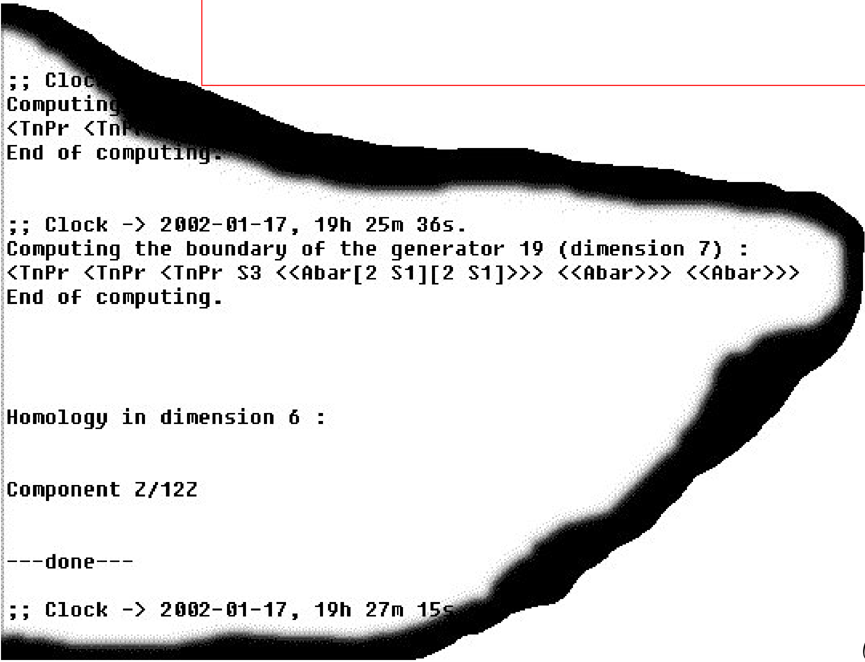
A Case Study in Constructive Algebraic Topology



Methods of Proof Theory in Mathematics
Max Planck Institute for Mathematics, June 4-9, 2007

(Francis Sergeraert, Institut Fourier, Grenoble, France)

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Talk theme = Quotation (from Encyclopedic Dictionary of Mathematics):

> **As yet we are ignorant of an effective method of computing the cohomology of a Postnikov complex from π_n and k_{n+1}.**

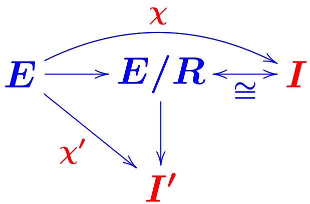
Rough plan of the talk:

1. **What does classification means?**
2. **Classification Problem in Algebraic Topology.**
3. **"Solution" based on Postnikov Towers.**
4. **Which is not constructive in standard Algebraic Topology.**
5. **Making Algebraic Topology constructive.**
6. **⇒ A fundamental new Computability Problem in Arithmetic.**

What is a classification?

Let E be a large set (class).

Classifying the elements of E consists in constructing:



where:

- **R = classifying equivalence relation between E-elements;**
- **E/R = associated quotient set;**
- **I (resp. I') = appropriate complete (resp. partial) invariants;**
- **χ (resp. χ') = classification process.**
- **I (I') and χ (χ') satisfy some recursiveness properties.**

Typical successful example.

- E = set of square K-matrices (K some commutative field).
- R = similarity relation ($A \overset{R}{\sim} B \Leftrightarrow A = SBS^{-1}$).
- $\chi : E \to I$ = invariant factors process.

Then a correct classification of matrices modulo similarity is:

$$\chi : E \longrightarrow I \cong E/R$$

$$\chi(M) = (p_1, p_2, \ldots, p_k)$$

where $p_1, \ldots, p_k \in {}^1K[X]$, satisfying $p_1|p_2|\ldots|p_k$, are the invariant factors of M.

