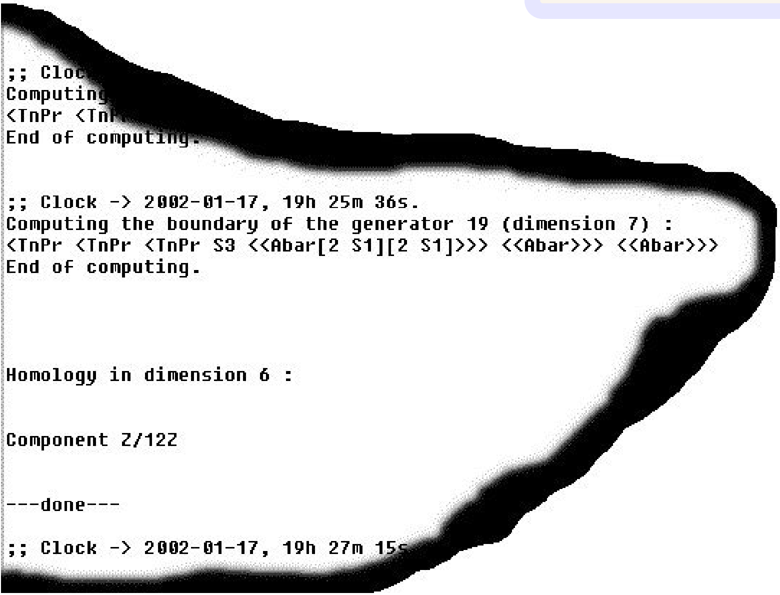
Constructive
Homological Algebra
I - The Problem



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Workshop: Formal Methods in Commutative Algebra,
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Semantics of colours:

Blue	=	**"Standard" Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**
Dark Orange	=	**Fuzzy objects.**
Pale grey	=	**Hyper-Fuzzy objects.**

1953 = Birth date of

the computational problem in Algebraic Topology.

<u>Theorem</u> (Serre): In simply connected Algebraic Topology,

the homology and homotopy groups

of the “reasonable” spaces

are $\mathbb{Z}$-modules of finite type.

⇒ <u>Problem</u>: ∃? algorithms:

$$\boldsymbol{X} \mapsto H_*(X) \qquad ???$$

$$\boldsymbol{X} \mapsto \pi_*(X) \qquad ???$$

Example of "reasonable" space:

$$X = \Omega(D^3 \cup_2 \Omega(D^4 \cup_4 \Omega(P^\infty(\mathbb{R})/P^3(\mathbb{R}))))$$

Example of problem: $H_* X =$???

Kenzo program + 2 months of computation ⇒

$$\begin{aligned}
H_0(X) &= \mathbb{Z} \\
H_1(X) &= \mathbb{Z}/2 \\
H_2(X) &= (\mathbb{Z}/2)^2 + \mathbb{Z} \\
H_3(X) &= (\mathbb{Z}/2)^4 + \mathbb{Z}/8 \\
H_4(X) &= (\mathbb{Z}/2)^{10} + \mathbb{Z}/4 + \mathbb{Z}^2 \\
H_5(X) &= (\mathbb{Z}/2)^{23} + \mathbb{Z}/8 + \mathbb{Z}/16 \\
H_6(X) &= (\mathbb{Z}/2)^{52} + (\mathbb{Z}/4)^3 + \mathbb{Z}^3 \\
H_7(X) &= (\mathbb{Z}/2)^{113} + \mathbb{Z}/4 + (\mathbb{Z}/8)^3 + \mathbb{Z}/16 + \mathbb{Z}/32 + \mathbb{Z}
\end{aligned}$$

First important result:

Theorem (Edgar Brown, 1956):

$\boldsymbol{X}$ **= finite simply connected simplicial set**

$\Rightarrow \boldsymbol{\pi_* X}$ **computable.**

Edgar Brown's warning:

> **It must be emphasized that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.**

Brown's warning still valid today!

