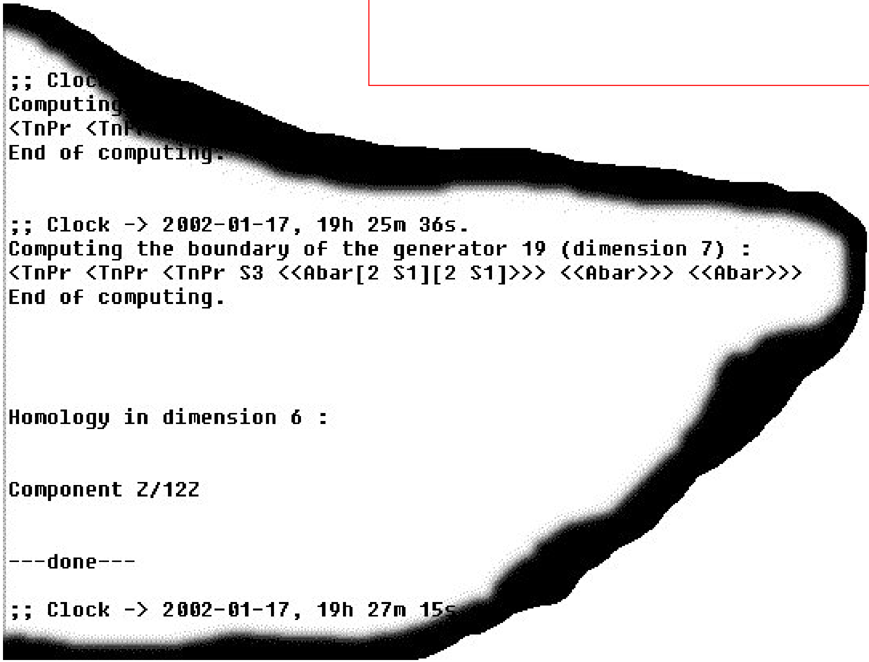
Constructive Algebraic Topology



Francis Sergeraert, Institut Fourier, Grenoble, France

Poitiers talk, April 13, 2006

(Pdf-file *www-reachable* @ `www-fourier.ujf-grenoble.fr/~sergerar/`)

Fundamental flaw of “classical” Algebraic Topology:

1. **Topology is difficult.**
2. **Algebraic topology consists in reducing topological problems to algebraic ones.**
3. **That is, to problems having an “automatic” (computable) solution.**
4. **But classical algebraic topology does not reach this point.**

Typical example:

“Theorem”. There exists an exact sequence:

$$0 \longrightarrow A \xrightarrow{f} X \xrightarrow{g} C \longrightarrow 0$$

allowing you to “ “compute” ” the unknown group X

when the groups A and C are known.

$$0 \rightarrow A \xrightarrow{f} X \xrightarrow{g} C \rightarrow 0$$

But such a statement is non-sense!

The pair (A, C) is clearly non-sufficient to determine the unknown group because of the *extension problem*.

The missing information to determine the *right* group X is a cohomology class $\tau \in H^2(C, A)$, but which needs, to be defined, the X group *itself* and the maps f and g!

Example:

J.-P. Serre " "computing" " $\pi_6(S^3)$ in 1950.

Serre spectral sequence $\Rightarrow$ there exists an exact sequence:

$$0 \to \mathbb{Z}_2 \to \pi_6(S^3) \to \mathbb{Z}_6 \to 0$$

But two different extensions are possible ($\mathbb{Z}_2 \oplus \mathbb{Z}_6$ or $\mathbb{Z}_{12}$?); the right one is determined by $\tau \in H^2(\mathbb{Z}_6, \mathbb{Z}_2) = \mathbb{Z}_2$ where:

1. **The class τ is mathematically well defined;**
2. **The class τ is computationally *unreachable***

   in the framework of the Serre spectral sequence.

Corollary: The group $\pi_6(S^3)$ remained *unknown* in Serre's work in 1950.

Finally determined by Barratt and Paechter in 1952 thanks to *new specific methods* ($= \mathbb{Z}_{12}$).

Now "stupidly" computed by the Kenzo program in one minute.

