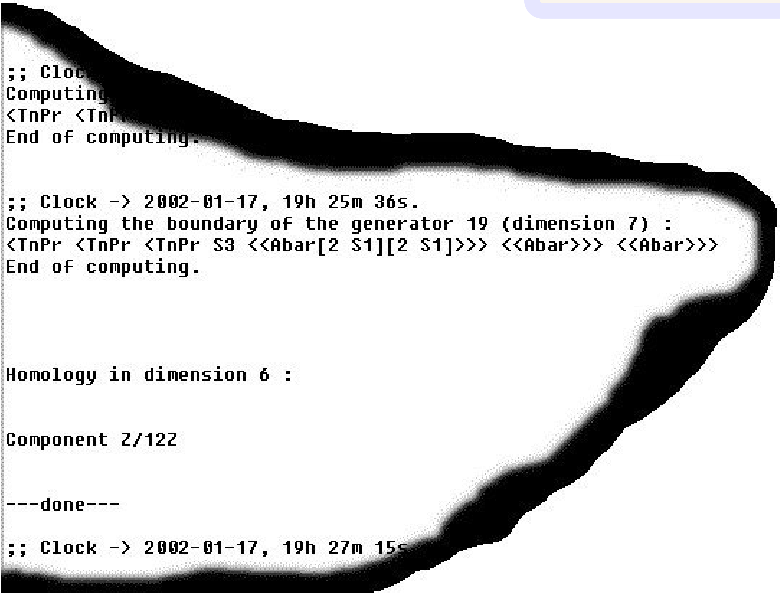
Discrete Vector Fields

Constructive Algebraic Topology

1. Basic Facts



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Meeting Constructive Algebraic Topology
Cirm, Luminy, January 24-28, 2011

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, . . .
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, . . .
Green	=	**Solution, essential point,**
		mathematicians, . . .
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

Plan.

1. **Introduction.**
2. **Discrete vector fields.**
3. **Homological Reductions.**
4. **Admissible Algebraic Vector Field $\Rightarrow$ Homological Reduction.**

1/4. Introduction.

Algebraic Topology is a translator:

1/4. Introduction.

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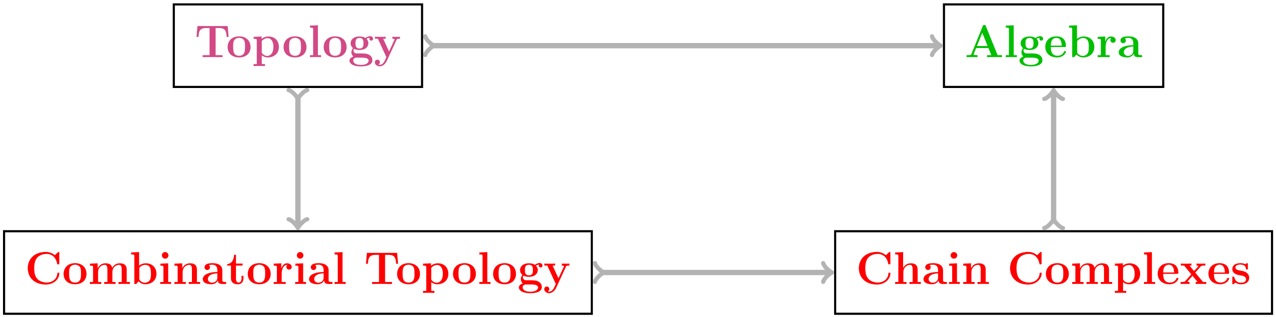


Serre (1950): Up to homotopy

any map can be transformed into a fibration.

Fibration = Twisted Product

		Topology $\longmapsto$ Algebra
Product		Eilenberg-Zilber Theorem
Twisted product		Serre Spectral Sequence

Discrete vector fields

$\Rightarrow$ **New understanding of the Eilenberg-Zilber Theorem**

$\Rightarrow$ **An effective version of the Serre Spectral Sequence**

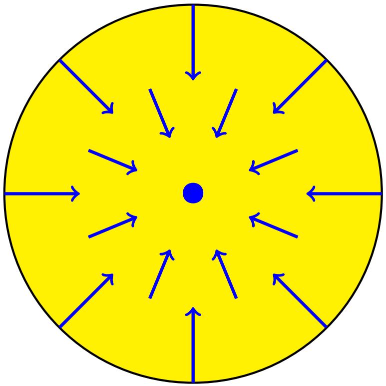
as a direct consequence of this version of Eilenberg-Zilber.

2/4. Discrete vector fields

Ordinary vector fields → **Discrete vector fields** → **Algebraic vector fields**

Ordinary vector field:

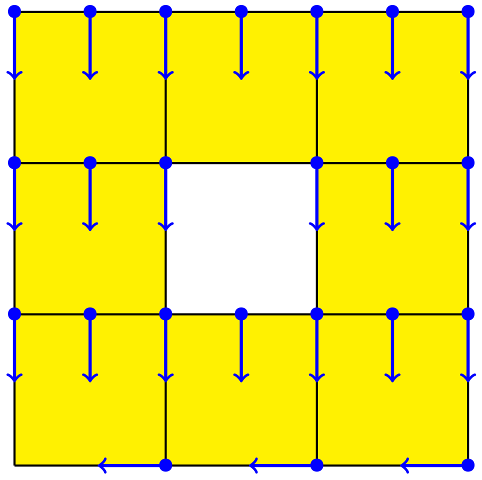
$D^2 =$

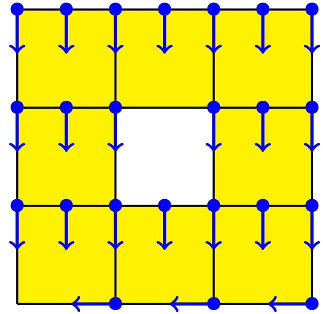


$\Rightarrow$ $D^2 \sim \bullet$

Discrete vector field in a cellular complex.

Example for a cubical complex.