Main currently available solutions:

1. **Edgar Brown $\Rightarrow$ Rolf Schön $\Rightarrow$ Alain Clément $\Rightarrow$???**
2. **Operadic solution (Justin Smith $+ \ldots +$ Michael Mandell)**

   Implementation ???
3. **Effective Homology I ($\Rightarrow$ Kenzo) + II (2008).**

Main obstacle: Infinite intermediate objects.

Typical simple example: $\pi_4(S^3) = \ ?$

Cartan-Serre-Whitehead method:

$H_2(S^3) = 0 + H_3(S^3) = \mathbb{Z} \Rightarrow$ **fibration:**

$$K(\mathbb{Z}, 2) \hookrightarrow X_4 \longrightarrow S^3$$

with $\pi_p(X_4) = \pi_p(S^3)$ **for every** $p \neq 3$ **and** $\pi_3(X^4) = 0$.

$\Rightarrow \pi_p(X_4) = 0$ **for** $p < 4$

$\Rightarrow$ **(Hurewicz' theorem)** $\pi_4(X_4) = H_4(X_4)$

but the standard model for $K(\mathbb{Z}, 2)$ **is infinite**

and X_4 **cannot be "totally" installed on a machine.**

Main methods invented by the topologists

to overcome this obstacle:

Exact sequences and spectral sequences

Typically, if $F \hookrightarrow E \rightarrow B$ is a fibration,

the Serre spectral sequence is

a(n enormous) set of (very sophisticated) relations

connecting the groups $H_p(F)$, $H_p(E)$ and $H_p(B)$.

In some particular cases this can be a tool:

$$\{H_p(B), H_p(F)\}_{p\in\mathbb{N}} \mapsto \{H_p(E)\}_{p\in\mathbb{N}}$$

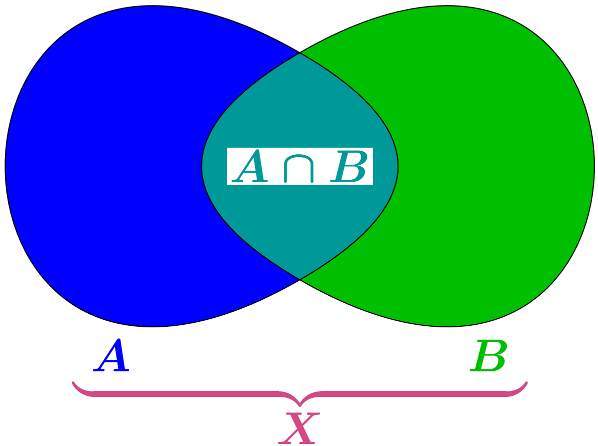
In general not!

Typical example:

<u>Theorem</u> (Mayer-Vietoris): X = space covered by $\{A, B\}$.

$\Rightarrow$ canonical exact sequence:

$$\cdots \longrightarrow H_p(A \cap B) \xrightarrow{\alpha_p} H_p(A) \oplus H_p(B) \longrightarrow H_p(X) \longrightarrow$$
$$\longrightarrow H_{p-1}(A \cap B) \xrightarrow{\alpha_{p-1}} H_{p-1}(A) \oplus H_{p-1}(B) \longrightarrow \cdots$$



How to use Mayer-Vietoris?

Long exact sequence:

$$A_* \xrightarrow{\alpha} B_* \to C_* \to D_* \xrightarrow{\beta} E_*$$

A_*, B_*, D_*, E_* **given** $\quad \Rightarrow C_* = ???$

Long exact sequence $\Rightarrow$ **short exact sequence:**

$$0 \to \text{Coker}\,\alpha \to C_* \to \text{Ker}\,\beta \to 0$$

But most often α **and** β **unknown!**

If α **and** β **are known,**

the extension class $\tau \in H^2(\text{Ker}\,\beta; \text{Coker}\,\alpha)$

giving the right extension $C_* = \text{Coker}\,\alpha \boxed{\times_\tau} \text{Ker}\,\beta$

can be very hard to be computed.

Analysis of the obstacle:

Standard Algebraic Topology is not constructive!