Properties of $\chi : E \longrightarrow I \cong E/R$:

1. **The classification map χ is surjective.**
2. **The invariant set I is not too large ($\mathrm{id} : E \longrightarrow I = E = (E/=)$?!?!).**
3. **The invariant set I is not too small ($* : E \longrightarrow \{*\} = E/E$?!?!).**
4. **The equivalence relation R on E is worthwhile (R = similarity).**
5. **The invariant set I is recursive ($\pi : E \longrightarrow \boxed{I :=}\ E/R$?!?!).**
6. **The invariant map χ is recursive.**
7. **A recursive section of χ can be given.**

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1. **What does classification means?**
2. **Classification Problem in Algebraic Topology.**
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5. **Making Algebraic Topology constructive.**
6. **$\Rightarrow$ A fundamental new Computability Problem in Arithmetic.**

Classification in Algebraic Topology?

Classification of spaces up to homeomorphism = Topology.

Much too difficult!

Defining a rougher equivalence relation $\Rightarrow$ Homotopy type.

Two maps $f_0, f_1 : A \longrightarrow B$ are homotopic if there exists a continuous deformation $f_0 \sim f_1$.

Two spaces A and B have the same homotopy type ($A \sim B$) if:

$$A \underset{f}{\overset{g}{\rightleftarrows}} B$$

with $gf \sim \mathrm{id}_A$ and $fg \sim \mathrm{id}_B$.

Typically: Hollow circle $\sim$ Hollow cylinder $\sim$ Solid torus.

Known invariants of the homotopy type:

Homotopy Groups (Poincaré + Hurewicz).

Poincaré group $= \pi_1 A = \{\gamma : (S^1, *) \longrightarrow (A, *)\}/ \sim$

More generally:

Hurewicz groups $= \pi_n A = \{\gamma : (S^n, *) \longrightarrow (A, *)\}/ \sim$

Abelian groups for $n \geq 2$.

The isomorphism class of $\pi_n A$

depends only on the homotopy type of A.

First definitive obstacle for effective Algebraic Topology:

Spaces with $\pi_1 \neq 0$ too close to Novikov's word problem.

Theorem (Rabin): There exists a finitely triangulated complex K of dimension 2 such that:

- **$\pi_1 K \neq 0$;**
- **There exists no proof of this fact;**
- **Such a triangulated complex K cannot be identified as such.**

$\Rightarrow$ Most common Algebraic Topology limited to $\pi_1 = 0$.

From now on, all spaces K are assumed satisfying $\pi_1(K) = 0$.

<u>Question</u>: Is $(\pi_2(K), \pi_3(K), \ldots, \pi_n(K), \ldots)_{n\geq 2}$

a complete set of invariants for the homotopy type?

<u>Answer</u>: In particular cases, yes, in general no.

Simplest case: only one $n \geq 2$ with $\pi_n(K) \neq 0$.

Then the homotopy type of K is

entirely determined by the isomorphism class of $\pi_n(K)$.

<u>Example</u>: The homotopy type of $P^\infty(\mathbb{C})$

is entirely determined by

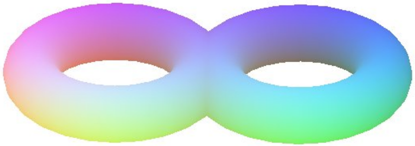
its sequence of homotopy groups $(\pi_2 = \mathbb{Z}, 0, 0, \ldots)$.

Eilenberg-MacLane space $K(\pi, n)$.

Definition: $K(\pi, n)$ is the unique space K satisfying $\pi_m(K) = 0$ for every m except $\pi_n(K) = \pi$.

NB: Unique up to homotopy equivalence.

Examples:

- $K(\mathbb{Z}, 1) = S^1$.
- $K(\mathbb{Z}_2, 1) = P^\infty\mathbb{R} = \mathbb{R}^\infty_* / \sim_{\text{proj}}$.
- $K(\langle a, b, c, d; [a, b][c, d] \rangle, 1) = S_2 =$ 
- $K(\mathbb{Z}, 2) = P^\infty\mathbb{C} = \mathbb{C}^\infty_* / \sim_{\text{proj}}$.

Eilenberg-MacLane spaces =

elementary components of the homotopy world.

In fact $K(\pi, n)$ = topological group.

Spaces with two non-null homotopy groups?

