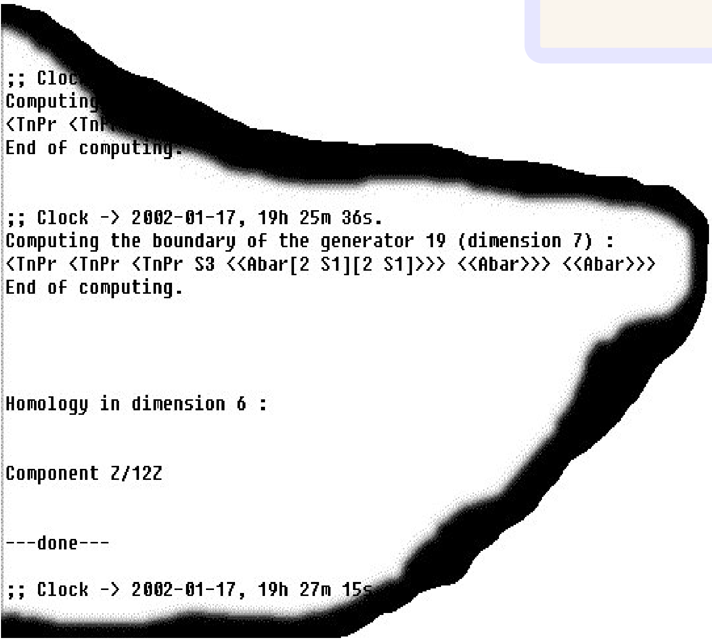
Constructive

Algebraic Topology



Ana Romero, Universidad de La Rioja
Julio Rubio, Universidad de La Rioja
Francis Sergeraert, Institut Fourier, Grenoble
Paris, ATMCS III, July 7-11, 2008

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective,**
		algorithm, machine object, . . .
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Main result:

Constructive Algebraic Topology

is Constructive (and simpler).

Important steps in Algebraic Topology:

Euclid, Euler, Riemann, Poincaré, Serre.

Serre: Every homology or homotopy group of a "reasonable" simply connected space is of finite type.

$\Rightarrow$ Could be output by a computer:

$$\mathbb{Z}_2^4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z} \;\leftrightarrow\; (2, 2, 2, 2, 6, 0)$$

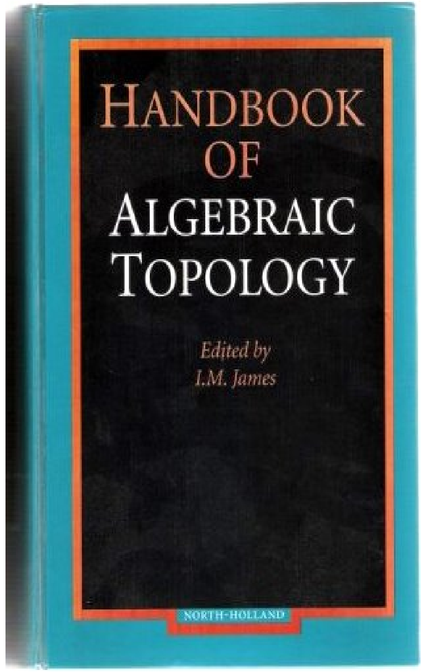
But can be computed by a computer?

Typical example

extracted from

the encyclopedy:

(Ioan James editor).

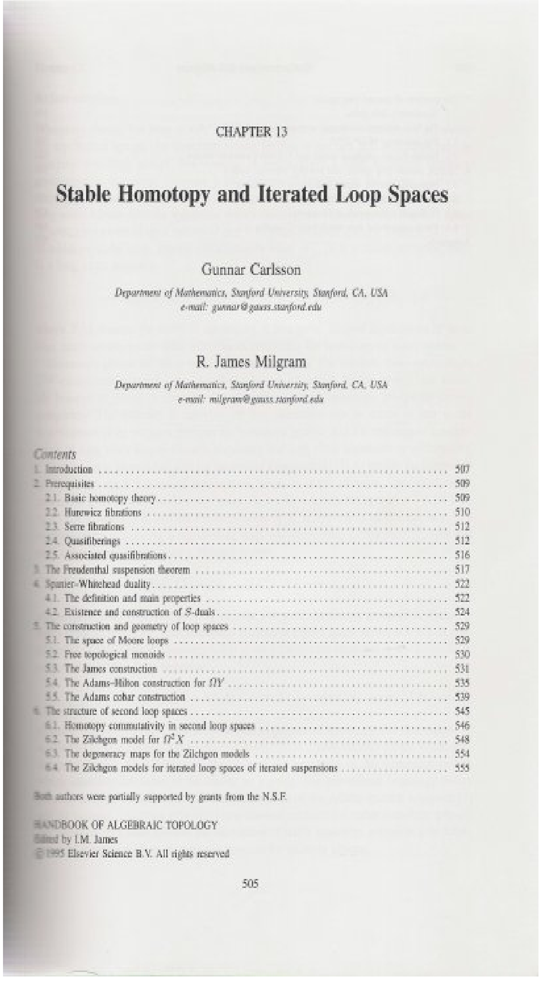


Chapter 13

Stable Homotopy and Iterated Loop Spaces

Gunnar Carlsson

James Milgram



CHAPTER 13

Stable Homotopy and Iterated Loop Spaces

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Contents

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HANDBOOK OF ALGEBRAIC TOPOLOGY
Edited by I.M. James

6. The structure of second loop spaces

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of X, which is homotopy equivalent to ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume X has no i-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adams' solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

Translation: No known algorithm using these methods computes $H_*(\Omega^n X)$ for $n \geq 3$ except when X is an n-suspension $X = \Sigma^n Y$.

Typical example: $H_*(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})) =$???

Adams: There exists a finite-type CW-complex with the homotopy type of $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$.

Dimension	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
Cell-#	1	1	2	5	13	33	84	214	545	1388	3535	$\cdots$

But what about the homological boundary matrices ???

