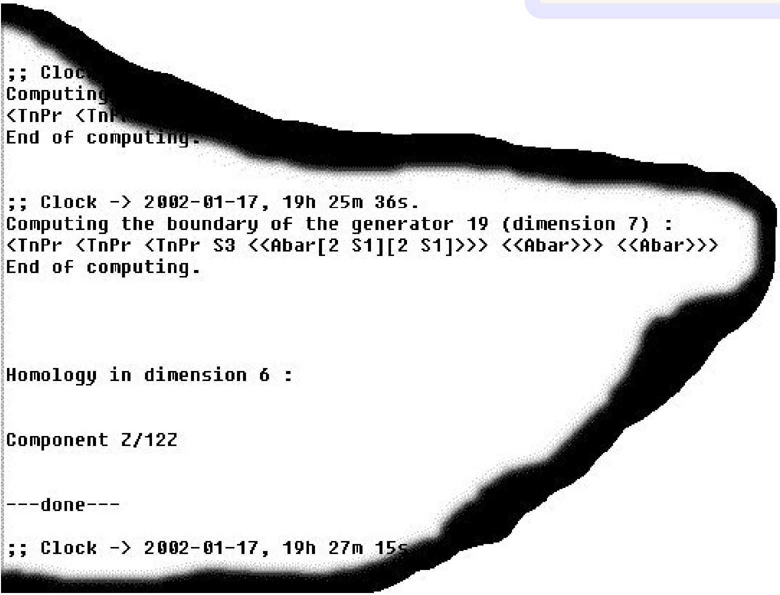
Discrete Vector Fields

and

Basic Algebraic Topology



Francis Sergeraert, Institut Fourier, Grenoble
Cirm-Luminy, March 2010

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, ...
Violet	=	**Problem, difficulty,**
		obstacle, disadvantage, ...
Green	=	**Solution, essential point,**
		mathematicians, ...
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

Plan:

1. **Algebraic discrete vector fields.**

2. **Products of simplices.**

3. **Eilenberg-Zilber Theorem.**

4. **Vector Fields $\Rightarrow$ Eilenberg-Zilber.**

5. **Vector Fields $\Rightarrow$ Twisted Eilenberg-Zilber.**

6. **Vector-Fields $\Rightarrow$ Basic Spectral Sequences.**

Definition: A(n algebraic) cellular chain complex C_* is a pair: $C_* = ((\beta_p)_{p\in\mathbb{Z}}, (d_p)_{p\in\mathbb{Z}})$ satisfying:

- β_p is the distinguished basis of a free $\mathbb{Z}$-module $C_p = \mathbb{Z}[\beta_p]$.
- $d_p : C_p \to C_{p-1}$ is a differential ($d^2 = 0$).

Examples: Chain complexes coming from:

- Simplicial complexes, simplicial sets, CW-complexes.
- Digital images.
- Chain complexes of the very kernel of standard Algebraic Topology.

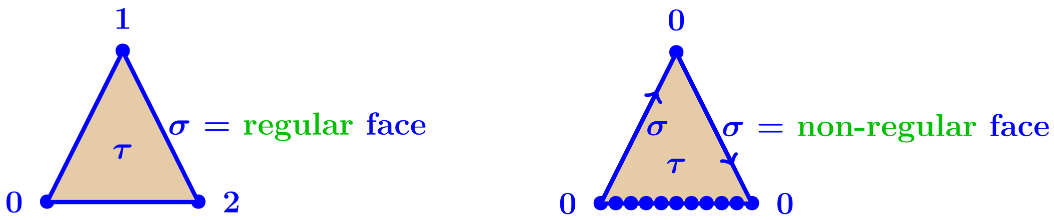
$C_* = ((\beta_p)_{p\in\mathbb{Z}}, d)$ = Cellular chain complex.

Definition: A p-cell is an element of β_p.

Definition: If $\sigma \in \beta_{p-1}$ and $\tau \in \beta_p$,
then $\varepsilon(\sigma,\tau) :=$ coefficient of σ in $d\tau$.

Definition: σ is a face of τ if $\varepsilon(\sigma,\tau) \neq 0$.

Definition: σ is a regular face of τ if $\varepsilon(\sigma,\tau) = \pm 1$.