<u>Theorem</u>: Let E be a space with two non-null homotopy groups:

$$\pi_m(E) = \pi_m,\ \pi_n(E) = \pi_n,\ \pi_k(E) = 0 \text{ otherwise},\ 2 \leq m < n.$$

Then there exists a version E' of E (same homotopy type):

$$E' = \boxed{\text{twisted}} \text{ product} = E' = K(\pi_m, m) \times_{\boxed{\tau}} K(\pi_n, n).$$

Simplest examples of twisted products:

Non-twisted $S^1 \times \mathbb{Z}$ = stack of circles.

1-twisted $S^1 \times_{\boxed{1}} \mathbb{Z} = \mathbb{R}$.

The canonical projection $\mathbb{R} = S^1 \times_{\boxed{1}} \mathbb{Z} \longrightarrow S^1$
is the complex exponential map $t \mapsto e^{2\pi i t}$.

Twist $\Rightarrow$
Problems with multiple "branches" in complex logarithms.

Hopf fibration: $S^3 = S^2 \times_{\boxed{1}} S^1$
$\Leftrightarrow P^1(\mathbb{C}) = S^2 = S^3/\sim_{\text{proj}}$ with $S^3 \subset \mathbb{C}^2$.

General fibration theory $\Rightarrow$

Theorem: A fibration:

$$K(\pi, n) \hookrightarrow \boxed{E = B \times_\tau K(\pi, n)} \longrightarrow B$$

is classified by an element $k_\tau \in H^{n+1}(B, \pi)$.

Examples:

$\pi = \mathbb{Z}$, $n = 1$, $K(\mathbb{Z}, 1) = S^1$, $B = S^2 \Rightarrow H^2(S^2, \mathbb{Z}) = \mathbb{Z}$

$k = 0 \Rightarrow S^2 \times_0 S^1 = S^1 \times S^2$.

$k = 1 \Rightarrow S^2 \times_1 S^1 = S^3 =$ Hopf fibration.

$k = 2 \Rightarrow S^2 \times_2 S^1 = P^3\mathbb{R} =$ "semi-Hopf fibration"

$\cdots \quad \cdots$

Generalization = Fundamental Postnikov's result:

E = arbitrary space with $\pi_1 = 0$.

Then $E \stackrel{\simeq}{=} \lim_{\leftarrow} E_n$ with $(E_n)_{n\geq 1}$ defined by a sequence:

$$(\pi_2, k_2 = 0, \pi_3, k_3, \pi_4, k_4, \ldots, \pi_n, k_n, \ldots)$$

by the recursive formula:

$$E_1 = * \quad \text{and} \quad E_n = E_{n-1} \times_{k_n} K(\pi_n, n) \text{ if } n \geq 2.$$

with $k_n \in H^{n+1}(E_{n-1})$.

$$\mathrm{E} = \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{K(\pi_2, 2)}_{E_2} \times_{k_3} K(\pi_3, 3)}_{E_3} \times_{k_4} K(\pi_4, 4)}_{E_4} \times_{k_5} \cdots}_{\cdots} \cdots}_{E}$$

Postnikov System

Now two essential problems:

> **As yet we are ignorant of an effective method of computing the cohomology of a Postnikov system from π_n and k_n.**

$$k_5 \in H^6(K(\pi_2, 2) \times_{k_3} K(\pi_3, 3) \times_{k_4} K(\pi_4, 4)) = ?????$$

Now three theoretical solutions, one machine-implemented.

- **Rolf Schön, Effective Algebraic Topology, AMS Memoir #451, 1991.**
- **Operadic Solutions (Justin Smith, Michael Mandell, Clemens Berger + Benoit Fresse, . . .).**
- **Effective Homology (FS, Julio Rubio, Ana Romero, . . .)**

Example: the beginning of the simplest non-trivial Postnikov tower with homotopy groups = $\mathbb{Z}_2$.