Kenzo computing $d_5 : [C_5(\Omega^3) = \mathbb{Z}^{33}] \rightarrow [C_4(\Omega^3) = \mathbb{Z}^{13}]$:

========== MATRIX 13 lines + 33 columns =====

L1=[C1=-2]

L2=[C1=-1]

L3=[C1=-4][C2=1][C3=-1][C4=-2]

L4=[C2=1][C3=-1][C6=2]

L5=[C1=6][C4=1][C6=1]

L6=[C1=4][C4=4][C6=4][C7=3]

L7=[C1=4][C12=-2][C14=2]

L8=[C1=6][C4=1][C6=1]

L9=[C1=4][C4=4][C6=4][C7=3]

L10=[C8=4][C10=1][C11=-1][C14=-4][C15=-2][C20=-2]

L11=[C1=4][C8=4][C10=1][C11=-1][C16=-4][C18=-1][C19=1][C23=-2]

L12=[C12=4][C13=2][C16=-4][C18=-1][C19=1][C27=-2]

L13=[C1=-1][C20=4][C21=2][C23=-4][C24=-2][C27=4][C28=2]

========== END-MATRIX

Computing in the same way:

$$d_6 : [C_6(\Omega^3) = \mathbb{Z}^{84}] \to [C_5(\Omega^3) = \mathbb{Z}^{33}] :$$

+ Elementary matrix Smith reductions

$$\Rightarrow H_5(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})) \; = \; \mathbb{Z}_2^4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}.$$

How it is possible?

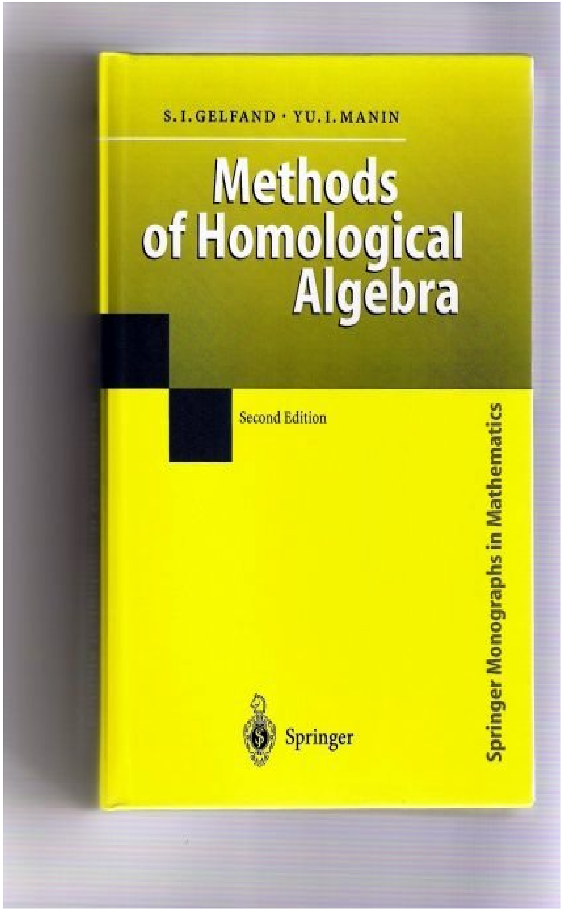
Solution = Effective Homology.

Typical example

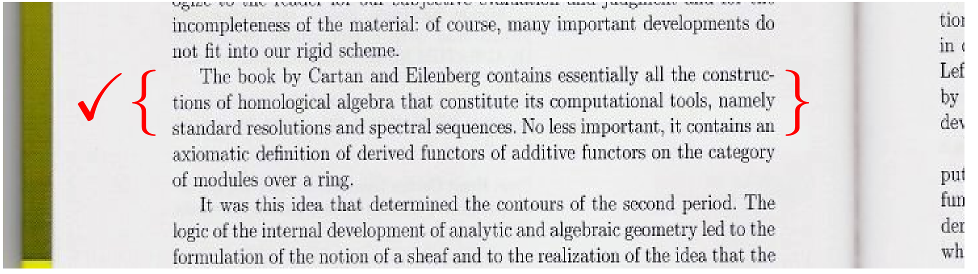
of erroneous statement

in a (very good)

classical book.



In the foreword:



incompleteness of the material: of course, many important developments do not fit into our rigid scheme.

The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences. No less important, it contains an axiomatic definition of derived functors of additive functors on the category of modules over a ring.

It was this idea that determined the contours of the second period. The logic of the internal development of analytic and algebraic geometry led to the formulation of the notion of a sheaf and to the realization of the idea that the

"The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences."

Essentially false !!

Typical problem not computationnally solved by exact sequences.

J.-P. Serre (1950) computing (?) sphere homotopy groups.

Serre's result: Exact sequence:

$$0 \longleftarrow \mathbb{Z}_6 \longleftarrow \pi_6(S^3) \longleftarrow \mathbb{Z}_2 \longleftarrow 0$$

$$\Rightarrow \quad \pi_6(S^3) = \mathbb{Z}_{12} \ \boxed{\text{or}} \ \mathbb{Z}_2 \oplus \mathbb{Z}_6 \ ???$$

"Solution": "compute" the cohomology class

$$\varepsilon \in H^2(\mathbb{Z}_6, \mathbb{Z}_2) = \mathbb{Z}_2 \text{ classifying the extension.}$$

Needs a representant of the generator of $\mathbb{Z}_6$

in an esoteric chain group $C_6(X_6)$

with X_6 the total space of a terrible fibration

+ a final terrible computation.

Solved one year later by Barrat and Paechter,

thanks to a very specific study:

A NOTE ON $\pi_r(V_{n,m})$

BY M. G. BARRATT AND G. F. PAECHTER

MAGDALEN COLLEGE, OXFORD

Communicated by S. Lefschetz, November 28, 1951

Introduction.—Let $k \geqslant 3$. We shall prove

THEOREM 1.1. *$\pi_{k+3}(S^k)$ has an element of order four.*

Let $V_{k+m,\,m}$ be the Stiefel Manifold of all orthogonal m-frames in real Euclidean $(k+m)$-space.

