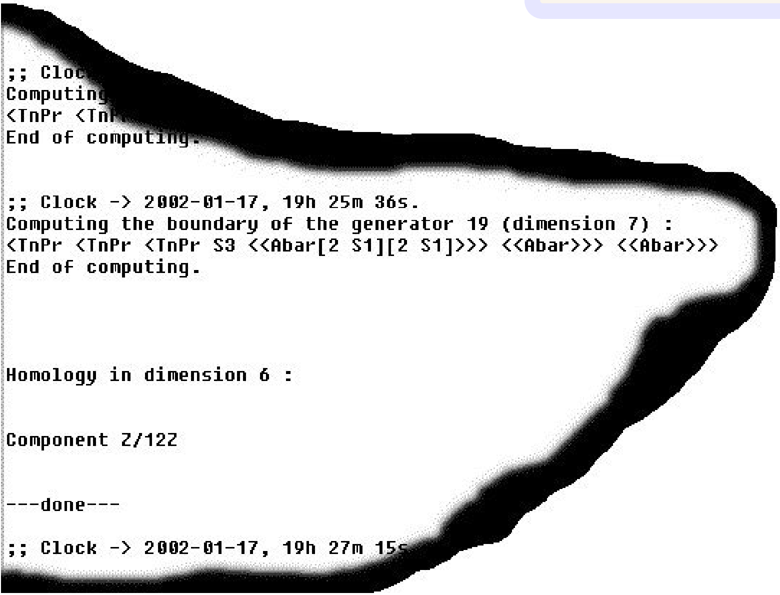
Constructive Postnikov Towers



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Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

Notion of Eilenberg-MacLane space.

π = abelian group + $n \in \mathbb{N}$

$\Rightarrow K(\pi, n)$ = space well defined

up to homotopy equivalence.

Characteristic property:

- $p \neq n \Rightarrow \pi_p(K(\pi, n)) = 0$.
- $\pi_n(K(\pi, n)) = \pi$.

Simple examples:

$$K(\mathbb{Z}, 1) = S^1, \;\; K(\mathbb{Z}/2, 1) = P^\infty\mathbb{R}, \;\; K(\mathbb{Z}, 2) = P^\infty\mathbb{C}.$$

Standard model for $K(\pi, n)$ = abelian topological group.

Realization of the simplicial set:

$$K(\pi, n)_p = Z^n(\Delta^p, \pi)$$

Other important simplicial set = $E(\pi, n-1)_p = C^{n-1}(\Delta^p, \pi)$.

The coboundary operator $\delta : C^{n-1} \to Z^n \subset C^n$

induces a canonical projection:

$$p : E(\pi, n-1) \to K(\pi, n)$$

which is a Kan fibration of fiber space $K(\pi, n-1)$:

$$K(\pi, n-1) \hookrightarrow E(\pi, n-1) \to K(\pi, n)$$

$$\boxed{\boxed{K(\pi, n-1) \hookrightarrow E(\pi, n-1) \longrightarrow K(\pi, n)}}$$

is a universal fibration:

$$E(\pi, n-1) \text{ contractible} \Rightarrow \ BK(\pi, n-1) \ = \ K(\pi, n)$$
$$K(\pi, n-1) \ \sim \ \Omega K(\pi, n).$$

$K(\pi, n)$ **represents the functor** $X \mapsto H^n(X, \pi)$.

Canonical bijection: $H^n(X, \pi) \stackrel{\cong}{\longleftrightarrow} [X, K(\pi, n)]$.

Canonical bijection: $Z^n(X, \pi) \stackrel{\cong}{\longleftrightarrow} \mathrm{SimpMap}(X, K(\pi, n))$.

$K(\pi, n+1) = BK(\pi, n)$

$\Rightarrow K(\pi, n+1)$ classifies the principal $K(\pi, n)$-fibrations.

$$\begin{array}{ccc}
K(\pi,n) & \xleftrightarrow{\;=\;} & K(\pi,n) \\
\downarrow & & \downarrow \\
X \times_f K(\pi,n) & \longrightarrow & E(\pi,n) = K(\pi,n+1) \times_{\mathrm{id}} K(\pi,n) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & K(\pi,n+1)
\end{array}$$

Canonical bijections:

$$\begin{aligned}
K(\pi,n)\text{-pr.-fibrations on } X &\xleftrightarrow{\;\cong\;} [X, K(\pi, n+1)] \\
&\xleftrightarrow{\;\cong\;} H^{n+1}(X, \pi)
\end{aligned}$$

Every space is assumed to be simply connected.