$$E_2 = K(\mathbb{Z}_2, 2) \Rightarrow \boxed{H^4(E_2, \mathbb{Z}_2) = \mathbb{Z}_2} \Rightarrow k_3.$$

$$E_3 = E_2 \times_{k_3} K(\mathbb{Z}_2, 3) \Rightarrow \boxed{H^5(E_3, \mathbb{Z}_2) = \mathbb{Z}_2^2} \Rightarrow k_4.$$

$$E_4 = E_3 \times_{k_4} K(\mathbb{Z}_2, 4) \Rightarrow \boxed{H^6(E_4, \mathbb{Z}_2) = \mathbb{Z}_2^4} \Rightarrow k_5.$$

$$E_5 = E_4 \times_{k_5} K(\mathbb{Z}_2, 5) \Rightarrow \boxed{H^7(E_5, \mathbb{Z}_2) = \mathbb{Z}_2^5} \Rightarrow k_6.$$

$$E_6 = E_5 \times_{k_6} K(\mathbb{Z}_2, 6) \Rightarrow \cdots$$

Standard terminology:

The cohomology class k_n is usually called

Postnikov invariant or k-invariant.

Which implicitly implies $(\pi_2, 0, \pi_3, k_3, \pi_4, k_4, \ldots)$

classifies a homotopy type?

False. Key observation:

$S^1 \times_0 \mathbb{Z}$ = **stack of circles.**

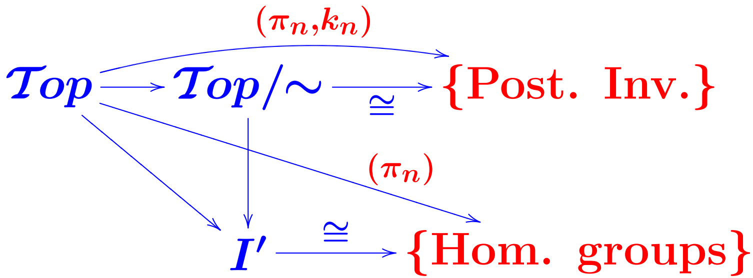
$S^1 \times_{\boxed{1}} \mathbb{Z} = \mathbb{R}$.

$S^1 \times_{\boxed{-1}} \mathbb{Z} = \mathbb{R}$ **as well!!**

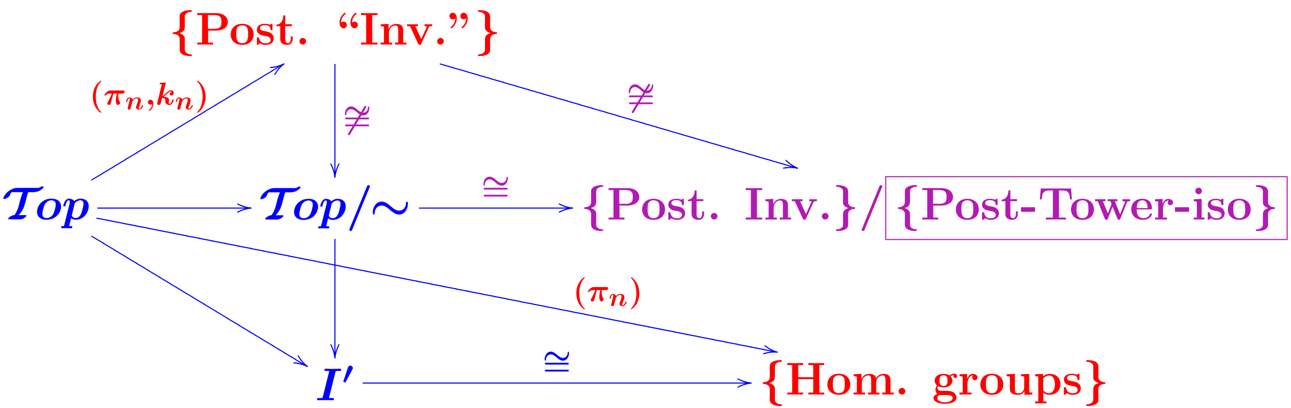
The "invariant" k of $S^1 \times_k \mathbb{Z}$ is an invariant of the fibration structure, not an invariant of the homotopy type of the total space of the fibration.