Definition:

A Discrete Vector Field is a pairing:

$$V = \{(\sigma_i, \tau_i)\}_{i \in v}$$

satisfying:

- $\forall i \in v, \quad \tau_i =$ **some k_i-cell and $\sigma_i =$ some $(k_i - 1)$-cell.**
- $\forall i \in v, \quad \sigma_i$ **is a regular face of τ_i.**
- $\forall i \neq j \in v, \quad \sigma_i \neq \sigma_j \neq \tau_i \neq \tau_j$.
- **The vector field V is admissible.**

Definition: A(n algebraic) cellular chain complex C_* is a triple $C_* = \{C_p, \beta_p, d_p\}_{p \in \mathbb{Z}}$ satisfying:

- β_p is the distinguished basis of the free $\mathbb{Z}$-module $C_p = \mathbb{Z}[\beta_p]$.
- $d_p : C_p \rightarrow C_{p-1}$ is a differential ($d^2 = 0$).

Examples: Chain complexes coming from:

- Simplicial complexes, cubical complexes, simplicial sets, CW-complexes...
- Digital images.
- Chain complex defining some Koszul homology ($\mathbb{Z} \mapsto \mathfrak{k}$).
- ... ...

$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = Cellular chain complex.

Definition: A p-cell is an element of β_p.

Definition: If $\tau \in \beta_p$ and $\sigma \in \beta_{p-1}$,

then $\varepsilon(\sigma, \tau) :=$ coefficient of σ in $d\tau$

is called the incidence number between σ and τ.

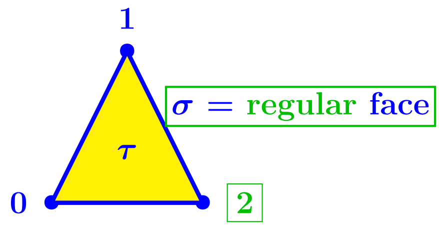
Definition: σ is a face of τ if $\varepsilon(\sigma, \tau) \neq 0$.

Definition: σ is a regular face of τ if $\varepsilon(\sigma, \tau) = \pm 1$.

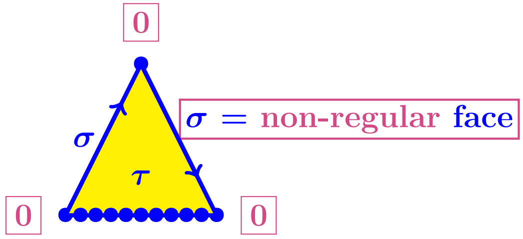
[More generally if $\mathbb{Z} \mapsto \boldsymbol{R}$,

regular face $\Leftrightarrow$ $\varepsilon(\sigma, \tau)$ invertible]

Geometrical example of non-regular face:



Triangle



$P^2\mathbb{R}$

$$C_*(\text{Triangle}) = \{0 \longleftarrow \mathbb{Z}^3 \xleftarrow{\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & \boxed{1} \end{bmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{bmatrix} \boxed{1} \\ -1 \\ 1 \end{bmatrix}} \mathbb{Z} \longleftarrow 0\}$$

$$C_*(P^2\mathbb{R}) = \{0 \longleftarrow \mathbb{Z} \xleftarrow{\begin{bmatrix} \boxed{0} \end{bmatrix}} \mathbb{Z} \xleftarrow{\begin{bmatrix} \boxed{2} \end{bmatrix}} \mathbb{Z} \longleftarrow 0\}$$

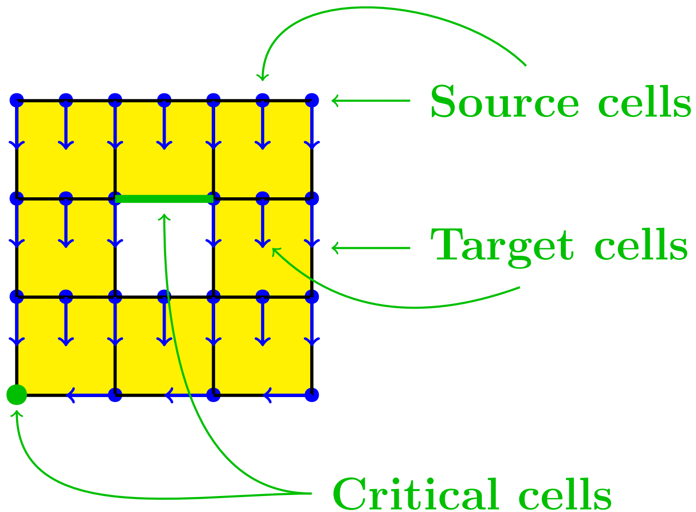
$C_* = \{C_p, \beta_p, d_p\}_{p \in \mathbb{Z}}$ = **Cellular chain complex.**

$V = \{(\sigma_i, \tau_i)\}_{i \in v}$ = **Vector field.**

Definition: A critical p-cell is an element of β_p which does not occur in V.

Other cells divided in source cells and target cells.

Example:



$C_* = \{C_p, \beta_p, d_p\}_{p\in\mathbb{Z}}$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)\}_{i\in v}$ = Vector field.

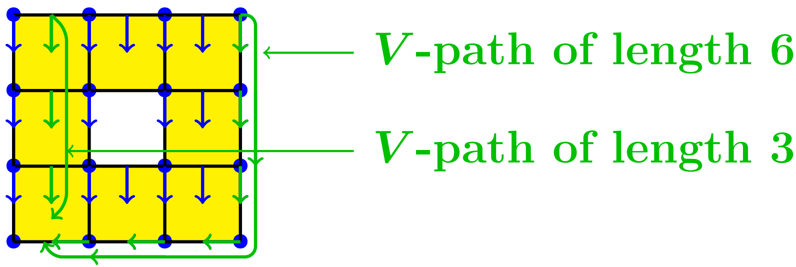
Definition: V-path = sequence $(\sigma_{i_1}, \tau_{i_1}, \sigma_{i_2}, \tau_{i_2}, \ldots, \sigma_{i_n}, \tau_{i_n})$

satisfying:

1. $(\sigma_{i_j}, \tau_{i_j}) \in V$.
2. σ_{i_j} face of $\tau_{i_{j-1}}$.
3. $\sigma_{i_j} \neq \sigma_{i_{j-1}}$.

Remark: σ_{i_j} not necessarily regular face of $\tau_{i_{j-1}}$.