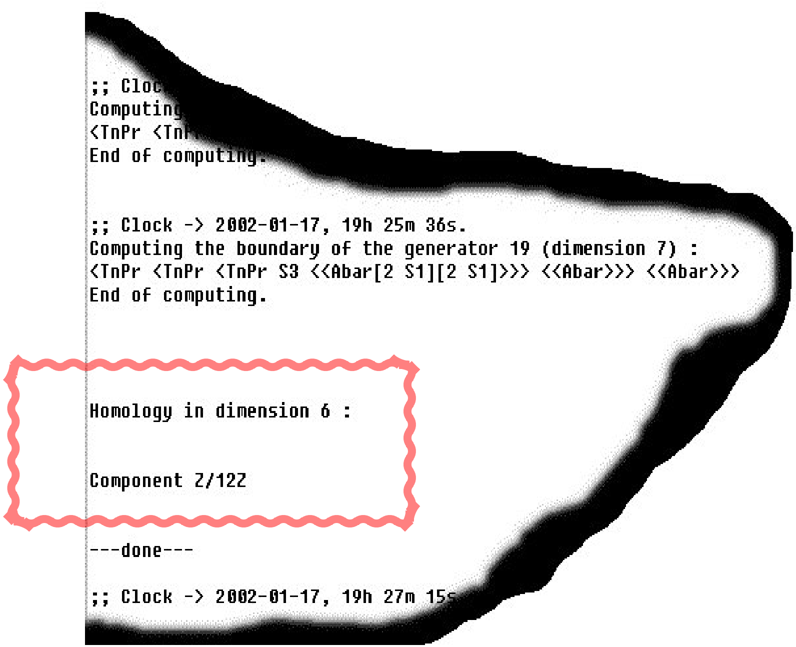
THEOREM 1.2. *The groups $\pi_{k+2}(V_{k+m,\,m})$ are given by the following table, in which Z_p, Z_∞, are cyclic groups of order p, ∞, respectively.*

$\pi^2_{k,m}$	$m = 1$	$m = 2$	$m = 3$	$m \geq 4$
$k = 1$	0	Z_∞	$Z_\infty + Z_\infty$	Z_∞
$k = 4s - 2$	Z_2	$Z_2 + Z_2$	Z_2	0
$k = 4s$	Z_2	$Z_2 + Z_2$	$Z_2 + Z_2$	$Z_2 + Z_2$
$k = 4s - 1$	Z_2	Z_4	$Z_2 + Z_\infty$	Z_2
$k = 4s + 1$	Z_2	Z_4	$Z_4 + Z_\infty$	Z_8

Let Y^{n+1} be the $(n - 1)$-fold suspension of the real projective plane, so that Y^{n+1} consists of an n-sphere S^n and an $(n + 1)$-cell e^{n+1} attached to S^n by a map of degree 2. We prove

THEOREM 1.3 $\pi_{n+2}(Y^{n+1}) = Z_4$ *if* $n \geqslant 3$.

Now "stupidly" obtained by the Kenzo program:


;; Cloc
Computing
<TnPr <TnP
End of computing.

;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.

Homology in dimension 6 :

Component Z/12Z

---done---

;; Clock -> 2002-01-17, 19h 27m 15s

Analysis of the problem:

"Standard" homological algebra is not constructive.

Typical statement:

The sequence $A \overset{\alpha}{\longleftarrow} B \overset{\beta}{\longleftarrow} C$ **is exact.**

Common translation:

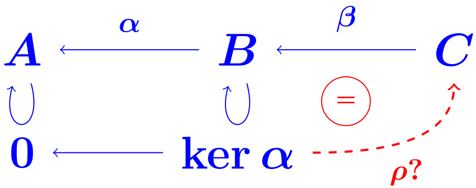
$$(\forall b \in B) \;\; [(\alpha(b) = 0) \Rightarrow (\boxed{\exists c \in C} \; \underline{\text{st}} \; b = \beta(c))]$$

with $\boxed{\exists c \in C}$ **most often non-constructive.**

Constructive exactness:

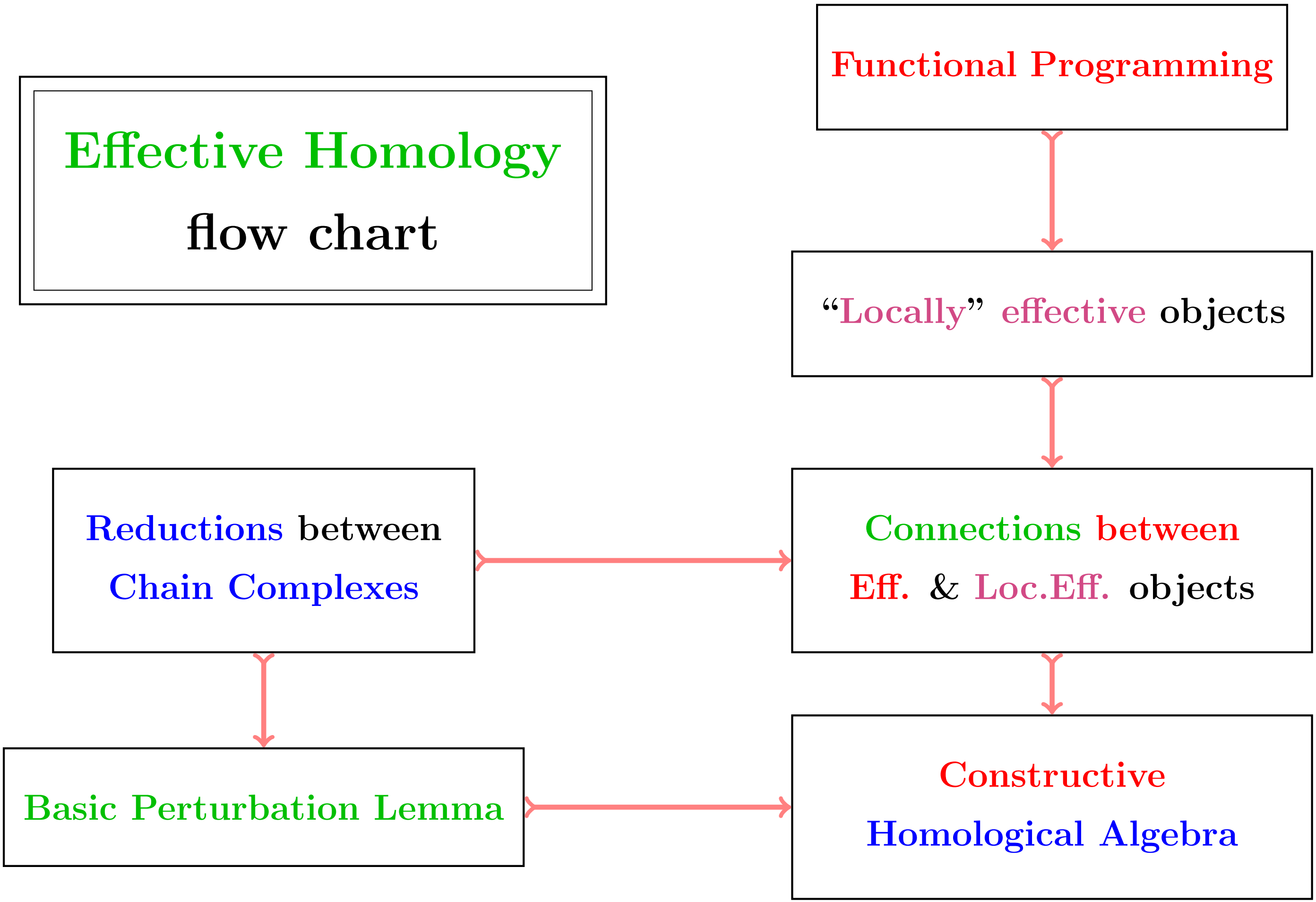
$$A \xleftarrow{\alpha} B \xleftarrow{\beta} C \text{ constructively exact}$$