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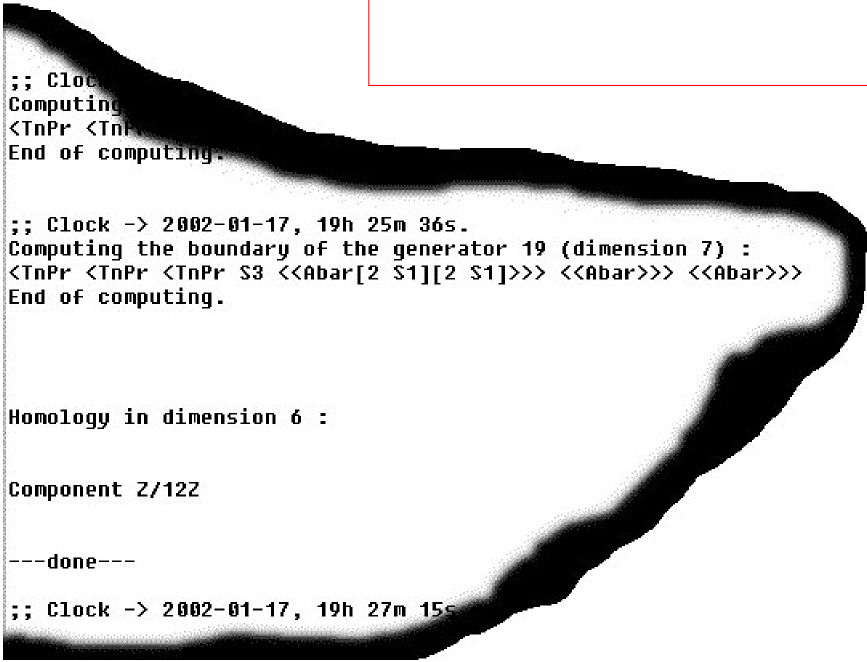

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Typical example of confusion among algebraic topologists:

In the book "Methods of Homological Algebra" (Gelfand + Manin), it is explained in the preface:

> ***The book by Cartan and Eilenberg*** **[Homological Algebra]** ***contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences.***

Serre's problem is a good counterexample

to Gelfand-Manin's claim:

The Serre spectral sequence does not allow its user to compute $\pi_6(S^3)$; it gives only a partial information: the unknown object is between $\mathbb{Z}_2$ and $\mathbb{Z}_6$ in a short exact sequence, so that $\pi_6(S^3) = \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_6$.

To terminate the computation,

please invent a new method...

From the point of view of proof theory,

the situation is there quite terrible.

The traditional style of the texts in algebraic topology makes the naive reader believe the exact or spectral sequence which is described is a computation tool.

But the reader which is aware of what an algorithm actually is immediately sees these claimed computational tools are erroneously qualified.

And experience shows most algebraic topologists are,

from this point of view, naive readers. . .

Main ingredient for the Rubio-S. solution

for Constructive Algebraic Topology.

Topology $\Longleftrightarrow$ **Algebra**

Loc. effective objects $\Longleftrightarrow$ **Effective objects**

Loc. computable world $\Longleftrightarrow$ **Computable world**

Roughly speaking:

An effective object is an object which is essentially entirely known.

In particular the standard global information concerning this object is reachable (= *computable*).

A locally effective object is most often a quite infinite object.

For any "local" ingredient of this object, any necessary information is reachable.

But in general *no global information* for the underlying object is reachable.

Typical example:

Loc. eff. chain complexes $\Leftrightarrow$ Eff. chain complexes.

Standard notion of chain complex in Algebraic Topology:

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \overset{d_n}{\leftarrow} C_n \overset{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

where:

1. C_n = free $\mathbb{Z}$-module with distinguished basis;
2. $d_n \circ d_{n+1} = 0$;
3. $H_n(C_*) := \ker(d_n)/\mathrm{im}(d_{n+1})$.

Notion of effective chain complex :

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \stackrel{d_n}{\leftarrow} C_n \stackrel{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\beta, d)$$

where:

1. β: $\mathbb{Z} \rightarrow \mathcal{L}\text{ist}$: $n \mapsto [g_1^n, \ldots, g_{k_n}^n]$ = distinguished basis of C_n.
2. d: $\mathbb{Z}\widetilde{\times}\mathbb{N}_* \rightarrow \mathcal{U}$: $(n, i) \mapsto d_n(g_i^n) \in C_{n-1}$ when g_i^n makes sense.

In particular every C_n is a free $\mathbb{Z}$-module with a finite distinguished basis.

$\Rightarrow$ Every $d_n : C_n \rightarrow C_{n-1}$ is computable.

$\Rightarrow$ Every homology group $H_n(C_*)$ is computable

(every global information is reachable).

Notion of locally effective chain complex:

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \overset{d_n}{\leftarrow} C_n \overset{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\chi, d)$$

where:

1. χ: $\mathcal{U} \times \mathbb{Z} \rightarrow \text{Bool} = \{\top, \bot\} : (\omega, n) \mapsto \top$
   if and only if ω is a generator of C_n;

2. d: $\mathcal{U} \widetilde{\times} \mathbb{Z} \rightarrow \mathcal{U} \ : (\omega, n) \mapsto d_n(\omega) \in C_{n-1}$
   when ω is a generator of C_n $(\Leftrightarrow \chi(\omega, n) = \top)$.