Examples:



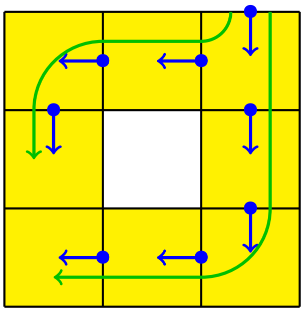
Definition: A vector field is admissible if

for every source cell $\boldsymbol{\sigma}$,

the length of any path starting from $\boldsymbol{\sigma}$

is bounded by a fixed integer $\boldsymbol{\lambda(\sigma)}$.

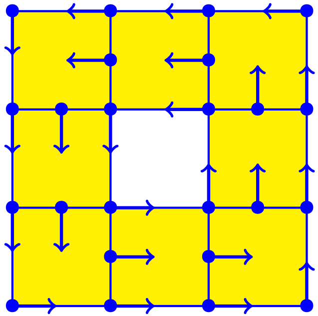
Example of two different paths with the same starting cell.



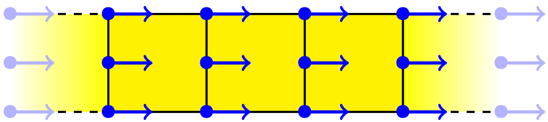
Remark: The paths from a starting cell

are not necessarily organized as a tree.

Typical examples of non-admissible vector fields.



???!!! $\sim$ $\emptyset$

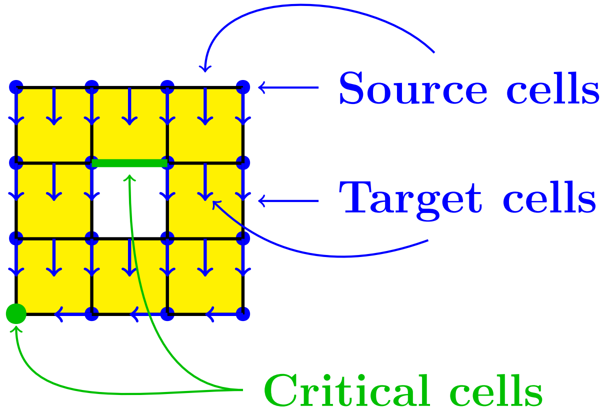


???!!! $\sim$ $\emptyset$

$\mathbb{R} \times \boldsymbol{I}$

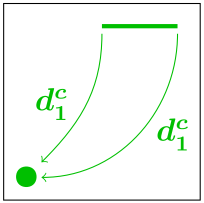
Main motivation.

$C_* =$



Fundamental Reduction Theorem $\Rightarrow$

$\rho : C_* \Rrightarrow C^c_* =$



$= \mathbb{Z} \xleftarrow{d^c_1=0} \mathbb{Z} = \text{Circle}$

$\text{Rank}(C_*) = (16, 24, 8)$ vs $\text{Rank}(C^c_*) = (1, 1, 0)$

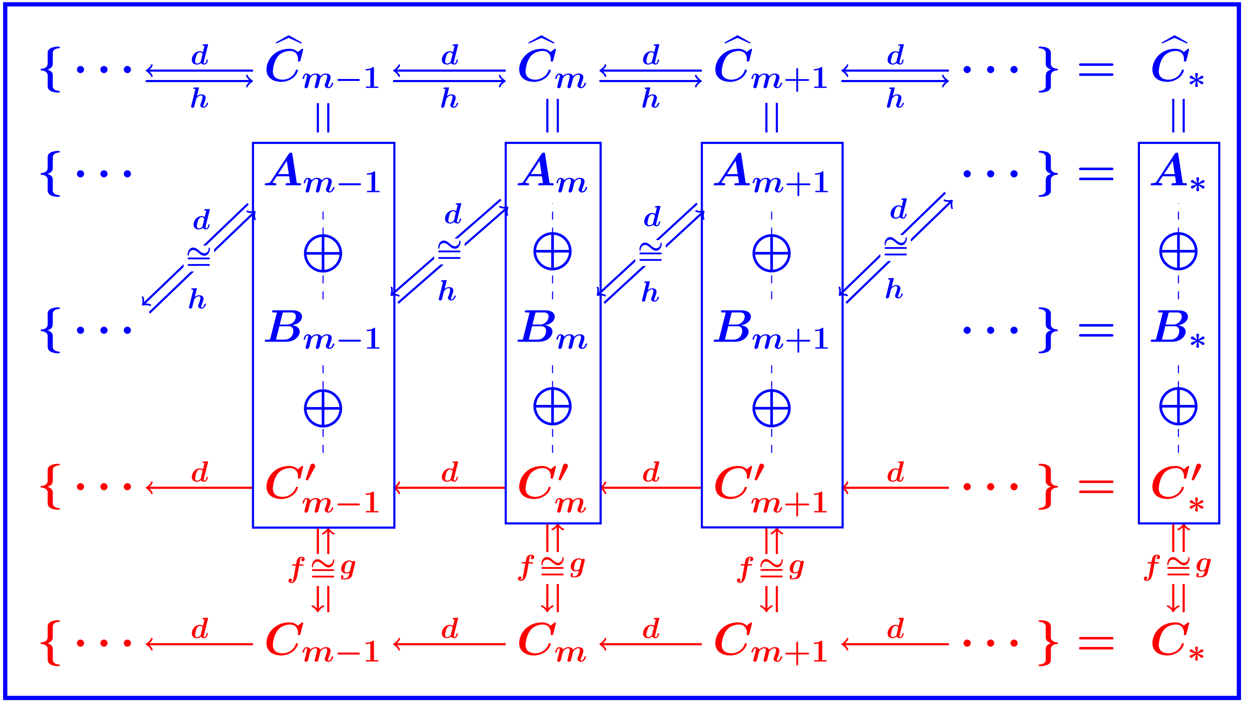
3/4. Homological Reductions.

Definition: A (homological) reduction is a diagram:

$$\rho: \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ **and** C_* **= chain complexes.**
2. f **and** g **= chain complex morphisms.**
3. h **= homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ **and** $hh = 0$.