if an algorithm $\rho : \ker \alpha \to C$ is given satisfying:



⇒ Organizational algebraic problems:

where ρ cannot be a group homomorphism.


Effective Homology
flow chart
Functional Programming
"Locally" effective objects
Reductions between
Chain Complexes
Connections between
Eff. & Loc.Eff. objects
Basic Perturbation Lemma
Constructive
Homological Algebra

Functional Programming:

The art of handling functional objects.

Examples of functional objects:

$$(\mathbb{Z}, +, -, \times) \quad (\mathbb{Z}[X], +, -, \times)$$

Other example:

Kan model for the loop space $\Omega S^3 := \mathcal{C}(S^1, S^3)$:

$$(\mathcal{S}_{\Omega S^3}, \{\partial_i^n\}_{n \geq 1, 0 \leq i \leq n}, \{\eta_i^n\}_{n \geq 0, 0 \leq i \leq n})$$

with $\mathcal{S}_{\Omega S^3}$ = the simplex set of the Kan model.

= "Locally" effective objects.

Main problem:

Designing programs $(f_1, \ldots, f_n) \mapsto f$.

Example:

$$(\mathfrak{R}, +_{\mathfrak{R}}, -_{\mathfrak{R}}, \times_{\mathfrak{R}}) \mapsto (\mathfrak{R}[X], +_{\mathfrak{R}[X]}, -_{\mathfrak{R}[X]}, \times_{\mathfrak{R}[X]})$$

Topological example. X = **topological space.**

$$(\mathcal{S}_X, \{\partial(X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(X)_i^n\}_{n\geq 0, 0\leq i\leq n})$$
$$\mapsto (\mathcal{S}_{\Omega X}, \{\partial(\Omega X)_i^n\}_{n\geq 1, 0\leq i\leq n}, \{\eta(\Omega X)_i^n\}_{n\geq 0, 0\leq i\leq n})$$

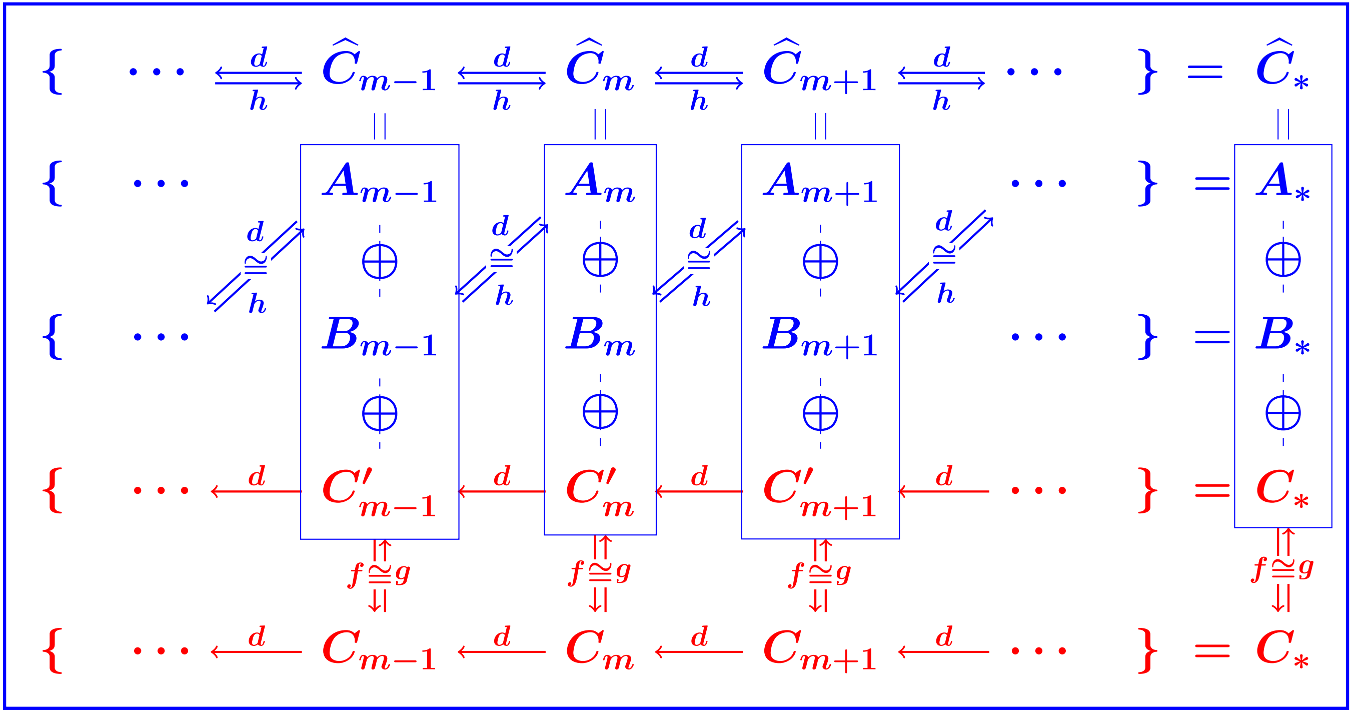
Solution = λ-calculus, Lisp, ML, Axiom, Haskell...