Triangle

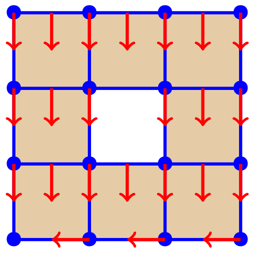
$P^2\mathbb{R}$

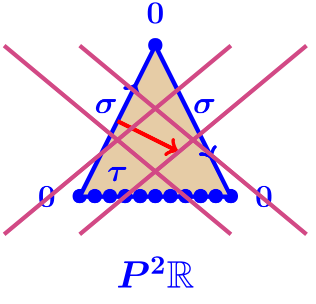
$C_* = ((\beta_p)_{p\in\mathbb{Z}}, d)$ = Cellular chain complex.

Definition: A vector field is a set $V = \{(\sigma_i, \tau_i)_{i\in v}\}$ satisfying:

- $(\forall i)$ σ_i is regular face of τ_i.
- $(\forall i \neq j)$ $\sigma_i \neq \sigma_j \neq \tau_i \neq \tau_j$.

Examples:





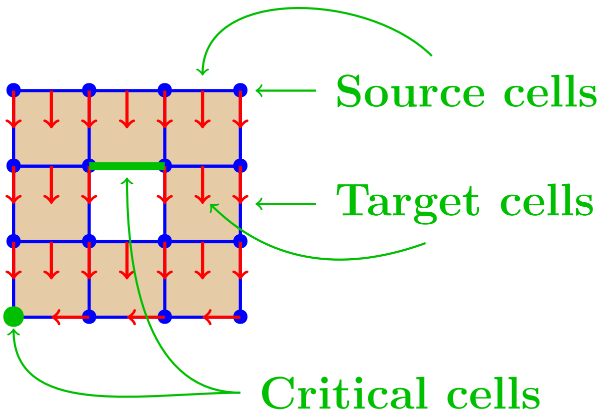
$C_* = ((\beta_p)_{p \in \mathbb{Z}}, d)$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)_{i \in v}\}$ = Vector field.

Definition: A critical p-cell is an element of β_p which does not occur in V.

Other cells divided in source cells and target cells.

Example:



$C_* = ((\beta_p)_{p\in\mathbb{Z}}, d)$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)_{i\in v}\}$ = Vector field.

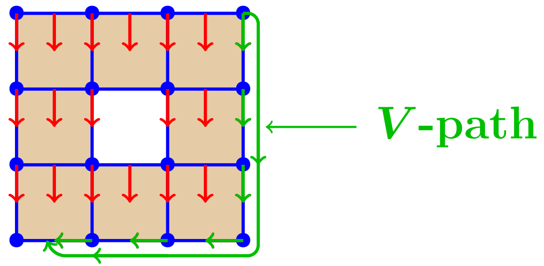
<u>Definition</u>: A V-path is a sequence:

$$(\sigma_{i_1}, \tau_{i_1}, \sigma_{i_2}, \tau_{i_2}, \ldots, \sigma_{i_n}, \tau_{i_n})$$

with $(\sigma_{i_j}, \tau_{i_j}) \in V$ and σ_{i_j} face of $\tau_{i_{j-1}}$.

Note: σ_{i_j} not necessarily regular face of $\tau_{i_{j-1}}$.

<u>Example</u>:



$C_* = ((\beta_p)_{p \in \mathbb{Z}}, d)$ = **Cellular chain complex.**

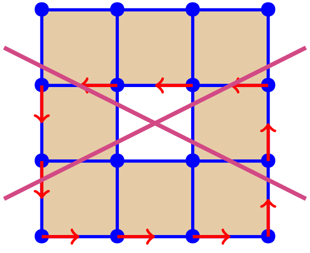
$V = \{(\sigma_i, \tau_i)_{i \in v}\}$ = **Vector field.**

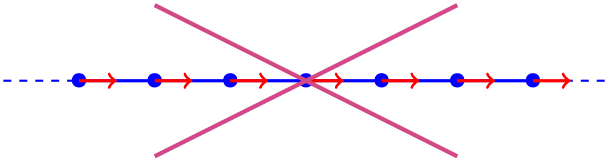
Definition: A vector field is admissible if,

for every source cell σ,

the length of the V-paths starting from σ

is bounded by $\lambda(\sigma) \in \mathbb{N}$.