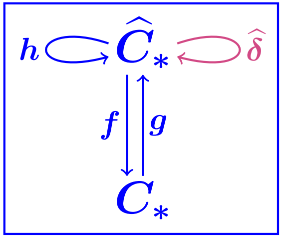


$A_* = \ker f \cap \ker h$ $\quad$ $B_* = \ker f \cap \ker d$ $\quad$ $C'_* = \mathrm{im}(g)$

$$\widehat{C}_* = A_* \oplus B_* \text{exact} \oplus C'_* \cong C_*$$

Basic Perturbation "Lemma" (BPL):

Given:

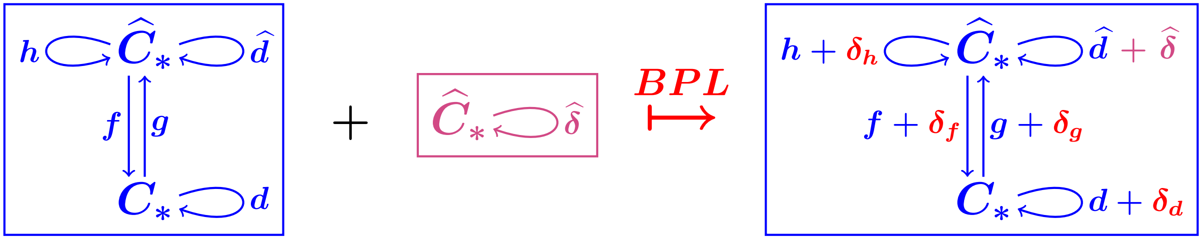


satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$ of $\widehat{C}_*$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:



4/4. Admissible Algebraic Vector Field ⇒ Homological Reduction.

Fundamental Theorem:

Given: $C_* = (C_p, \beta_p, d_p)_{p\in\mathbb{Z}}$ = Cellular chain complex.

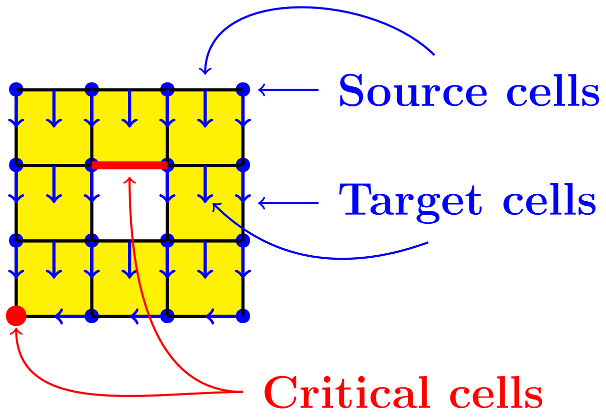
$V = (\sigma_i, \tau_i)_{i\in v}$ = Admissible Discrete Vector Field.

⇒ **Canonical Reduction:**

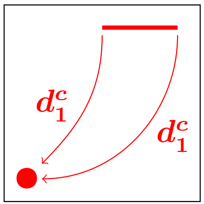
$$\rho_V = \boxed{h \circlearrowright (C_p, \beta_p, d_p)_{p\in\mathbb{Z}} \underset{f}{\overset{g}{\leftrightarrows}} (C^c_p, \beta^c_p, d^c_p)_{p\in\mathbb{Z}}}$$

$$\boxed{\text{Initial Complex}} \overset{\rho_V}{\Rrightarrow} \boxed{\text{Critical complex}}$$

Toy Example.

$C_* =$ 

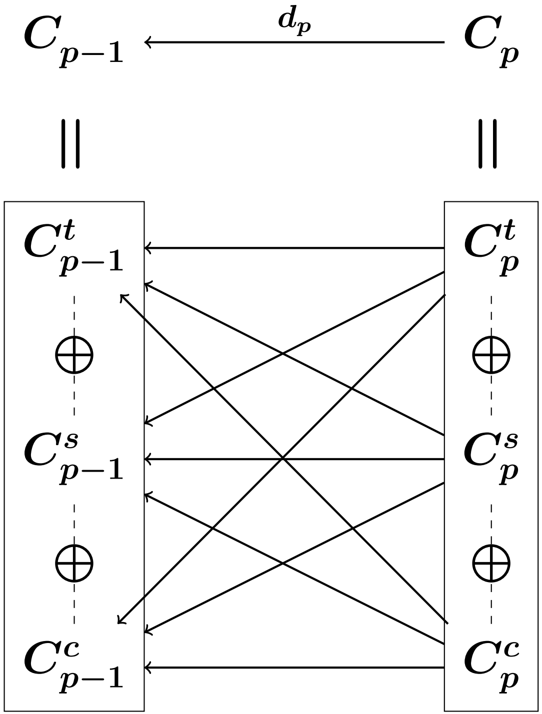
Fundamental Reduction Theorem $\Rightarrow$

$\rho : C_* \Rrightarrow C_*^c =$ 

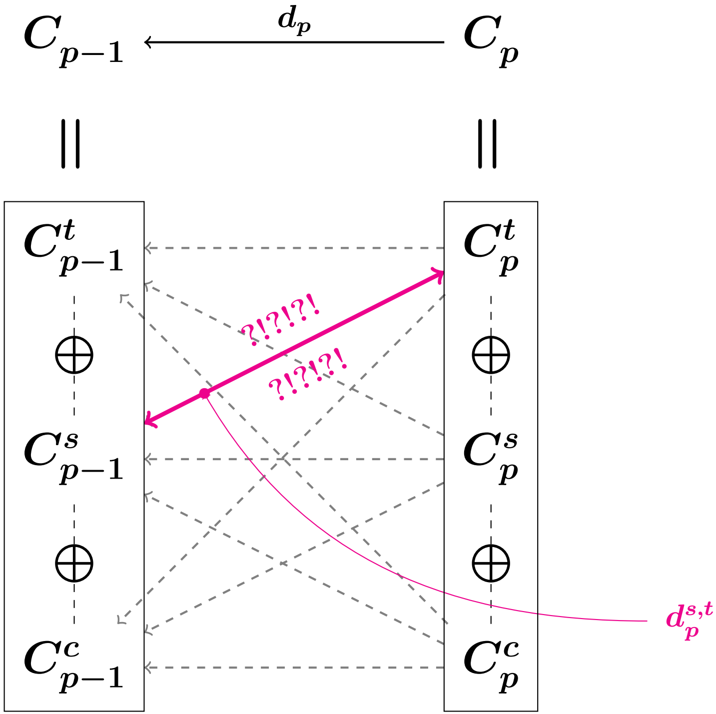
$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \text{Circle}$