Any finite set of pointwise computations may be done.

Gödel + Church + Turing + Post $\Rightarrow$ no global information is reachable;
in particular, the homology groups of C_* are *not computable*.

A typical example of locally effective object is the set $\mathbb{Z}$ in a pocket computer:

1. **If a computation like $(a + b) \times (c + d)$ concerning only four (arbitrary) integers is asked for, the pocket computer may do it.**

2. **But no global information is available for the set $\mathbb{Z}$.**

Quite infinite objects so can be handled. Such an object will be a (finite) set of functional objects implementing the various computations which possibly could be asked for some arbitrary finite collection of elements of the object.

Example of locally effective topological spaces.

The standard simplicial techniques allow to model reasonable topological spaces as set of simplices (points, edges, triangles, tetrahedrons, ...) + incidence relations.

$$E = (\{S_0, S_1, S_2, \ldots\}\ ,\ \{\partial_i^n : S_n \rightarrow S_{n-1}\})$$

where the ∂_i^n must satisfy compatibility relations.

Most often, it is *mandatory* to consider simplicial objects where the S_i's are highly *infinite*.

So that in this case, a locally effective simplicial set is:

$$S = (\chi, \partial)$$

where:

1. $\chi : \mathcal{U} \times \mathbb{N} \to \{\top, \bot\} : (\sigma, n) \mapsto \top$
   if and only if σ is an n-simplex of S;

2. $\partial : \mathcal{U} \widetilde{\times} \mathbb{N}_* \widetilde{\times} \mathbb{N} \to \mathcal{U} : (\sigma, n, i) \mapsto \partial_i^n(\sigma) \in S_{n-1}$.

Allows to implement

all the classical topological spaces of algebraic topology

in a locally effective way.

Ingredients used by R. & S. to transform

the pseudo-computational tools of traditional Alg. Topology

into actual computational tools:

1) Functional Programming.

From a theoretical point of view,

something similar to lambda-calculus is necessary.

For concrete applications,

an efficient functional programming language

must be used.

2) Homological perturbation theory (Shih, Ronnie Brown).

Definition: An object **with effective homology** is a 4-tuple:

$$(X, C_*(X), EC_*^X, \varepsilon^X)$$

where:

1. X = the underlying object is **locally effective**;
2. $C_*(X)$ = the chain complex defining its homology is locally effective;
3. EC_*^X = an **effective** chain complex ;
4. $\varepsilon^X : C_*(X) \Longleftrightarrow EC_*^X$ is a chain equivalence.

Fundamental result of Effective Homology Theory:

If $(X, C_*(X), EC_*^X, \varepsilon^X)$

is an object with effective homology,

then this object is an algebraic model

for the underlying mathematical object X.

In particular if some “reasonable” construction

starting from such mathematical objects is done,

then a version of this construction “with effective homology”

can be produced.

Meta-theorem: Let $X_{1*}, \ldots, X_{n*}$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_{1*}, \ldots, X_{n*}) \mapsto X_*.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}\colon \left(\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}\right) \mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is perfectly stable

and can be again used with X for further calculations.

Typical application.

Problem: Given X, computing $H_*(\Omega^n X) = H_*(\mathcal{C}(S^n, X))$?

Frank Adams (1956) : Solution for $n = 1$.

Hans Baues (1980) : Solution for $n = 2$.

Solution for $n > 2$?? Known as Adams' problem.

Effective homology produces

a very easy solution for Adam's problem.

With a much wider scope.

So easy that concrete computer implementation is not hard.

If $(X, C_*(X), EC_*^X, \varepsilon^X)$ is a simplicial set with effective homology which is n-connected, if $k \leq n$, then an algorithm produces the appropriate iterated Cobar construction:

$$(X, C_*(X), EC_*^X, \varepsilon^X) \longmapsto \Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X}$$

$$\Omega X, C_*(\Omega X), EC_*^{\Omega X}, \varepsilon^{\Omega X} \longmapsto \Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X}$$

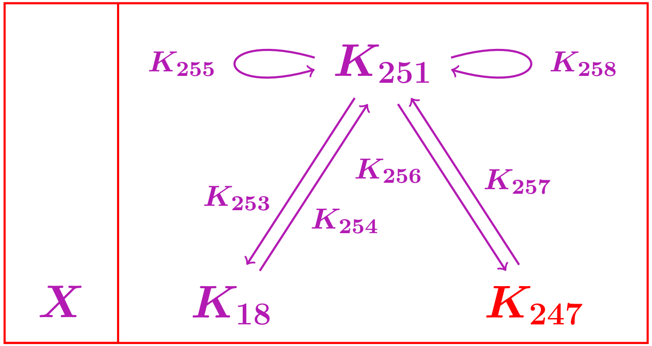
$$\Omega^2 X, C_*(\Omega^2 X), EC_*^{\Omega^2 X}, \varepsilon^{\Omega^2 X} \longmapsto \Omega^3 X, C_*(\Omega^3 X), EC_*^{\Omega^3 X}, \varepsilon^{\Omega^3 X}$$