Definition: **A (homological) reduction is a diagram:**

$$\rho: \boxed{h \circlearrowleft \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ **and** C_* **= chain complexes.**
2. f **and** g **= chain complex morphisms.**
3. h **= homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$**.**
5. $fh = 0$, $hg = 0$ **and** $hh = 0$**.**



$$\boxed{A_* = \ker f \cap \ker h} \quad \boxed{B_* = \ker f \cap \ker d} \quad \boxed{C'_* = \operatorname{im} g}$$

$$\widehat{C}_* = \boxed{\boxed{A_* \oplus B_* \text{ exact}} \oplus \boxed{C'_* \cong C_*}}$$

Let ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$ **be a reduction.**

Frequently:

1. $\widehat{C}_*$ **is a locally effective chain complex:**

   its homology groups are unreachable.

2. C_* **is an effective chain complex:**

   its homology groups are computable.

3. **The reduction** ρ **is an entire description of**

   the homological nature of $\widehat{C}_*$**.**

4. **Any homological problem in** $\widehat{C}_*$ **is solvable**

   thanks to the information provided by ρ**.**

$$\rho: \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

1. What is $H_n(\widehat{C}_*)$? Solution: Compute $H_n(C_*)$.

2. Let $x \in \widehat{C}_n$. Is x a cycle? Solution: Compute $d_{\widehat{C}_*}(x)$.

3. Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?

Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. Let $x, x' \in \widehat{C}_n$ be homologous cycles.

Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

Solution:

(a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

(b) $y = g(z) + h(x - x')$.

Definition: (C_*, d) = given chain complex.

A perturbation $\delta : C_* \rightarrow C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

Problem: Let ρ: $\boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

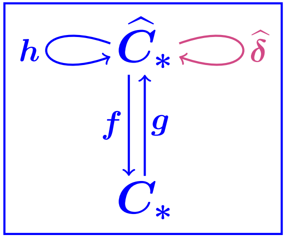
$$???: \boxed{h{+}? \circlearrowleft (\widehat{C}_*, \widehat{d}{+}\widehat{\delta}) \underset{f+?}{\overset{g+?}{\leftrightarrows}} (C_*, d{+}?)}$$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma" (BPL):

Given:

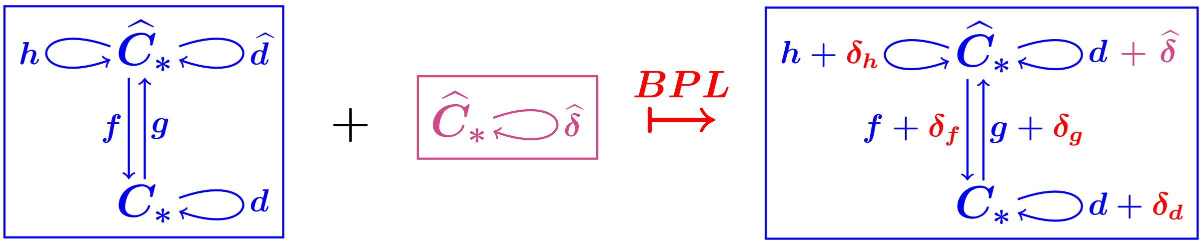


satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$ of $\widehat{C}_*$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:



Serre: "Everything" in Algebraic Topology can be reduced to Fibration problems.

Examples: Loop spaces, Classifying spaces, Homogeneous spaces, Whitehead tower, Postnikov tower, . . .

Remark: Fibration = Twisted Product = Perturbation of Trivial Product.

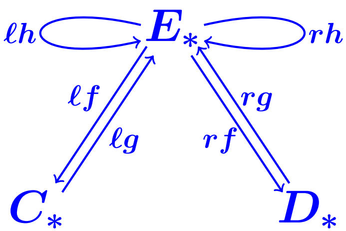
Corollary: BPL is effective

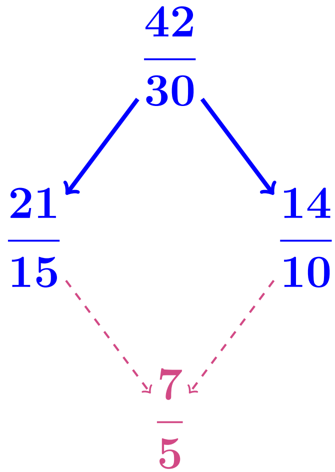
+ Fibration = Perturbation of Trivial Product

+ Everything is Fibration

$\Rightarrow$ Alg. Topology becomes Constructive.

<u>Definition</u>: A (strong chain-) equivalence $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$ is a pair of reductions $C_* \overset{\ell\rho}{\Lleftarrow} E_* \overset{r\rho}{\Rrightarrow} D_*$:





Normal form problem ??

More structure often necessary in C_*.

Most often: no possible choice for C_*.

<u>Definition</u>: An object with effective homology X is a 4-tuple:

$$X = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = "<u>the</u>" chain complex "traditionally" associated with X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \overset{\varepsilon}{\Longleftrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_1, \ldots, X_n$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_1, \ldots, X_n) \mapsto X.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}\colon \left(\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}\right) \mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is **perfectly stable**

and can be again used with X for further calculations.

Example:

Julio Rubio's solution of Adams' problem.

$$\boxed{\boldsymbol{X} = (X,\ C_*(X),\ EC_*^X,\ \varepsilon^X)}$$

$$\Downarrow\Downarrow\Downarrow \qquad \textbf{Eil.-Moore}_{EH}$$

$$\boxed{\Omega\boldsymbol{X} = (\Omega X,\ C_*(\Omega X),\ EC_*^{\Omega X},\ \varepsilon^{\Omega X})}$$

$\Longrightarrow$ **Trivial iteration now available.**

$\Rightarrow$ **Very simple solution of Adam's problem :**

Indefinite iteration of the Cobar construction ???

