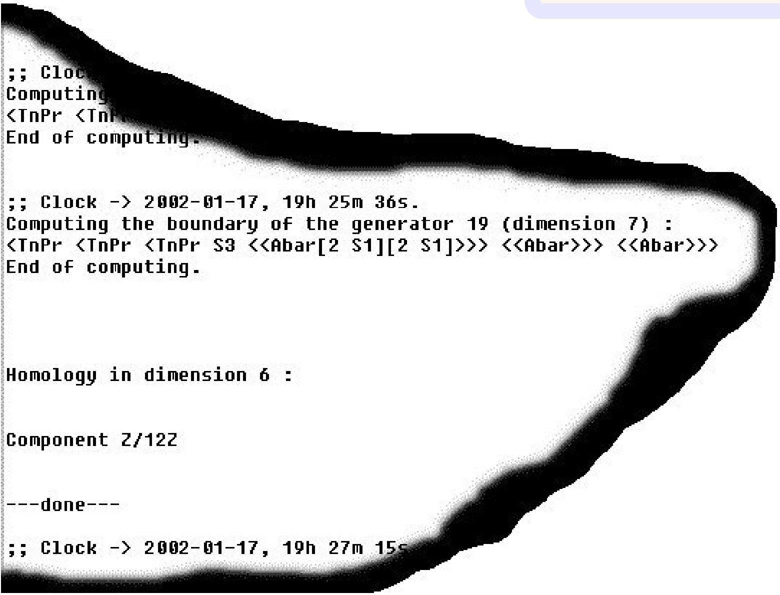
Constructive Homological Algebra

II - Homological Perturbations



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Workshop: Formal Methods in Commutative Algebra
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Problem when implementing SHPs:

$$\begin{tikzcd}
C_{p-1} & C_p \arrow[l, "d"'] & C_{p+1} \arrow[l, "d"'] \arrow[ddddl, bend left, "d"] \\
\cup & \cup & \\
0 & Z_p \arrow[l, "d"'] \arrow[ddddddd, bend right=60, "f"'] & \\
& \cup & \\
& B_p & \\
& \rotatebox{90}{$=$}\ ?? & \\
& \operatorname{Ker} f \arrow[d, "f"'] \arrow[uuuuuur, bend right, "h"'] & \\
& 0 & \\
& \cap & \\
& H'_p \arrow[uuuuuuu, bend right=60, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 & \mathbb{Z}/2 \arrow[l] & \mathbb{Z} \arrow[l, "\mathrm{pr}"'] \arrow[ddl, "\mathrm{pr}"'] \\
\| & \| & \\
0 & \mathbb{Z}/2 \arrow[l] \arrow[dddddd, bend right=60, "f"'] & \\
& \| & \\
& \mathbb{Z}/2 & \\
& \| & \\
& \mathbb{Z}/2 \arrow[d, "f"'] \arrow[uuuuuur, bend right, "h"'] & \\
& 0 & \\
& 0 \arrow[uuuuuu, bend right=60, "g"'] &
\end{tikzcd}$$

Implementation of $h : \mathbb{Z}/2 \longrightarrow \mathbb{Z}$???

The Basic Perturbation Lemma (BPL)

