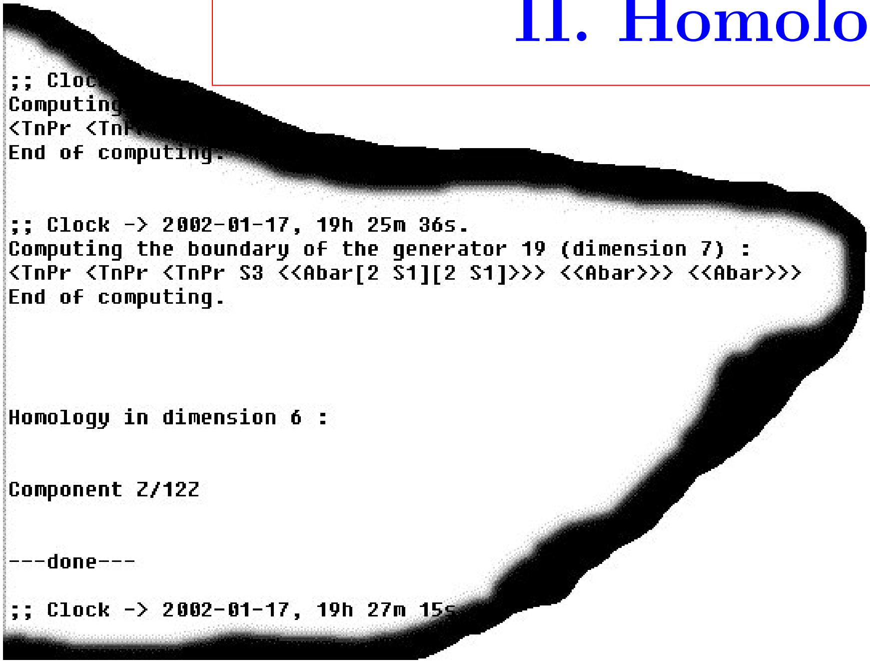
Algebraic Topology

(Castro-Urdiales tutorial)

II. Homological algebra



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Computing
<TnPr <TnP
End of computing.


;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.



Homology in dimension 6 :


Component Z/12Z


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Castro-Urdiales, January 9-13, 2006

General motivation of Algebraic Topology:

1. **Topology is very complicated.**
2. **Algebra is easier.**
3. **Is it possible to transform a topological problem into an algebraic one?**

Prototype example:

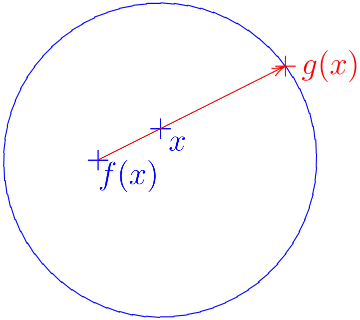
Theorem (Brouwer): $f : D^n \longrightarrow D^n$ continuous

$$\Rightarrow \exists x \in D^n \text{ st } f(x) = x.$$

1. Otherwise $\exists g : D^n \longrightarrow S^{n-1}$ st $g(x) = x$ if $x \in S^{n-1}$.

2. $\Leftrightarrow \exists g : D^n \longrightarrow S^{n-1}$ st :

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "\text{id}"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$



3. Translation through the H_{n-1}-functor:

$$\begin{tikzcd} H_{n-1}(S^{n-1}) = \mathbb{Z} \arrow[r, "0"] \arrow[rr, bend right, "\text{id}"'] & H_{n-1}(D^n) = 0 \arrow[r, "0"] & H_{n-1}(S^{n-1}) = \mathbb{Z} \end{tikzcd}$$

4. Impossible!

Main technique:

Topological Object ↦ Combinatorial Object ↦ Chain Complex

Chain complex:

1. “Algebraic” object associated to a combinatorial object.
2. Intermediate object to produce homology groups describing some fundamental properties of the initial topological object.
3. Many further structures can be installed on it.