$C_* = ((\beta_p)_{p\in\mathbb{Z}}, d)$ = Cellular chain complex.

$V = \{(\sigma_i, \tau_i)_{i\in v}\}$ = Admissible vector field.

Theorem: A "critical" chain complex $C_*^c = ((\beta_p^c)_{p\in\mathbb{Z}}, d^c)$ can be constructed:

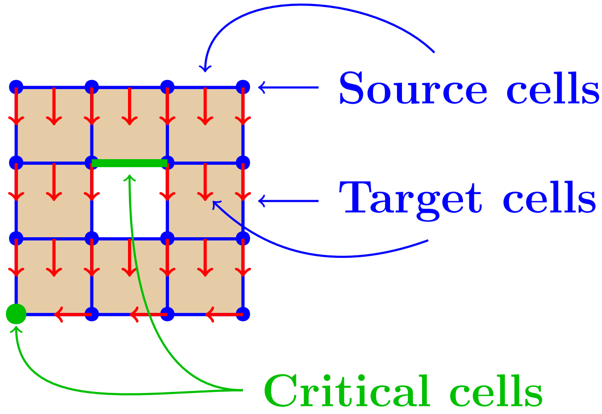
- β_p^c = the set of critical p-cells of V.
- $d_p^c : \mathbb{Z}[\beta_p^c] \rightarrow \mathbb{Z}[\beta_{p-1}^c]$ an appropriate "critical" differential deduced from the initial differential d and the vector field V.

Also a canonical reduction $\rho : C_* \Rrightarrow C_*^c$ is provided.

$\Rightarrow$ Any homological problem in C_* can be solved in C_*^c.

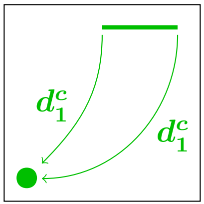
Simple example.

$$C_* =$$



Theorem $\Rightarrow$

$$\rho : C_* \Rrightarrow C^c_* =$$



$$= \mathbb{Z} \xleftarrow{d_1^c = 0} \mathbb{Z} = \text{Circle}$$

More sophisticated example:

$K = K(\mathbb{Z}, 1) =$ **Kan minimal (!) model of** $B\mathbb{Z}$.

$K_n = \mathbb{Z}_*^n \;\Rightarrow\; C_n(K) = \mathbb{Z}[\mathbb{Z}_*^n]$

$d[1|2|3|4] := [2|3|4] - [3|3|4] + [1|5|4] - [1|2|7] + [1|2|3]$

Represents the functor $X \mapsto H^1(X, \mathbb{Z})$

in the simplicial world.

$K(\mathbb{Z}, 1) =$ **the fundamental base**

of the algebraic topology of the fibrations.

What about the homological nature of $K(\mathbb{Z}, 1)$ **??**

Solution = **Vector Field V.**

Recipe:

Every $[1|a_2|a_3|\cdots]$ with $a_2 > 0$

is the target of the vector $([a_2+1|a_3|\cdots]\ ,\ [1|a_2|a_3|\cdots])$.

Every $[1|a_2|a_3|\cdots]$ with $a_2 < 0$

is the target of the vector $([a_2|a_3|\cdots]\ ,\ [1|a_2|a_3|\cdots])$.

Exercise: The critical cells are $\beta_0^c = \{[\,]\}$ and $\beta_1^c = \{[1]\}$.

$\Rightarrow$ $K(\mathbb{Z}, 1)$ has the homology type of the circle S^1

and also the homotopy type.

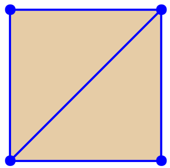
Continuation of the story:

Vector Fields $\Rightarrow$ Eilenberg-Zilber

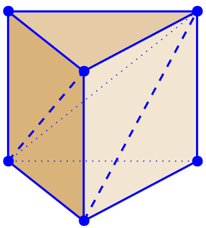
$\Rightarrow$ Twisted Eilenberg-Zilber $\Rightarrow$ Serre spectral sequence

$\Rightarrow$ Eilenberg-Moore spectral sequence.

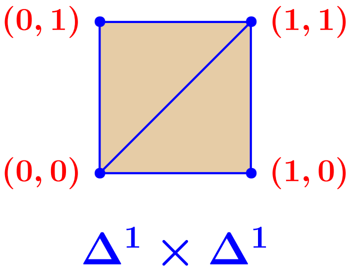
Main problem: Triangulation of $\boldsymbol{\Delta}^p \times \boldsymbol{\Delta}^q$???