Definition: A Postnikov tower T is a triple of sequences

$$T = ((P_p)_{p\geq 1}, (\pi_p)_{p\geq 2}, (k_p)_{p\geq 2}) \text{ with:}$$

- P_p = space = p-th stage of the tower.

  In particular $P_1 = *$.

- π_p = abelian group = homotopy group.
- $k_p \in H^{p+1}(P_{p-1}, \pi_p)$ = Postnikov (pseudo-)invariant.
- $P_p = K(\pi_p, p) \times_{k_p} P_{p-1}$ for $p \geq 2$ (redundant).

$$P = \lim P_p = ((K(\pi_2, 2) \times_{k_3} K(\pi_3, 3)) \times_{k_4} K(\pi_4, 4)) \times_{k_5} \cdots$$

Postnikov morphism:

$$\Phi : ((P_p), (\pi_p), (k_p)) \xrightarrow{???} ((P'_p), (\pi'_p), (k'_p))$$

$$\Phi = ((\alpha_p : P_p \to P'_p), (\beta_p : \pi_p \to \pi'_p)) \text{ satisfying:}$$

$$k_p \in H^{p+1}(P_{p-1}, \pi_p) \xrightarrow{\beta_{p*}} H^{p+1}(P_{p-1}, \pi'_p) \xleftarrow{\alpha^*_{p-1}} H^{p+1}(P'_{p-1}, \pi'_p) \ni k'_p$$

Condition: $(\forall p)\ \ \beta_{p*}(k_p) = \alpha^*_{p-1}(k'_p)$

$\Rightarrow$ **Notion of isomorphism of Postnikov towers.**

Remark: in $\Phi = ((\alpha_p : P_p \longrightarrow P'_p), (\beta_p : \pi_p \longrightarrow \pi'_p))$,

the $(\alpha_p)_p$ component is consequence of $(\beta_p)_p$:

$$\begin{array}{ccccc}
 & & \overset{\alpha_p}{\curvearrowright} & & \\
P_{p-1} \times_{k_p} K(\pi_p, p) & \xrightarrow[\beta_{p*}]{} & P_{p-1} \times_{k''_p} K(\pi'_p, p) & \xrightarrow[\alpha_{(p-1)*}]{} & P'_{p-1} \times_{k'_p} K(\pi'_p, p) \\
\downarrow & & \downarrow & & \downarrow \\
P_{p-1} & \xleftrightarrow{=} & P_{p-1} & \xrightarrow{\alpha_{p-1}} & P'_{p-1}
\end{array}$$

$$\begin{array}{ccccc}
k_p & \longmapsto & k''_p & \longleftarrow\!\shortmid & k'_p \\
H^{n+1}(P_{p-1}, \pi_p) & \xrightarrow[\beta_{p*}]{} & H^{n+1}(P_{p-1}, \pi'_p) & \xleftarrow[\alpha^*_{p-1}]{} & H^{n+1}(P'_{p-1}, \pi'_p)
\end{array}$$

Corollary: $\Phi : ((P_p), (\pi_p), (k_p)) \longrightarrow ((P'_p), (\pi'_p), (k'_p))$
is nothing but
a compatible collection $(\beta_p : \pi_p \to \pi'_p)_{p \geq 2}$.

Postnikov Theorem: Every simply connected space X
has a Postnikov model
$P = \lim((P_p), (\pi_p), (k_p))$
defined up to isomorphism.

Corollary: Calling k_p a Postnikov invariant or a k-invariant
is erroneous.

Construction of some Postnikov model for a space X.

Recursive process starting from $P_1 = *$.

Assume $((f_p), (P_p), (\pi_p), (k_p))_{p \leq n-1}$ known

$$\text{with } f_{n-1} : X \to P_{n-1}$$

$$\text{inducing isomorphisms } \pi_p(X) \xrightarrow{\cong} \pi_p \text{ for } p \leq n-1.$$

The map $f_{n-1} : X \to P_{n-1}$ induces an extension:

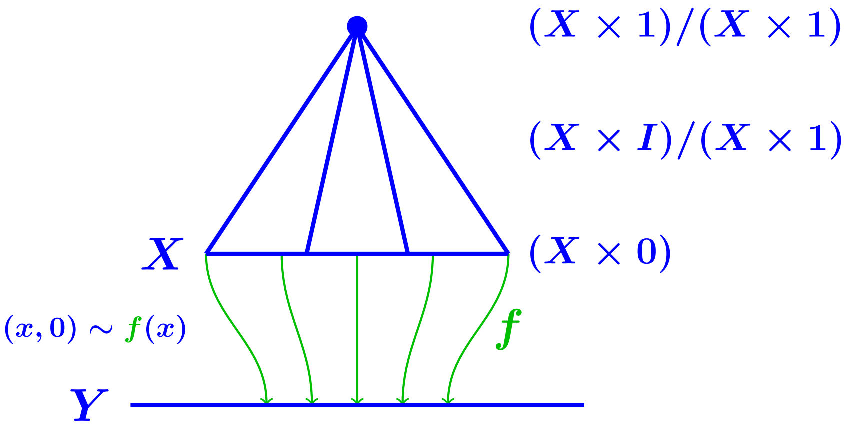
$$X \xrightarrow{f_{n-1}} P_{n-1} \xrightarrow{f'_{n-1}} \mathrm{Cone}(f_{n-1})$$

$$\mathrm{Cone} = \left\{ \begin{array}{ccccccc} \dashleftarrow & C_{p-2}(X) & \xleftarrow{-d} & C_{p-1}(X) & \xleftarrow{-d} & C_p(X) & \dashleftarrow \\ & & \swarrow_{f_{n-1}} & & \swarrow_{f_{n-1}} & & \\ \dashleftarrow & C_{p-1}(P_{n-1}) & \xleftarrow{d} & C_p(P_{n-1}) & \xleftarrow{d} & C_{p+1}(P_{n-1}) & \dashleftarrow \end{array} \right\}$$

Geometrical interpretation.

$f : X \to Y$ = continuous map.

$\mathrm{Cone}(f) = (Y \coprod (X \times I)) \,/\, ((X \times 1) \;\&\; (x,0) \sim f(x))$.



Properties of $f_{n-1} : \pi_p(X) \xrightarrow{\cong} \pi_p$ for $p \leq n - 1$.

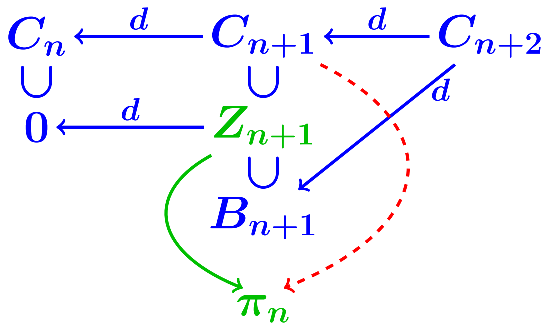
+ Long homotopy exact sequence of Cone(f_{p-1})

+ relative Hurewicz Theorem

$\Rightarrow H_p(\text{Cone}(f_{n-1})) = 0$ if $p < n + 1$.

$H_{n+1}(\text{Cone}(f_{p-1})) = \pi_n(X) =: \pi_n$.

In Cone(f_{p-1}):



Z_{n+1} direct summand of $C_{n+1} \Rightarrow$ non-canonical extension

$Z_{n+1} \longrightarrow \pi_n$ to $(z : C_{n+1} \longrightarrow \pi_n) \in Z^{n+1}(\text{Cone}(f_{p-1}), \pi_n)$.

$$\Rightarrow X \xrightarrow{f_{n-1}} P_{n-1} \hookrightarrow \mathrm{Cone}(f_{n-1}) \xrightarrow{Z_{n+1}} K(\pi, n+1)$$

$$\Rightarrow [\tau_1 \in H^{n+1}(\mathrm{Cone}(f_{n-1}), n+1)] +$$
$$+[\tau_2 \in H^{n+1}(P_{n-1}, \pi)] + [\tau_3 \in H^{n+1}(X)]$$

$$\Rightarrow \begin{array}{ccccccc}
X \times_{\tau_3} K_n & \xrightarrow{\overline{f}_{n-1}} & P_{n-1} \times_{\tau_2} K_n & \rightarrow & \mathrm{Cone} \times_{\tau_1} K_n & \longrightarrow & E(\pi, n) \\
\downarrow \uparrow s & \nearrow f_n & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow[f_{n-1}]{} & P_{n-1} & \longrightarrow & \mathrm{Cone}(f_{n-1}) & \xrightarrow[Z_{n+1}]{} & K(\pi, n+1)
\end{array}$$

with $K_n = K(\pi, n)$.

