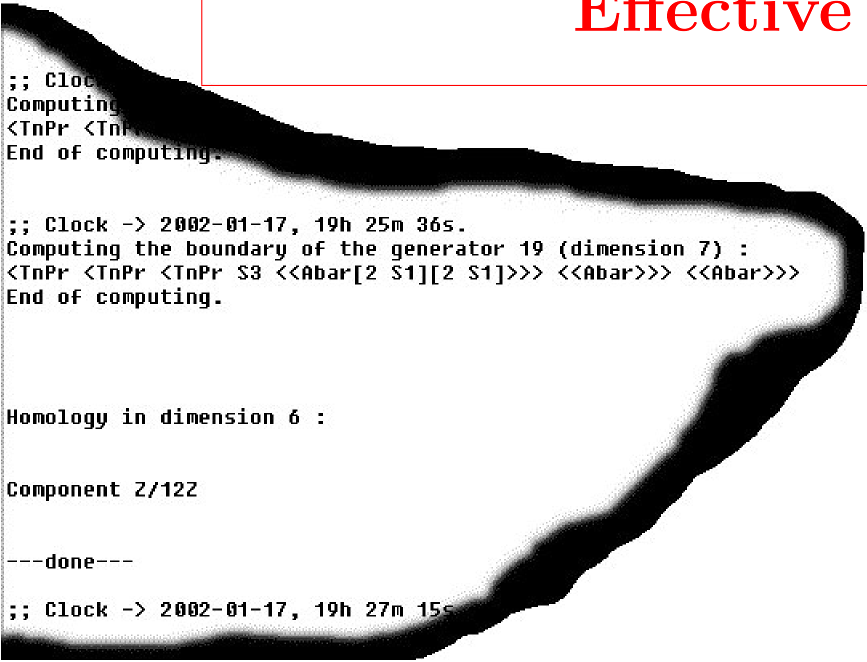
Koszul Homology and Resolutions

revisited through

Effective Homology

Talk in honour of Jacques Calmet



Francis Sergeraert, Institut Fourier, Grenoble, France
Logroño, February 28, 2008

Plan

1. Simplest example for an application of spectral sequence.

2. Rewriting it as a BPL application.

   $\Rightarrow$ effective version of the same result.

3. Doing the same for the Aramova-Herzog bicomplex.

4. Computing effective resolutions in Commutative Algebra.

Easy and convenient application of spectral sequences:

the Bicomplex Spectral Sequence.

Particular case:

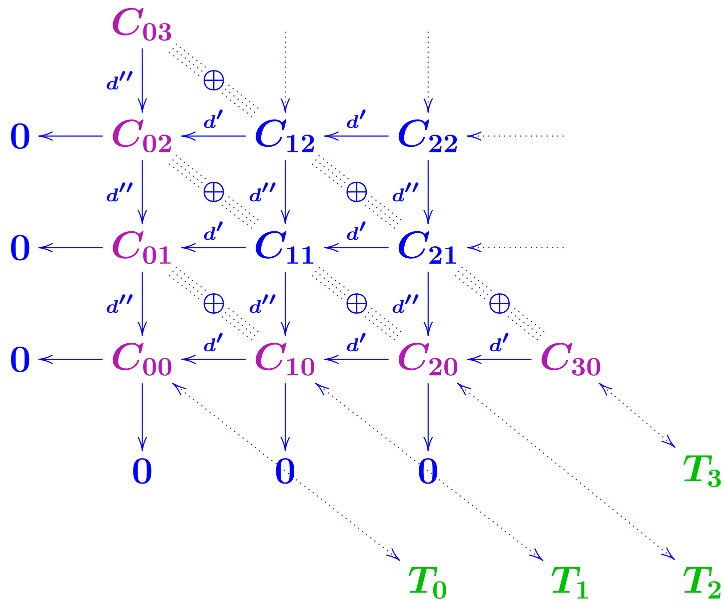
$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & C_{02} & \xleftarrow{d'} & C_{12} & \xleftarrow{d'} & C_{22} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \\
0 & \longleftarrow & C_{01} & \xleftarrow{d'} & C_{11} & \xleftarrow{d'} & C_{21} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} & & \\
0 & \longleftarrow & C_{00} & \xleftarrow{d'} & C_{10} & \xleftarrow{d'} & C_{20} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with every row and column exact

except maybe at C_{*0} and C_{0*}.

Totalization of a bicomplex.

$T_i := C_{i0} \oplus C_{i-1,1} \oplus \cdots \oplus C_{0i}$



Double-Complex Property:

$$d'd'' + d''d' = 0$$

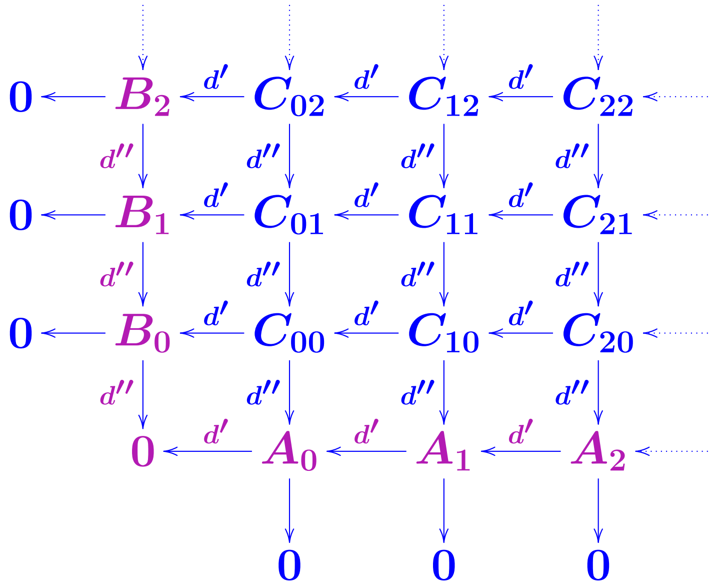
$\Rightarrow d := d' \boxed{\oplus} d'' =$ differential

$\Rightarrow (T_i, d) =$ Chain complex.

