# Effective Constructive

# Algebraic Topology

```
;; Cloc
Computing
<TnPr <TnP
End of computing.

;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.



Homology in dimension 6 :

Component Z/12Z

---done---

;; Clock -> 2002-01-17, 19h 27m 15s
```

*Ana Romero, Universidad de La Rioja*
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*Map Ictp Conference, Trieste, August 25-29, 2008*

**Semantics of colours:**

| | | |
|---:|:---:|:---|
| **Blue** | = | **"Standard" Mathematics** |
| **Red** | = | **Constructive, effective,** |
| | | **algorithm, machine object, . . .** |
| | | |
| **Violet** | = | **Problem, difficulty, obstacle, disadvantage, . . .** |
| **Green** | = | **Solution, essential point, mathematicians, . . .** |

Three solutions for **Constructive Algebraic Topology**:

> 1. **Rolf Schön** (**Inductive methods**).
>
> 2. **Effective Homology**.
>
> 3. **Operadic Algebraic Topology**.

**Only the second one** so far

> led to **concrete computer programs**.

<u>Plan of the talk</u>: 1. **Computer** illustration

> around **CW-complexes**.
>
> 2. **Constructive** statement of
>
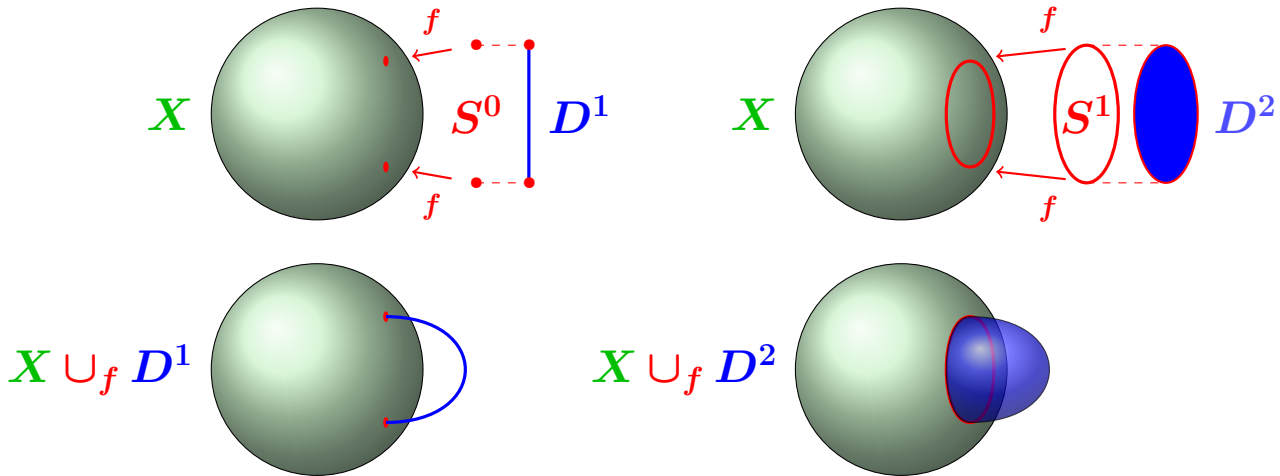> the **homological problem**.
>
> 3. Other **computer** illustrations.

**Attaching a cell $D^n$ to a topological space $X$**

**along the boundary $S^{n-1}$:**

$X =$ **Topological space.**

$f : S^{n-1} \to X =$ **continuous map.**

$\Rightarrow X \cup_f D^n := (X \coprod D^n)/(X \ni f(x) \sim x \in S^{n-1}).$

**Notion of CW-Complex $X$:**

$$X = \varinjlim\{X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_n \subset \cdots\}_{n \in \mathbb{N}}$$

with $X_0 = $ discrete space and

the $n$-skeleton $X_n$ is obtained

from the $(n-1)$-skeleton $X_{n-1}$

by attaching $n$-disks $D_1^n, D_2^n, \cdots$ to $X_{n-1}$

according to attaching maps $f_1^n, f_2^n, \cdots$

Every reasonable space can be presented

up to homotopy equivalence

as a CW-complex of finite type.