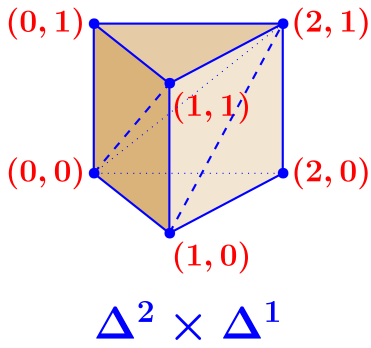


$\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$



$\boldsymbol{\Delta}^2 \times \boldsymbol{\Delta}^1$





Two $\boldsymbol{\Delta}^2$ **in** $\boldsymbol{\Delta}^1 \times \boldsymbol{\Delta}^1$**:** $(0,0) < (0,1) < (1,1)$

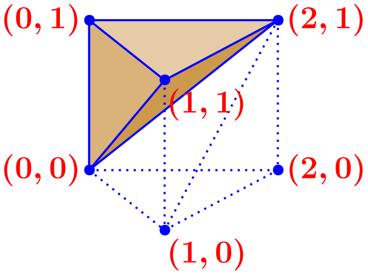
$(0,0) < (1,0) < (1,1)$

Three $\boldsymbol{\Delta}^3$ **in** $\boldsymbol{\Delta}^2 \times \boldsymbol{\Delta}^1$**:** $(0,0) < (0,1) < (1,1) < (2,1)$

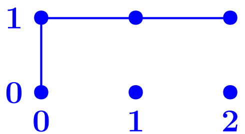
$(0,0) < (1,0) < (1,1) < (2,1)$

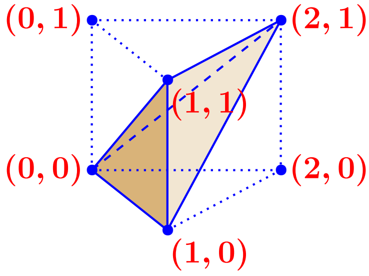
$(0,0) < (1,0) < (2,0) < (2,1)$

Rewriting the triangulation of $\Delta^2 \times \Delta^1$.

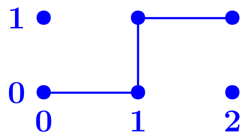


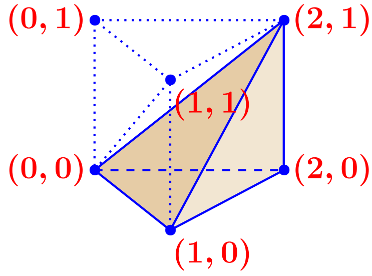
$(0,0) < (0,1) < (1,1) < (2,1)$



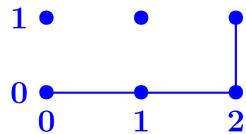


$(0,0) < (1,0) < (1,1) < (2,1)$



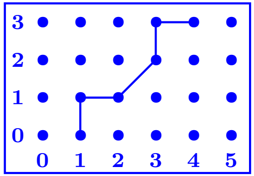


$(0,0) < (1,0) < (2,0) < (2,1)$



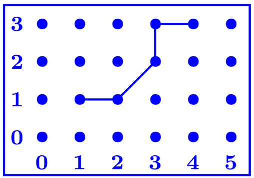
“Seeing” the triangulation of $\Delta^5 \times \Delta^3$.

Example of 5-simplex :