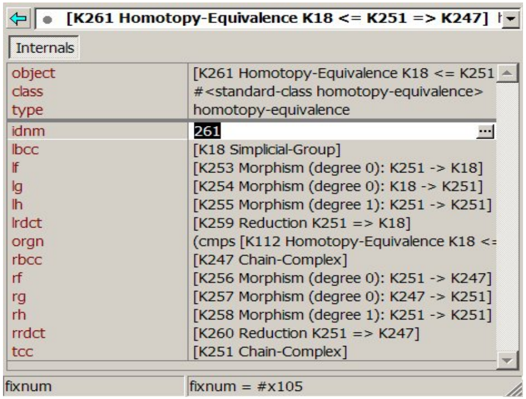
$$\cdots$$

$$\cdots \longmapsto \Omega^n X, C_*(\Omega^n X), EC_*^{\Omega^n X}, \varepsilon^{\Omega^n X}$$

??? $\mathrm{Cobar}^n(EC_*X)$ $\longleftarrow$

Exemple




[K261 Homotopy-Equivalence K18 <= K251 => K247]

Internals

object	[K261 Homotopy-Equivalence K18 <= K251
class	#<standard-class homotopy-equivalence>
type	homotopy-equivalence
idnm	261
lbcc	[K18 Simplicial-Group]
lf	[K253 Morphism (degree 0): K251 -> K18]
lg	[K254 Morphism (degree 0): K18 -> K251]
lh	[K255 Morphism (degree 1): K251 -> K251]
lrdct	[K259 Reduction K251 => K18]
orgn	(cmps [K112 Homotopy-Equivalence K18 <=
rbcc	[K247 Chain-Complex]
rf	[K256 Morphism (degree 0): K251 -> K247]
rg	[K257 Morphism (degree 0): K247 -> K251]
rh	[K258 Morphism (degree 1): K251 -> K251]
rrdct	[K260 Reduction K251 => K247]
tcc	[K251 Chain-Complex]

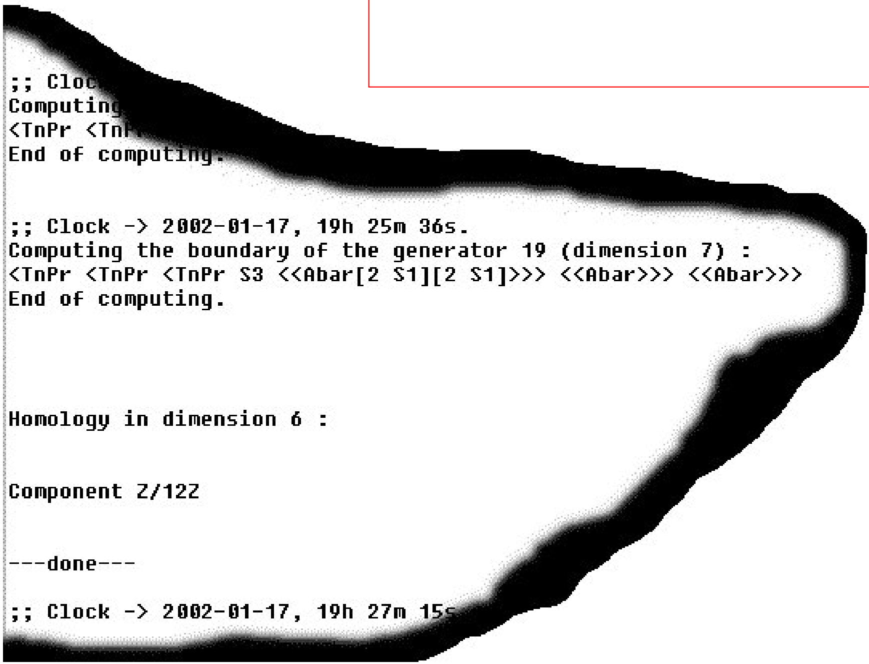
fixnum | fixnum = #x105

$$P_4 = P^\infty\mathbb{R}/P^3\mathbb{R}$$

$$X = O^2 P_4 = \Omega^2(P_4) = \mathbb{C}(S^2, P_4)$$

$$C_* X = K_{18}$$

The END


;; Cloc
Computing
<TnPr <TnP
End of computing.
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
Homology in dimension 6 :
Component Z/12Z
---done---
;; Clock -> 2002-01-17, 19h 27m 15s