Example: Construction of $\alpha : H_p(A \cap B) \rightarrow H_p(A) \oplus H_p(B)$**?**

Let us assume $H_p(A \cap B) = \mathbb{Z}/6$, $H_p(A) = \mathbb{Z}/7$, $H_p(B) = \mathbb{Z}/8$.

Meaning of $H_p(A)$ "=" $\mathbb{Z}/7$ **??**

$\boxed{\exists}$ **isomorphism** $H_p(A) \xrightarrow{\cong} \mathbb{Z}/7$.

Most often, this existence is not constructive!!

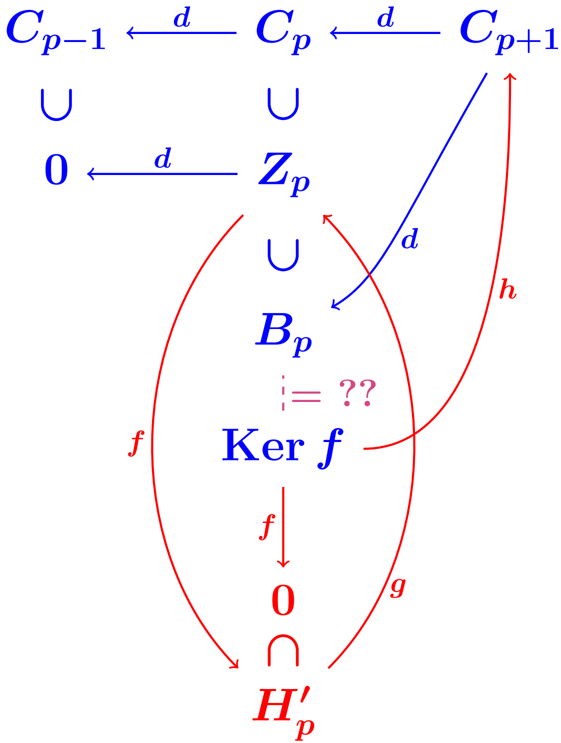
How to organize standard Homological Algebra

to construct and manipulate

such constructive isomorphisms?

“Actual” Homology Group = subquotient: $H_p = Z_p / B_p$.

Critical diagram:



Definition:

Constructive isomorphism

= (f,g,h) with:

f morphism satisfies $fd = 0$.

g map satisfies $fg = \mathrm{id}$.

h map satisfies $dh = \mathrm{id}$.

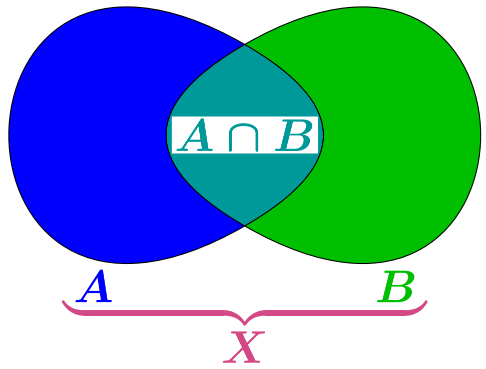
H'_p hyp. $\cong H_p$.

Chain complex: $C_* = \{\cdots \leftarrow C_{q-1} \leftarrow C_q \leftarrow C_{q-1} \leftarrow \cdots\}$

Definition: A SHP (Solution for the Homological Problem) for C_* is a family $(H'_p, f_p, g_p, h_p)_{p\in\mathbb{Z}}$ satisfying:

- $\{H'_p\}_{p\in\mathbb{Z}}$ = family of effective groups.
- $\{f_p : Z_p \rightarrow H'_p\}_{p\in\mathbb{Z}}$ = morphism family with $f_p d_{p+1} = 0$.
- $\{g_p : H'_p \rightarrow Z_p\}_{p\in\mathbb{Z}}$ = map family with $f_p g_p = \mathrm{id}$.
- $\{h_p : \mathrm{Ker}\, f_p \rightarrow C_{p+1}\}_{p\in\mathbb{Z}}$ = map family satisfies $d_{p+1} h_p = \mathrm{id}$.

Mayer-Vietoris revisited.