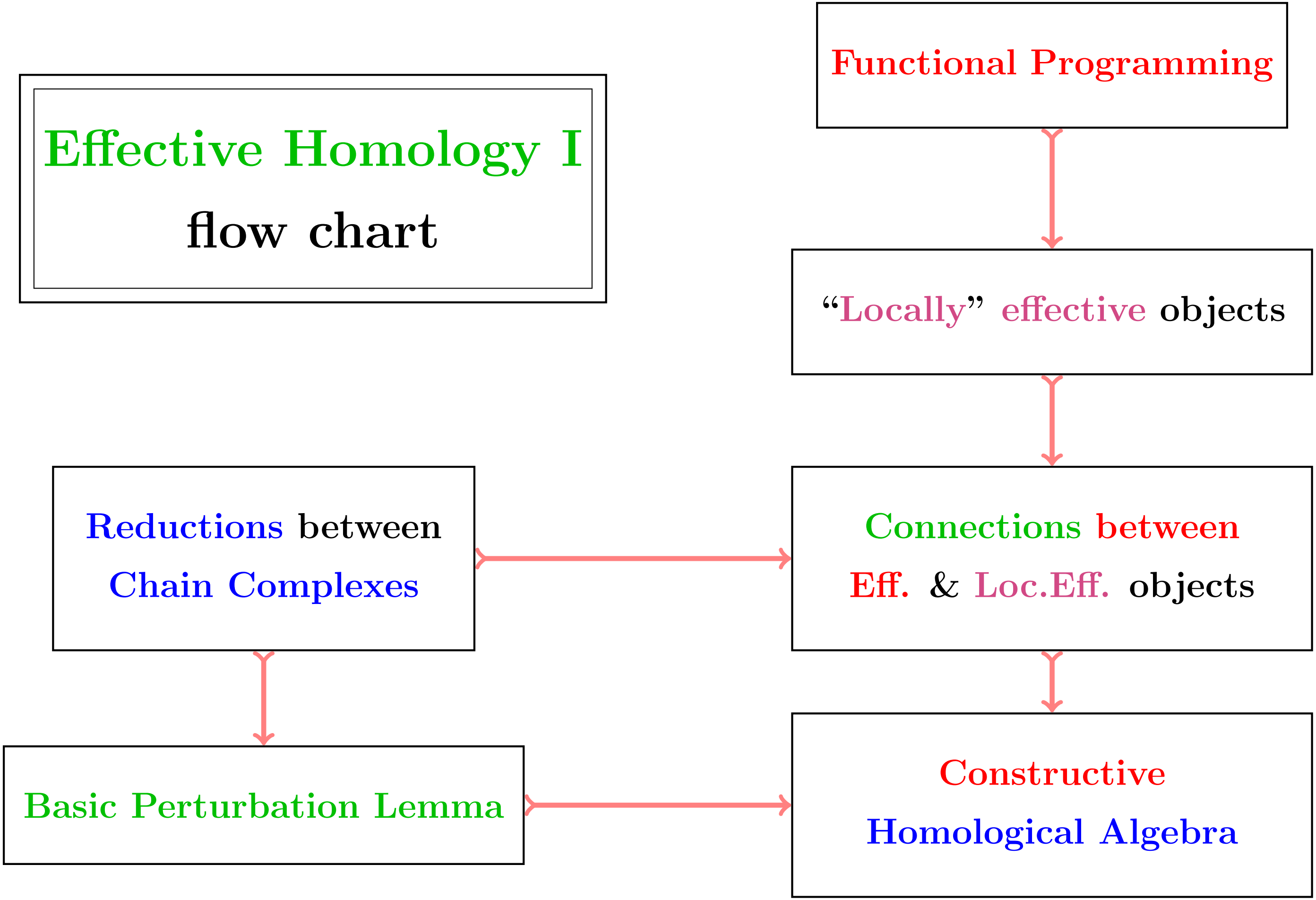
frequently allows to efficiently overcome this difficulty.

$\Rightarrow$ Effective Homology I :

- **Notion of object with effective homology.**
- **Most exact and spectral sequences become effective.**
- **But spectral sequences coming from exact couples fail!**

$\Rightarrow$ Effective homology II $=$ SHPs only.

Effective Homology I
flow chart



Functional Programming:

The art of handling functional objects.

Examples of functional objects:

$$(\mathbb{Z}, +, -, \times) \quad (\mathbb{Z}[X], +, -, \times)$$

Other example:

Kan model for the loop space $\Omega S^3 := \mathcal{C}(S^1, S^3)$:

$$(\mathcal{S}_{\Omega S^3}, \{\partial_i^n\}_{n \geq 1, 0 \leq i \leq n}, \{\eta_i^n\}_{n \geq 0, 0 \leq i \leq n})$$

with $\mathcal{S}_{\Omega S^3}$ = the simplex set of the Kan model.

= "Locally" effective objects.

Main problem:

Designing programs $(f_1, \ldots, f_n) \mapsto f$.

Example:

$$(\mathfrak{R}, +_{\mathfrak{R}}, -_{\mathfrak{R}}, \times_{\mathfrak{R}}) \mapsto (\mathfrak{R}[X], +_{\mathfrak{R}[X]}, -_{\mathfrak{R}[X]}, \times_{\mathfrak{R}[X]})$$

Topological example. X = topological space.

$$\begin{aligned}(\mathcal{S}_X, \{\partial(X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(X)_i^n\}_{n\geq 0, 0\leq i\leq n})& \\ \mapsto (\mathcal{S}_{\Omega X}, \{\partial(\Omega X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(\Omega X)_i^n\}_{n\geq 0, 0\leq i\leq n})&\end{aligned}$$

Solution = λ-calculus, Lisp, ML, Axiom, Haskell...