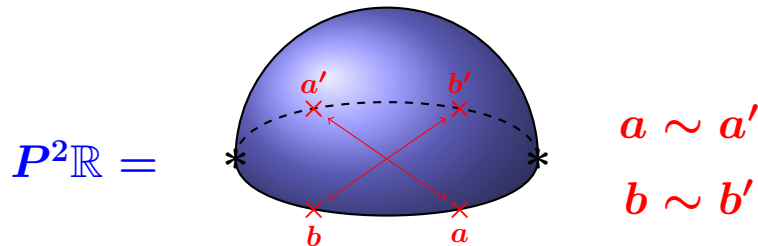**Example 1. Presentation of $X = P^2\mathbb{R}$ as a CW-complex.**

$$X_0 = *$$

$$D^1 \supset S^0 \xrightarrow{f^1} *$$

$$\Rightarrow X_1 = X_0 \cup_{f^1} D^1 = X_1 = S^1 = \{Z \in \mathbb{C} \underline{\text{st}} \ |z| = 1\}$$

$$D^2 \supset S^1 \xrightarrow{f^2} S^1 : z \mapsto z^2$$

$$\Rightarrow X = X_2 = X_1 \cup_{f^2} D^2 = P^2\mathbb{R}$$

$$P^2\mathbb{R} =$$



$$a \sim a'$$

$$b \sim b'$$

**Example 2.** More generally:

$$\text{Presentation of } X = P^\infty\mathbb{R} \text{ as a CW-complex.}$$

1. $X_0 = P^0\mathbb{R} = S^0/\sim = *$.

2. Let us assume $X_n = P^n\mathbb{R}$ constructed.

3. $D^{n+1} \supset S^n \xrightarrow{f^{n+1}} P^n\mathbb{R}$

$$\text{with } f^{n+1} = \text{the canonical projection.}$$

4. $\Rightarrow X_{n+1} = D^{n+1} \cup_{f^{n+1}} X_n = P^{n+1}\mathbb{R}$.

$(+\!\!+\, n)$ ; goto 2.

5. $X = \lim_{\rightarrow} X_n = P^\infty\mathbb{R}$.

**Example 3. Simplicial complexes and simplicial sets.**

$X$ = simplicial set.

<u>Definition</u>: The $n$-skeleton $X_n$ of $X$ is obtained from $X$ by keeping the non-degenerate simplices of dimension $\leq n$ (and their degeneracies), throwing away the non-degenerate simplices of dimension $> n$ (and their degeneracies).

$|X_n|$ obtained from $|X_{n-1}|$

by attaching $n$-simplices = $n$-disks.

$\Rightarrow X$ = CW-complex with $|X| = \lim_{\to} |X_n|$.

**Simplicial version of $P^\infty\mathbb{R}$:**

$$P^\infty\mathbb{R} = X = K(\mathbb{Z}_2, 1)$$

$$\Rightarrow X_n^{ND} = \{\sigma_n\}$$

$$\partial_i\sigma_n = \quad \sigma_{n-1} \ \text{ if } i = 0, n;$$
$$= \eta_{i-1}\sigma_{n-2} \ \text{ if } 0 < i < n.$$

$$\Rightarrow C_*X = \{\cdots \xleftarrow{\quad} \overset{0}{\mathbb{Z}} \xleftarrow{0} \overset{1}{\mathbb{Z}} \xleftarrow{\times 2} \overset{2}{\mathbb{Z}} \xleftarrow{0} \overset{3}{\mathbb{Z}} \xleftarrow{\times 2} \overset{4}{\mathbb{Z}} \xleftarrow{0} \overset{5}{\mathbb{Z}} \xleftarrow{\times 2} \cdots\}$$

$$\Rightarrow H_i(P^\infty\mathbb{R}) = \mathbb{Z} \ \ \text{ if } i = 0;$$
$$\mathbb{Z}_2 \ \text{ if } i > 0 \text{ odd};$$
$$0 \ \ \text{ if } i > 0 \text{ even}.$$

The same for $P^\infty\mathbb{C}$ ?

Topological version ? Easy.