Definition: A chain complex C_* is a sequence:

$$C_* = (\{C_m\}_{m\in\mathbb{N}}, \{d_m\}_{m\in\mathbb{N}}) \text{ where:}$$

1. C_m is an Abelian group ($= \mathbb{Z}$-module).
2. $d_m : C_m \rightarrow C_{m-1}$ is the differential (or boundary map), a $\mathbb{Z}$-linear operator.
3. $\forall m \in \mathbb{Z}$ the composition $d_m d_{m+1} : C_{m+1} \rightarrow C_{m-1}$ is null.

$$\begin{tikzcd}
\cdots & C_{m-2} \arrow[l, "d_{m-2}"'] & C_{m-1} \arrow[l, "d_{m-1}"'] \arrow[ll, bend left, "0"] & C_m \arrow[l, "d_m"'] \arrow[ll, bend left, "0"] & C_{m+1} \arrow[l, "d_{m+1}"'] \arrow[ll, bend left, "0"] & \cdots \arrow[l, "d_{m+2}"'] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

General organization of a chain complex.

$$\begin{tikzcd}
 & C_{m-1} \arrow[dddl] & C_m \arrow[dddl, "d_m"'] & C_{m+1} \arrow[dddl, "d_{m+1}"'] & {} \arrow[dddl] \\
 & \cup & \cup & \cup & \\
 & Z_{m-1} \arrow[ddddl] & Z_m \arrow[ddddl, "d_m"'] & Z_{m+1} \arrow[ddddl, "d_{m+1}"'] & {} \arrow[ddddl] \\
\cdots & \cup & \cup & \cup & \cdots \\
 & B_{m-1} & B_m & B_{m+1} & \\
 & \cup & \cup & \cup & \\
{} & 0 & 0 & 0 &
\end{tikzcd}$$

where: $Z_m = \ker d_m = d_m^{-1}(0)$ (**m-cycles**)

and $B_m = \operatorname{im} d_{m+1} = d_{m+1}(C_{m+1})$ (**m-boundaries**).

Finally: $H_m(C_*) = \dfrac{Z_m}{B_m} = \dfrac{\ker d_m}{\operatorname{im} d_{m+1}} = \dfrac{\{m\text{-cycles}\}}{\{m\text{-boundaries}\}}$.

Remark: $B_m \subset Z_m \Leftrightarrow d_m d_{m+1} = 0$.

Main example:

X = given simplicial set.

$\Rightarrow C_*(X)$ = chain complex canonically associated to X.

$C_m(X) := \mathbb{Z}[X_m]$ and $d(\sigma) := \sum_{i=0}^m (-1)^m \partial_i^m(\sigma)$.

$H_m(X) := H_m(C_*(X))$.

Equivalent version: $C_*^{ND}(X)$ with:

$C_m^{ND}(X) := \mathbb{Z}[X_m^{ND}]$ and $d(\sigma) := \sum_{i=0}^m (-1)^m \partial_i^m(\sigma \bmod ND)$.

$H_m^{ND}(X) := H_m(C_*^{ND}(X)) \overset{\text{thr}}{=} H_m(X)$.

Extremal situations.

Pseudo-chain complex = Zero differential $\Leftrightarrow$ $H_m = C_m$.

$$\begin{tikzcd}
 & C_{m-1} = Z_{m-1} & C_m = Z_m \arrow[ddl, "d_m=0"'] & C_{m+1} = Z_{m+1} \arrow[ddl, "d_{m+1}=0"'] & \\
\cdots & & & & \cdots \\
 & 0 = B_{m-1} & 0 = B_m & 0 = B_{m+1} &
\end{tikzcd}$$

Exact sequence = Chain complex with $Z_m = B_m$ $\Leftrightarrow$ $H_m = 0$.

$$\begin{tikzcd}
 & C_{m-1} & C_m \arrow[dl, "d_m"'] & C_{m+1} \arrow[dl, "d_{m+1}"'] & \\
\cdots & Z_{m-1} = B_{m-1} & Z_m = B_m \arrow[dl, "d_m"] & Z_{m+1} = B_{m+1} \arrow[dl, "d_{m+1}"] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

H_m measures the lack of exactness.