$\text{Rank}(C_*) = (16, 24, 8)$ vs $\text{Rank}(C_*^c) = (1, 1, 0)$

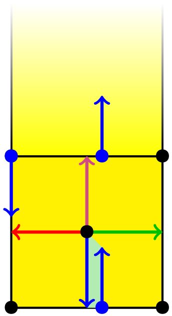
Proof 1:

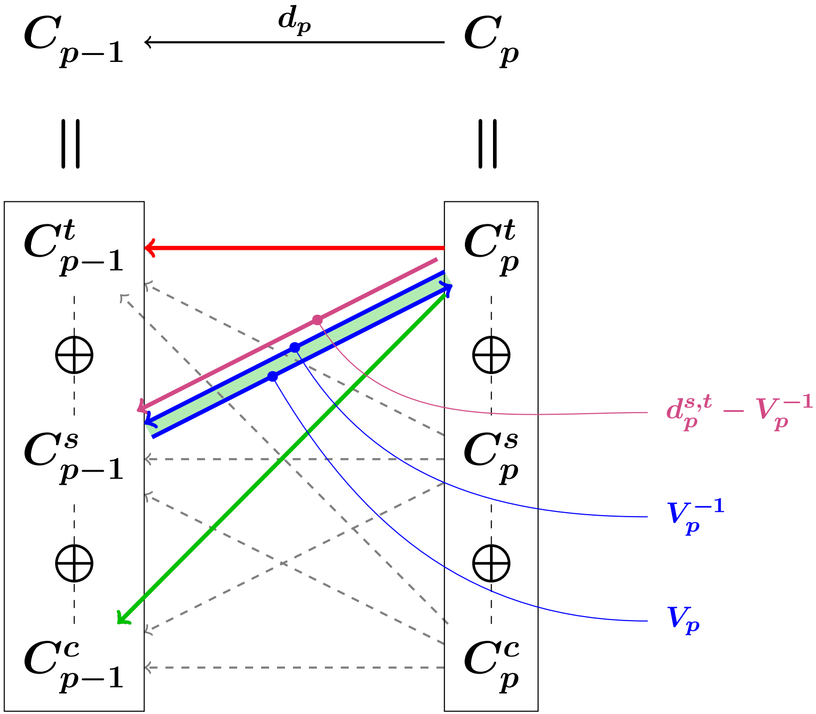


Proof 2:

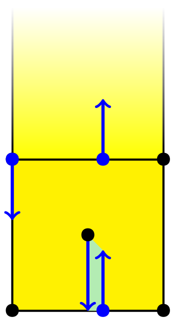


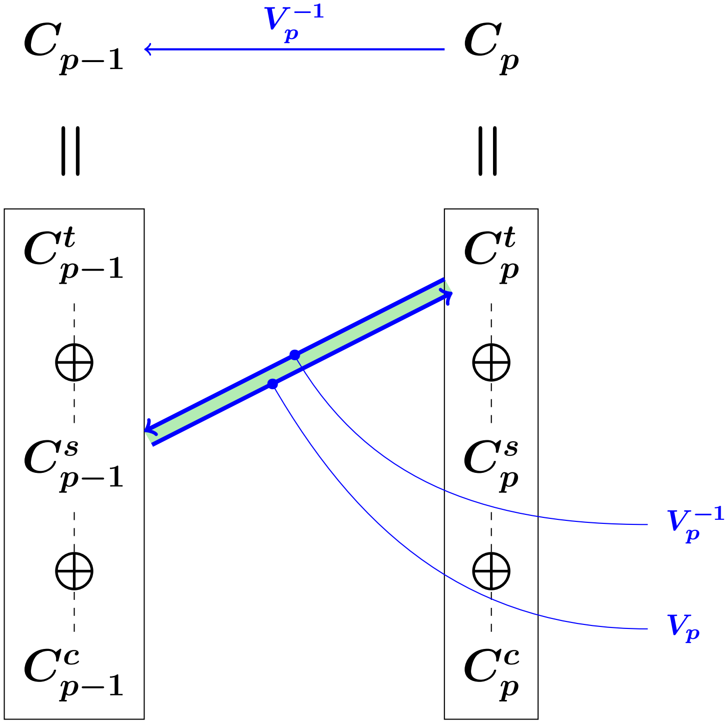
Proof 3:



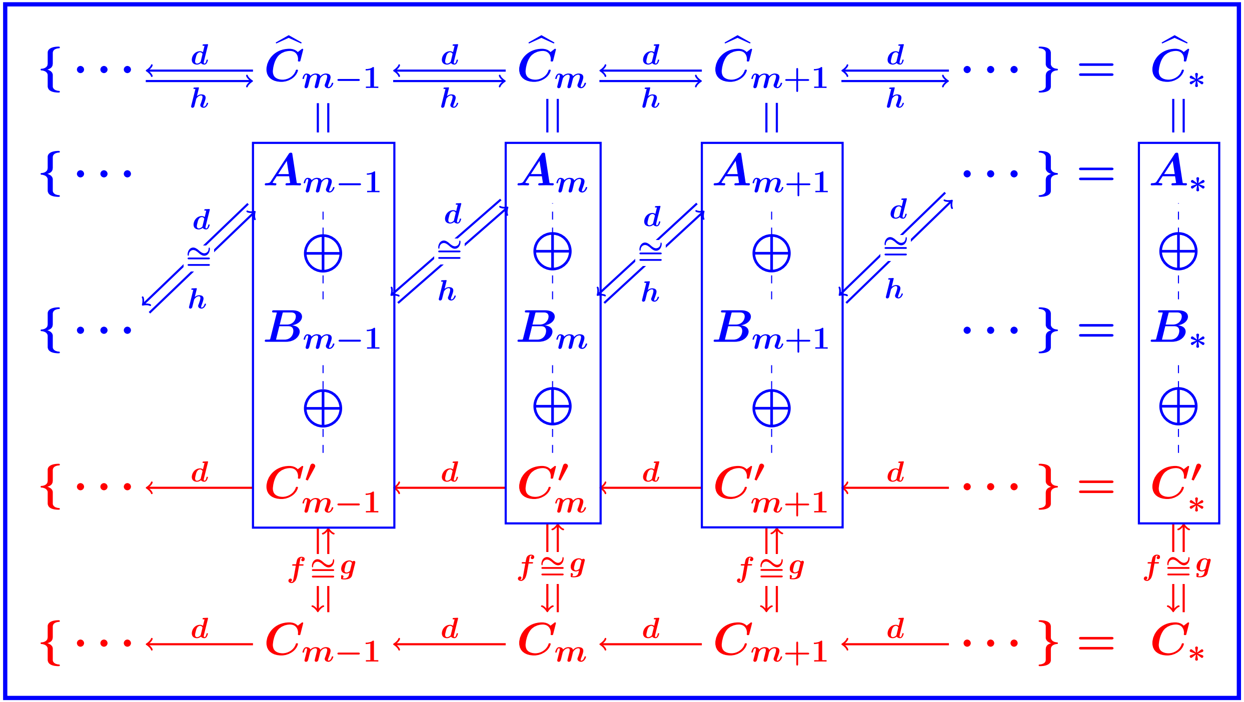

C_{p-1}
d_p
C_p
C^t_{p-1}
$\oplus$
C^s_{p-1}
$\oplus$
C^c_{p-1}
C^t_p
$\oplus$
C^s_p
$\oplus$
C^c_p
$d^{s,t}_p - V^{-1}_p$
V^{-1}_p
V_p

Proof 4:





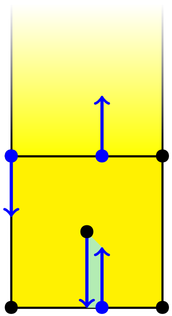
C_{p-1}
V_p^{-1}
C_p
$\|$
$\|$
C_{p-1}^t
$\oplus$
C_{p-1}^s
$\oplus$
C_{p-1}^c
C_p^t
$\oplus$
C_p^s
$\oplus$
C_p^c
V_p^{-1}
V_p

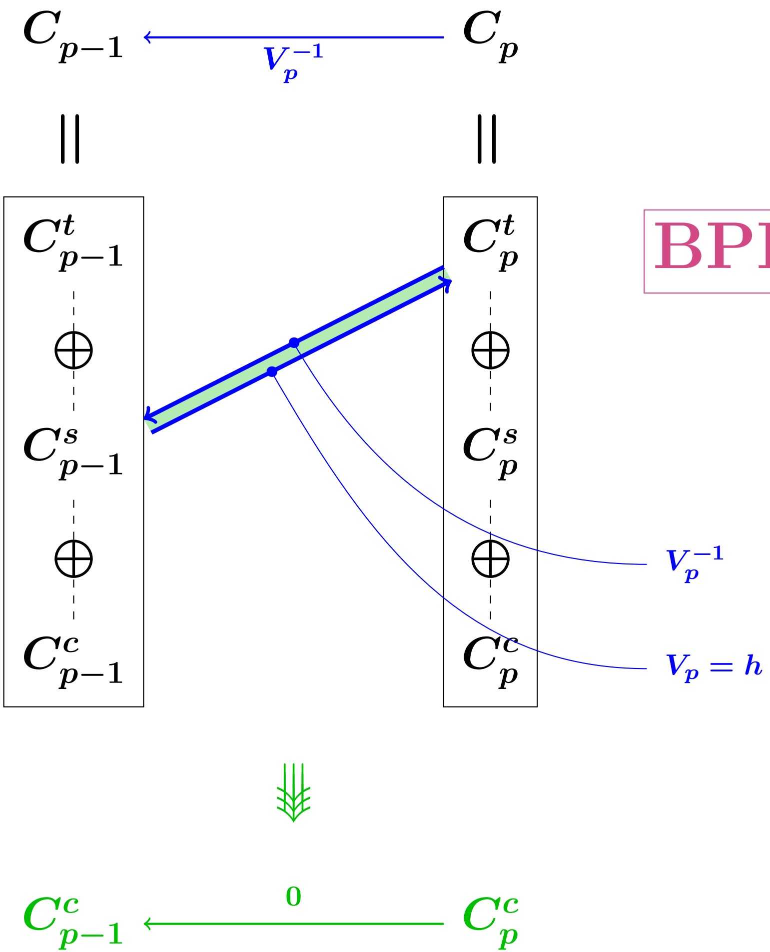


$A_* = \ker f \cap \ker h$ $B_* = \ker f \cap \ker d$ $C'_* = \mathrm{im}(g)$

$\widehat{C}_* = A_* \oplus B_* \text{exact} \oplus C'_* \cong C_*$

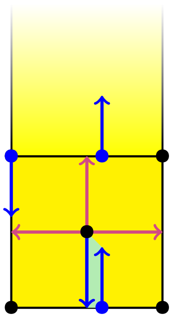
Proof 5:

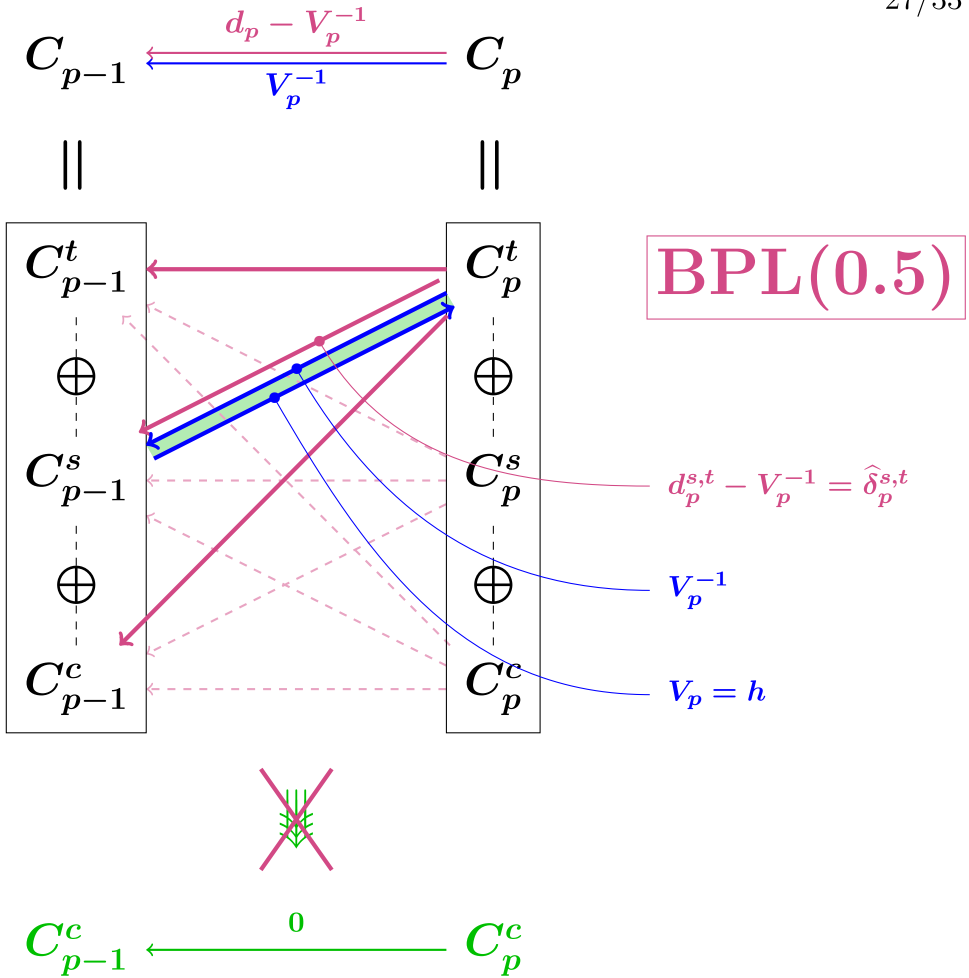




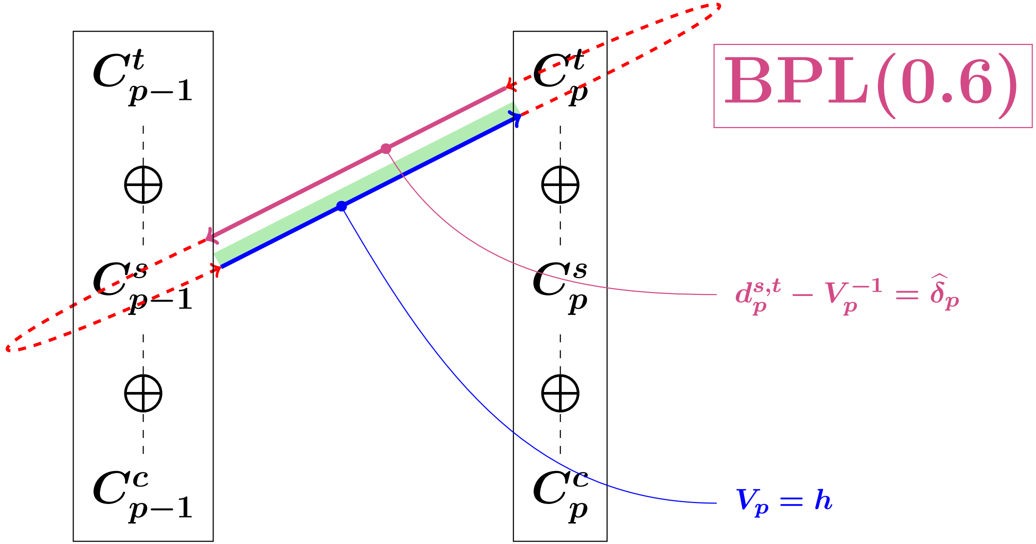
BPL(0.0)

Proof 6:



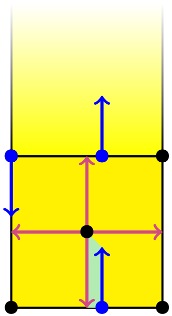

C_{p-1}
C_p
$d_p - V_p^{-1}$
V_p^{-1}
C_{p-1}^t
C_{p-1}^s
C_{p-1}^c
C_p^t
C_p^s
C_p^c
BPL(0.5)
$d_p^{s,t} - V_p^{-1} = \widehat{\delta}_p^{s,t}$
V_p^{-1}
$V_p = h$
C_{p-1}^c
0
C_p^c

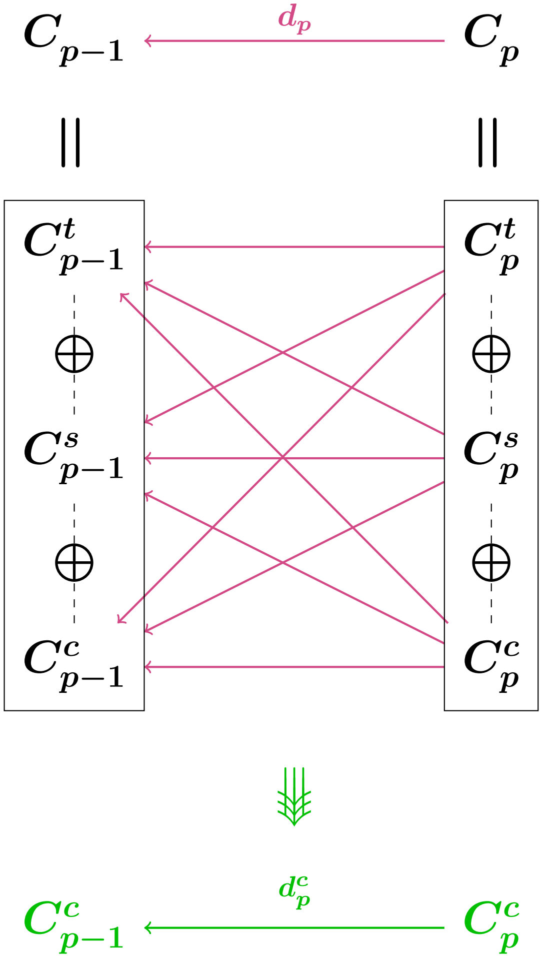
Proof 7:



Nilpotency Hypothesis ??? = Admissibility of Vector Field

Proof 8:



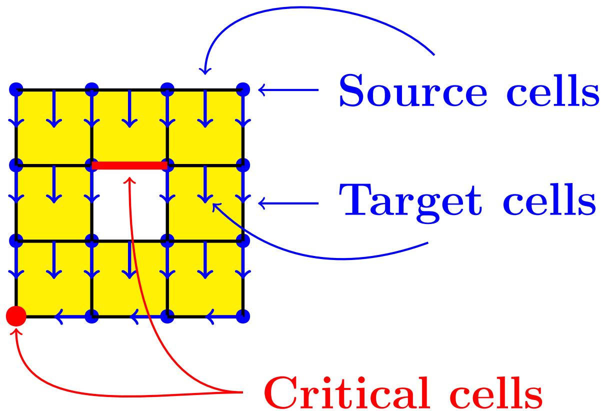


BPL(1.0)

QED !

