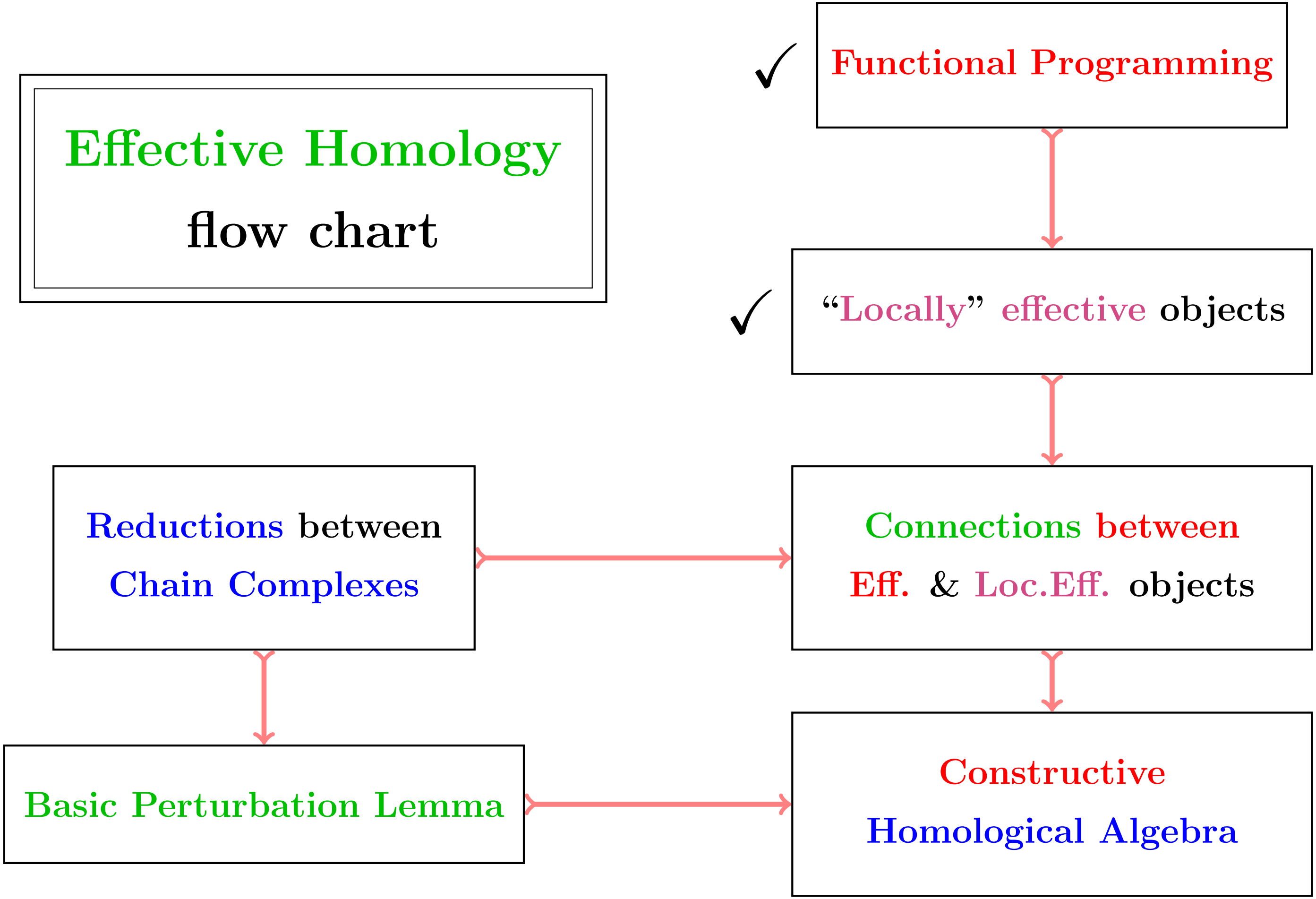
1. χ: $\mathcal{U} \times \mathbb{Z} \rightarrow \mathrm{Bool} = \{\top, \bot\} : (\omega, n) \mapsto \top$
   if and only if ω is a generator of C_n;

2. d: $\mathcal{U} \widetilde{\times} \mathbb{Z} \rightarrow \mathcal{U} \ : (\omega, n) \mapsto d_n(\omega) \in C_{n-1}$
   when ω is a generator of C_n ($\Leftrightarrow \chi(\omega, n) = \top$).

Any finite set of pointwise computations may be done.

Gödel + Church + Turing + Post $\Rightarrow$ no global information is reachable; in particular, the homology groups of C_* are *not computable*.