$C_* A$, $C_* B$, $C_*(A \cap B)$

with SHPs given.

$H_* X = ???$

$$\cdots \longrightarrow H_p(A \cap B) \xrightarrow{\alpha_p} H_p(A) \oplus H_p(B) \longrightarrow H_p(X) \longrightarrow$$
$$\longrightarrow H_{p-1}(A \cap B) \xrightarrow{\alpha_{p-1}} H_{p-1}(A) \oplus H_{p-1}(B) \longrightarrow \cdots$$

Constructing: $\alpha_p : H_p(A \cap B) \rightarrow H_p A \oplus H_p B$ **???**

Solution:

$$H'_p(A \cap B) \xrightarrow{g_p} Z_p(A \cap B) \xrightarrow{i_p} Z_p A \oplus Z_p B \xrightarrow{f_p} H'_p A \oplus H'_p B$$

$\Rightarrow$ **OK !!**

Moral:

Constructive Homological Algebra
is a matter of being able to
appropriately compute, handle and "proof"
representants for homology classes.

Effective Homology I (BPL) ⇒

"Simple" exact and spectral sequences become effective.

Effective Homology II (SHPs) ⇒

Sophisticated spectral sequences

(Bockstein, Bousfielkd-Kan, . . .)

defined through exact couples

become effective.

The main ingredients of Homological Algebra are

a ground ring R

and chain complexes made of R-modules and R-morphisms.

Three classes of modules:

- **Effective modules.**
- **Locally Effective modules.**
- **Fuzzy modules.**

1. Effective module M.

The membership property of an arbitray object $a \in ?\ M$ is decidable.

The module is discrete: equality between objects is decidable.

Ordinary computations can be executed.
If $\alpha \in R$ and $a, b \in M$, algorithms compute $a + b$ and αa.

The global structure of M is known.
"Reasonable" questions about M can be answered.

Depending on R, the isomorphism problem between two effective R-modules M and M' is or is not decidable.

R = Cramer ring (= strongly discrete and coherent):

- Any linear system $LV = 0$,

  with $L \in \mathrm{Hom}(R^m, R^n)$ given, $V \in R^m$ unknown,

  has a complete "solution" $S \in \mathrm{Hom}(R^k, R^m)$:

  $\mathrm{Ker}\, L = \mathrm{Im}\, S$.

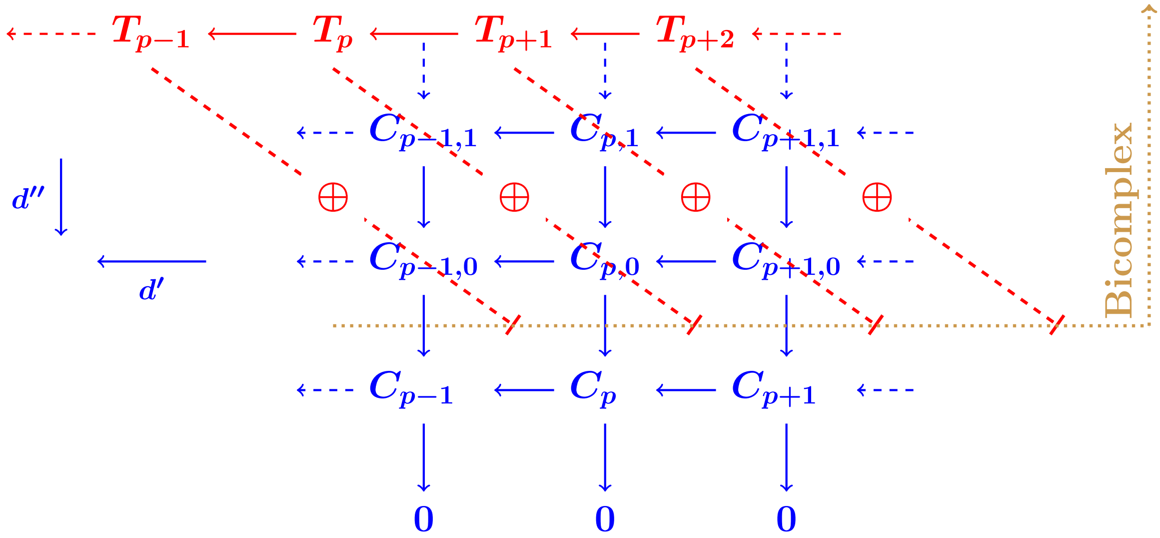
$$R^n \xleftarrow{L} R^m \xleftarrow{S} R^k$$

