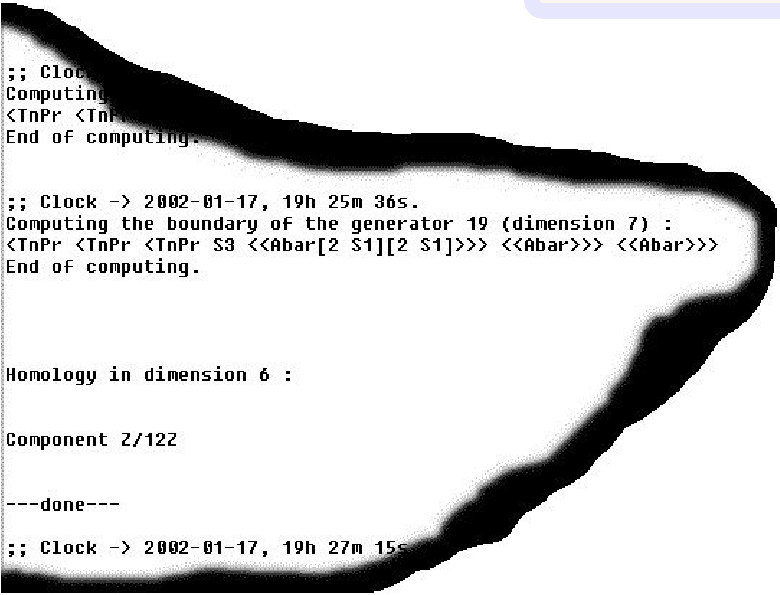
Discrete Vector Fields
Constructive Algebraic Topology
2. E-Zilber, Serre, E-Moore



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Meeting Constructive Algebraic Topology
Cirm, Luminy, January 24-28, 2011

Continuation of the story:

Vector Fields ⇒ Eilenberg-Zilber

⇒ Twisted Eilenberg-Zilber

⇒ Serre spectral sequence

⇒ Eilenberg-Moore spectral sequence.

Product problem in Combinatorial Topology.

1. **Simplicial organisation necessary**
   for example for Eilenberg-MacLane spaces.

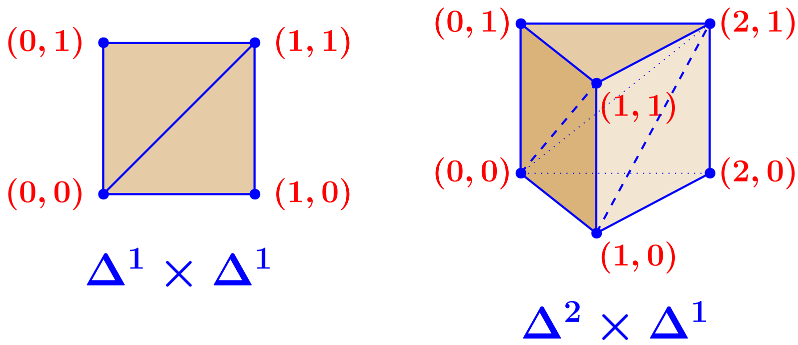
2. $\Rightarrow$ **Elementary models** $= \boldsymbol{\Delta}^n$ **for** $\boldsymbol{n} \in \mathbb{N}$.

3. **Fact:**
   No direct simplicial structure for a product $\boldsymbol{\Delta}^p \times \boldsymbol{\Delta}^q$.

4. **What about twisted products = Fibrations ??**

5. **Classical solution = Eilenberg-Zilber + Kan + Shih**
   + Serre and Eilenberg-Moore Spectral sequences.