Definition: **A (homological) reduction is a diagram:**

$$\rho\colon\ \boxed{\begin{tikzcd} \widehat{C}_* \arrow[loop left, "h"] \arrow[r, shift right, "f"'] & C_* \arrow[l, shift right, "g"'] \end{tikzcd}}$$

with:

1. $\widehat{C}_*$ **and** C_* **= chain complexes.**
2. f **and** g **= chain complex morphisms.**
3. h **= homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ **and** $hh = 0$.

$$\boxed{\begin{array}{cccccccccccc}
\{ & \cdots & \underset{h}{\overset{d}{\leftrightarrows}} & \widehat{C}_{m-1} & \underset{h}{\overset{d}{\leftrightarrows}} & \widehat{C}_m & \underset{h}{\overset{d}{\leftrightarrows}} & \widehat{C}_{m+1} & \underset{h}{\overset{d}{\leftrightarrows}} & \cdots & \} = & \widehat{C}_* \\
 & & & \| & & \| & & \| & & & & \| \\
\{ & \cdots & & A_{m-1} & & A_m & & A_{m+1} & & \cdots & \} = & A_* \\
 & & \overset{d}{\underset{h}{\cong}} & \oplus & \overset{d}{\underset{h}{\cong}} & \oplus & \overset{d}{\underset{h}{\cong}} & \oplus & \overset{d}{\underset{h}{\cong}} & & & \oplus \\
\{ & \cdots & & B_{m-1} & & B_m & & B_{m+1} & & \cdots & \} = & B_* \\
 & & & \oplus & & \oplus & & \oplus & & & & \oplus \\
\{ & \cdots & \xleftarrow{d} & C'_{m-1} & \xleftarrow{d} & C'_m & \xleftarrow{d} & C'_{m+1} & \xleftarrow{d} & \cdots & \} = & C'_* \\
 & & & f \,\big\downarrow\!\!\big\uparrow\, g \;\cong & & f \,\big\downarrow\!\!\big\uparrow\, g \;\cong & & f \,\big\downarrow\!\!\big\uparrow\, g \;\cong & & & & f \,\big\downarrow\!\!\big\uparrow\, g \;\cong \\
\{ & \cdots & \xleftarrow{d} & C_{m-1} & \xleftarrow{d} & C_m & \xleftarrow{d} & C_{m+1} & \xleftarrow{d} & \cdots & \} = & C_*
\end{array}}$$

$$\boxed{\boxed{A_* = \ker f \cap \ker h} \quad \boxed{B_* = \ker f \cap \ker d} \quad \boxed{C'_* = \operatorname{im} g}}$$

$$\widehat{C}_* = \boxed{\boxed{A_* \oplus B_* \text{ exact}} \oplus \boxed{C'_* \cong C_*}}$$

Let ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$ be a reduction.

Frequently:

1. **$\widehat{C}_*$ is a locally effective chain complex:**

   its homology groups are unreachable.

2. **C_* is an effective chain complex:**

   its homology groups are computable.

3. **The reduction ρ is an entire description of**

   the homological nature of $\widehat{C}_*$.

4. **Any homological problem in $\widehat{C}_*$ is solvable**

   thanks to the information provided by ρ.

ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$

1. **What is $H_n(\widehat{C}_*)$?** **Solution: Compute $H_n(C_*)$.**

2. **Let $x \in \widehat{C}_n$. Is x a cycle?** **Solution: Compute $d_{\widehat{C}_*}(x)$.**

3. **Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?**

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. **Let $x, x' \in \widehat{C}_n$ be homologous cycles.**

   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) **Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.**

   (b) $y = g(z) + h(x - x')$.

<u>Definition</u>: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

<u>Problem</u>: Let ρ: $\boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftleftarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

$$???: \boxed{h+? \circlearrowleft (\widehat{C}_*, \widehat{d}+\widehat{\delta}) \underset{f+?}{\overset{g+?}{\leftleftarrows}} (C_*, d+?)}$$

describing in the same way the homology of

the chain complex with the $\boxed{\text{perturbed}}$ differential?

Basic Perturbation "Lemma" (BPL):

Given:

$$\boxed{\begin{tikzcd} \widehat{C}_* \arrow[loop left, "h"] \arrow[loop right, "\widehat{\delta}"] \arrow[d, shift right, "f"'] \\ C_* \arrow[u, shift right, "g"'] \end{tikzcd}}$$

satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$ of $\widehat{C}_*$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:

$$\boxed{\begin{tikzcd} \widehat{C}_* \arrow[loop left, "h"] \arrow[loop right, "\widehat{d}"] \arrow[d, shift right, "f"'] \\ C_* \arrow[u, shift right, "g"'] \arrow[loop right, "d"] \end{tikzcd}} + \boxed{\begin{tikzcd} \widehat{C}_* \arrow[loop right, "\widehat{\delta}"] \end{tikzcd}} \overset{BPL}{\longmapsto} \boxed{\begin{tikzcd} \widehat{C}_* \arrow[loop left, "h+\delta_h"] \arrow[loop right, "d+\widehat{\delta}"] \arrow[d, shift right, "f+\delta_f"'] \\ C_* \arrow[u, shift right, "g+\delta_g"'] \arrow[loop right, "d+\delta_d"] \end{tikzcd}}$$

Proof:

$\phi := \sum_{i=1}^{\infty}(-1)^i(h\widehat{\delta})^i$ **and** $\psi := \sum_{i=1}^{\infty}(-1)^i(\widehat{\delta}h)^i$ **are defined.**

Then:

- $\delta_d := f\widehat{\delta}(\mathrm{id}_{\widehat{C}} + \phi)g = f(\mathrm{id}_{\widehat{C}} + \psi)\widehat{\delta}g$
- $\delta_f := f\psi$
- $\delta_g := \phi g$
- $\delta_h := \phi h = h\psi$

is the solution.