Actual result about Postnikov "invariants":

Wished:



Obtained:



<u>Definition</u>:

Given two Postnikov towers:

$$\boldsymbol{P = (\pi_n, k_n)_{n\geq 2} \quad , \quad P' = (\pi'_n, k'_n)_{n\geq 2}.}$$

A Postnikov tower morphism $f : P \to P'$

is a collection $\{f_n : \pi_n \to \pi'_n\}$

compatible with the k_n's and k'_n's.

$\Rightarrow$ Natural definition of isomorphism

between Postnikov towers.

Example 1:

$$\textbf{Isomorphisms: } ((\mathbb{Z},0),(\mathbb{Z},k_3)) \xrightarrow{\cong} ((\mathbb{Z},0),(\mathbb{Z},k_3')) \ ??$$

$$k_3, k_3' \in H^4(K(\mathbb{Z},2),\mathbb{Z}) = \mathbb{Z}{c_1}^2$$

$$\pi_2 = \mathbb{Z} \xrightarrow{f_2=\pm 1} \mathbb{Z} = \pi_2'$$

$$\pi_3 = \mathbb{Z} \xrightarrow{f_3=\pm 1} \mathbb{Z} = \pi_3'$$

$\Rightarrow$ k_3, k_3', f_2, f_3 $\boxed{\text{compatible}}$ $\iff$ $k_3' = f_3 k_3$.

Remark: $k_3 \in \mathbb{Z}[c_1]^{[2]}$.

Example 2:

$$\textbf{Isomorphisms: } ((\mathbb{Z}^n, 0), (\mathbb{Z}, k_3)) \xrightarrow{\cong} ((\mathbb{Z}^n, 0), (\mathbb{Z}, k_3')) \ ??$$

$$k_3, k_3' \in H^4(K(\mathbb{Z}^n, 2), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n]^{[2]} = Q_{\mathbb{Z}}(c_1, \ldots, c_n)$$

$$\pi_2 = \mathbb{Z}[c_1, \ldots, c_n]^{[1]} \xrightarrow{f_2 \in GL_n(\mathbb{Z})} \mathbb{Z}[c_1, \ldots, c_n]^{[1]} = \pi_2'$$

$$\pi_3 = \mathbb{Z} \xrightarrow{f_3 = \pm 1} \mathbb{Z} = \pi_3'$$

$\Rightarrow k_3, k_3', f_2, f_3$ $\boxed{\text{compatible}}$ $\iff f_{3*}(k_3) = {f_2}^*(k_3')$.

$\Leftrightarrow$ **classification of $\mathbb{Z}$-quadratic forms**

up to $\mathbb{Z}$-linear equivalence.

Solution = Gauss + Serre.

Example 3:

$$((\mathbb{Z}^n,0),(0,0),(0,0),(\mathbb{Z},k_5)) \xrightarrow{\overset{??}{\cong}} ((\mathbb{Z}^n,0),(0,0),(0,0),(\mathbb{Z},k_5'))$$

$$k_5, k_5' \in H^6(K(\mathbb{Z}^n,2),\mathbb{Z}) = \mathbb{Z}[c_1,c_2,\ldots,c_n]^{[3]} = C_{\mathbb{Z}}(c_1,\ldots,c_n)$$

$$\pi_2 = \mathbb{Z}[c_1,\ldots,c_n]^{[1]} \xrightarrow{f_2 \in GL_n(\mathbb{Z})} \mathbb{Z}[c_1,\ldots,c_n]^{[1]} = \pi_2'$$

$$\pi_5 = \mathbb{Z} \xrightarrow{f_5 = \pm 1} \mathbb{Z} = \pi_5'$$

$\Rightarrow k_5, k_5', f_2, f_5$ $\boxed{\text{compatible}}$ $\Longleftrightarrow f_{5*}(k_5) = {f_2}^*(k_5')$.

$\Leftrightarrow$ **classification of $\mathbb{Z}$- $\boxed{\text{cubic}}$ forms**

up to $\mathbb{Z}$-linear equivalence.

Solution = ?????

Conclusion:

The general decision problem of isomorphism
between Postnikov towers
is a terrible arithmetical problem !!