Effective Homology flow chart



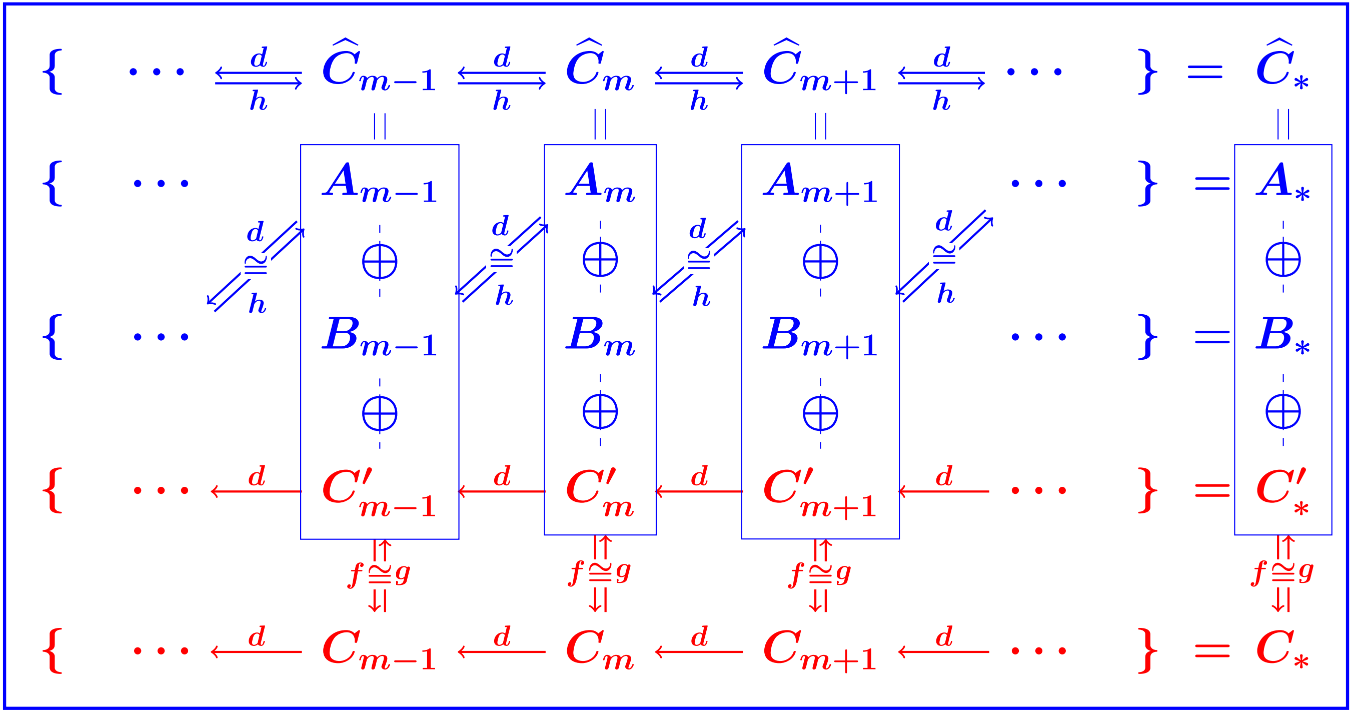
6/8. Homological Reductions.

Definition: A (homological) reduction is a diagram:

$$\rho: \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

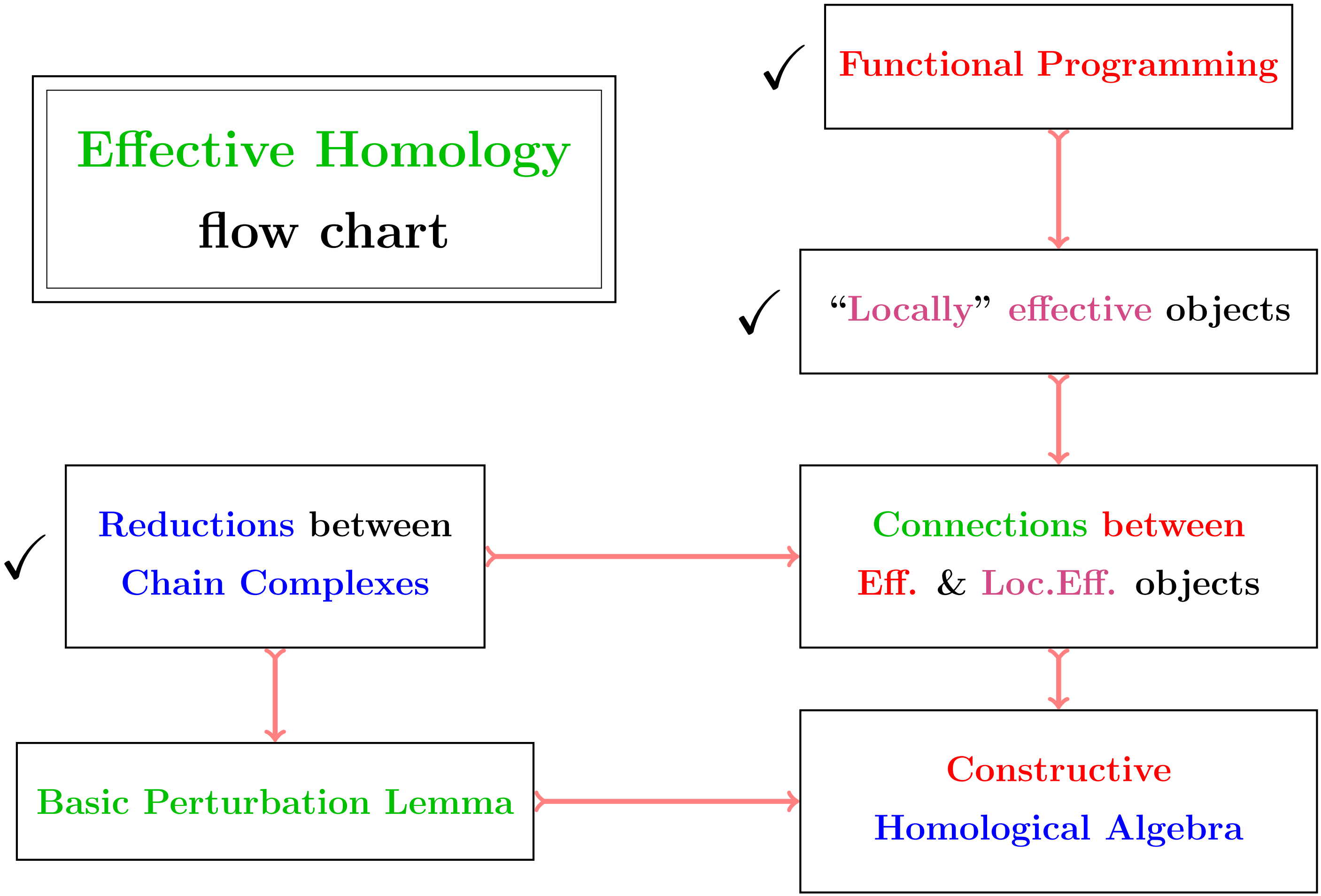
with:

1. $\widehat{C}_*$ and C_* = chain complexes.
2. f and g = chain complex morphisms.
3. h = homotopy operator (degree +1).
4. $fg = \mathrm{id}_{C_*}$ and $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ and $hh = 0$.



$A_* = \ker f \cap \ker h$ $\quad$ $B_* = \ker f \cap \ker d$ $\quad$ $C'_* = \operatorname{im} g$

$$\widehat{C}_* = A_* \oplus B_* \text{ exact} \oplus C'_* \cong C_*$$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Let ρ: $\boxed{h \circlearrowleft \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$ **be a reduction.**

Frequently:

1. $\widehat{C}_*$ **is a locally effective chain complex:**

   its homology groups are unreachable.

2. C_* **is an effective chain complex:**

   its homology groups are computable.

3. **The reduction** ρ **is an entire description of**

   the homological nature of $\widehat{C}_*$**.**

4. **Any homological problem in** $\widehat{C}_*$ **is solvable**

   thanks to the information provided by ρ**.**

ρ: $h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*$

1. **What is $H_n(\widehat{C}_*)$?** **Solution: Compute $H_n(C_*)$.**

2. **Let $x \in \widehat{C}_n$. Is x a cycle?** **Solution: Compute $d_{\widehat{C}_*}(x)$.**