<u>Definition</u> :

$$H_i(\{C_{jk}, d'_{jk}, d''_{jk}\}) = H_i^T := H_i\{0 \leftarrow T_0 \xleftarrow{d} T_1 \xleftarrow{d} T_2 \xleftarrow{d} \cdots\}$$

Other construction:

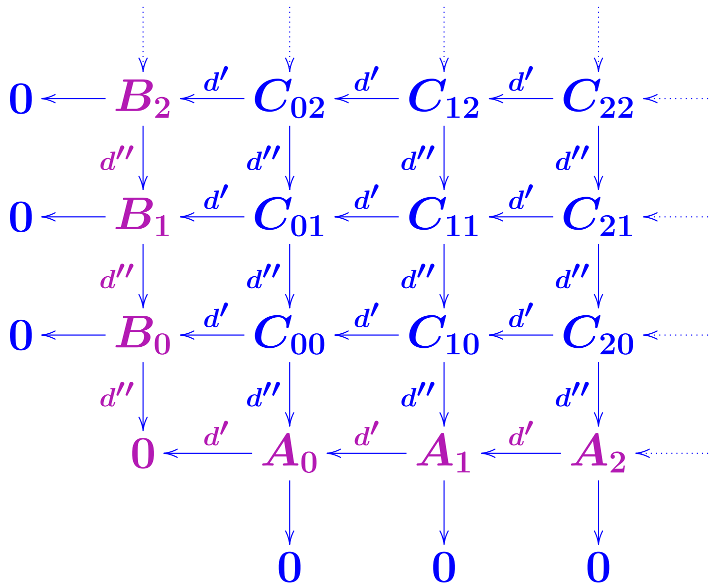


with: $B_i := C_{0i}/d'(C_{1i})$

$A_i := C_{i0}/d''(C_{i1})$

But new problem!

⇒



The violet lines are chain complexes in general non-exact.

⇒ Homology groups:

$$H_i^A := H_i\{0 \leftarrow A_0 \overset{d'}{\leftarrow} A_1 \overset{d'}{\leftarrow} A_2 \overset{d'}{\leftarrow} \cdots\}$$
$$H_i^B := H_i\{0 \leftarrow B_0 \overset{d''}{\leftarrow} B_1 \overset{d''}{\leftarrow} B_2 \overset{d''}{\leftarrow} \cdots\}$$

Theorem :

In the special case of a first quadrant bicomplex

with rows and columns exact

except maybe at C_{*0} and C_{0*},

then there are canonical isomorphisms:

$$H_i^B \cong H_i^T \cong H_i^A$$

Usual proof = Filtering the totalization

$\Rightarrow$ **Spectral sequence.**

Vertical filtration:

$F_i(T) :=$

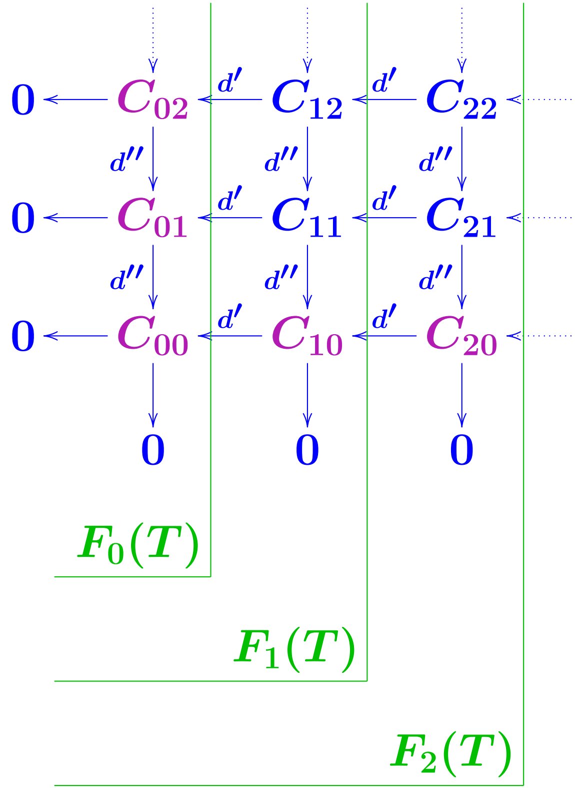
$\{C_{jk}\}$ **vanished for** $j > i$.

$\Rightarrow$

$F_i(T)/F_{i-1}(T) = C_{i*}$

$=$ **Column** i

Acyclic except in C_{i0}.