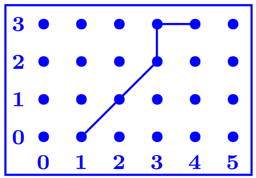


$= \sigma \in (\Delta^5 \times \Delta^3)_5$

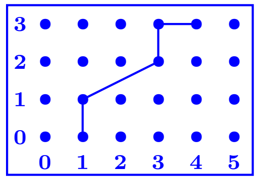
$\Rightarrow$ **6 faces:**



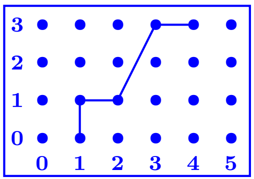
$\partial_0 \sigma$



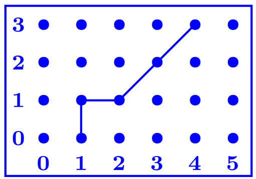
$\partial_1 \sigma$



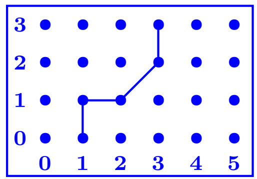
$\partial_2 \sigma$



$\partial_3 \sigma$

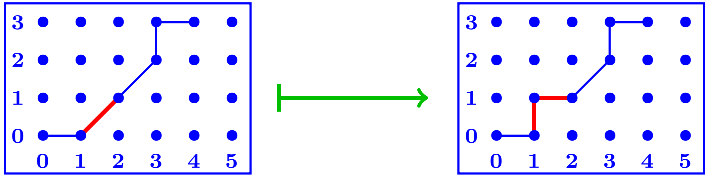


$\partial_4 \sigma$

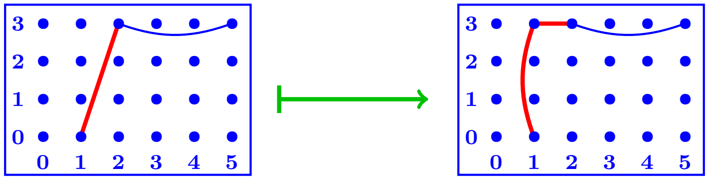


$\partial_5 \sigma$

⇒ Canonical discrete vector field for $\Delta^5 \times \Delta^3$.

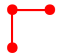


Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

Recipe: First "event" = Diagonal step =  ⇒ Source cell.

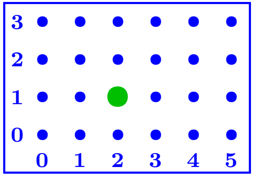
= (-90°)-bend =  ⇒ Target cell.

Critical cells ??

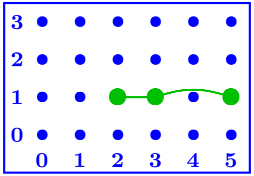
Critical cell = cell without any "event"

= without any diagonal or $-90°$-bend.

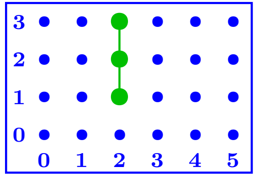
Examples.



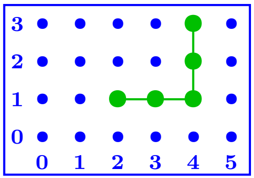
$\Delta^0_2 \times \Delta^0_1$



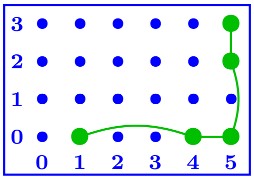
$\Delta^2_{2,3,5} \times \Delta^0_1$



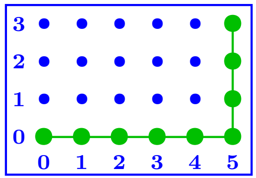
$\Delta^0_2 \times \Delta^2_{1,2,3}$



$\Delta^2_{2,3,4} \times \Delta^2_{1,2,3}$



$\Delta^2_{1,4,5} \times \Delta^2_{0,2,3}$



$\Delta^5_{0,1,2,3,4,5} \times \Delta^3_{0,1,2,3}$

Conclusion:

$$C^c_* = C_*(\Delta^5) \otimes C_*(\Delta^3)$$

Fundamental theorem of vector fields $\Rightarrow$

Canonical Homological Reductions:

$$\rho : C_*(\Delta^5 \times \Delta^3) \Rrightarrow C_*(\Delta^5) \otimes C_*(\Delta^3)$$

$$\rho : C_*(\Delta^p \times \Delta^q) \Rrightarrow C_*(\Delta^p) \otimes C_*(\Delta^q)$$

$p = q = 10 \quad \Rightarrow \quad$ **16,583,583,743 vs 4,190,209**

More generally: X and Y = simplicial sets.

An admissible discrete vector field

is canonically defined on $C_*(X \times Y)$.

$\Rightarrow$ Critical chain complex $C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$.