3. **Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?**

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

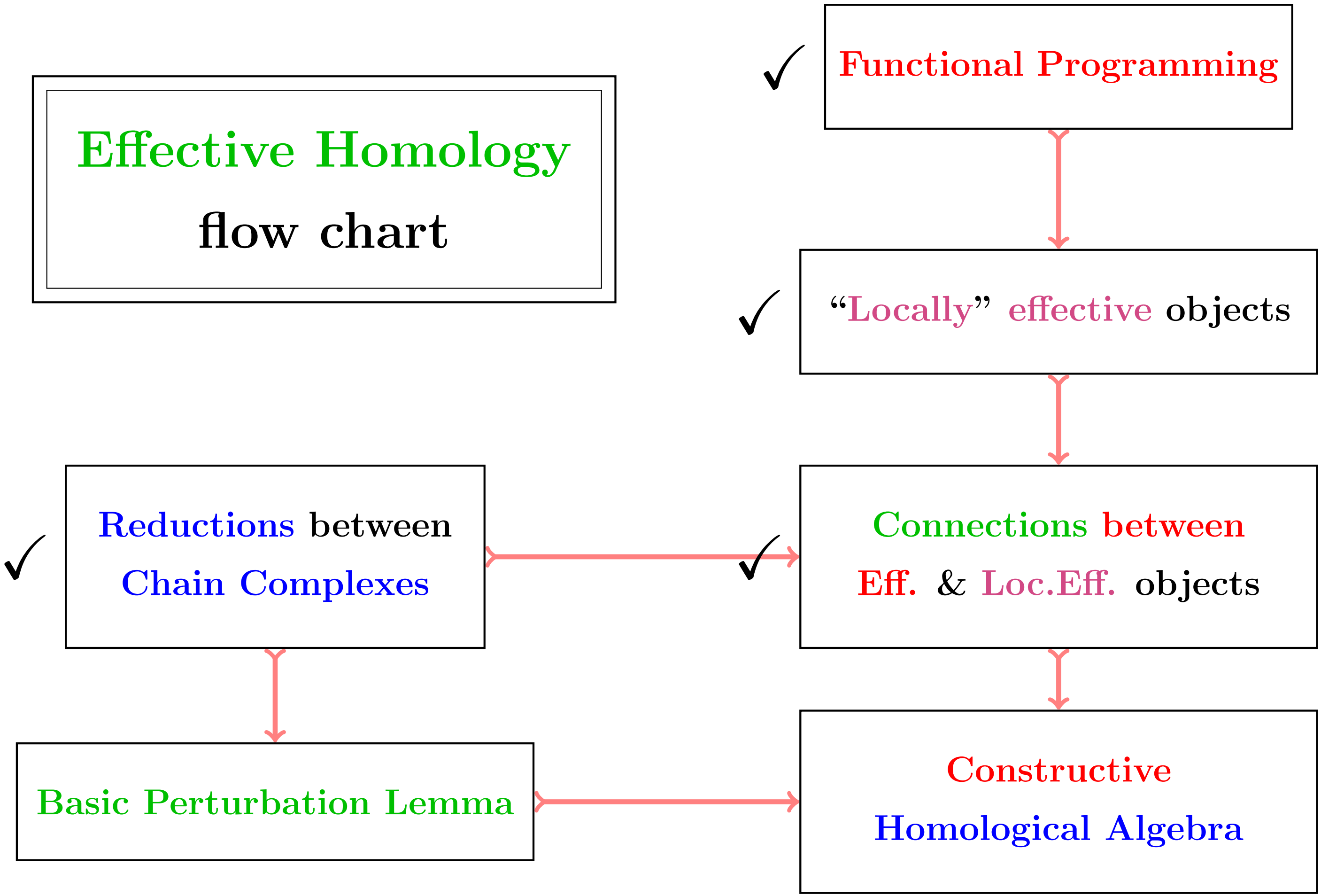
4. **Let $x, x' \in \widehat{C}_n$ be homologous cycles.**

   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.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7/8. Basic Perturbation Lemma.

Definition: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d{+}\delta).$$

Problem: Let ρ: $h \circlearrowleft (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d)$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

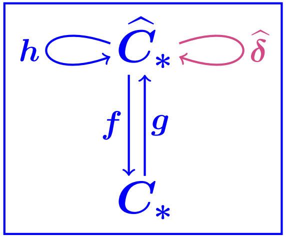
???: $h{+}? \circlearrowleft (\widehat{C}_*, \widehat{d}{+}\widehat{\delta}) \underset{f+?}{\overset{g+?}{\leftrightarrows}} (C_*, d{+}?)$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma" (BPL):

Given:

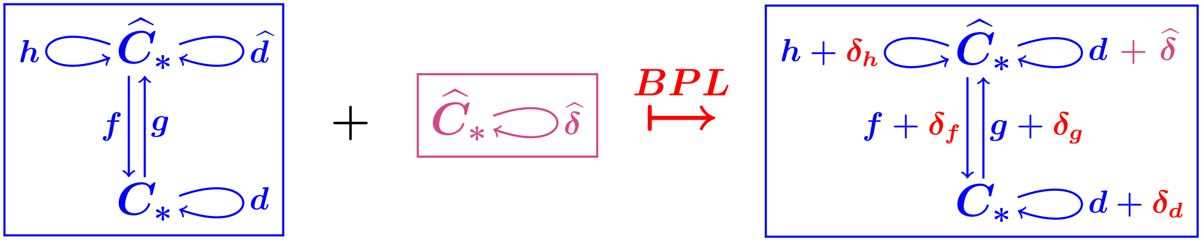


satisfying:

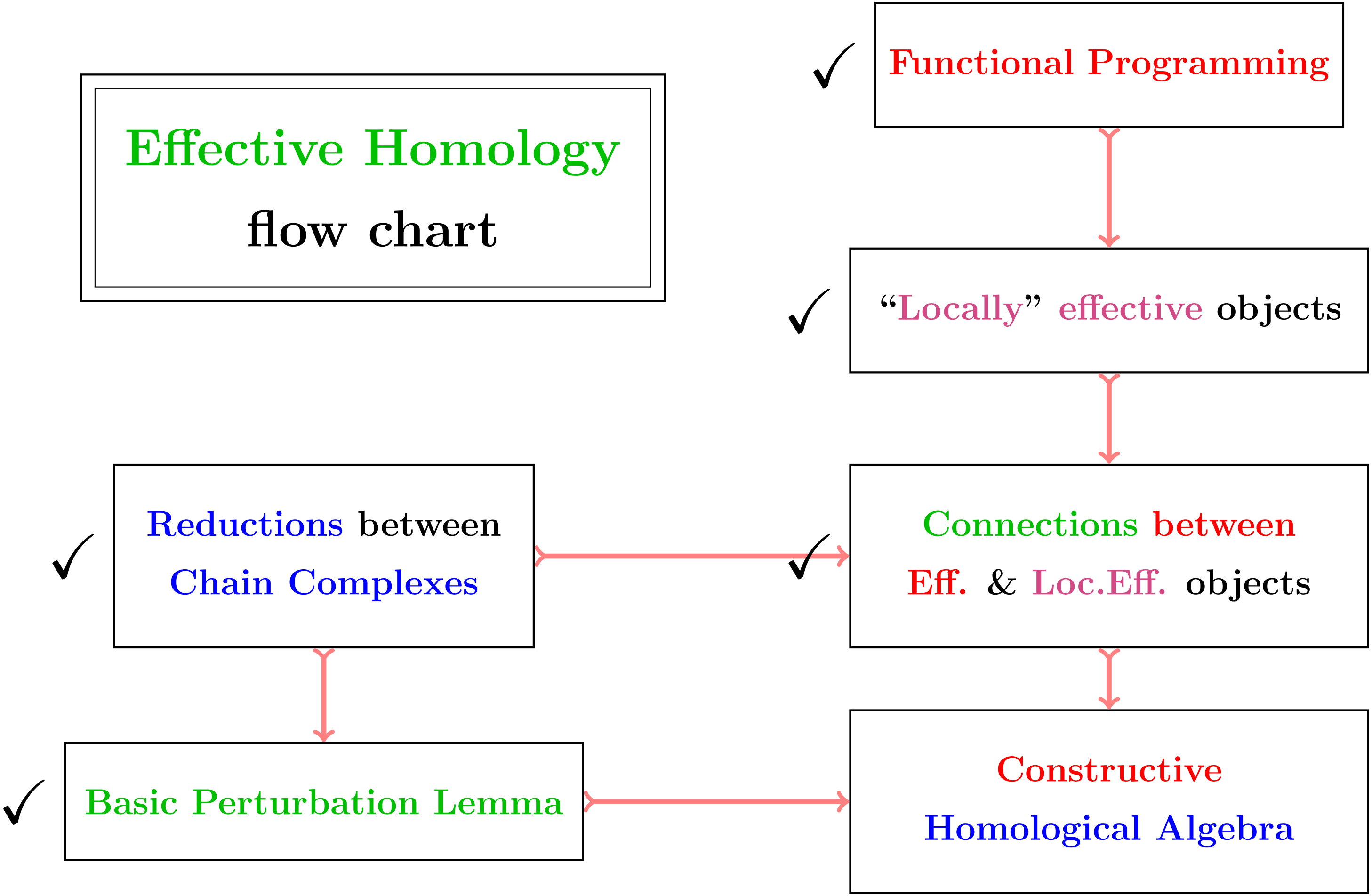
1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$ of $\widehat{C}_*$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:



Effective Homology flow chart



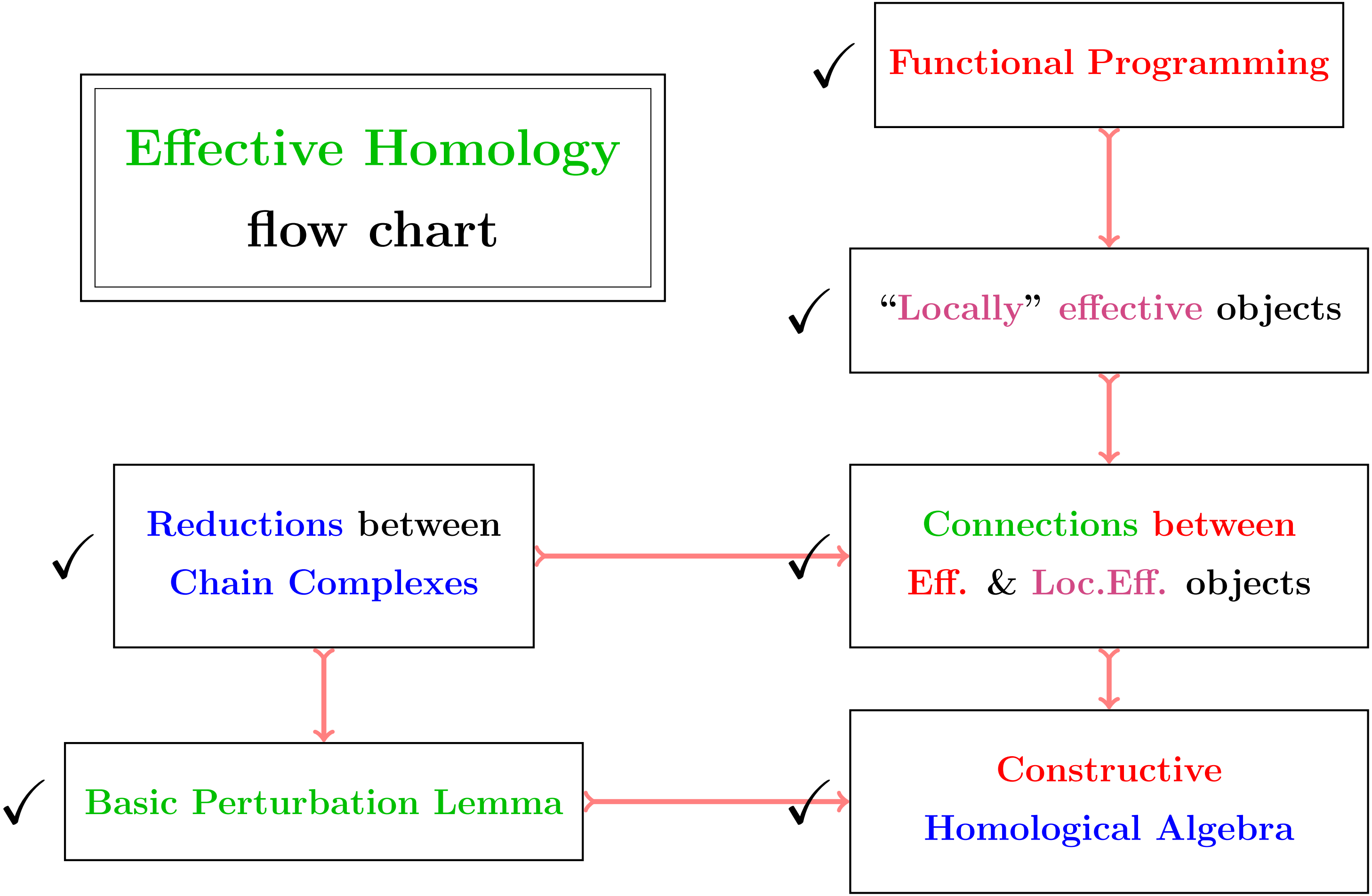
8/8. ⇒ Algebraic Topology becomes Constructive

Serre: "Everything" in Algebraic Topology can be reduced to Fibration problems.

Examples: Loop spaces, Classifying spaces, Homogeneous spaces, Whitehead tower, Postnikov tower, ...

Remark: Fibration = Twisted Product = Perturbation of Trivial Product.