$F_i(T)/F_{i-1}(T)$ acyclic except along the $*0$ axis

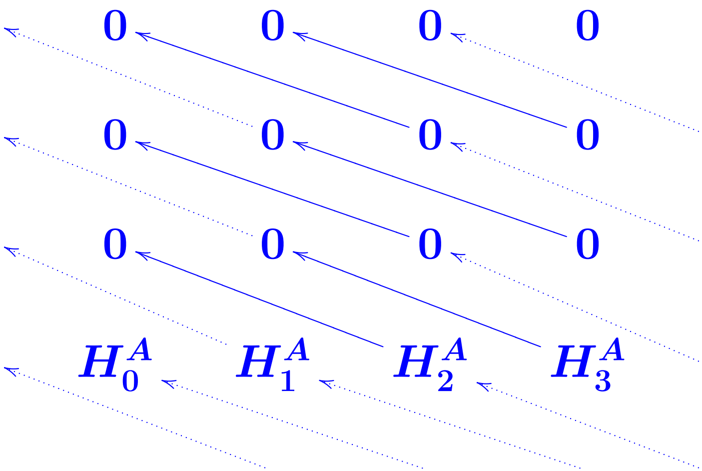
$\Rightarrow$ the spectral sequence "degenerates".

E^1-page of the spectral sequence:

$$\begin{array}{ccccccccc}
\leftarrow\cdots & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \leftarrow\cdots \\
\leftarrow\cdots & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \leftarrow\cdots \\
\leftarrow\cdots & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \leftarrow\cdots \\
\leftarrow\cdots & A_0 & \xleftarrow{d'} & A_1 & \xleftarrow{d'} & A_2 & \xleftarrow{d'} & A_3 & \leftarrow\cdots
\end{array}$$

$\Rightarrow$

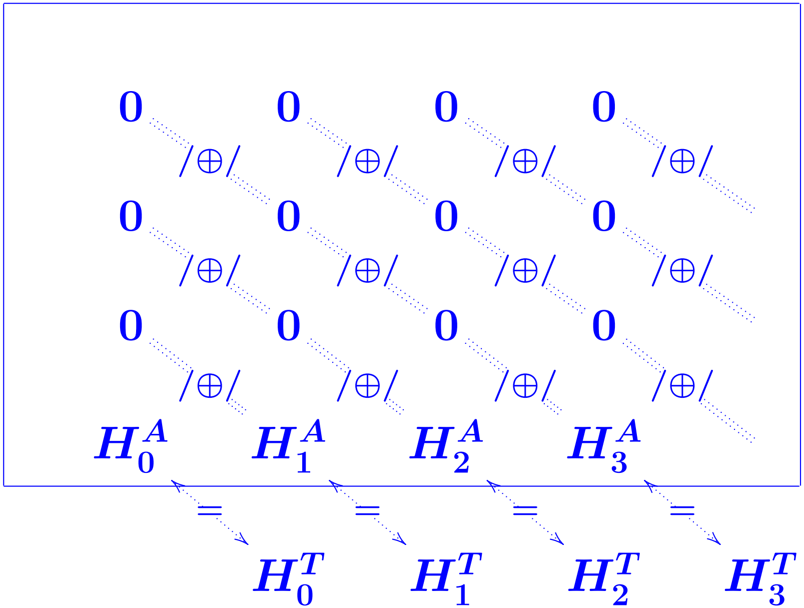
E^2-page of the spectral sequence:



$\Rightarrow r \geq 2 \Rightarrow$ $E^r_{pq} = 0$ **for** $q > 0$

and canonical isomorphism $E^r_{p0} \cong H^A_p$.

⇒ E^{∞}-page of the spectral sequence:



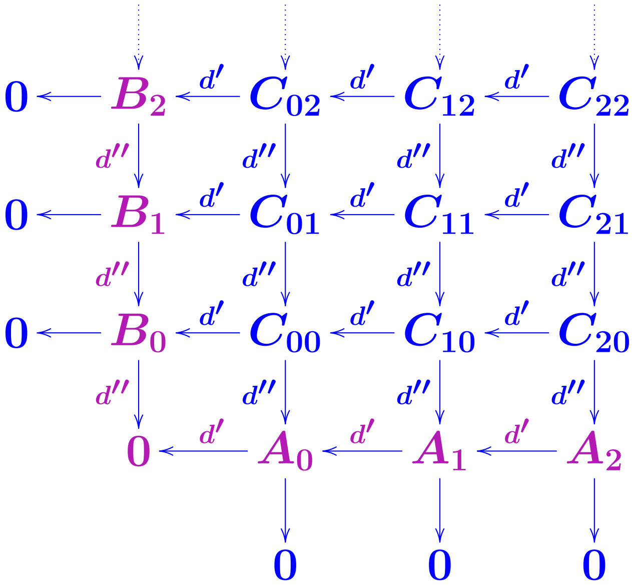
⇒ $$H_i^A \cong H_i^T \cong H_i^B$$

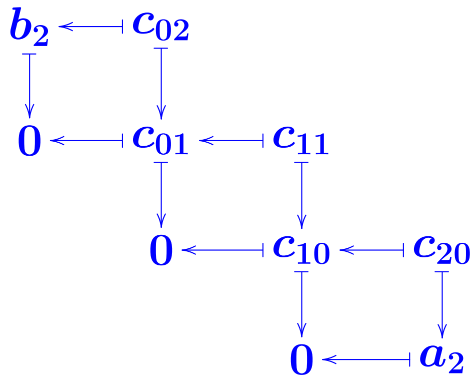
QED

Spectral sequence proof easy but not effective!

What about an effective proof?

Diagram chasing proof: $H_2^A \ni h_2^A \overset{???}{\longmapsto} h_2^B \in H_2^B$





Proof details: terribly awful!!