$$\boldsymbol{X} \;=\; (X, C_*(X), EC_*^X, \varepsilon^X)$$

$$\Downarrow \;\Omega_{EH}$$

$$\Omega X \;=\; (\Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X})$$

$$\Downarrow \;\Omega_{EH}$$

$$\Omega^2 X \;=\; (\Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X})$$

$$\Downarrow \;\Omega_{EH}$$

$$\Omega^3 X \;=\; (\Omega^3 X, C_*(\Omega^3 X), EC_*^{\Omega^3 X}, \varepsilon^{\Omega^3 X})$$

$$\Downarrow \;\Omega_{EH}$$

$$\Omega^4 X \;=\; \ldots$$

"Cobar" $^{3}(EC_*^X)$

Example: **Effective homology version of the Serre spectral sequence.**

$$\boldsymbol{F} = (F,\ C_*(F),\ \boldsymbol{EC}_*^{\boldsymbol{F}},\ \varepsilon^F)$$

$$+\ \boldsymbol{B} = (B,\ C_*(B),\ \boldsymbol{EC}_*^{\boldsymbol{B}},\ \varepsilon^B)$$

$$+\ \boldsymbol{\tau} : B \rightarrow \boldsymbol{F}$$

$$\Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \text{Serre}_{EH}$$

$$\boldsymbol{E} = \boldsymbol{F} \times_{\boldsymbol{\tau}} \boldsymbol{B} = (E,\ C_*(E),\ \boldsymbol{EC}^{\boldsymbol{E}},\ \varepsilon^E)$$

(Serre + G. Hirsch + H. Cartan + Shih W. + Szczarba + Ronnie Brown + J. Rubio + FS)

Proof.

$$C_*(F \times B) \stackrel{\text{id}}{\Lleftarrow} C_*(F \times B) \stackrel{EZ}{\Rrightarrow} C_*F \otimes C_*B$$

$$C_*F \otimes C_*B \stackrel{\otimes}{\Lleftarrow} \widehat{C}^F \otimes \widehat{C}^B \stackrel{\otimes}{\Rrightarrow} EC^F \otimes EC^B$$

$\Downarrow\Downarrow\Downarrow\Downarrow\Downarrow$ Serre_{EH}

$$C_*(F \times_{\boxed{\tau}} B) \stackrel{\text{id}}{\Lleftarrow} C_*(F \times_{\boxed{\tau}} B) \stackrel{\text{Shih}}{\Rrightarrow} C_*F \otimes_{\boxed{t}} C_*B$$

$$C_*F \otimes_{\boxed{t}} C_*B \stackrel{EPL}{\Lleftarrow} \widehat{C}^F \otimes_{\boxed{t'}} \widehat{C}^B \stackrel{BPL}{\Rrightarrow} EC^F \otimes_{\boxed{t''}} EC^B$$

+ **Composition of equivalences** $\Longrightarrow$ **O.K.**

Combining these ingredients $\Rightarrow$

Homological Algebra becomes constructive.

Corollary: The "standard" exact and spectral sequences of Homological Algebra really become computational tools.

$\Rightarrow$ Concrete computer programs (EAT, Kenzo).

Warning about the right chronology. Example, let:

$$\phi: \; F \hookrightarrow E \longrightarrow B$$

be a fibration with B simply connected.

1. **The ordinary Serre Spectral Sequence is not constructive.**

2. **Methods of Effective Homology give an algorithm:**

$$[EH_*(F) + EH_*(B) + \phi] \mapsto EH_*(E).$$

3. **Methods of Effective Homology can then compute:**

$$[EH_*(F) + EH_*(B) + \phi + EH_*(E) \mapsto \mathrm{SSS}(\phi)].$$

That is, the SSS is a byproduct of Effective Homology.

(Ana Romero)

Are your computer programs efficient?

What about benchmarks?

Do you compute new sphere homotopy groups?

Non-relevant question!

To be compared with prime number chasing.

Two different activities:

1. **Searching for very big prime numbers.**

2. **Designing methods applicable to arbitrary numbers.**

Most efficient current methods for big prime numbers:

Specific tests (Lucas-Lehmer):

$\Rightarrow (2^{32,582,657} - 1)$ **prime (2006).**

Most "efficient" general method:

Agrawal-Kayal-Saxena n^{12}-algorithm.

Both methods have totally different scopes.

Analogous situation in Algebraic Topology.

In case of spheres ($\sim$ Mersenne numbers),

specific methods go very far.

But these methods are inapplicable in general situations.

Typical situation:

Modifying a loop space by "pre-attaching" a cell.

What influence about

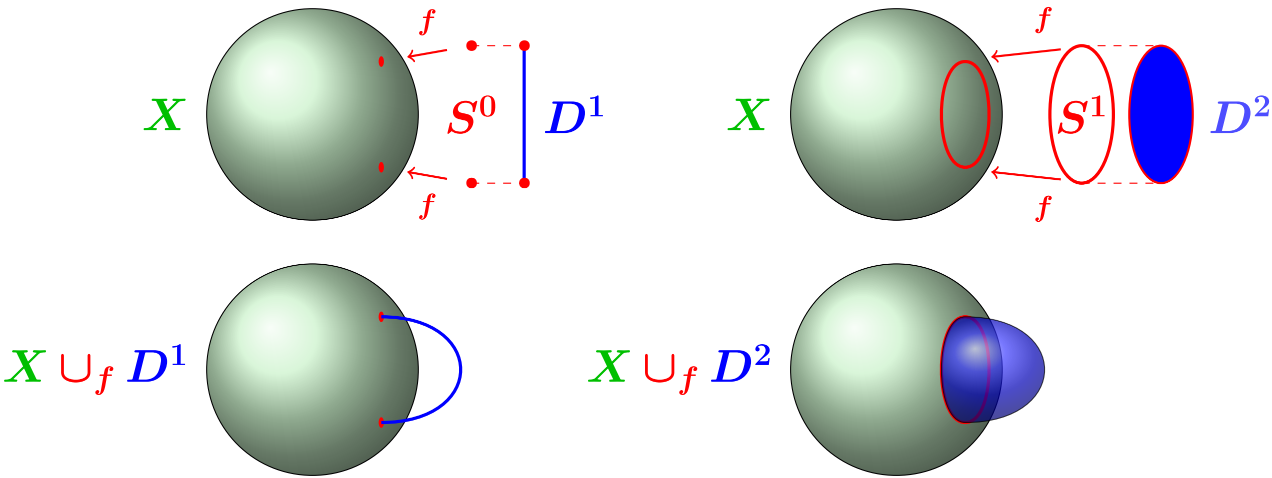
the homology groups of the new loop space?

Attaching a cell D^n to a topological space along the boundary S^{n-1}:

X = Topological space.

$f : S^{n-1} \to X$ = continuous map.

$\Rightarrow X \cup_f D^n := (X \coprod D^n)/(X \ni f(x) \sim x \in S^{n-1})$.



Given:

- X = Topological space.
- $H_*(X)$ = Homology of X.
- $H_*(\Omega X)$ = Homology of the loop space ΩX.
- $f : S^{n-1} \longrightarrow X$ continue.