Corollary: BPL is effective
+ Fibration = Perturbation of Trivial Product
+ Everything is Fibration
⇒ Alg. Topology becomes Constructive.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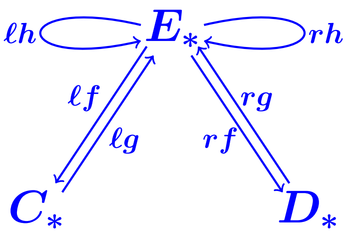
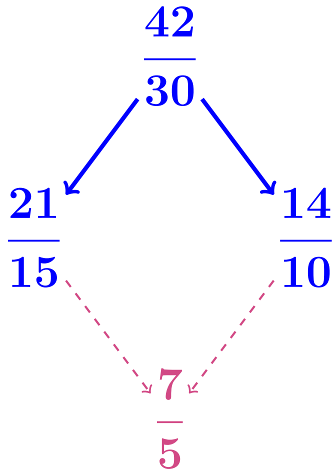
<u>Definition</u>: A (strong chain-) equivalence $\varepsilon : C_* \Longleftrightarrow\!\!\!\!\!\!\Longleftrightarrow D_*$

is a pair of reductions $C_* \overset{\ell\rho}{\Lleftarrow} E_* \overset{r\rho}{\Rrightarrow} D_*$**:**





Normal form problem ??

More structure often necessary in C_***.**

Most often: no possible choice for C_***.**

Definition: An object with effective homology $\boldsymbol{X}$ is a 4-tuple:

$$\boldsymbol{X} = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = "the" chain complex "traditionally" associated with X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \stackrel{\varepsilon}{\Longleftrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_1, \ldots, X_n$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_1, \ldots, X_n) \mapsto X.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}: (\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}) \mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is **perfectly stable** and can be **again used** with X for **further calculations**.

Example:

Julio Rubio's solution of Adams' problem.

$$\boxed{\boldsymbol{X} = (X,\ C_*(X),\ EC_*^X,\ \varepsilon^X)}$$

$$\Downarrow\Downarrow\Downarrow \qquad \textbf{Eil.-Moore}_{EH}$$

$$\boxed{\Omega\boldsymbol{X} = (\Omega X,\ C_*(\Omega X),\ EC_*^{\Omega X},\ \varepsilon^{\Omega X})}$$

$\Longrightarrow$ **Trivial iteration now available.**

$\Rightarrow$ Very simple solution of Adam's problem :

Indefinite iteration of the Cobar construction ???

$$
\begin{aligned}
\boldsymbol{X} &= (X, C_*(X), \boldsymbol{EC}_*^{\boldsymbol{X}}, \varepsilon^X) \\
&\Downarrow \ \Omega_{EH} \\
\boldsymbol{\Omega X} &= (\Omega X, C_*(\Omega X), \boldsymbol{EC}_*^{\boldsymbol{\Omega X}}, \varepsilon^{\Omega X}) \\
&\Downarrow \ \Omega_{EH} \\
\boldsymbol{\Omega^2 X} &= (\Omega^2 X, C_*(\Omega^2 X), \boldsymbol{EC}_*^{\boldsymbol{\Omega^2 X}}, \varepsilon^{\Omega^2 X}) \\
&\Downarrow \ \Omega_{EH} \\
\boldsymbol{\Omega^3 X} &= (\Omega^3 X, C_*(\Omega^3 X), \boldsymbol{EC}_*^{\boldsymbol{\Omega^3 X}}, \varepsilon^{\Omega^3 X}) \\
&\Downarrow \ \Omega_{EH} \\
\boldsymbol{\Omega^4 X} &= \ldots
\end{aligned}
$$

"Cobar" $\boxed{3}$ $(\boldsymbol{EC}_*^{\boldsymbol{X}})$

Example: **Effective homology version of the Serre spectral sequence.**

$$F = (F,\ C_*(F),\ EC_*^F,\ \varepsilon^F)$$

$$+\ \ B = (B,\ C_*(B),\ EC_*^B,\ \varepsilon^B)$$

$$+\ \ \tau : B \rightarrow F$$

$$\Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \ \mathrm{Serre}_{EH}$$

$$E = F \times_\tau B = (E,\ C_*(E),\ EC^E,\ \varepsilon^E)$$

(Serre + G. Hirsch + H. Cartan + Shih W. + Szczarba + Ronnie Brown + J. Rubio + FS)

Proof.