More convenient:

Use the homological perturbation "lemma".

1. An exact sequence:

$$0 \leftarrow A_i \leftarrow C_{i0} \leftarrow C_{i1} \leftarrow \cdots$$

can be viewed as a reduction:

$$\rho_i : \boxed{h_i \circlearrowright C_{i*} \underset{f_i}{\overset{g_i}{\rightleftarrows}} A_i}$$

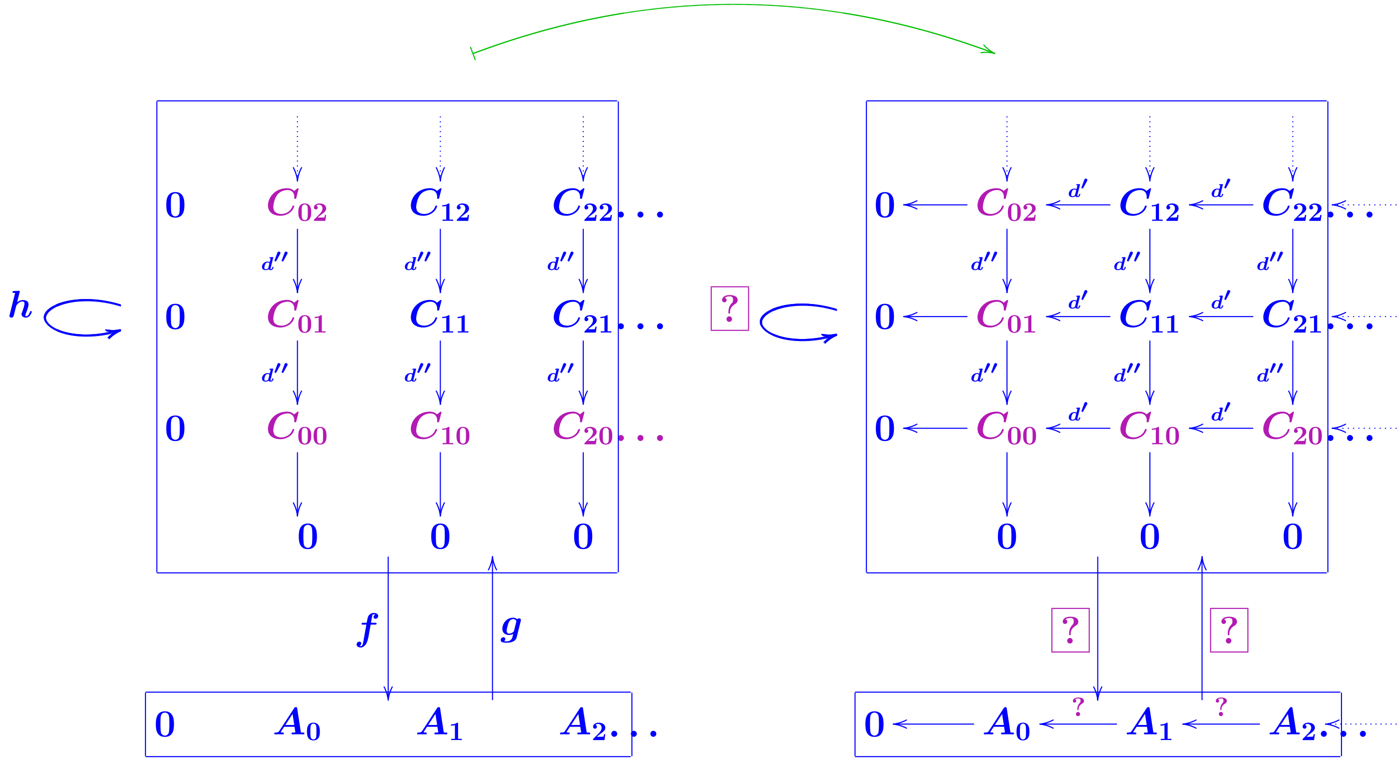
2. Direct sum with respect to i:

$$\rho : \boxed{h \circlearrowright \bigoplus_i C_{i*} \underset{f}{\overset{g}{\rightleftarrows}} \bigoplus_i A_i}$$

$\Rightarrow \ldots / \ldots$

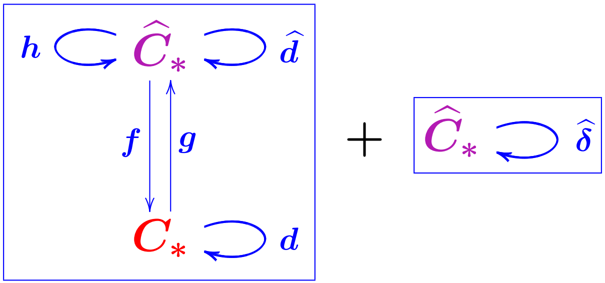
$\Rightarrow \ldots / \ldots$

Homological Perturbation



Basic Perturbation "Lemma" (BPL):

Given:

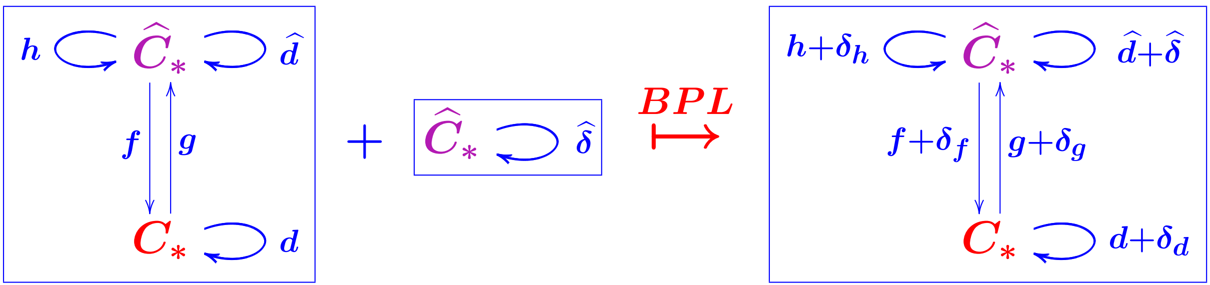


satisfying:

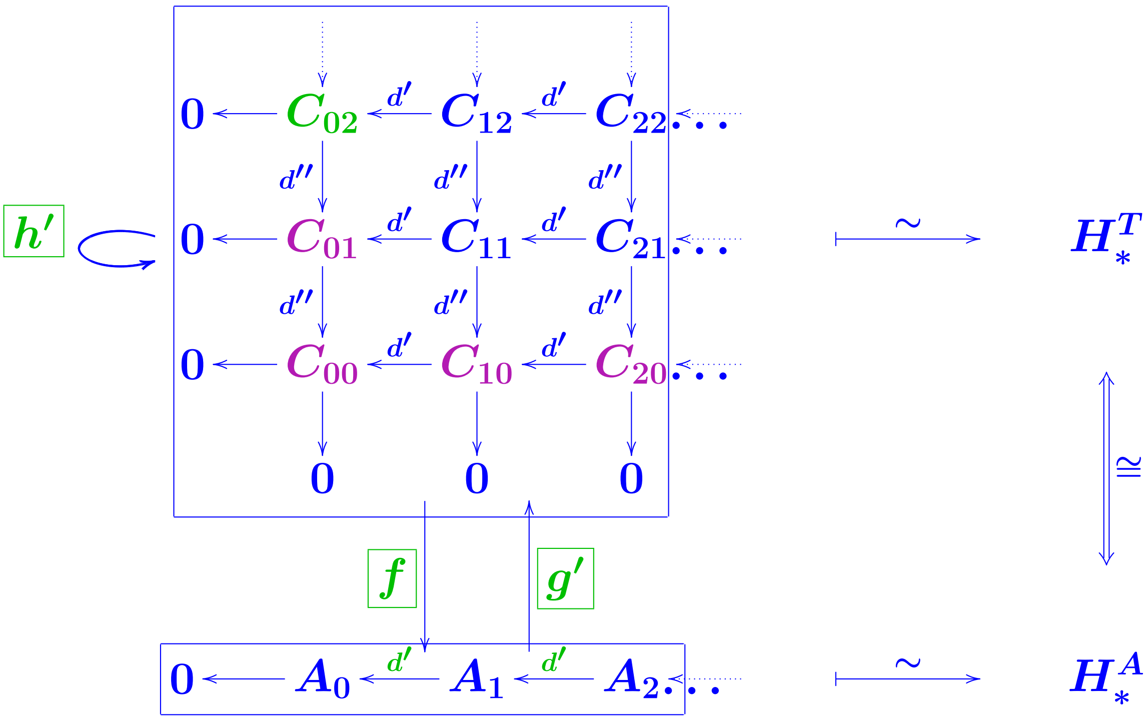
1. $\widehat{\delta}$ **is a perturbation of the differential** $\widehat{d}$;

2. **The operator** $h \circ \widehat{\delta}$ **is pointwise nilpotent.**

Then a general algorithm BPL **constructs:**



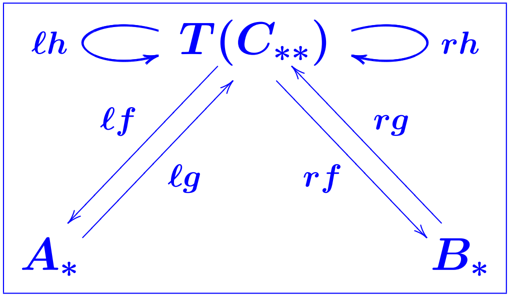
$\Rightarrow$



Doing the symmetric work

in the horizontal direction

constructs an equivalence:



$$= \quad \boxed{A_* \Longleftrightarrow B_*}$$

which effectively induces an isomorphism:

$$H_*^A \xLeftrightarrow[{[\ell f, \ell g, rf, rg]}]{\cong} H_*^B$$

QED

Example of Application in Commutative Algebra:

Let $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]$ = common polynomial ring.

Let $M = \mathfrak{R}$-module of finite type.

Strong known relations between:

1. Free $\mathfrak{R}$-resolution of M:

$$0 \leftarrow M \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

2. Homology of the Koszul complex $\mathrm{Ksz}(M)$:

$$H_*(0 \leftarrow M \leftarrow M \otimes \mathfrak{k}^m \leftarrow M \otimes \wedge^2 \mathfrak{k}^m \leftarrow M \otimes \wedge^3 \mathfrak{k}^m \leftarrow \cdots)$$

Usually: Resolution $\mapsto$ Homology of Koszul Complex:

$$\mathrm{Rsl}(M) \mapsto \mathrm{Rsl}(M) \otimes \mathfrak{k} \mapsto H_*(\mathrm{Rsl}(M) \otimes \mathfrak{k}) = H_*(\mathrm{Ksz}(M)).$$

Effective Homology produces a reverse process.