Cone homology exact sequence $\Rightarrow \tau_3 = 0$

$\Rightarrow$ τ_3 fibration is trivial $\Rightarrow$ section s $\Rightarrow$ $f_n := \overline{f}_{n-1} s$.

$$\begin{array}{ccccccc}
X \times_{\tau_3} K_n & \xrightarrow{\overline{f}_{n-1}} & P_{n-1} \times_{\tau_2} K_n & \rightarrow & \text{Cone} \times_{\tau_1} K_n & \longrightarrow & E(\pi, n) \\
\downarrow \uparrow s & \nearrow f_n & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow[f_{n-1}]{} & P_{n-1} & \longrightarrow & \text{Cone}(f_{n-1}) & \xrightarrow[Z_{n+1}]{} & K(\pi, n+1)
\end{array}$$

Postnikov Theorem: With this construction,

$$f_n : X \rightarrow P_{n-1} \times_{\tau_2} K(\pi, n)$$

induces isomorphisms:

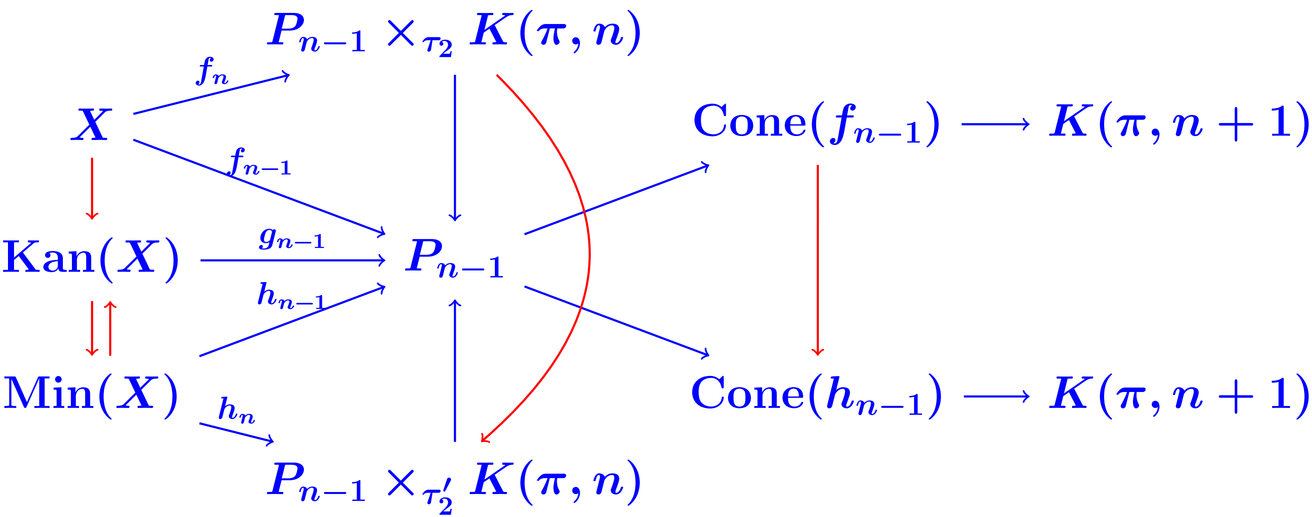
$$\pi_p(X) \cong \pi_p(P_{n-1} \times_{\tau_2} K(\pi, n)).$$

$\Rightarrow$ Continuation of the Postnikov tower:

$$\pi_n = H_{n+1}(\text{Cone}(f_{n-1})) \qquad\qquad P_n = P_{n-1} \times K(\pi, n)$$

$$k_n = \tau_2 = Z_{n+1}|P_{n-1}$$

Hint for the proof.

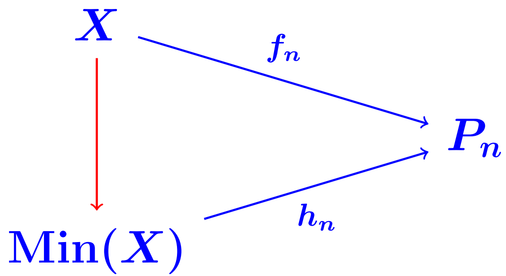


Red arrow $\downarrow$ = Homotopy equivalence.

Min(X) minimal $\Rightarrow$ Proof simplex by simplex.

+ Homotopy equivalences $\Rightarrow$ QED.

Remark: In the sub-diagram:



the h_n map is

an isomorphism of simplicial set in dimension $\leq n$.

$\Rightarrow$ the process is also a recursive construction of Min(X).

But is this process effective ???