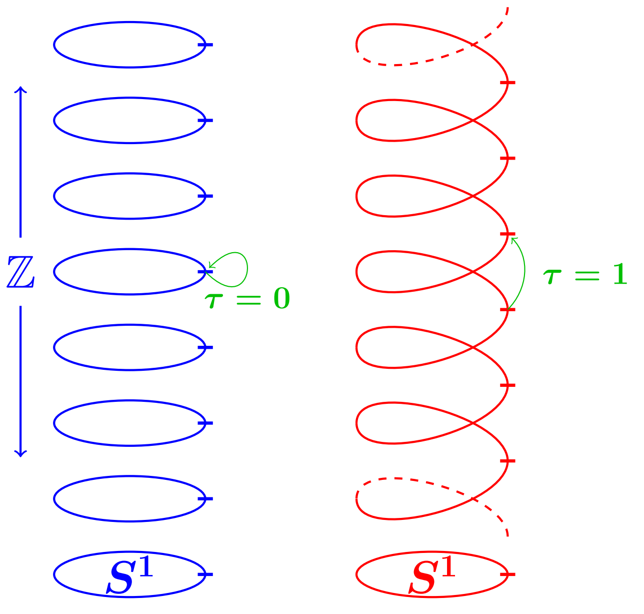
Eilenberg-Zilber Theorem: Canon. homological reduction:

$$\rho_{EZ} : C_*(X \times Y) \ggg C^c_*(X \times Y) = C_*(X) \otimes C_*(Y)$$

$\Rightarrow$ Künneth theorem to compute $H_*(X \times Y)$.

Notion of twisted product.

Simplest example: $\mathbb{Z} \times \boldsymbol{S^1}$ **vs** $\mathbb{Z} \times_{\boldsymbol{\tau}} \boldsymbol{S^1} = \mathbb{R}$**:**



General notion of twisted product: B = base space.

F = fibre space.

G = structural group.

Action $G \times F \to F$.

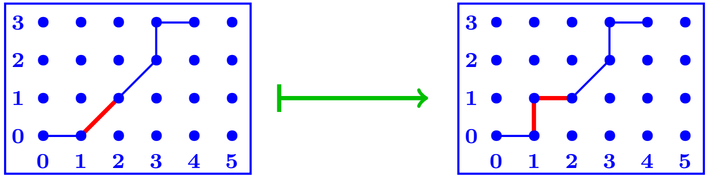
$\tau : B \to G$ = Twisting function.

Structure of $F \times_\tau B$:

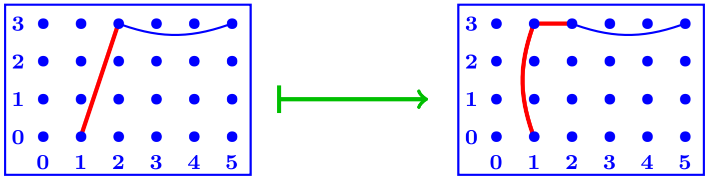
$$\partial_i(\sigma_f, \sigma_b) = (\partial_i \sigma_f, \partial_i \sigma_b) \quad \text{for } i > 0$$
$$\partial_0(\sigma_f\ ,\ \sigma_b) = (\tau(\sigma_b).\partial_0 \sigma_f\ ,\ \partial_0 \sigma_b)$$

$\Rightarrow$ **Only the 0-face is modified in the twisted product.**

Reminder about the EZ-vector field of $\Delta^5 \times \Delta^3$.



Source $= \partial_2($ Target$)$



Source $= \partial_1($ Target$)$

The vector field is concerned by faces ∂_i only if $i > 0$.

1. The twisting function τ modifies only $\boxed{0}$-faces.

2. The EZ-vector field V_{EZ} of $X \times Y$

uses only $\boxed{i}$-faces with $i \geq 1$.

$\Rightarrow$ V_{EZ} is defined and admissible as well on $X \times_{\boxed{\tau}} Y$.

Fundamental theorem of admissible vector fields $\Rightarrow$

$$V_{EZ} \Rightarrow \quad \begin{array}{c} C_*(X \times Y) \\ \Downarrow \\ C_*(X) \otimes C_*(Y) \end{array} \qquad\qquad V_{EZ} \Rightarrow \quad \begin{array}{c} C_*(X \times_{\boxed{\tau}} Y) \\ \Downarrow \\ C_*(X) \otimes_{\boxed{t}} C_*(Y) \end{array}$$

Known as the twisted Eilenberg-Zilber Theorem.