Theorem: A simple algorithm:

***Input*: Effective Homology $H_*^{\mathrm{Eff}}(\mathrm{Ksz}(M))$.**

***Output*: An effective resolution $\mathrm{Rsl}^{\mathrm{Eff}}(M)$.**

Combined with elementary methods of Effective Homology:

$$\mathrm{GB}(M) \mapsto H_*^{\mathrm{Eff}}(\mathrm{Ksz}(M)) \mapsto \mathrm{Rsl}^{\mathrm{Eff}}(M)$$

New simple understanding

Koszul complex and its differential:

$$\mathrm{Ksz}(M) := \{\cdots \leftarrow M \otimes \wedge^{i-1}\mathfrak{k}^m \leftarrow M \otimes \wedge^{i}\mathfrak{k}^m \leftarrow \cdots\}$$

$$d(\mu.dx_2 \wedge dx_3 \wedge dx_5) :=$$
$$x_2\mu.dx_3 \wedge dx_5 - x_3\mu.dx_2 \wedge dx_5 + x_5\mu.dx_2 \wedge dx_3$$

Two $\mathfrak{R}$-modules $M_1, M_2 \mapsto$ a "double" Koszul complex:

$$M_1 \otimes \wedge\mathfrak{k}^m \otimes M_2$$

with two differentials:

$$d'(\mu_1.(dx_2 \wedge dx_5).\mu_2) := x_2\mu_1.dx_5.\mu_2 - x_5\mu_1.dx_2.\mu_2$$
$$d''(\mu_1.(dx_2 \wedge dx_5).\mu_2) := -\mu_1.dx_5.(x_2\mu_2) + \mu_1.dx_2.(x_5\mu_2)$$

$\Rightarrow$ Double-complex.

M = $\mathfrak{R}$-module of finite type.

$H_*^{\text{Eff}}(\text{Ksz}(M))$ given:

Equivalence: $\boxed{d \circlearrowright M \otimes \wedge \mathfrak{k}^m} \Longleftarrow\!\!\!\Longrightarrow \boxed{EC_M \circlearrowleft d}$

EC_M = chain complex of finite dimensional $\mathfrak{k}$-vector spaces.

Particular case: $M = \mathfrak{R}$. Koszul's theorem:

Reduction: $\boxed{d \circlearrowright \wedge \mathfrak{k}^m \otimes \mathfrak{R}} \Rrightarrow \boxed{\mathfrak{k} \circlearrowleft 0}$

We intend to play with both equivalences

in the double Koszul complex:

$$M \otimes \wedge \mathfrak{k}^m \otimes \mathfrak{R}$$

Detailed description of $M \otimes \wedge\mathfrak{k}^m \otimes \mathfrak{R}$ = Aramova-Herzog bicomplex:

$\boldsymbol{AH(M)} :=$

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M \otimes \wedge^3 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^2 \otimes \mathfrak{R}_1 & \xrightarrow{d''} & M \otimes \wedge^1 \otimes \mathfrak{R}_2 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_3 & \longrightarrow & 0 \\
{\scriptstyle d'}\downarrow & & {\scriptstyle d'}\downarrow & & {\scriptstyle d'}\downarrow & & \downarrow & & \\
M \otimes \wedge^2 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^1 \otimes \mathfrak{R}_1 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_2 & \longrightarrow & 0 & & \\
{\scriptstyle d'}\downarrow & & {\scriptstyle d'}\downarrow & & \downarrow & & & & \\
M \otimes \wedge^1 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_1 & \longrightarrow & 0 & & & & \\
{\scriptstyle d'}\downarrow & & \downarrow & & & & & & \\
M \otimes \wedge^0 \otimes \mathfrak{R}_0 & \longrightarrow & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

$\mathfrak{R}_p = \mathfrak{k}[x_1, \ldots, x_m]^{[p]}$

$\wedge^q = \wedge^q \mathfrak{k}^m = \wedge^q(\mathfrak{m}/\mathfrak{m}^2)$

$M = \mathfrak{R}$-module

$\otimes = \otimes_{\mathfrak{k}}$

Horizontal $= M \otimes \mathrm{Ksz}(\mathfrak{R})_q$

Vertical $= \mathrm{Ksz}(M) \otimes \mathfrak{R}_p$

1. **Horizontal reduction.**

$\mathbf{Row}_q =$

$$\{\cdots \to M \otimes \wedge^{q-p} \otimes \mathfrak{R}_p \to M \otimes \wedge^{q-p-1} \otimes \mathfrak{R}_{p+1} \to \cdots\}$$

$$= q\text{-homogeneous component } M \otimes [\wedge \otimes \mathfrak{R}]_q$$

$$= M \otimes [\mathrm{Ksz}(\mathfrak{R})]_q$$

$\Rightarrow$ **acyclic!**

except for $q = 0$ **where** $\mathrm{Row}_0 = M \otimes \wedge^0 \otimes \mathfrak{R}_0 = M$.

$\Rightarrow$ **Reduction:** $\oplus_q \mathrm{Row}_q \Rrightarrow M$

Adding the vertical differentials = BPL

$\Rightarrow$ **Reduction** $AH(M) \Rrightarrow M$.