<u>Theorem</u>: Simplicial set $\boldsymbol{X}$ with effective homology $\Rightarrow$

Postnikov tower's construction for $\boldsymbol{X}$ is effective.

<u>Proof</u>.

Assume $f_{n-1} : X \longrightarrow P_{n-1}$ available.

Homotopy groups $\pi_n X$ of finite type $\Rightarrow$

P_{n-1} with effective homology.

X and P_{n-1} with effective homology $\Rightarrow$

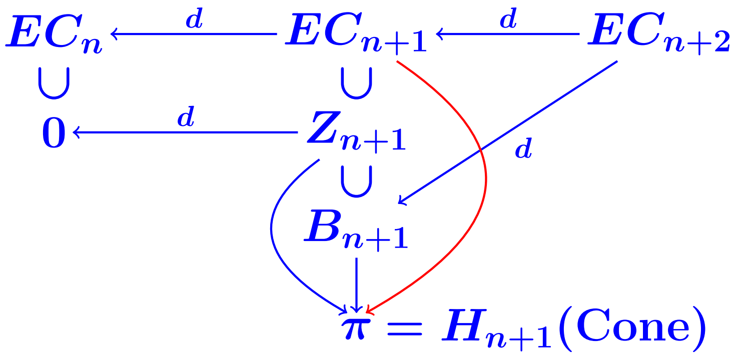
$\mathrm{Cone}(f_{n-1})$ with effective homology.

$\mathrm{Cone}(f_{n-1})$ with effective homology $\Rightarrow$

$$C_*(\mathrm{Cone}(f_{n-1})) \Lleftarrow\!\!\Rrightarrow EC_*$$

with EC_* free $\mathbb{Z}$-chain complex of finite type.

$\Rightarrow$



with $EC_{n+1} = \mathbb{Z}^r$ and $Z_{n+1} = \mathbb{Z}^s$ direct summand in EC_{n+1}.

$\Rightarrow$ Extension of $[Z_{n+1} \to \pi]$ to $[EC_{n+1} \to \pi]$

non-canonical but effective.

$$\Rightarrow \quad C_{n+1}(X) \to C_{n+1}(P_{n-1}) \to C_{n+1}(\text{Cone}) \to EC_{n+1} \to \pi$$

$$\Rightarrow \quad \tau_3 \in H^{n+1}(X) \text{ and } \tau_2 \in H^{n+1}(P_{n-1}) \text{ effective.}$$

$$\Rightarrow \quad X \xrightarrow{f_n} P_n \xrightarrow{p} P_{n-1} \text{ effective.}$$

QED

Example to illustrate

the erroneous nature of Postnikov invariants.

$$X = K(\mathbb{Z}^n, 2) \times_k K(\mathbb{Z}, 2d-1)$$

$$k \in H^{2d}(K(\mathbb{Z}^n), \mathbb{Z})$$

$K(\mathbb{Z}^n, 2) = K(\mathbb{Z}, 2)^n$ **and** $H^*(K(\mathbb{Z}), 2) = \mathbb{Z}[c_1]$

$$\Rightarrow \quad H^*(K(\mathbb{Z}, 2)^n) = \mathbb{Z}[c_1]^{\otimes n}$$

$$\Rightarrow H^{2d}(K(\mathbb{Z}^n, 2)) \cong \mathbb{Z}_d[x_1, \ldots, x_n]$$

$\Rightarrow k =$ **homogenous polynomial of degree d with n variables.**

$$X = K(\mathbb{Z}^n, 2) \times_k K(\mathbb{Z}, 2d-1)$$

Automorphism of $X \Leftrightarrow \{[\alpha_2 : \mathbb{Z}^n \xrightarrow{\cong} \mathbb{Z}^n] + [\alpha_{2d-1} : \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}]\}$

$$\Leftrightarrow \{[\alpha_2 \in GL_n\mathbb{Z}] + [\alpha_{2d-1} = \pm 1]\}$$

$$k(x_1, \dots, x_n) \in \mathbb{Z}_{2d}[x_1, \dots, x_n]$$

$\Rightarrow$ **Every linear isomorphism** α_2 + **every sign change** α_{2d-1}

$\Rightarrow k'(x_1, \dots, x_n) := \alpha_{2d-1} k(\alpha(x_1, \dots, x_n))$

$\Rightarrow X' = K(\mathbb{Z}^n, 2) \times_{k'} K(\mathbb{Z}, 2d-1)$ **has**

the same homotopy type as $X = K(\mathbb{Z}^n, 2) \times_k K(\mathbb{Z}, 2d-1)$

$\Rightarrow k \in H^{2d-1}(K(\mathbb{Z}^n, 2), \mathbb{Z})$

is not **an invariant of the homotopy type.**

Application to obstruction computations.