Problem:

- Determine $H_*(\Omega(X \cup_f D^n)) = ???$

Example:

$H_*(\Omega S^3)$ computed by J.-P. Serre (1950).

$H_*(\Omega^2 S^3 := \Omega(\Omega S^3))$ computed by W. Browder (1958).

Modifying $\Omega S^3 \mapsto \Omega S^3 \cup_2 D^3$.

$\Rightarrow$ Problem:

$$H_*(\Omega(\Omega S^3 \cup_2 D^3)) = ???$$

Remark: $H_*(\Omega S^3 \cup_2 D^3)$ direct consequence of Serre's result.

"Standard" Algebraic Topology $\Rightarrow$???

In Effective Homology:

$\boldsymbol{S^3}$ **= Finite simplicial set** $\Rightarrow$ $\boldsymbol{S^3}$ **= OEH** [1)].

EMSSEH [2)] $\Rightarrow$ $\boldsymbol{\Omega S^3}$ **= OEH** [1)].

MVESEH [3)] $\Rightarrow$ $\boldsymbol{\Omega S^3 \cup_2 D^3}$ **= OEH** [1)].

EMSSEH [2)] $\Rightarrow$ $\boldsymbol{\Omega(\Omega S^3 \cup_2 D^3)}$ **= OEH** [1)].

$\boldsymbol{\Omega(\Omega S^3 \cup_2 D^3)}$ **= OEH** [1)] $\Rightarrow$ $\boldsymbol{H_*(\Omega(\Omega S^3 \cup_2 D^3))}$ **computable**.

$\Rightarrow$ **OK.**

1) OEH = Object with Effective Homology.
2) EMSSEH = Eilenberg-Moore Spectral Sequence with Effective Homology.
3) MVESEH = Mayer-Vietoris Exact Sequence with Effective Homology

A more complicated analogous computation:

$X = \Omega(\Omega(\Omega(P^\infty\mathbb{R}/P^3\mathbb{R}) \cup_4 D^4) \cup_2 D^3))$ $H_*X = ???$

$H_0(X) = \mathbb{Z}.$

$H_1(X) = \mathbb{Z}/2\mathbb{Z}.$

$H_2(X) = (\mathbb{Z}/2\mathbb{Z})^2 + \mathbb{Z}.$

$H_3(X) = (\mathbb{Z}/2\mathbb{Z})^4 + \mathbb{Z}/8\mathbb{Z}.$

$H_4(X) = (\mathbb{Z}/2\mathbb{Z})^{10} + \mathbb{Z}/4\mathbb{Z} + \mathbb{Z}^2.$

$H_5(X) = (\mathbb{Z}/2\mathbb{Z})^{23} + \mathbb{Z}/8\mathbb{Z} + \mathbb{Z}/16\mathbb{Z}.$

$H_6(X) = (\mathbb{Z}/2\mathbb{Z})^{52} + (\mathbb{Z}/4\mathbb{Z})^3 + \mathbb{Z}^3.$

$H_7(X) = (\mathbb{Z}/2\mathbb{Z})^{113} + \mathbb{Z}/4\mathbb{Z} + (\mathbb{Z}/8\mathbb{Z})^3 + \mathbb{Z}/16\mathbb{Z} + \mathbb{Z}/32\mathbb{Z} + \mathbb{Z}.$

The longest Kenzo computation (2 months).

Two main steps in a Kenzo calculation $H(d) =$???

1) "Automatic" writing of a sophisticated highly functional program P.

2) Using program P to compute $P(d) = H(d)$.

1) $\Rightarrow$ Always very fast ($<$ 1 sec.).

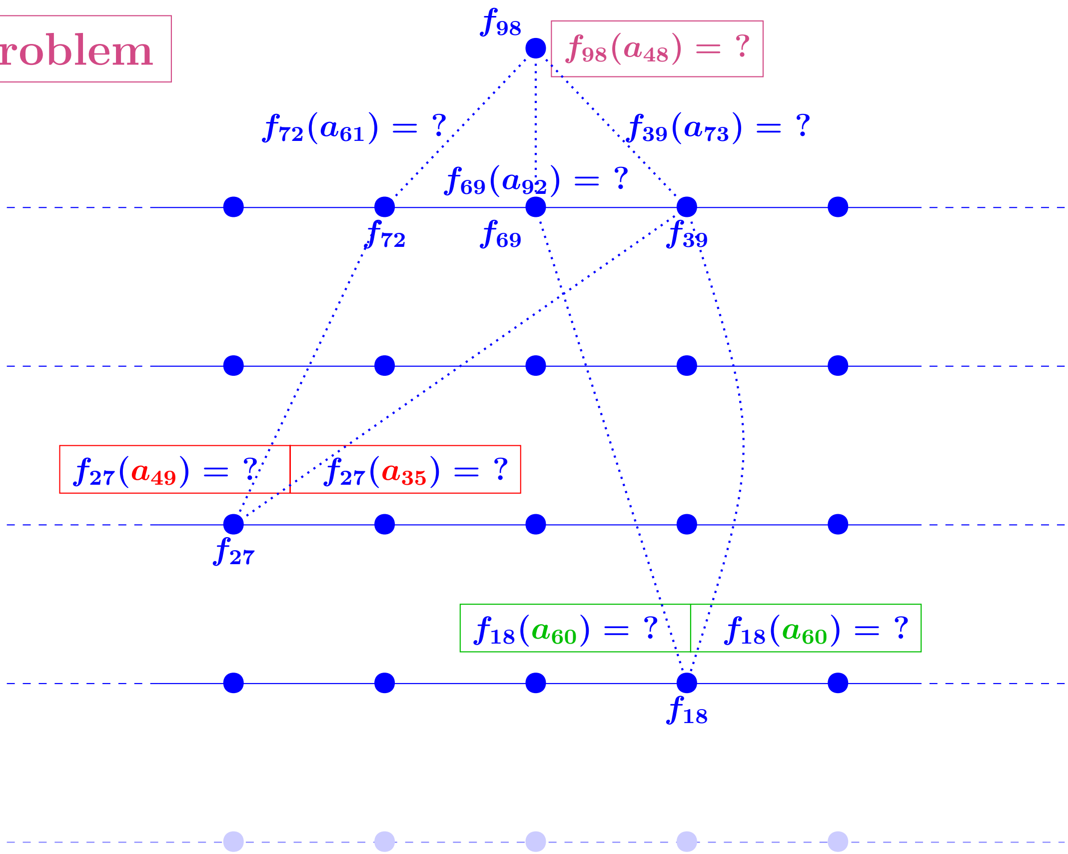
2) $\Rightarrow$ Can be very long (hours, days, months, years, centuries, ...)