$$C_*(F \times B) \overset{\mathrm{id}}{\Lleftarrow} C_*(F \times B) \overset{EZ}{\Rrightarrow} C_*F \otimes C_*B$$

$$C_*F \otimes C_*B \overset{\otimes}{\Lleftarrow} \widehat{C}^F \otimes \widehat{C}^B \overset{\otimes}{\Rrightarrow} EC^F \otimes EC^B$$

$$\Downarrow\Downarrow\Downarrow\Downarrow\Downarrow \ \mathrm{Serre}_{EH}$$

$$C_*(F \times_{\boxed{\tau}} B) \overset{\mathrm{id}}{\Lleftarrow} C_*(F \times_{\boxed{\tau}} B) \overset{\mathrm{Shih}}{\Rrightarrow} C_*F \otimes_{\boxed{t}} C_*B$$

$$C_*F \otimes_{\boxed{t}} C_*B \overset{EPL}{\Lleftarrow} \widehat{C}^F \otimes_{\boxed{t'}} \widehat{C}^B \overset{BPL}{\Rrightarrow} EC^F \otimes_{\boxed{t''}} EC^B$$

$+$ **Composition of equivalences** $\Longrightarrow$ **O.K.**

Combining these ingredients $\Rightarrow$

Homological Algebra becomes constructive.

<u>Corollary:</u> The "standard" exact and spectral sequences of Homological Algebra really become computational tools.

$\Rightarrow$ Concrete computer programs (EAT, Kenzo).

Example of computation.

$$P^2\mathbb{R} \subset P^3\mathbb{R} \subset P^4\mathbb{R} \subset \cdots \subset P^\infty\mathbb{R}$$

$$\Rightarrow \boxed{\mathtt{P}} = P^\infty\mathbb{R}/P^3\mathbb{R} \text{ is defined.}$$

$$\boxed{\mathtt{OOP}} = \Omega^2\boxed{\mathtt{P}} = \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})$$

$$\pi_2(\boxed{\mathtt{OOP}}) = H_2(\boxed{\mathtt{OOP}}) = \mathbb{Z}$$

$$\Rightarrow f : S^2 \longrightarrow \boxed{\mathtt{OOP}} \text{ of degree 2 defined.}$$

$$\Rightarrow \boxed{\mathtt{DOOP}} = D^3{}_2\cup \boxed{\mathtt{OOP}} \text{ defined.}$$

$$\boxed{\mathtt{ODOOP}} = \Omega\boxed{\mathtt{DOOP}} = \mathrm{Cont}(S^1, \quad D^3{}_2\cup \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})).$$

Exercise: $H_4(\boxed{\mathtt{ODOOP}}) = ??$

Example of computation.

$$P^2\mathbb{R} \subset P^3\mathbb{R} \subset P^4\mathbb{R} \subset \cdots \subset P^\infty\mathbb{R}$$

$$\Rightarrow \boxed{\mathtt{P}} = P^\infty\mathbb{R}/P^3\mathbb{R} \text{ is defined.}$$

$$\boxed{\mathtt{OOP}} = \Omega^2\boxed{\mathtt{P}} = \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})$$

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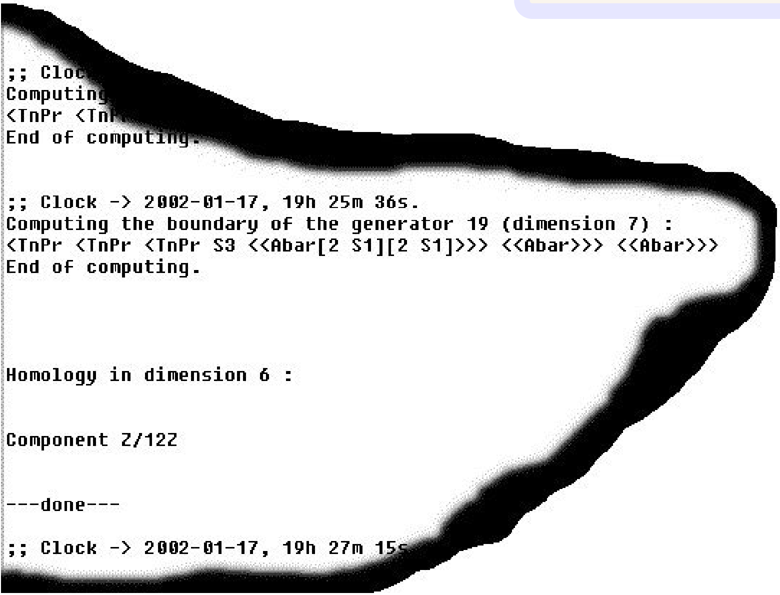
$$\boxed{\mathtt{ODOOP}} = \Omega\boxed{\mathtt{DOOP}} = \mathrm{Cont}(S^1,\quad D^3 \cup_2 \mathrm{Cont}(S^2, P^\infty\mathbb{R}/P^3\mathbb{R})).$$

Solution: $H_4(\boxed{\mathtt{ODOOP}}) = (\mathbb{Z}/2)^8 + \mathbb{Z}$

The END



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