2. Vertical equivalence:

$\mathrm{Col}_p =$

$$\{\cdots \to M \otimes \wedge^{q-p} \otimes \mathfrak{R}_p \to M \otimes \wedge^{q-p-1} \otimes \mathfrak{R}_p \to \cdots\}$$

$$= \mathrm{Ksz}(M) \otimes \mathfrak{R}_p$$

Available equivalence: $\mathrm{Ksz}(M) \Lleftarrow\!\!\Rrightarrow C_*^{\mathrm{Eff}} =$

= chain complex of finite-dimensional $\mathfrak{k}$-vector spaces.

$\Rightarrow$ **Equivalence:** $\oplus_p \mathrm{Col}_p \Lleftarrow\!\!\Rrightarrow C_*^{\mathrm{Eff}} \otimes \mathfrak{R}$

Adding the horizontal differentials = BPL

$\Rightarrow$ **Equivalence:** $AH(M) \Lleftarrow\!\!\Rrightarrow C_*^{\mathrm{Eff}} \otimes \mathfrak{R} \circlearrowleft d$

where now $C_*^{\mathrm{Eff}} \otimes \mathfrak{R} \circlearrowleft d =$

= Chain complex of free $\mathfrak{R}$-modules.

3. Composing Reduction 1 + Equivalence 2:

$$\boldsymbol{M \lll AH(M) \Lleftarrow\!\!\!\Rrightarrow C_*^{\mathrm{Eff}} \otimes \mathfrak{R}}$$

gives an equivalence:

$$\boldsymbol{M \Lleftarrow\!\!\!\Rrightarrow C_*^{\mathrm{Eff}} \otimes \mathfrak{R}}$$

Interpreting this equivalence ⇒

$$\boldsymbol{C_*^{\mathrm{Eff}} \otimes \mathfrak{R} = \boxed{\text{free effective } \mathfrak{R}\text{-resolution of } M}.}$$

With **simple natural explicit formulas**

for the differentials and the contraction of $C_*^{\mathrm{Eff}} \otimes \mathfrak{R}$.

= Hodge decomposition of $C_*^{\mathrm{Eff}} \otimes \mathfrak{R}$

Toy example:

$$\mathfrak{R} = \mathbb{Q}[x, y, t] \qquad\qquad M = \mathfrak{R}/< x - t^3,\ y - t^5 >$$

Groebner basis $= < xt^2 - y,\ t^3 - x,\ x^2 - yt >$

$\Rightarrow$ **Approximate monomial module:**

$$M' = \mathfrak{R}/< xt^2,\ t^3,\ x^2 >$$

$\Rightarrow$ **a recursive process produces an equivalence:**

$$\mathrm{Ksz}(M') \Lleftarrow\!\!\!\Longrightarrow\!\!\!\!\!\!\Rrightarrow \{\mathbb{Q} \stackrel{0}{\longleftarrow} \mathbb{Q}^3 \stackrel{0}{\longleftarrow} \mathbb{Q}^3 \stackrel{\neq 0}{\longleftarrow} \mathbb{Q}\}$$

Applying BPL $\Rightarrow$ **Equivalence:**

$$\mathrm{Ksz}(M) \Lleftarrow\!\!\!\Longrightarrow\!\!\!\!\!\!\Rrightarrow \{\mathbb{Q} \stackrel{0}{\longleftarrow} \mathbb{Q}^3 \stackrel{\neq 0}{\longleftarrow} \mathbb{Q}^3 \stackrel{\neq 0}{\longleftarrow} \mathbb{Q}\}$$

Aramova-Herzog bicomplex + Two BPL applications:

⇒ Resolution:

$$M \xleftarrow{\varepsilon} \mathfrak{R} \xleftarrow{d_1} \mathfrak{R}^3 \xleftarrow{d_2} \mathfrak{R}^3 \xleftarrow{d_3} \mathfrak{R} \longleftarrow 0$$

$$\overset{d_1}{[x^2 - yt, -t^3 + x, -xt^2 + y]} \quad \overset{d_2}{\begin{bmatrix} 0 & 1 & t^2 \\ 0 & -x & -y \\ 0 & t & x \end{bmatrix}} \quad \overset{d_3}{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}$$

Final obvious simplifications ⇒ Minimal resolution:

$$M \xleftarrow{\varepsilon} \mathfrak{R} \xleftarrow{d_1'} \mathfrak{R}^2 \xleftarrow{d_2'} \mathfrak{R} \longleftarrow 0$$

$$\overset{d_1'}{[-t^3 + x, -xt^2 + y]} \quad \overset{d_2'}{\begin{bmatrix} xt^2 - y \\ -t^3 + x \end{bmatrix}}$$

First Stillman example: $\mathfrak{R} = \mathbb{Q}[a, b, c, d, e, f, g, h, i]$

$$M = \mathfrak{R} / < fh - ei, ch - bi, \ldots >$$

defined by the 9 2×2-minors of the 3×3-matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$M' = \mathfrak{R} / < fh, ch, \ldots >$$

Equivalence: $\text{Ksz}(M') \Longleftrightarrow \{0 \leftarrow \mathbb{Q} \overset{0}{\leftarrow} \mathbb{Q}^9 \overset{0}{\leftarrow} \mathbb{Q}^{18} \overset{1}{\leftarrow} \mathbb{Q}^{14} \overset{1}{\leftarrow} \mathbb{Q}^4 \overset{0}{\leftarrow} 0\}$