6. **Other solution = Discrete Vector Fields.**



Two Δ^2 in $\Delta^1 \times \Delta^1$: $(0,0) < (0,1) < (1,1)$

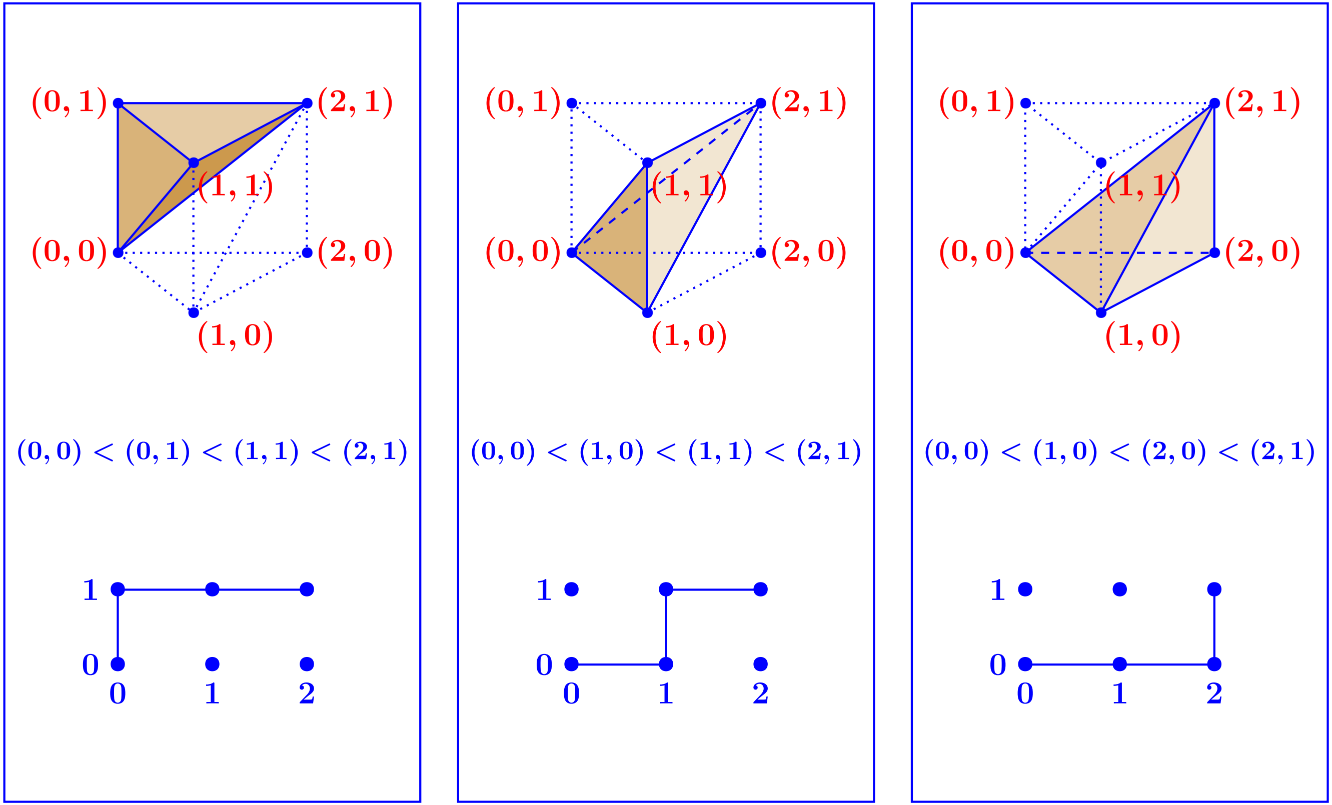
$(0,0) < (1,0) < (1,1)$

Three Δ^3 in $\Delta^2 \times \Delta^1$: $(0,0) < (0,1) < (1,1) < (2,1)$

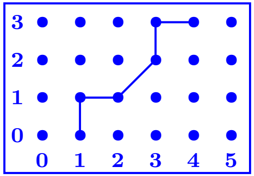
$(0,0) < (1,0) < (1,1) < (2,1)$

$(0,0) < (1,0) < (2,0) < (2,1)$

Rewriting the triangulation of $\Delta^2 \times \Delta^1$.

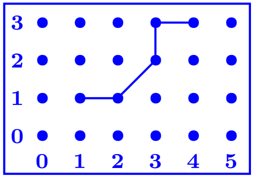


"Seeing" the triangulation of $\boldsymbol{\Delta}^5 \times \boldsymbol{\Delta}^3$.