$P^\infty\mathbb{C} = X = \lim_\to X_{2n}$ where:

$$X_{2n} = X_{2n-2} \cup_{f^{2n}} D^{2n}$$

with: $D^{2n} \supset S^{2n-1} \to P^{n-1}\mathbb{C}$ the canonical projection.

Simplicial version?

Much harder!

Easy **up to homotopy**.

Easiest solution $= K(\mathbb{Z}, 2)$.

Justification $=$ two principal fibrations:

$$S^1 \hookrightarrow S^\infty \longrightarrow P^\infty\mathbb{C}$$

$$K(\mathbb{Z}, 1) \hookrightarrow E(\mathbb{Z}, 1) \longrightarrow K(\mathbb{Z}, 2)$$

$+ (K(\mathbb{Z}, 1) \sim S^1) + (S^\infty \text{ contractible}) + (E(\mathbb{Z}, 1) \text{ contractible})$
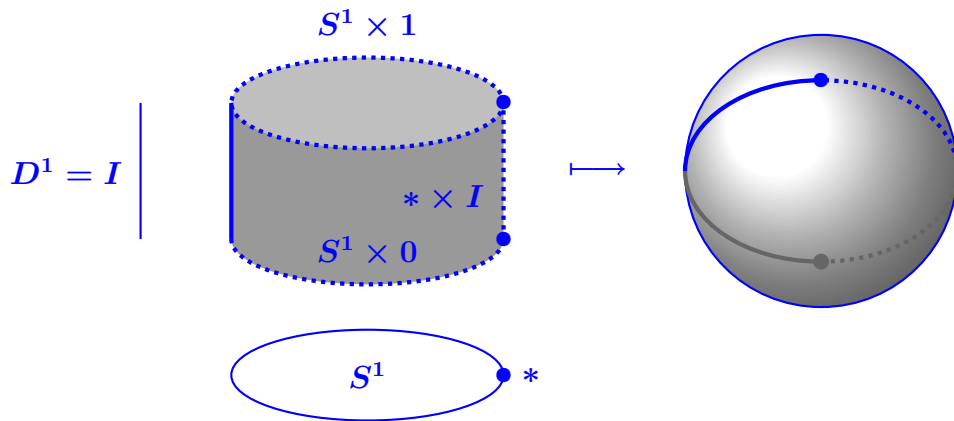
$$\Rightarrow P^\infty\mathbb{C} \sim K(\mathbb{Z}, 2)$$

Remark: $K(\mathbb{Z}, 2)$ not of finite type!

Simplicial model of finite type for $P^2\mathbb{C}$??

**Cellular homology.**

$$S^n = S^1 \times D^{n-1}/\sim \text{ with } (z, x) \sim (z', x') \text{ if } x = x' \in \partial D^{n-1}.$$



**Canonical self-map of degree $k$ for $S^n$:**

$$\alpha_k : S^n \to S^n : (z, x) \mapsto (z^n, x).$$

<u>**Theorem**</u> (**Hopf**): $\mathcal{C}(S^n, S^n)/\sim \;\cong\; \mathbb{Z}.$

**CW-complex:**

$$X = \lim_{\rightarrow} X_n = \{(D_i^n, f_i^n : S^{n-1} \to X_{n-1})_{1 \leq i \leq m_n}\}_{n \in \mathbb{N}}$$

Associated **cellular chain complex:**

$$\cdots \longleftarrow \mathbb{Z}^{(m_{n-1})} \xleftarrow{d_n} \mathbb{Z}^{(m_n)} \longleftarrow \cdots$$

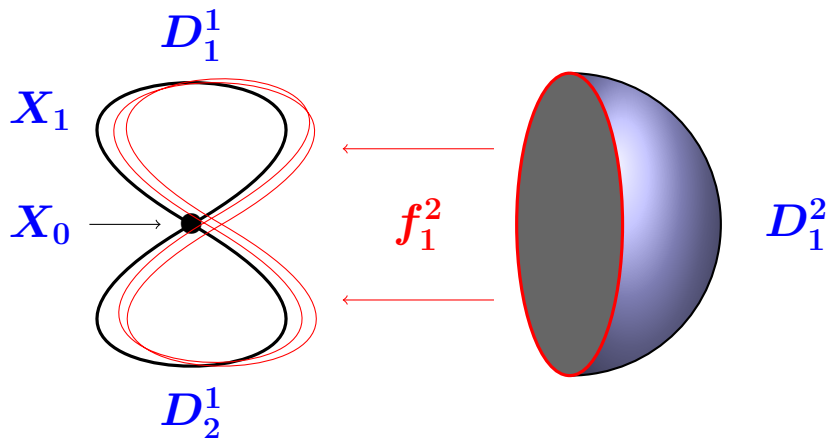**Coefficient $\alpha_{1,1}$ of $d_n$ in column 1 and row 1**