BPL application $\Rightarrow$

Equivalence: $\text{Ksz}(M) \Longleftrightarrow \{0 \leftarrow \mathbb{Q} \overset{0}{\leftarrow} \mathbb{Q}^9 \overset{0}{\leftarrow} \mathbb{Q}^{18} \overset{2}{\leftarrow} \mathbb{Q}^{14} \overset{3}{\leftarrow} \mathbb{Q}^4 \overset{0}{\leftarrow} 0\}$

$\Rightarrow$ **Betti numbers** $= \{1, 9, 16, 9, 1\}$

Aramova-Herzog bicomplex + 2×BPL ⇒ Resolution:

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R}^{18} \overset{2}{\longleftarrow} \mathfrak{R}^{14} \overset{3}{\longleftarrow} \mathfrak{R}^4 \longleftarrow 0$$

Obvious simplifications ⇒ Minimal resolution:

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R}^{16} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R} \longleftarrow 0$$

But the methods of effective homology produce

an effective resolution:

with a Hodge decomposition:

$$\mathrm{id} = dh + hd$$

.

Interpretation.

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \underset{h}{\overset{d}{\leftrightarrows}} \mathfrak{R}^{16} \underset{h}{\overset{d}{\leftrightarrows}} \mathfrak{R}^9 \longleftarrow \mathfrak{R} \longleftarrow 0$$

Let $\mu \in \mathfrak{R}^{16} \in \mathrm{Rsl}_2(M)$**.**

Then:

$\mu - hd\mu$ **is the projection of** μ

over the syzygy subspace of $\mathfrak{R}^{16}$

$h\mu \in \mathfrak{R}^9 = \mathrm{Rsl}_3(M)$ **is the preimage of this syzygy in** $\mathfrak{R}^9$**.**

Hodge decomposition of μ**:**

$$\mu = hd\mu + dh\mu$$

Example of calculation.

$b_{2,1}$ = first generator of $\mathrm{Rsl}_2 = \mathfrak{R}^{16}$.

$$\mu = b_{2,1} \Rightarrow \mu - hd\mu = 0$$

$$\mu' = fg \cdot b_{2,1} \Rightarrow \mu' - hd\mu' = \\ -di \cdot g_{2,1} - ci \cdot g_{2,3} + ei \cdot g_{2,6} - fi \cdot g_{2,7} + i^2 \cdot g_{2,15}.$$

$$h\mu' = h(\mu' - hd\mu') = i \cdot g_{3,4}$$

Hodge decomposition:

$$\mu' = fg \cdot b_{2,1} = hd\mu' + dh\mu' = \\ (fg + di) \cdot b_{2,1} + ci \cdot b_{2,3} - ei \cdot b_{2,6} + fi \cdot b_{2,7} - i^2 \cdot b_{2,15} \\ -di \cdot b_{2,1} - ci \cdot b_{2,3} + ei \cdot b_{2,6} - fi \cdot b_{2,7} + i^2 \cdot b_{2,15}$$

Final scheme for computing a resolution by this method.

1. **Given M = finite type $\mathfrak{R}$-module.**

2. $\Rightarrow$ **Groebner basis** $\mathrm{Gb}(M)$.

3. $\Rightarrow$ **Monomial approximation** $M' \sim M$.

4. $\Rightarrow$ $H_*^{\mathrm{Eff}}(\mathrm{Ksz}(\mathrm{M'}))$.

5. **BPL** $\Rightarrow$ $H_*^{\mathrm{Eff}}(\mathrm{Ksz}(\mathrm{M}))$.

6. **Aramova-Herzog bicomplex** + **BPL** $\Rightarrow$ $\mathrm{Rsl}^{\mathrm{Eff}}(M)$.

The END

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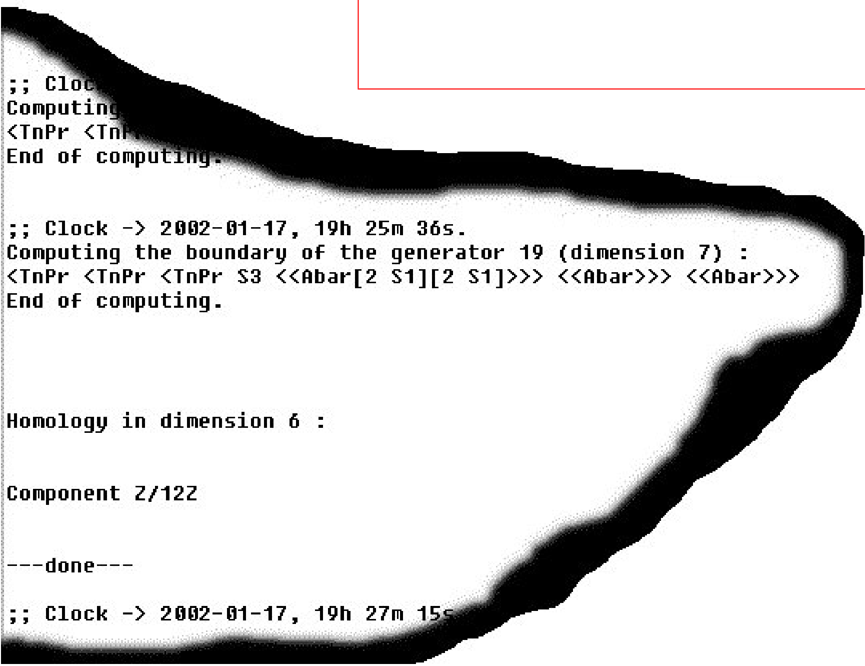

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Logroño, February 28, 2008