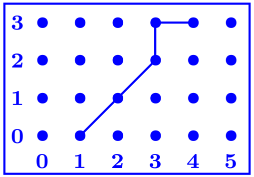
Example of 5-simplex : 

$= \boldsymbol{\sigma} \in (\boldsymbol{\Delta}^5 \times \boldsymbol{\Delta}^3)_5$

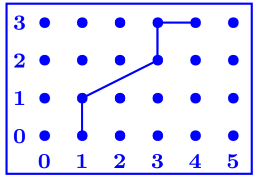
$\Rightarrow$ **6 faces:**



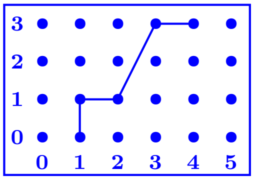
$\partial_0\sigma$



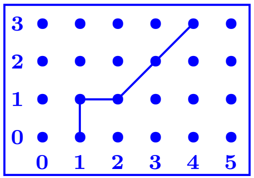
$\partial_1\sigma$



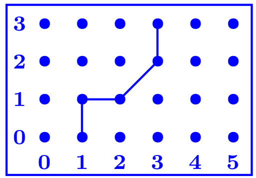
$\partial_2\sigma$



$\partial_3\sigma$

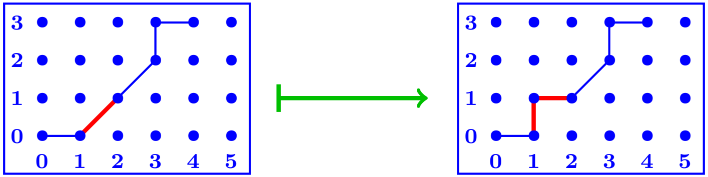


$\partial_4\sigma$

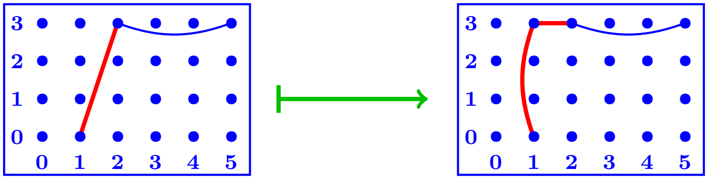


$\partial_5\sigma$

⇒ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

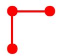


Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

<u>Recipe</u>: First "event" = Diagonal step =  ⇒ Source cell.

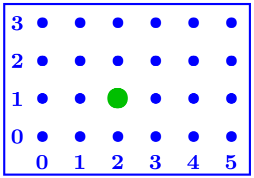
= (-90°)-bend =  ⇒ Target cell.

Critical cells ??

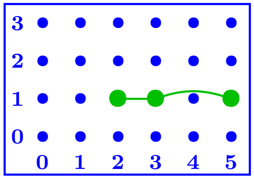
Critical cell = cell without any "event"

= without any diagonal or −90°-bend.

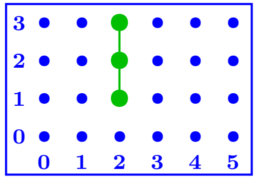
Examples.



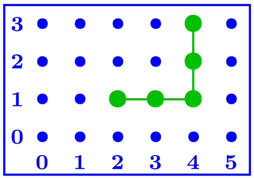
$\Delta^0_2 \otimes \Delta^0_1$



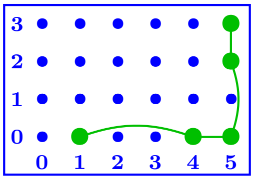
$\Delta^2_{2,3,5} \otimes \Delta^0_1$



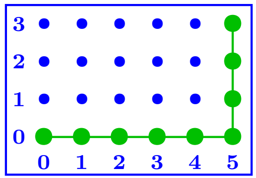
$\Delta^0_2 \otimes \Delta^2_{1,2,3}$



$\Delta^2_{2,3,4} \otimes \Delta^2_{1,2,3}$



$\Delta^2_{1,4,5} \otimes \Delta^2_{0,2,3}$



$\Delta^5_{0,1,2,3,4,5} \otimes \Delta^3_{0,1,2,3}$

Conclusion:

$$C_*^c = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

Fundamental theorem of vector fields $\Rightarrow$

Canonical Homological Reductions:

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow$ **16,583,583,743 vs 4,190,209**

More generally: X and Y = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C_*^c(X \times Y) = C_*(X) \otimes C_*(Y)$.