Contractible chain complex ($\sim$ exact but $\boxed{\neq \text{ exact}}$).

Contractible := $\exists\{h_m : C_m \to C_{m+1}\}_{m\in\mathbb{Z}}$ **with** $dh + hd = \text{id}$.

1. **Contractible** $\Rightarrow$ **Exact.**

$c \in C_m \Rightarrow c = (dh + hd)c = dhc + hdc$.

$c \in Z_m \Rightarrow dc = 0 \Rightarrow c = dhc \in B_m \Rightarrow c \in B_m$.

$\Rightarrow Z_m = B_m \Leftrightarrow H_m = 0$.

2. **Exact** $\boxed{\boxed{\not\Rightarrow}}$ **Contractible.**

Example: $\cdots \longleftarrow 0 \longleftarrow \mathbb{Z}_2 \xleftarrow{\text{pr}} \boxed{\mathbb{Z}} \xleftarrow{\times 2} \mathbb{Z} \longleftarrow 0 \cdots$

But 'pr' has no section $\Rightarrow$ Chain complex non-contractible.

Computation of homology groups ?

Elementary when the chain complexes are of finite type.

But very frequently, chain complexes are not ($S^1 \sim K(\mathbb{Z}, 1)$).

$\Rightarrow$ **Homological Algebra**

“Methods of Homological Algebra” (Gelfand + Manin)

Preface extract:

> ***The book by Cartan and Eilenberg* [Homological Algebra] *contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences.***

Typical "computational tool" of homological algebra:

Theorem: Let $\boxed{0 \to A_* \xrightarrow{\{f_m\}} B_* \xrightarrow{\{g_m\}} C_* \to 0}$ be a short exact sequence of chain complexes. Then there is a canonical exact sequence:

$$\cdots \to H_{m+1}(C_*) \to H_m(A_*) \to H_m(B_*) \to H_m(C_*) \to H_{m-1}(A_*) \to \cdots$$

"Application": $H_m(A_*)$ and $H_m(C_*)$ known $\forall m$.

In particular $H_2(C_*) = H_0(A_*) = 0$, $H_1(A_*) = H_1(C_*) = \mathbb{Z}_2$.

Then $H_1(B_*) = ???$

$$\cdots \to 0 \to \mathbb{Z}_2 \to H_1(B_*) \to \mathbb{Z}_2 \to 0 \to \cdots$$

$\Rightarrow H_1(B_*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$???

Find something else!!

Example:

Jean-Pierre Serre " "computing" " $\pi_6(S^3)$ in 1950.

Serre spectral sequence $\Rightarrow$ there exists an exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_6(S^3) \longrightarrow \mathbb{Z}_6 \longrightarrow 0$$

Two different extensions are possible ($\mathbb{Z}_2 \oplus \mathbb{Z}_6$ $\boxed{\text{or}}$ $\mathbb{Z}_{12}$?). The right one is determined by $\tau \in H^2(\mathbb{Z}_6, \mathbb{Z}_2) = \mathbb{Z}_2$ where:

1. **The class τ is mathematically well defined;**
2. **The class τ is computationally *unreachable* in the framework of the Serre spectral sequence.**

Corollary: The group $\pi_6(S^3)$ remained *unknown* in Serre's work in 1950.

Finally determined by Barratt and Paechter in 1952 thanks to *new specific methods* ($= \mathbb{Z}_{12}$).

Now "stupidly" $\boxed{\boxed{\text{computed}}}$ by the Kenzo program in one minute.

The simplest example

to understand the nature of the problem.