High efficiency in functional programming with Common-Lisp

$\Rightarrow$ No technical difficulty in 1).

Terrible problem of memory management in 2).

Functional Problem



Only heuristic methods available:

EAT-1: No result stored ⇒ poor time computing efficiency.

EAT-2: "Non-trivial" results stored

⇒ Computing time divided by ∼10.

Kenzo-1: Strong improvement in storing-searching methods.

⇒ Computing time divided by ∼10.

Kenzo-2: For overcoming space complexity:

Periodic cleaning of stored results.

⇒ Computing time divided by ∼10.

Theoretical framework for a rational study ??? Open !!!

"Vertical" vs "Horizontal" time complexity.

$$\boxed{\text{Computing } \boldsymbol{H_n(X)}, \boldsymbol{\pi_n(X)}\ldots}$$

Two parameters: n and X.

"Horizontal" complexity := wrt X.

"Vertical" complexity := wrt n.

Effective homology $\Rightarrow$ Horizontal complexity $= P$.

David Annick (1986) $\Rightarrow$

Vertical complexity $\geq NP$-complete.

Back to "standard" Mathematics.

Traditional main problem of Algebraic Topology:

Classifying the homotopy types.

1. **Only "reasonable" spaces: CW-complexes $\cong$ Simp. sets.**

2. **Non-simply connected topology excluded (word problem).**

3. **Classification "up to homeomorphism" out of scope $\Rightarrow$**

   Only classification "up to homotopy equivalence".

4. **"Standard" solution = Postnikov "invariants".**

Main problem of Algebraic Topology:

Algebraic Models for Topological Spaces ?

Main idea: Topology is difficult, Algebra is easy (!?).

Subquestion: what does the word "Algebra" means?

Answer: No meaning at all, only a "cultural" tradition.

Correct question:

Computable Models for Topological Spaces ?

Three solutions:

1. **Rolf Schön.**
2. **Effective Homology.**
3. **Operads.**

Schön's solution =

Intensive use of inductive limits

to approximate infinite objects.

Only one computer application:

Alain Clément, Lausanne, Haskell program.

Comparison: Effective Homology $\leftrightarrow$ Operads ???

Object with effective homology = Triple: (X, HX, ε) with:

X = locally effective version of the object.

HX = Effective chain complex

describing the ordinary homology of X.

ε = Strong connection $X \stackrel{\varepsilon}{\longleftrightarrow} HX$.

<u>Theorem</u>: The triple (X, HX, ε) is a computable model

of the homotopy type of X.

Operadic model for a topological space X:

$$(HX, \mathcal{M})$$

with:

HX = Effective chain complex describing the ordinary homology of X.

$\mathcal{M}$ = E_∞-operadic structure over HX.

Theorem (Mandell): The pair $(HX, \mathcal{M})$ is an "algebraic" model of the homotopy type of X.

Connection between (X, HX, ε) and $(HX, \mathcal{M})$?

<u>Theorem</u>: There exists a canonical correspondance:

$$(X, HX, \varepsilon) \longleftrightarrow (HX, \mathcal{M})$$

1. **"$\longrightarrow$" = Berger-Fresse.**

2. **"$\longleftarrow$" = S.-Mandell.**

Far from concrete implementations ! !

Main open problems in Effective Homology:

1. Eilenberg-Zilber = “$\square \longleftrightarrow \boxslash$”.

<u>Problem</u>: General formula

of unavoidable exponential complexity.

How to design an efficient algorithm

for concrete particular cases?

2. Twisted Eilenberg-Zilber.

New important results experimentally discovered in 98.

Not yet proved!

3. Spectral sequences.

Filtrated chain complexes vs Exact Couples.

Particular cases of Bousfield-Kan, Adams, May, Adams-Novikov... spectral sequences.

Cf recent thesis by Ana Romero.

4. Commutative Algebra.

Recent result:

Canonical correspondance between:

Effective resolution of $K[x_1, \ldots, x_m]$-modules

$$\updownarrow \cong$$

Effective homology of Koszul complex.

$\Rightarrow$ New algorithms producing effective resolutions.

5. Concrete implementation of the canonical correspondance:

$$(X, HX, \varepsilon) \longleftrightarrow (HX, \mathcal{M})$$

6. Efficient memory management

for high level functional programming ???

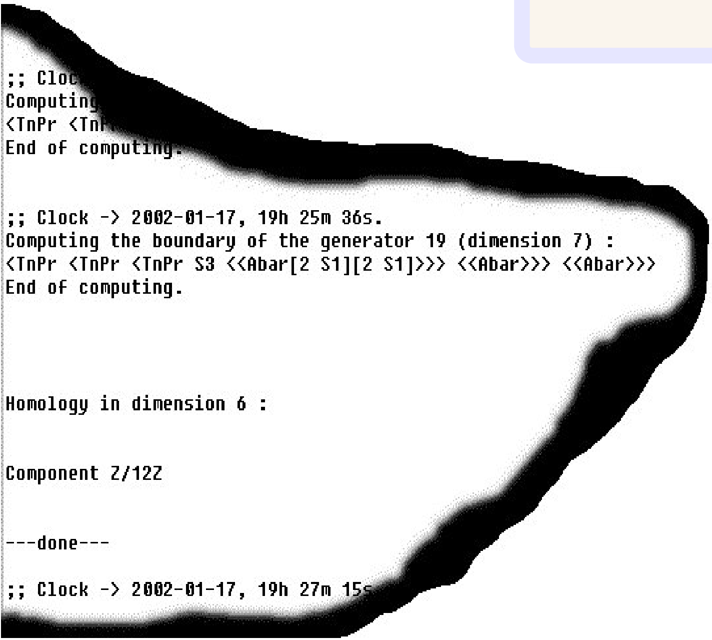
7. Program proof, theorem proving.

Recent result (Jesus Aransay):

Isabelle-certified proof of the Basic-Perturbation-Lemma.

Competing work by Coq people.

The END



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