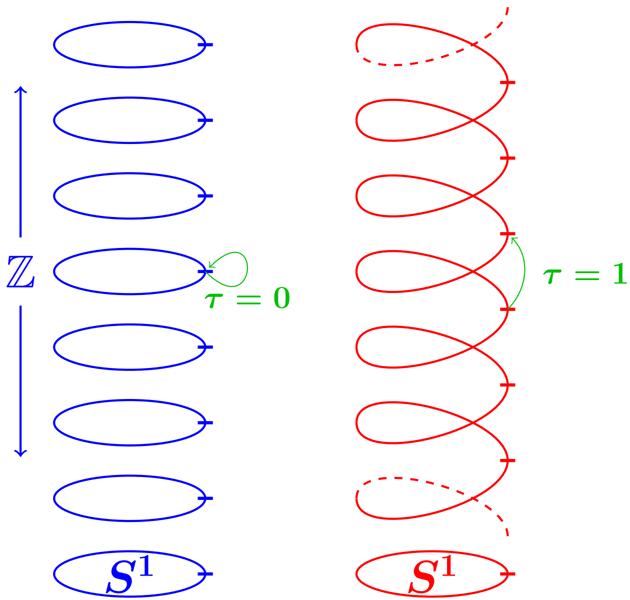
Eilenberg-Zilber Theorem: Canon. homological reduction:

$$\rho_{EZ} : C_*(X \times Y) \Rrightarrow C_*^c(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

Notion of twisted product.

Simplest example: $\mathbb{Z} \times \boldsymbol{S^1}$ **vs** $\mathbb{Z} \times_{\boldsymbol{\tau}} \boldsymbol{S^1} = \mathbb{R}$**:**



General notion of twisted product: B = base space.

F = fibre space.

G = structural group.

Action $G \times F \to F$.

$\tau : B \to G$ = Twisting function.

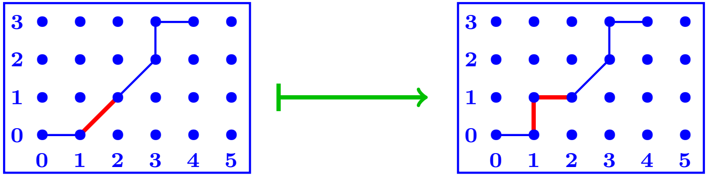
Structure of $F \times_\tau B$:

$$\partial_i(\sigma_f, \sigma_b) = (\partial_i \sigma_f, \partial_i \sigma_b) \quad \text{for } i > 0$$

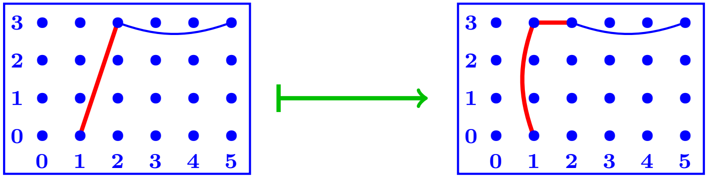
$$\partial_0(\sigma_f \ , \ \sigma_b) = (\tau(\sigma_b).\partial_0 \sigma_f \ , \ \partial_0 \sigma_b)$$

$\Rightarrow$ Only the **0-face** is modified in the twisted product.

Reminder about the EZ-vector field of $\Delta^5 \times \Delta^3$.



Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

The vector field is concerned by faces ∂_i only if $i > 0$.

1. The twisting function τ modifies only $\boxed{0}$-faces.

2. The EZ-vector field V_{EZ} of $X \times Y$

uses only $\boxed{i}$-faces with $i \geq 1$.

$\Rightarrow V_{EZ}$ is defined and admissible as well on $X \times_{\boxed{\tau}} Y$.

Fundamental theorem of admissible vector fields $\Rightarrow$

$$C_*(X \times Y) \qquad\qquad C_*(X \times_{\boxed{\tau}} Y)$$

$$V_{EZ} \Rightarrow \Downarrow\!\!\Downarrow \qquad\qquad V_{EZ} \Rightarrow \Downarrow\!\!\Downarrow$$

$$C_*(X) \otimes C_*(Y) \qquad\qquad C_*(X) \otimes_{\boxed{t}} C_*(Y)$$

Known as the twisted Eilenberg-Zilber Theorem.