QED

Serre: "Everything" in **Algebraic Topology** can be reduced to **Fibration problems**.

Examples: **Loop spaces**, **Classifying spaces**, **Homogeneous spaces**, **Whitehead tower**, **Postnikov tower**, ...

Remark: **Fibration** = **Twisted Product** = **Perturbation** of **Trivial Product**.

Corollary: **BPL** is **effective**
+ **Fibration** = **Perturbation** of **Trivial Product**
+ **Everything is Fibration**
$\Rightarrow$ **Alg. Topology** becomes $\boxed{\textbf{Constructive}}$.

<u>**Definition**</u>**: A (strong chain-) equivalence** $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$
is a pair of reductions $C_* \overset{\ell\rho}{\Lleftarrow} E_* \overset{r\rho}{\Rrightarrow} D_*$**:**

$$\begin{tikzcd}
 & E_* \arrow[loop left, "\ell h"] \arrow[loop right, "rh"] \arrow[dl, shift right, "\ell f"'] \arrow[dr, shift left, "rf"] & \\
C_* \arrow[ur, shift right, "\ell g"'] & & D_* \arrow[ul, shift left, "rg"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \frac{42}{30} \arrow[dl] \arrow[dr] & \\
\frac{21}{15} \arrow[dr, dashed] & & \frac{14}{10} \arrow[dl, dashed] \\
 & \frac{7}{5} &
\end{tikzcd}$$

Normal form problem ??

More structure often necessary in C_***.**

Most often: no possible choice for C_***.**

<u>Definition</u>: An $\boxed{\text{object with effective homology}}$ $\boldsymbol{X}$

is a 4-tuple:

$$\boldsymbol{X} = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = "<u>the</u>" chain complex "traditionally" associated with X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \overset{\varepsilon}{\Longleftrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_1, \ldots, X_n$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_1, \ldots, X_n) \mapsto X.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}\colon\ (\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n})$$
$$\mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is $\boxed{\text{perfectly stable}}$

and can be again used with X for further calculations.

Example:

Julio Rubio's solution of Adams' problem.

$$\boxed{\boldsymbol{X} = (X,\ C_*(X),\ \boldsymbol{EC}_*^{\boldsymbol{X}},\ \varepsilon^X)}$$

$$\Downarrow\Downarrow\Downarrow \qquad \textbf{Eil.-Moore}_{\boldsymbol{EH}}$$

$$\boxed{\boldsymbol{\Omega X} = (\Omega X,\ C_*(\Omega X),\ \boldsymbol{EC}_*^{\boldsymbol{\Omega X}},\ \varepsilon^{\Omega X})}$$

$\Longrightarrow$ **Trivial iteration now available.**

$\Rightarrow$ Very simple solution of $\boxed{\text{Adam's problem}}$:

$\boxed{\boxed{\text{Indefinite}}\ \text{iteration}}$ of the Cobar construction ???

$$
\begin{aligned}
X &= (X, C_*(X), EC_*^X, \varepsilon^X) \\
&\Downarrow \Omega_{EH} \\
\Omega X &= (\Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^2 X &= (\Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^3 X &= (\Omega^3 X, C_*(\Omega^3 X), EC_*^{\Omega^3 X}, \varepsilon^{\Omega^3 X}) \\
&\Downarrow \Omega_{EH} \\
\Omega^4 X &= \ldots
\end{aligned}
$$

"Cobar" $\boxed{\boxed{3}}(EC_*^X)$

<u>Example</u>: Effective homology version of

the Serre spectral sequence.

$$
\begin{aligned}
& \boldsymbol{F} = (F,\ C_*(F),\ \boldsymbol{EC}_*^{\boldsymbol{F}},\ \varepsilon^F) \\
+ \ & \boldsymbol{B} = (B,\ C_*(B),\ \boldsymbol{EC}_*^{\boldsymbol{B}},\ \varepsilon^B) \\
+ \ & \boldsymbol{\tau} : B \longrightarrow F \\
& \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \ \mathbf{Serre}_{\boldsymbol{EH}} \\
& \boldsymbol{E} = \boldsymbol{F} \times_{\boldsymbol{\tau}} \boldsymbol{B} = (E,\ C_*(E),\ \boldsymbol{EC}^{\boldsymbol{E}},\ \varepsilon^E)
\end{aligned}
$$

(Serre + G. Hirsch + H. Cartan + Shih W.