obtained from $g_{1,1}^n$:

$$f_1^n : S^{n-1} \to X^{n-1}$$

$$\Rightarrow g_{1,1}^n : S^{n-1} \to Y_1^{n-1} = X^{n-1}/[X^{n-2} \cup (\cup_{i \neq 1} D_i^{n-1})] = S^{n-1}$$

$$\Rightarrow \alpha_{1,1} = \deg(g_{1,1}^n).$$

**Example:** $X =$



**Cellular complex** $= \{0 \longleftarrow \mathbb{Z} \overset{d_1}{\longleftarrow} \mathbb{Z}^2 \overset{d_2}{\longleftarrow} \mathbb{Z} \longleftarrow 0\}$

**with** $d_1 = [0\ 0]$ **and** $d_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow H_* = \{\mathbb{Z}, \mathbb{Z}_2 + \mathbb{Z}, 0\}$

**Theorem** (**Adams**, 1956): Let $X$ be a **1-reduced CW-complex**

(one vertex, no 1-cell).

Then $\exists$ a **CW-model** for the **loop space** $\Omega X$, where every sequence $(\sigma_1, \ldots, \sigma_k)$ of **cells** of $X$ of respective **dimensions** $(d_1, \ldots, d_k)$ generate a **cell** of **dimension** $(d_1 + \cdots + d_k - k)$ in the **CW-model** of $\Omega X$.
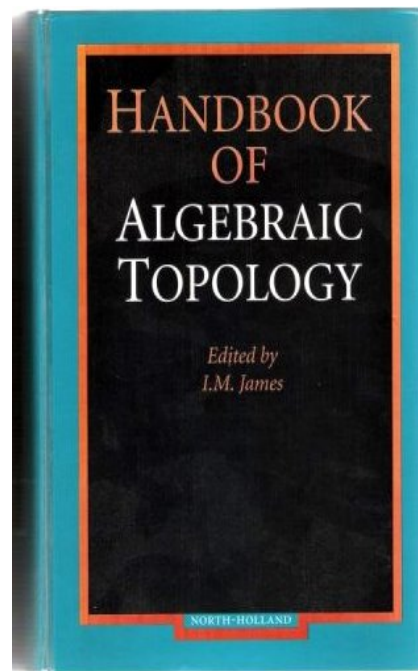
**Examples:**

$$S^3 = (*, 0, 0, 1) \implies \Omega S^3 = (*, 0, 1, 0, 1, 0, 1, \ldots).$$

$$P^2\mathbb{C} = (*, 0, 1, 0, 1) \implies$$

$$\Omega P^2\mathbb{C} = (*, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \ldots).$$

Typical example

extracted from

the encyclopedy:

(Ioan James editor).

# Chapter 13

## Stable Homotopy
### and Iterated Loop Spaces

### Gunnar Carlsson
### James Milgram

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CHAPTER 13

# Stable Homotopy and Iterated Loop Spaces

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*Contents*

## 6. The structure of second loop spaces

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of $X$, which is homotopy equivalent to $\Omega X$. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume $X$ has no $i$-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adams' solution of the problem in the special case that $X$ is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

<u>Translation</u>: **No known algorithm** using these methods

computes $H_*(\Omega^n X)$ for $n \geq 3$

except when $X$ is an $n$-suspension $X = \Sigma^n Y$.

Typical example: $H_*(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})) = $ ???