Theorem: **Let X, Y be simplicial sets with eff.,homology**

$f : Y_n \rightarrow X_n$ a simpl. map between n-skeletons.

The obstruction $\mathfrak{o}(f)$ to extend f

to $f' : Y_{n+1} \rightarrow X_{n+1}$ is computable.

Proof. **The Postnikov tower of X is computable.**

$$\Rightarrow \qquad X \xrightarrow{p} P_{n-1} \rightarrow \mathrm{Cone}(p) \rightarrow K(\pi_n, n+1)$$

Let $\sigma \in Y_{n+1}$ be an n-simplex.

Then $f|\partial\sigma$ defined.

$$\Rightarrow \qquad \begin{array}{ccccccc}
Y_n & \xrightarrow{f} & X & \xrightarrow{p} & P_{n-1} & \longrightarrow & \mathrm{Cone}(p) \\
\cup & & \cup & & \cup & & \\
\partial\sigma & \longmapsto & f\partial\sigma & \longmapsto & pf\partial\sigma & & \\
\cap & & & & & & \\
\sigma & \overset{???}{\longmapsto} & ??? & & & & \\
\Cap & & & & & & \\
Y_{n+1} & & & & & &
\end{array}$$

$\partial\sigma$, $f\partial\sigma$ and $pf\partial\sigma$ are n-spheres modelled on $\partial\Delta^{n+1}$.

$$\begin{array}{ccccccc} Y_n & \xrightarrow{f} & X & \xrightarrow{p} & P_{n-1} & \longrightarrow & \mathrm{Cone}(p) \\ \cup & & \cup & & \cup & & \\ \partial\sigma & \longmapsto & f\partial\sigma & \longmapsto & pf\partial\sigma & & \end{array}$$

P_{n-1} minimal and $\pi_n(P_{n-1}) = 0 \Rightarrow$

unique completion of $pf\partial\sigma$ as a full $(n+1)$-simplex.

$f\partial\sigma \in Z_n(X)$ + effective cone homology exact sequence $\Rightarrow$

Cycle $z_\sigma \in Z_{n+1}(\mathrm{Cone}(p))$ +

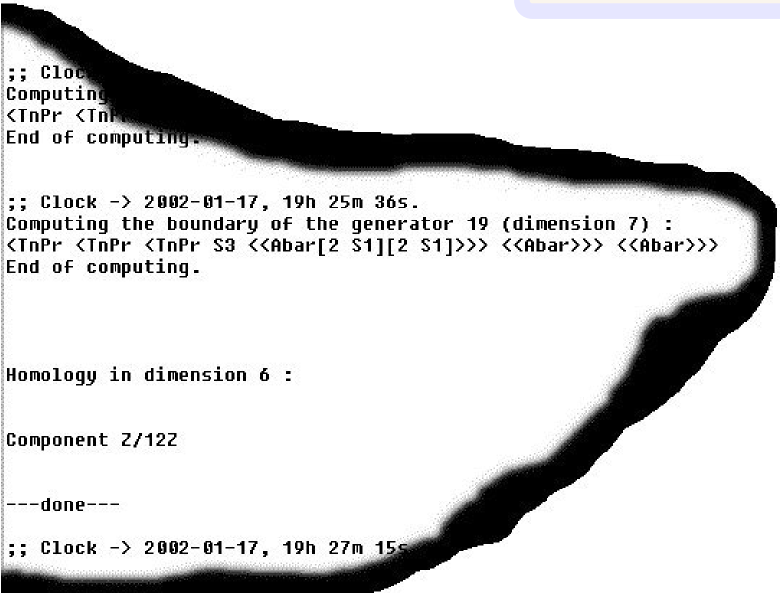
pre-Postnikov class $h \in H^{n+1}(\mathrm{Cone}, \pi_n)$

$\Rightarrow h(z_\sigma) \in \pi_n$ is the homotopy class of $\partial\sigma$.

The map $Y_{n+1} \ni \sigma \mapsto h(z_\sigma) \in \pi_n$

is the looked-for obstruction $\mathfrak{o}(f)$. QED

The END



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;; Cloc
Computing
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End of computing.

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Computing the boundary of the generator 19 (dimension 7) :
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End of computing.

Homology in dimension 6 :

Component Z/12Z

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