+ Szczarba + Ronnie Brown + J. Rubio + FS)

Proof.

$$\boxed{\begin{array}{c} C_*(F \times B) \xLeftarrow{\text{id}} C_*(F \times B) \xRightarrow{EZ} C_*F \otimes C_*B \\ C_*F \otimes C_*B \xLeftarrow{\otimes} \widehat{C}^F \otimes \widehat{C}^B \xRightarrow{\otimes} EC^F \otimes EC^B \end{array}}$$

$$\Downarrow\Downarrow\Downarrow\Downarrow\Downarrow \ \text{Serre}_{EH}$$

$$\boxed{\begin{array}{c} C_*(F \times_{\boxed{\tau}} B) \xLeftarrow{\text{id}} C_*(F \times_{\boxed{\tau}} B) \xRightarrow{\text{Shih}} C_*F \otimes_{\boxed{t}} C_*B \\ C_*F \otimes_{\boxed{t}} C_*B \xLeftarrow{EPL} \widehat{C}^F \otimes_{\boxed{t'}} \widehat{C}^B \xRightarrow{BPL} EC^F \otimes_{\boxed{t''}} EC^B \end{array}}$$

+ Composition of equivalences $\Longrightarrow$ O.K.

Combining these ingredients $\Rightarrow$

Homological Algebra becomes constructive.

Corollary: The "standard" exact and spectral sequences of Homological Algebra really become computational tools.

$\Rightarrow$ Concrete computer programs (EAT, Kenzo).

Warning about the right chronology. Example, let:

$$\phi: \ F \hookrightarrow E \longrightarrow B$$

be a fibration with B simply connected.

1. **The ordinary Serre Spectral Sequence $\boxed{\text{is not}}$ constructive.**

2. **Methods of Effective Homology give an algorithm:**

$$[EH_*(F) + EH_*(B) + \phi] \mapsto EH_*(E).$$

3. **Methods of Effective Homology can $\boxed{\text{then}}$ compute:**

$$[EH_*(F) + EH_*(B) + \phi + EH_*(E) \mapsto \text{SSS}(\phi)].$$

That is, the SSS is a byproduct of Effective Homology.

(Ana Romero)

Are your computer programs efficient?

What about benchmarks?

Do you compute new sphere homotopy groups?

Non-relevant question!

To be compared with prime number chasing.

Two different activities:

1. **Searching for very big prime numbers.**
2. **Designing methods applicable to arbitrary numbers.**

Most efficient current methods for big prime numbers:

$\boxed{\text{Specific}}$ tests (Lucas-Lehmer):

$\Rightarrow (2^{32,582,657} - 1)$ **prime (2006).**

Most "efficient" $\boxed{\text{general}}$ method:

Agrawal-Kayal-Saxena n^{12}-algorithm.

Both methods have $\boxed{\text{totally different scopes}}$.

Analogous situation in Algebraic Topology.

In case of spheres ($\sim$ Mersenne numbers),

specific methods go very far.

But these methods are inapplicable in general situations.

Typical situation:

Modifying a loop space by "pre-attaching" a cell.

What influence about

the homology groups of the new loop space?

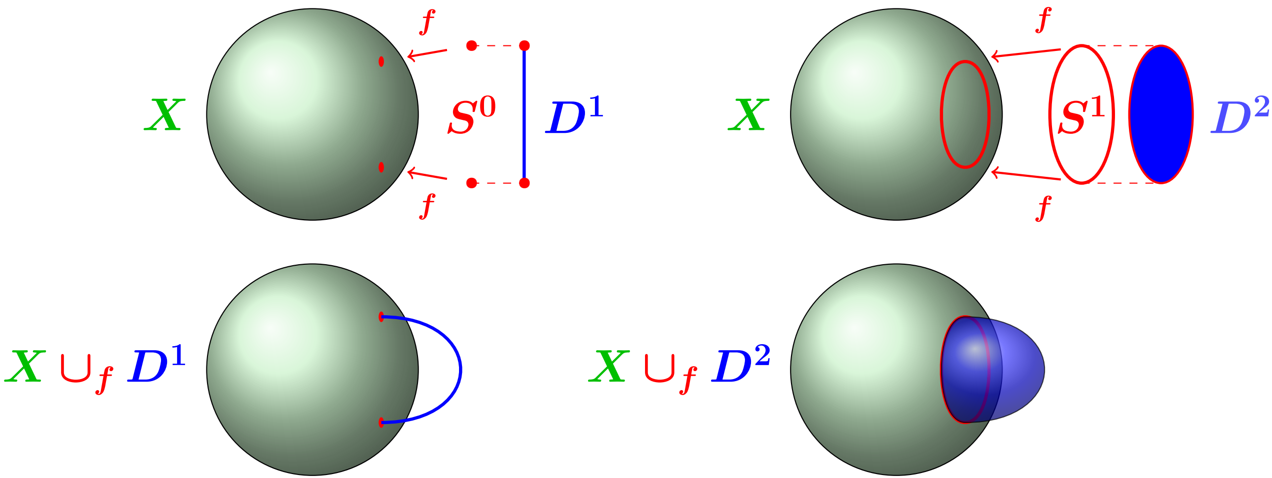
Attaching a cell D^n to a topological space