- For any $L \in \mathrm{Hom}(R^m, R^n)$,

  an algorithm $\sigma : R^n \to \{\perp\} \coprod R^m$ satisfies:

  - $\sigma(V) = \perp \Leftrightarrow V \notin \mathrm{Im}\, L$.
  - $\sigma(V) = U \in R^m \Leftrightarrow V = LU$.

Bicomplex theorem:



$$\left.\begin{array}{r}\text{Every column exact}\\ T_p = \oplus_{a+b=p} C_{a,b}\\ d'd' = d''d'' = d'd'' + d''d' = 0\end{array}\right\} \Rightarrow H_*(C_*, d') \cong H_*(T_*, d' \oplus d'')$$

R = Cramer ring.

Definition: An effective module M is an R-module

with a finite presentation:

$$M = (R^{b_0}, R^{b_1}, d) \;\Leftrightarrow\; 0 \leftarrow M \leftarrow R^{b_0} \xleftarrow{d} R^{b_1}$$

R^{b_0} = type of the elements of M.

$m \sim m'$ mod $d(R^{b_1})$ decidable $\Rightarrow$ M discrete.

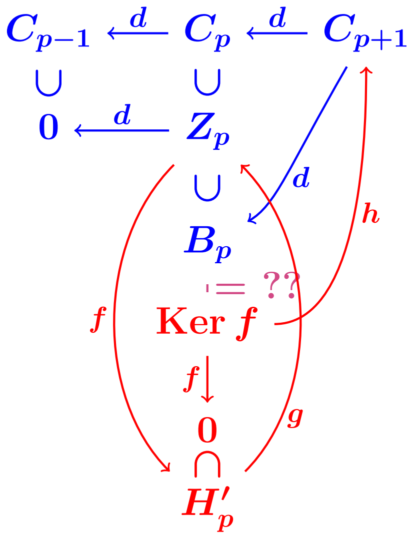
Proposition: M = effective module.

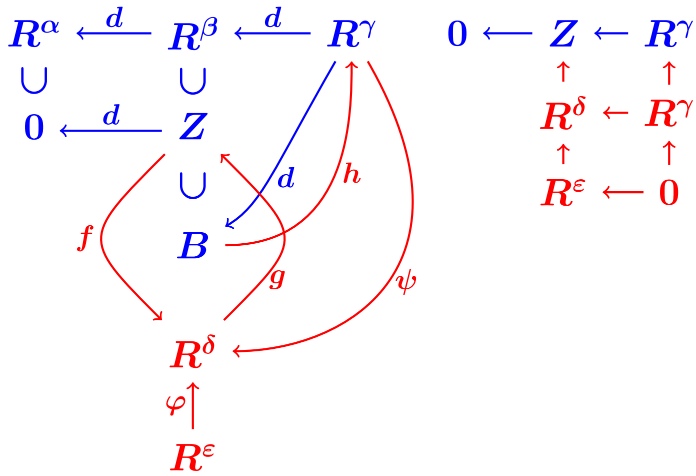
$\Rightarrow$ M admits a free resolution of finite type.

$$0 \leftarrow M \leftarrow R^{b_0} \leftarrow R^{b_1} \leftarrow R^{b_2} \leftarrow R^{b_3} \leftarrow \cdots$$

Proof: Cramer. QED

Proposition: An algorithm produces a SHP for a chain complex of free ft modules.