Corollary: Base B 1-reduced $\Rightarrow$ Algorithm:

$$[(F, C_*(F), EC_*^F, \varepsilon_F) + (B, C_*(B), EC_*^B, \varepsilon_B) + G + \tau]$$
$$\longmapsto (F \times_\tau B, C_*(F \times_\tau B), EC_*^{F \times_\tau B}, \varepsilon_{F \times_\tau B}).$$

Version of F with effective homology

$+$ Version of B with effective homology

$+$ $G + \tau$ describing the fibration $F \hookrightarrow F \times_\tau B \to B$

$\Rightarrow$ Version with effective homology of the total space $F \times_\tau B$.

$=$ Version with effective homology

of the Serre Spectral Sequence

Analogous result for the Eilenberg-Moore spectral sequence.

Key results:

G = Simplicial group $\Rightarrow$ BG = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

X = Simplicial set $\Rightarrow$ KX = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

Analogous process $\Rightarrow$ Algorithms:

$$(G, C_*G, EC_*^G, \varepsilon_G) \ \mapsto\ (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(G, C_*X, EC_*^X, \varepsilon_X) \ \mapsto\ (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

More generally:

$[\alpha : E \rightarrow B] + [\alpha' : E' \rightarrow B] + [\alpha \text{ fibration}]$

$\Rightarrow$ **algorithm:** $(B_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E \times_B E')_{EH}$.

$$\begin{array}{ccc} E' \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \alpha \\ E' & \xrightarrow[\alpha']{} & B \end{array}$$

= **Version with effective homology**

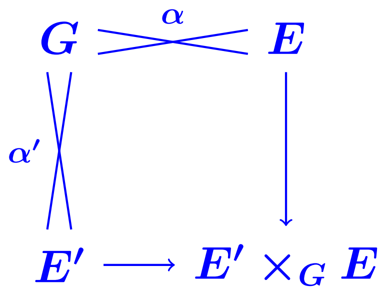
of Eilenberg-Moore spectral sequence I.

Also:

$[G \text{ simplicial group}] + [\alpha : G \times E \rightarrow E] +$

$[\alpha' : E' \times G \rightarrow E'] + [\alpha \text{ principal fibration}]$

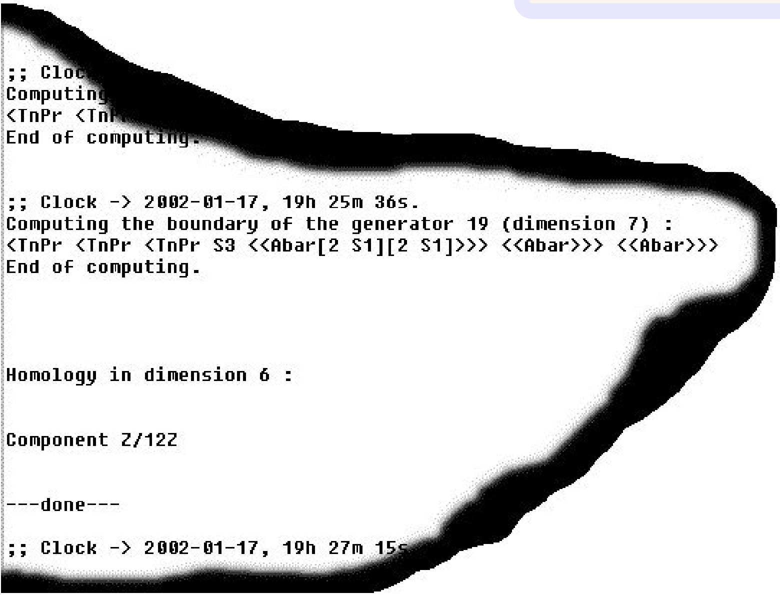
$\Rightarrow$ **algorithm:** $(G_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E' \times_G E)_{EH}$.



= Version with effective homology

of Eilenberg-Moore spectral sequence II.

The END



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