Transforming Postnikov "invariants" into actual invariants
is harder in the simplest cases than solving
the classification problem
of homogeneous $\mathbb{Z}$-forms of degree n
up to $\mathbb{Z}$-linear equivalence.

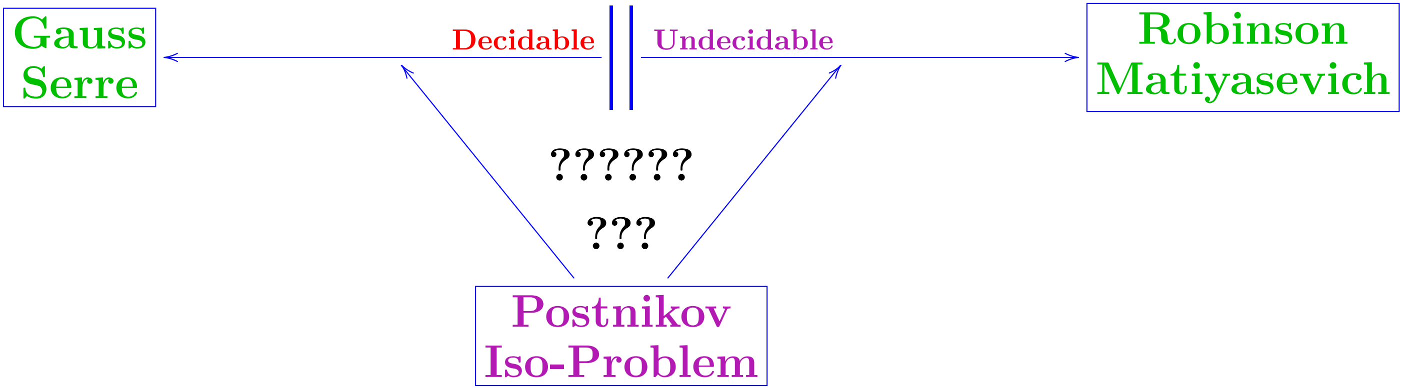
Tempting:

Use negative Robinson-Matiyasevich answer

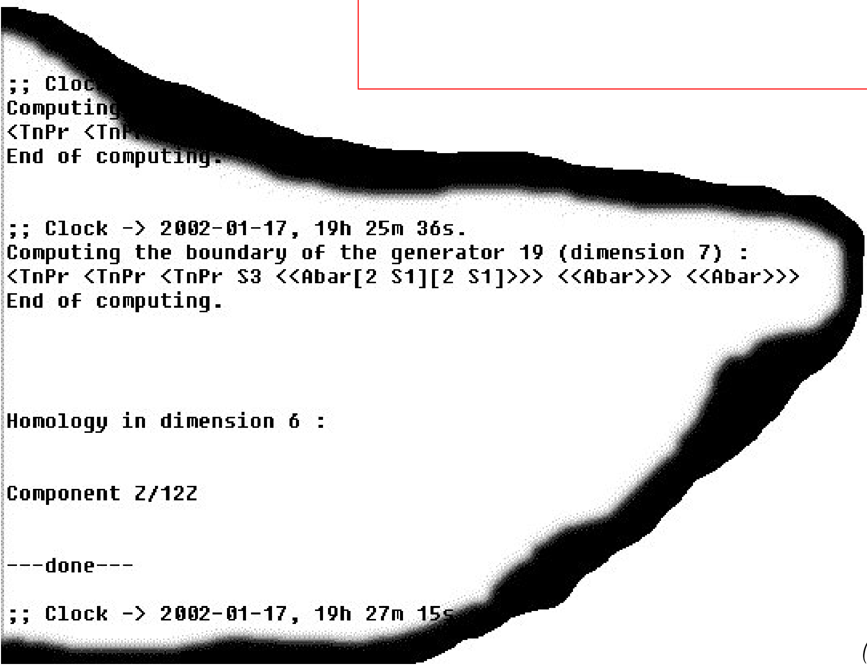
to 10th Hilbert problem to prove:

The general isomorphism problem between Postnikov towers

is undecidable.



The END



End of computing.
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
Homology in dimension 6 :
Component Z/12Z
---done---