<u>Adams</u>: There $\boxed{\text{exists}}$ a finite-type CW-complex

with the homotopy type of $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$.

| Dimension | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cell-# | 1 | 1 | 2 | 5 | 13 | 33 | 84 | 214 | 545 | 1388 | 3535 | $\cdots$ |

But **what about** the **homological boundary matrices** ???

**Kenzo computing $d_5 : [C_5(\Omega^3) = \mathbb{Z}^{33}] \to [C_4(\Omega^3) = \mathbb{Z}^{13}]$ :**

========== MATRIX 13 lines + 33 columns =====
L1=[C1=-2]
L2=[C1=-1]
L3=[C1=-4][C2=1][C3=-1][C4=-2]
L4=[C2=1][C3=-1][C6=2]
L5=[C1=6][C4=1][C6=1]
L6=[C1=4][C4=4][C6=4][C7=3]
L7=[C1=4][C12=-2][C14=2]
L8=[C1=6][C4=1][C6=1]
L9=[C1=4][C4=4][C6=4][C7=3]
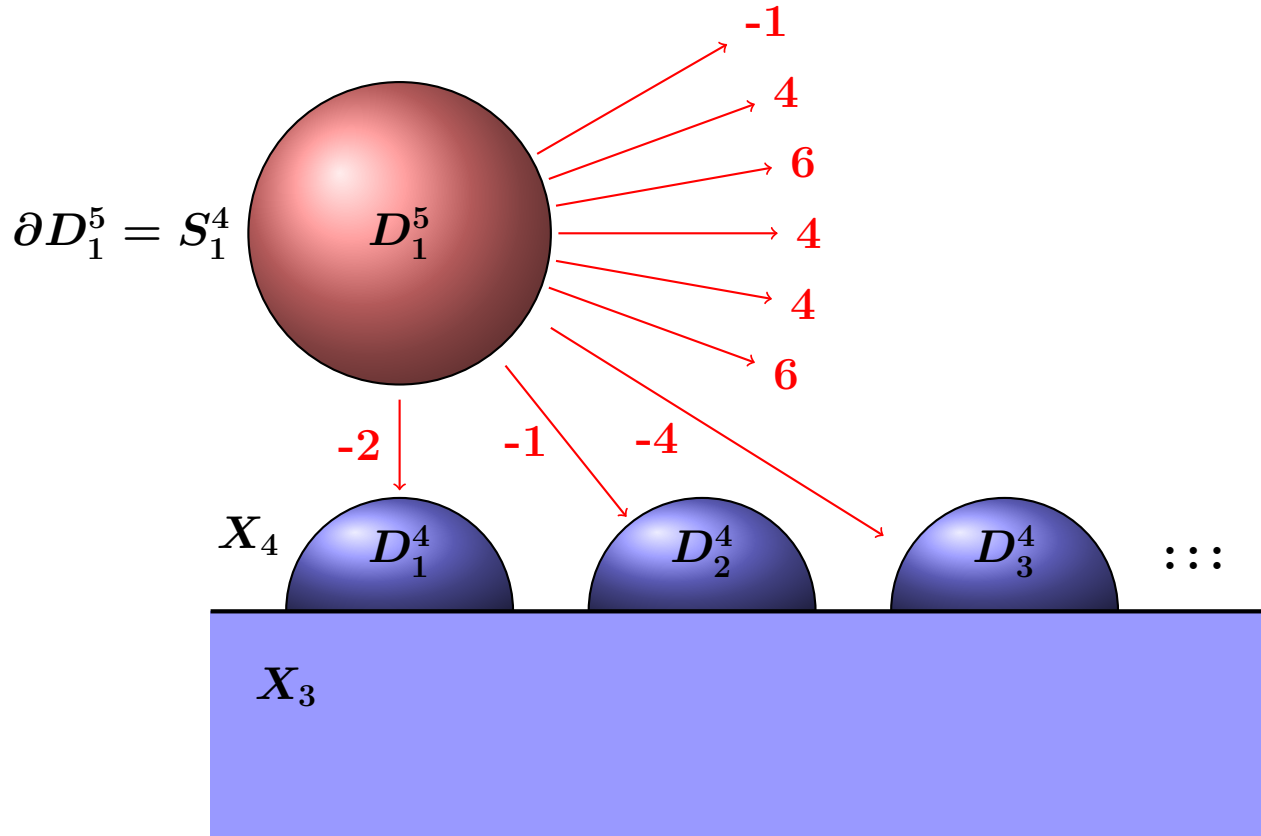L10=[C8=4][C10=1][C11=-1][C14=-4][C15=-2][C20=-2]
L11=[C1=4][C8=4][C10=1][C11=-1][C16=-4][C18=-1][C19=1][C23=-2]
L12=[C12=4][C13=2][C16=-4][C18=-1][C19=1][C27=-2]
L13=[C1=-1][C20=4][C21=2][C23=-4][C24=-2][C27=4][C28=2]
========== END-MATRIX