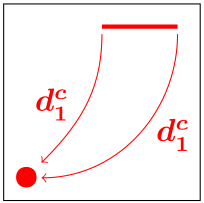
Toy Example.

$C_* =$



Fundamental Reduction Theorem ⇒

$\rho : C_* \Rrightarrow C_*^c =$



$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \text{Circle}$

$\text{Rank}(C_*) = (16, 24, 8)$ vs $\text{Rank}(C_*^c) = (1, 1, 0)$

More sophisticated example:

$K = K(\mathbb{Z}, 1) =$ Kan minimal (!) model of $B\mathbb{Z}$.

$K_n = \mathbb{Z}^n_* \;\Rightarrow\; C_n(K) = \mathbb{Z}[\mathbb{Z}^n_*]$

$d[1|2|3|4] := [2|3|4] - [3|3|4] + [1|5|4] - [1|2|7] + [1|2|3]$

Represents the functor $X \mapsto H^1(X, \mathbb{Z})$

in the simplicial world.

$K(\mathbb{Z}, 1) =$ the fundamental base

of the algebraic topology of the fibrations.

What about the homological nature of $K(\mathbb{Z}, 1)$??

Solution = **Vector Field V.**

Recipe:

Every $[1|a_2|a_3|\cdots]$ with $a_2 > 0$

is the target of the vector $([a_2+1|a_3|\cdots]\ ,\ [1|a_2|a_3|\cdots])$.

Every $[1|a_2|a_3|\cdots]$ with $a_2 < 0$

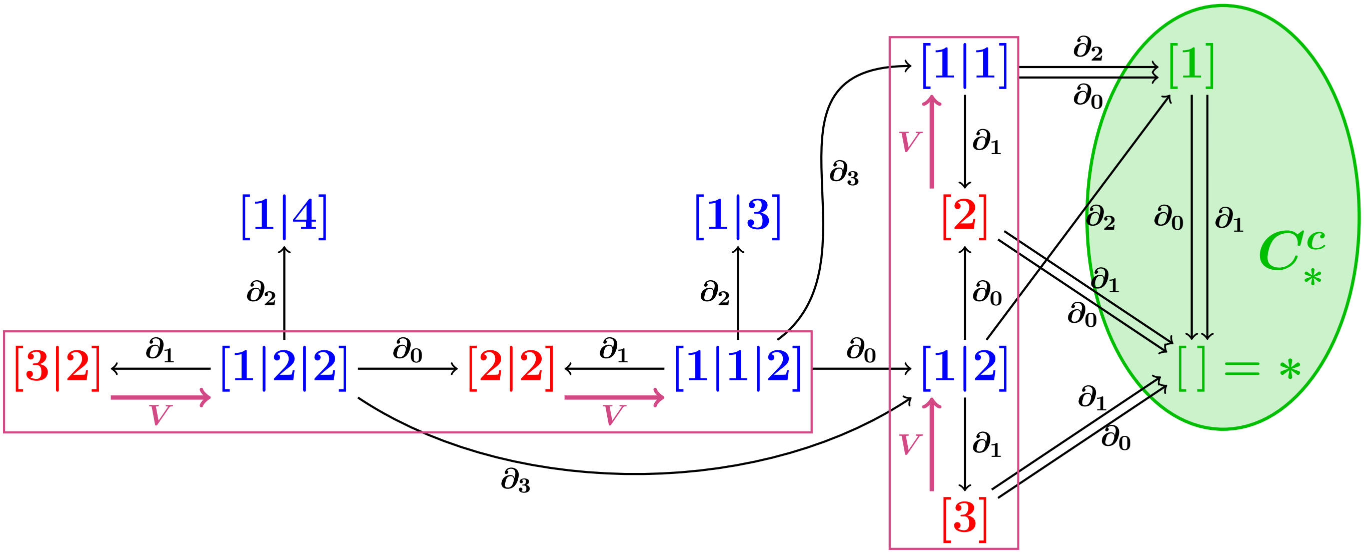
is the target of the vector $([a_2|a_3|\cdots]\ ,\ [1|a_2|a_3|\cdots])$.

Exercise: The critical cells are $\beta_0^c = \{[\,]\}$ and $\beta_1^c = \{[1]\}$.

$\Rightarrow$ $K(\mathbb{Z},1)$ has the homology type of the circle S^1

and also the homotopy type.

Typical examples of V-path:



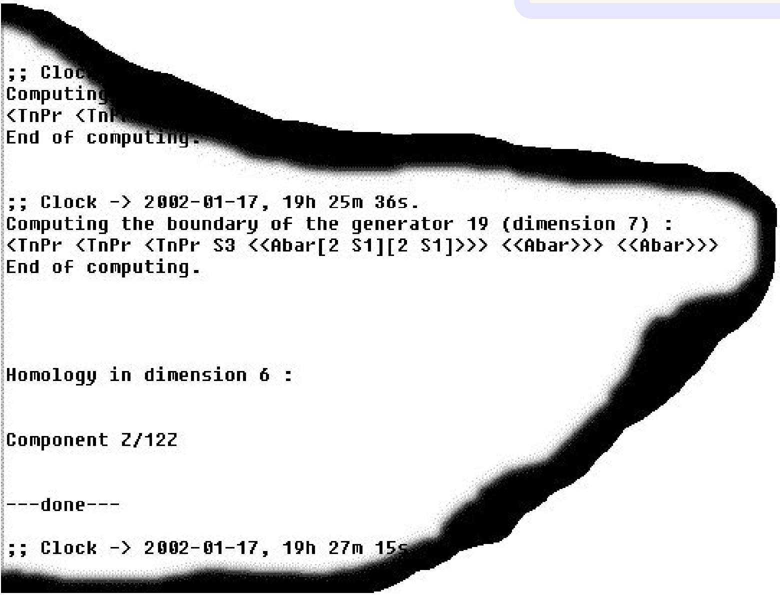
Red = Source cell

Blue = Target cell

Green = Critical cell

Violet = Vector Field

The END



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Logroño, Spain, 8-12 November, 2010