Corollary: Base B 1-reduced $\Rightarrow$ Algorithm:

$$[(F, C_*(F), EC_*^F, \varepsilon_F) + (B, C_*(B), EC_*^B, \varepsilon_B) + G + \tau]$$
$$\longmapsto (F \times_\tau B, C_*(F \times_\tau B), EC_*^{F \times_\tau B}, \varepsilon_{F \times_\tau B}).$$

Version of F with effective homology

$+$ Version of B with effective homology

$+$ $G + \tau$ describing the fibration $F \hookrightarrow F \times_\tau B \to B$

$\Rightarrow$ Version with effective homology of the total space $F \times_\tau B$.

$=$ Version with effective homology

of the Serre Spectral Sequence

Analogous result for the Eilenberg-Moore spectral result.

Key results:

G = Simplicial group $\Rightarrow$ BG = classifying space.

$$BG = \ldots(((SG \times_\tau SG) \times_\tau SG) \times_\tau SG) \times_\tau \cdots\cdots$$

X = Simplicial set $\Rightarrow$ KX = Kan loop space.

$$KX = \ldots(((S^{-1}X \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau S^{-1}X) \times_\tau \cdots$$

Analogous process $\Rightarrow$ Algorithms:

$$(G, C_*G, EC_*^G, \varepsilon_G) \;\mapsto\; (BG, C_*BG, EC_*^{BG}, \varepsilon_{BG})$$
$$(G, C_*X, EC_*^X, \varepsilon_X) \;\mapsto\; (KX, C_*KX, EC_*^{KX}, \varepsilon_{KX})$$

More generally:

$[\alpha : E \to B] + [\alpha' : E' \to B] + [\alpha \text{ fibration}]$

$\Rightarrow$ **algorithm:** $(B_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E \times_B E')_{EH}$.

$$\begin{array}{ccc} E' \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \alpha \\ E' & \underset{\alpha'}{\longrightarrow} & B \end{array}$$

$=$ **Version with effective homology**

of Eilenberg-Moore spectral sequence I.

Also:

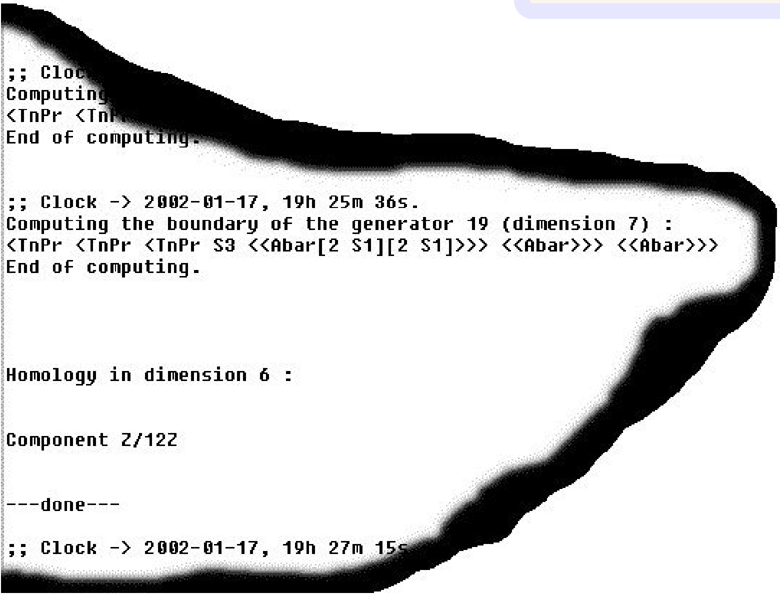
[G simplicial group] + [$\alpha : G \times E \rightarrow E$] + [$\alpha' : E' \times G \rightarrow E'$] + [$\alpha$ principal fibration]

$\Rightarrow$ algorithm: $(G_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E' \times_G E)_{EH}$.

$$\begin{array}{ccc} G & \overset{\alpha}{\longrightarrow} & E \\ {\scriptstyle \alpha'}\uparrow & & \downarrow \\ E' & \longrightarrow & E' \times_G E \end{array}$$

= Version with effective homology of Eilenberg-Moore spectral sequence II.

The END



Francis Sergeraert, Institut Fourier, Grenoble
Cirm-Luminy, March 2010