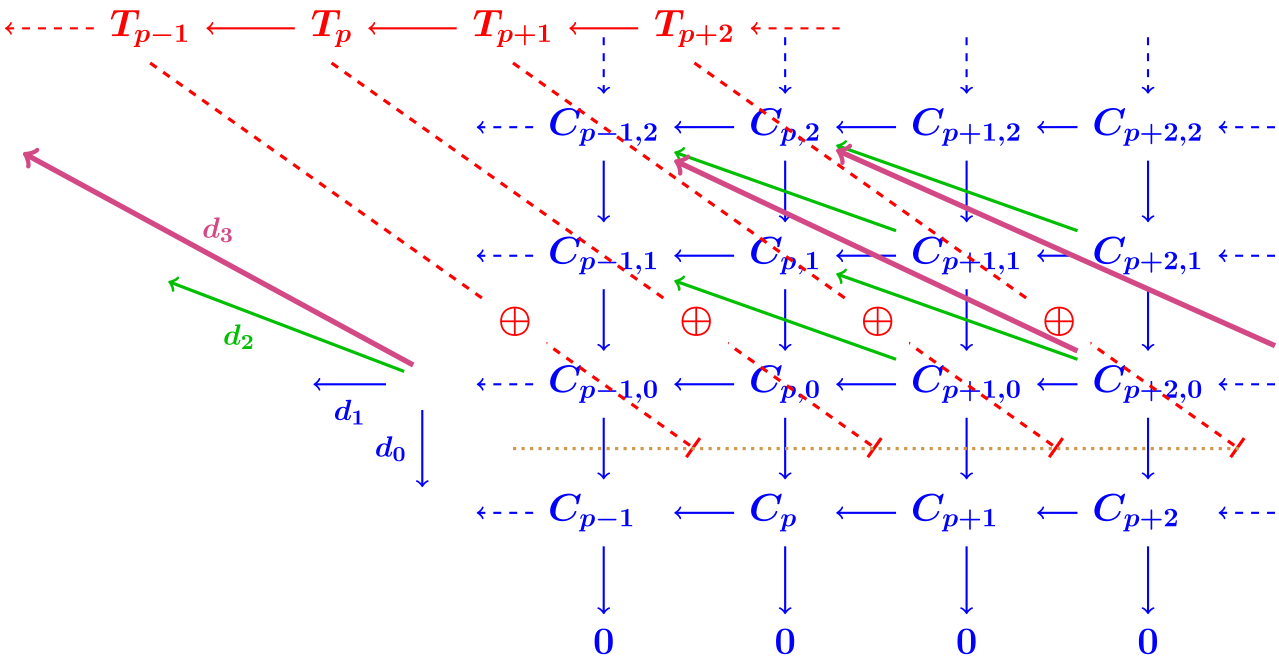




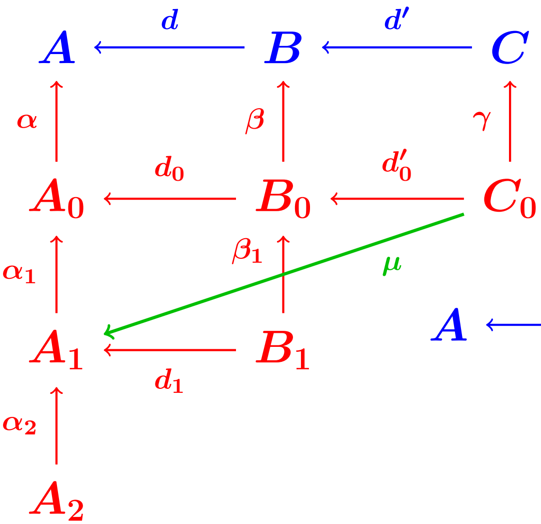
$$H' = (R^\delta, R^\gamma \oplus R^\varepsilon, \psi \oplus \phi)$$

Theorem: **An algorithm produces a SHP for a chain complex of modules with finite presentation.**

Multicomplex theorem:

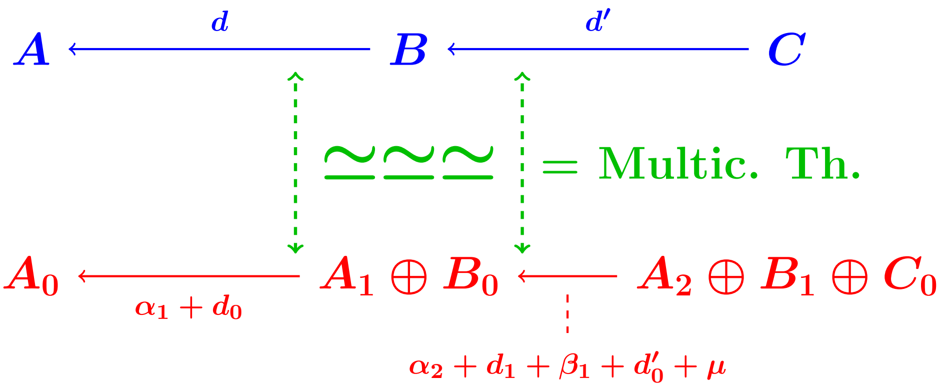


Proof.



Red = Free + finite type

Blue = Finite presentation



QED

2. Locally effective module M.

The membership property of an arbitray object $a \in ? M$

is decidable.

The module is discrete:

equality between objects is decidable.

Ordinary computations can be executed.

If $\alpha \in R$ and $a, b \in M$, algorithms compute $a + b$ and αa.

The global structure of M is unknown.

In particular M maybe is not of finite type.

No computation

involving the whole “knowledge” of M can be done.

3. Fuzzy module M.

The membership property of an arbitray object $a \in ? M$

is decidable.

The module is not necessarily discrete:

equality between objects is in general undecidable.

Ordinary computations can be executed.

If $\alpha \in R$ and $a, b \in M$, algorithms compute $a + b$ and αa.

The global structure of M is unknown.

In particular M maybe is not of finite type.

No computation

involving the whole "knowledge" of M can be done.

Typical example of fuzzy module.

Given a chain complex:

$$\cdots \leftarrow C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \leftarrow \cdots$$

made of locally effective modules.

Then $H_p = \operatorname{Ker} d_p / \operatorname{Im} d_{p+1}$ is a fuzzy module.

The element type for H_p is Z_p:

a homology class is implemented as a cycle.

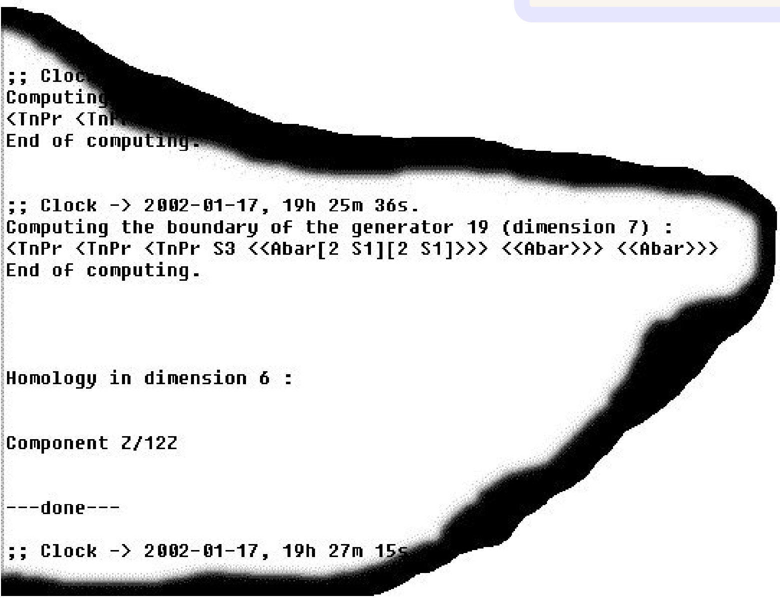
C_{p-1} discrete $\Rightarrow$ Membership to H_p decidable.

C_{p+1} not of finite type

$\Rightarrow$ Membership to $\operatorname{Im} d_{p+1}$ undecidable.

$\Rightarrow$ H_p not discrete.

The END



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