along the boundary S^{n-1}:

X = Topological space.

$f : S^{n-1} \to X$ = continuous map.

$$\Rightarrow X \cup_f D^n := (X \coprod D^n)/(X \ni f(x) \sim x \in S^{n-1}).$$



Given:

- $\boldsymbol{X}$ = **Topological space.**
- $\boldsymbol{H_*(X)}$ = **Homology of $\boldsymbol{X}$.**
- $\boldsymbol{H_*(\Omega X)}$ = **Homology of the loop space $\boldsymbol{\Omega X}$.**
- $\boldsymbol{f : S^{n-1} \longrightarrow X}$ **continue.**

Problem:

- Determine $H_*(\Omega(X \cup_f D^n)) = ???$

Example:

$H_*(\Omega S^3)$ computed by J.-P. Serre (1950).

$H_*(\Omega^2 S^3 := \Omega(\Omega S^3))$ computed by W. Browder (1958).

Modifying $\Omega S^3 \mapsto \Omega S^3 \cup_2 D^3$.

$\Rightarrow$ Problem:

$$H_*(\Omega(\Omega S^3 \cup_2 D^3)) = ???$$

Remark: $H_*(\Omega S^3 \cup_2 D^3)$ direct consequence of Serre's result.

"Standard" Algebraic Topology $\Rightarrow$???

In Effective Homology:

S^3 = **Finite simplicial set** $\Rightarrow$ S^3 = **OEH** [1].

EMSSEH [2] $\Rightarrow$ ΩS^3 = **OEH** [1].

MVESEH [3] $\Rightarrow$ $\Omega S^3 \cup_2 D^3$ = **OEH** [1].

EMSSEH [2] $\Rightarrow$ $\Omega(\Omega S^3 \cup_2 D^3)$ = **OEH** [1].

$\Omega(\Omega S^3 \cup_2 D^3)$ = **OEH** [1] $\Rightarrow$ $H_*(\Omega(\Omega S^3 \cup_2 D^3))$ $\boxed{\textbf{computable}}$.

$\Rightarrow$ **OK.**

1) OEH = Object with Effective Homology.
2) EMSSEH = Eilenberg-Moore Spectral Sequence with Effective Homology.
3) MVESEH = Mayer-Vietoris Exact Sequence with Effective Homology

A more complicated analogous computation:

$$X = \Omega(\Omega(\Omega(P^\infty\mathbb{R}/P^3\mathbb{R}) \cup_4 D^4) \cup_2 D^3)) \qquad H_*X = ???$$

$$H_0(X) = \mathbb{Z}.$$

$$H_1(X) = \mathbb{Z}/2\mathbb{Z}.$$

$$H_2(X) = (\mathbb{Z}/2\mathbb{Z})^2 + \mathbb{Z}.$$

$$H_3(X) = (\mathbb{Z}/2\mathbb{Z})^4 + \mathbb{Z}/8\mathbb{Z}.$$

$$H_4(X) = (\mathbb{Z}/2\mathbb{Z})^{10} + \mathbb{Z}/4\mathbb{Z} + \mathbb{Z}^2.$$

$$H_5(X) = (\mathbb{Z}/2\mathbb{Z})^{23} + \mathbb{Z}/8\mathbb{Z} + \mathbb{Z}/16\mathbb{Z}.$$

$$H_6(X) = (\mathbb{Z}/2\mathbb{Z})^{52} + (\mathbb{Z}/4\mathbb{Z})^3 + \mathbb{Z}^3.$$

$$H_7(X) = (\mathbb{Z}/2\mathbb{Z})^{113} + \mathbb{Z}/4\mathbb{Z} + (\mathbb{Z}/8\mathbb{Z})^3 + \mathbb{Z}/16\mathbb{Z} + \mathbb{Z}/32\mathbb{Z} + \mathbb{Z}.$$

The longest Kenzo computation (2 months).

Two main steps in a Kenzo calculation $H(d) =$???