**Meaning:**

**Analysis of the problem:**

"Standard" homological algebra is not $\boxed{\text{constructive}}$.

**Typical statement:**

The sequence $A \xleftarrow{\alpha} B \xleftarrow{\beta} C$ is exact.

**Common translation:**

$$(\forall b \in B) \ \left[(\alpha(b) = 0) \Rightarrow \left(\boxed{\exists c \in C} \ \underline{\text{st}} \ b = \beta(c)\right)\right]$$

with $\boxed{\exists c \in C}$ most often **non-constructive**.

**Constructive exactness:**

$$A \xleftarrow{\alpha} B \xleftarrow{\beta} C \text{ constructively exact}$$

if an **algorithm** $\rho : \ker \alpha \to C$ is given satisfying:

$$
\begin{array}{ccccc}
A & \xleftarrow{\alpha} & B & \xleftarrow{\beta} & C \\
\cup & & \cup & \scriptstyle(=) & \uparrow \\
0 & \longleftarrow & \ker \alpha & \dashrightarrow_{\rho?} &
\end{array}
$$

$\Rightarrow$ **Organizational algebraic problems:**

$$0 \longleftarrow \mathbb{Z}/2\mathbb{Z} \xleftarrow[\rho?]{\text{pr}} \mathbb{Z}$$

where $\rho$ $\boxed{\text{cannot be}}$ a **group homomorphism.**

**<u>Definition</u>: A (homological) reduction is a diagram:**

$$\rho: \quad \boxed{h \,\circlearrowright\, \widehat{C}_* \underset{f}{\overset{g}{\longleftarrow\!\!\!\longrightarrow}} C_*}$$

**with:**

1. $\widehat{C}_*$ **and** $C_* =$ **chain complexes.**

2. $f$ **and** $g =$ **chain complex morphisms.**

3. $h =$ **homotopy operator (degree $+1$).**

4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.

5. $fh = 0$, $hg = 0$ **and** $hh = 0$.

Let $\rho$: $\boxed{h \,\circlearrowright\, \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$ be a **reduction**.

**Frequently:**

1. $\widehat{C}_*$ is a **locally effective** **chain complex**:

   its **homology groups** are **unreachable**.

2. $C_*$ is an **effective** **chain complex**:

   its **homology groups** are **computable**.

3. The **reduction** $\rho$ is an entire description of

   the **homological nature** of $\widehat{C}_*$.

4. Any **homological problem** in $\widehat{C}_*$ is **solvable**

   thanks to the **information** provided by $\rho$.

$\rho$: $\quad h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*$

1. What is $H_n(\widehat{C}_*)$? $\qquad$ Solution: Compute $H_n(C_*)$.

2. Let $x \in \widehat{C}_n$. Is $x$ a **cycle**? $\quad$ Solution: Compute $d_{\widehat{C}_*}(x)$.

3. Let $x, x' \in \widehat{C}_n$ be **cycles**. Are they **homologous**?
   Solution: Look whether $f(x)$ and $f(x')$ are **homologous**.

4. Let $x, x' \in \widehat{C}_n$ be **homologous cycles**.
   $\qquad\qquad\qquad$ Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?
   Solution:
   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.
   (b) $y = g(z) + h(x - x')$.

# The END

```
;; Cloc
Computing
<TnPr <TnP
End of computing.

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<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.



Homology in dimension 6 :

Component Z/12Z


---done---

;; Clock -> 2002-01-17, 19h 27m 15s
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