1) "Automatic" writing of a sophisticated highly functional program P.

2) Using program P to compute $P(d) = H(d)$.

1) $\Rightarrow$ Always very fast ($<$ 1 sec.).

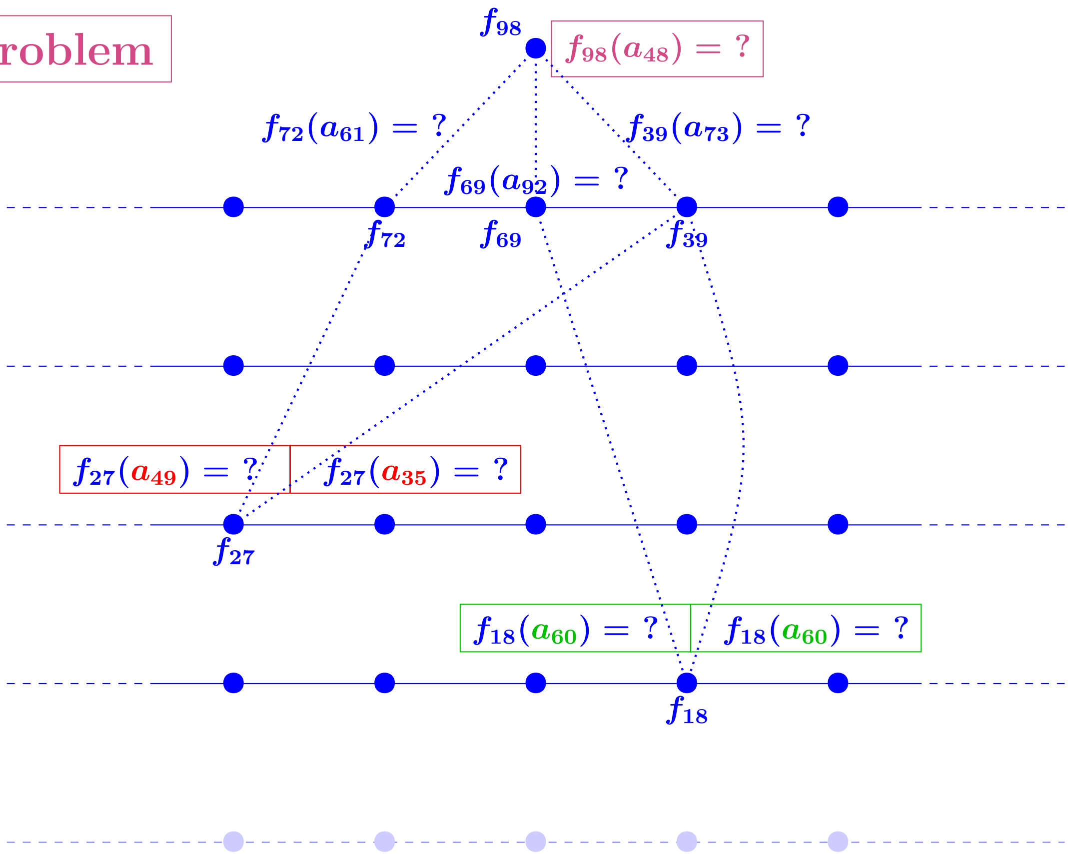
2) $\Rightarrow$ Can be very long (hours, days, months, years, centuries, ...)

High efficiency in functional programming with Common-Lisp

$\Rightarrow$ No technical difficulty in 1).

Terrible problem of memory management in 2).

Functional Problem



Only heuristic methods available:

EAT-1: No result stored $\Rightarrow$ poor time computing efficiency.

EAT-2: "Non-trivial" results stored

$\Rightarrow$ Computing time divided by $\sim$10.

Kenzo-1: Strong improvement in storing-searching methods.

$\Rightarrow$ Computing time divided by $\sim$10.

Kenzo-2: For overcoming space complexity:

Periodic cleaning of stored results.

$\Rightarrow$ Computing time divided by $\sim$10.

Theoretical framework for a rational study ??? Open !!!

"Vertical" vs "Horizontal" time complexity.

$$\boxed{\text{Computing } H_n(X),\ \pi_n(X)\ldots}$$

Two parameters: n and X.

"Horizontal" complexity := wrt X.

"Vertical" complexity := wrt n.

Effective homology $\Rightarrow$ Horizontal complexity $= P$.

David Annick (1986) $\Rightarrow$

Vertical complexity $\geq NP$-complete.

Back to “standard” Mathematics.

Traditional main problem of Algebraic Topology:

Classifying the homotopy types.

1. Only “reasonable” spaces: CW-complexes $\cong$ Simp. sets.

2. Non-simply connected topology excluded (word problem).

3. Classification “up to homeomorphism” out of scope $\Rightarrow$

Only classification “up to homotopy equivalence”.

4. “Standard” solution = Postnikov “invariants”.

Main problem of Algebraic Topology:

Algebraic Models for Topological Spaces ?

Main idea: Topology is difficult, Algebra is easy (!?).

Subquestion: what does the word "Algebra" means?

Answer: $\boxed{\text{No meaning at all}}$, only a "cultural" tradition.

Correct question:

Computable Models for Topological Spaces ?

Three solutions:

1. **Rolf Schön.**
2. **Effective Homology.**
3. **Operads.**

Schön's solution =

Intensive use of inductive limits

to approximate infinite objects.

Only one computer application:

Alain Clément, Lausanne, Haskell program.

Comparison: Effective Homology $\leftrightarrow$ Operads ???

Object with effective homology = Triple: (X, HX, ε) with:

X = locally effective version of the object.

HX = Effective chain complex

describing the ordinary homology of X.

ε = Strong connection $X \stackrel{\varepsilon}{\longleftrightarrow} HX$.

<u>Theorem</u>: The triple (X, HX, ε) is a $\boxed{\text{computable model}}$

of the homotopy type of X.

Operadic model for a topological space X:

$$(HX, \mathcal{M})$$

with:

HX = Effective chain complex describing the ordinary homology of X.

$\mathcal{M}$ = E_∞-operadic structure over HX.

<u>Theorem</u> (Mandell): The pair $(HX, \mathcal{M})$ is an "algebraic" model of the homotopy type of X.

Connection between (X, HX, ε) and $(HX, \mathcal{M})$?

Theorem: **There exists a canonical correspondance:**

$$(X, HX, \varepsilon) \longleftrightarrow (HX, \mathcal{M})$$

1. **"$\longrightarrow$" = Berger-Fresse.**

2. **"$\longleftarrow$" = S.-Mandell.**

Far from concrete implementations ! !

Main open problems in Effective Homology:

1. Eilenberg-Zilber = “ $\square \longleftrightarrow \boxslash$ ”.

Problem: General formula

of unavoidable exponential complexity.

How to design an efficient algorithm

for concrete particular cases?

2. Twisted Eilenberg-Zilber.

New important results experimentally discovered in 98.

Not yet proved!

3. Spectral sequences.

Filtrated chain complexes vs Exact Couples.

Particular cases of Bousfield-Kan, Adams,

May, Adams-Novikov... spectral sequences.

Cf recent thesis by Ana Romero.

4. Commutative Algebra.

Recent result:

Canonical correspondance between:

Effective resolution of $K[x_1, \ldots, x_m]$-modules

$$\updownarrow \cong$$

Effective homology of Koszul complex.

$\Rightarrow$ New algorithms producing effective resolutions.

5. **Concrete implementation of the canonical correspondance:**

$$(X, HX, \varepsilon) \longleftrightarrow (HX, \mathcal{M})$$

6. **Efficient memory management**

for high level functional programming ???

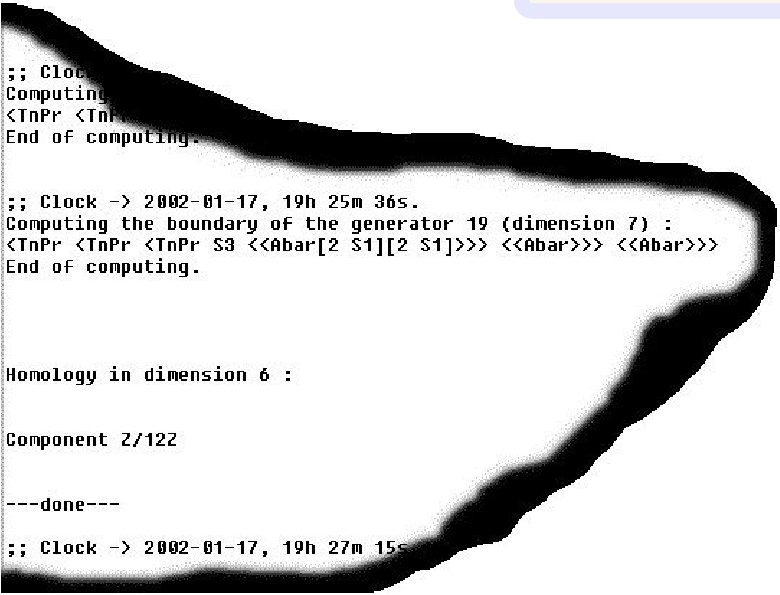
7. **Program proof, theorem proving.**

Recent result (Jesus Aransay):

Isabelle-certified proof of the Basic-Perturbation-Lemma.

Competing work by Coq people.

The END



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