Chain complex:

$$B_* \left\{ \begin{tikzcd} \cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] \arrow[dd, dotted, dash, "deg=0"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[ddl, "\times\alpha"] \arrow[dd, dotted, dash, "deg=1"] & 0 \arrow[l] & \cdots \arrow[l] & \}C_* \\ \cdots & & & & & \cdots & \\ \cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\times 2"] & 0 \arrow[l] & \cdots \arrow[l] & \}A_* \end{tikzcd} \right.$$

$$\Leftrightarrow \qquad \cdots \longleftarrow 0 \longleftarrow \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 2 & 0 \\ \alpha & 2 \end{bmatrix}} \mathbb{Z}^2 \longleftarrow 0 \longleftarrow \cdots$$

$\Rightarrow$ Short exact sequence of chain complexes:

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_*(= B_*/A_*) \longrightarrow 0$$

$$B_* \left\{ \begin{tikzcd} \cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] \arrow[dd, dash, dotted] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[dd, dash, dotted] \arrow[ddl, "\times\alpha"] & 0 \arrow[l] & \cdots \arrow[l] & \}C_* \\ \cdots & deg{=}0 & & & deg{=}1 & \cdots & \\ \cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\times 2"] & 0 \arrow[l] & \cdots \arrow[l] & \}A_* \end{tikzcd} \right.$$

Challenge: $H_*(A_*)$ and $H_*(C_*)$ known $\Rightarrow H_*(B_*) = ???$

$H_0(A_*) = H_0(C_*) = \mathbb{Z}_2$, $H_m(A_*) = H_m(C_*) = 0 \ \ \forall m \neq 0$.

Long exact sequence of homology $\Rightarrow$

$$\cdots \leftarrow 0 \leftarrow \mathbb{Z}_2 \leftarrow H_0(B_*) \leftarrow \mathbb{Z}_2 \leftarrow 0 \leftarrow \cdots$$

$\Rightarrow$ $\boxed{\text{Two}}$ possible $H_0(B_*)$: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$.

How to determine the right choice ???

Standard extension group theory:

$$0 \leftarrow \mathbb{Z}_2 \leftarrow E \leftarrow \mathbb{Z}_2 \leftarrow 0$$

The extension is determined by

a cohomology class $\tau \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.

$$\begin{tikzcd}
0 & 1 \arrow[l, mapsto] & a \arrow[l, mapsto] & \\
 & 0 & 2a \arrow[l, mapsto] & b \arrow[l, mapsto]
\end{tikzcd}$$

Rule: Consider $1 \in \mathbb{Z}_2$, then an arbitrary preimage $a \in E$;

Certainly the image of $2a$ is 0;

Exactness $\Rightarrow$ $2a$ is the image of a unique $b \in \mathbb{Z}_2$.

If $b = 0$, then $E = \mathbb{Z}_2 \oplus \mathbb{Z}_2$;

If $b = 1$, then $E = \mathbb{Z}_4$.

But E is unknown!

Solution: Instead of working with homology classes,

work with cycles representing them.

$$\begin{tikzcd}
\boxed{1} & {} \arrow[l, "\times 2"'] \arrow[dl, "\times \alpha"] \\
{} & {} \arrow[l, "\times 2"]
\end{tikzcd}$$

$1 \in H_0(C_*)$

$$\begin{tikzcd}
\boxed{1} \arrow[d, dash, dotted] & {} \arrow[l, "\times 2"'] \arrow[dl, "\times \alpha"] \\
\boxed{0} & {} \arrow[l, "\times 2"]
\end{tikzcd}$$

$a \in H_0(B_*)$?

$$\begin{tikzcd}
\boxed{2} \arrow[d, dash, dotted] & {} \arrow[l, "\times 2"'] \arrow[dl, "\times \alpha"] \\
\boxed{0} & {} \arrow[l, "\times 2"]
\end{tikzcd}$$

$2a \in H_0(B_*)$?

$$\begin{tikzcd}
\boxed{0} \arrow[d, dash, dotted] & \boxed{-1} \arrow[l, "\times 2"'] \arrow[dl, "\times \alpha"] \\
\boxed{-\alpha} & {} \arrow[l, "\times 2"]
\end{tikzcd}$$

$2a \in H_0(B_*)$?

$$\begin{tikzcd}
{} & {} \arrow[l, "\times 2"'] \arrow[dl, "\times \alpha"] \\
\boxed{-\alpha} & {} \arrow[l, "\times 2"]
\end{tikzcd}$$

$b \in H_0(A_*)$

Conclusion:

α **even** $\Rightarrow$

$$H_0(B_*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

α **odd** $\Rightarrow$

$$H_0(B_*) = \mathbb{Z}_4$$

$$\begin{tikzcd}
 & \cdots & \mathbf{0} \arrow[l] & \mathbb{Z} \arrow[l] \arrow[dd, dash, dotted, "deg=0"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[ddl, "\times\alpha"] \arrow[dd, dash, dotted, "deg=1"] & \mathbf{0} \arrow[l] & \cdots \arrow[l] & \}C_* \\
B_* \Bigg\{ & \cdots & & & & & \cdots & \\
 & \cdots & \mathbf{0} \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\times 2"] & \mathbf{0} \arrow[l] & \cdots \arrow[l] & \}A_*
\end{tikzcd}$$

Lifting homology classes to explicit cycles gives a solution.

A little more general situation:

$$\begin{tikzcd}
 & \cdots & \mathbf{0} \arrow[l] & \mathbb{Z}^\infty \arrow[l] \arrow[dd, dash, dotted, "deg=0"'] & \mathbb{Z}^\infty \arrow[l, "f"'] \arrow[ddl, "g"] \arrow[dd, dash, dotted, "deg=1"] & \mathbf{0} \arrow[l] & \cdots \arrow[l] & \}C_* \\
B_* \Bigg\{ & \cdots & & & & & \cdots & \\
 & \cdots & \mathbf{0} \arrow[l] & \mathbb{Z}^\infty \arrow[l] & \mathbb{Z}^\infty \arrow[l, "h"] & \mathbf{0} \arrow[l] & \cdots \arrow[l] & \}A_*
\end{tikzcd}$$

$$H_0 A_* = H_0(C_*) = \mathbb{Z}_2 \Rightarrow H_0(B_*) = ???$$

Same solution if it is possible to work in A_*, B_* and C_*.

Notion of effective (free $\mathbb{Z}$-) chain complex :

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \overset{d_n}{\leftarrow} C_n \overset{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\beta, d)$$

where:

1. β: $\mathbb{Z} \rightarrow \mathcal{L}\mathrm{ist}$: $n \mapsto [g_1^n, \ldots, g_{k_n}^n]$ = distinguished basis of C_n.
2. d: $\mathbb{Z} \widetilde{\times} \mathbb{N}_* \rightarrow \mathcal{U}$: $(n, i) \mapsto d_n(g_i^n) \in C_{n-1}$ when g_i^n makes sense.

In particular every C_n is a free $\mathbb{Z}$-module with a finite distinguished basis.

$\Rightarrow$ Every $d_n : C_n \rightarrow C_{n-1}$ is entirely computable.

$\Rightarrow$ Every homology group $H_n(C_*)$ is computable

(every global information is reachable).

Notion of $\boxed{\boxed{\text{locally}}\ \text{effective}}$ chain complex:

$$C_* = \boxed{\ldots \leftarrow C_{n-1} \overset{d_n}{\leftarrow} C_n \overset{d_{n+1}}{\leftarrow} C_{n+1} \leftarrow \ldots}$$

$$C_* = (\chi, d)$$

where:

1. χ: $\mathcal{U} \times \mathbb{Z} \rightarrow \text{Bool} = \{\top, \bot\} : (\omega, n) \mapsto \top$
   if and only if ω is a generator of C_n;

2. d: $\mathcal{U} \widetilde{\times} \mathbb{Z} \rightarrow \mathcal{U} \ : (\omega, n) \mapsto d_n(\omega) \in C_{n-1}$
   when ω is a generator of C_n $(\Leftrightarrow \chi(\omega, n) = \top)$.

Any finite set of “$\boxed{\text{generator-wise}}$” computations may be done.

Gödel + Church + Turing + Post ⇒ no global information is reachable.

In particular, the homology groups of C_* are $\boxed{\textit{not computable}}$.

Main ingredients of the solution $\boxed{\text{Effective Homology}}$ for Constructive Algebraic Topology:

✓ **Simplicial Topology.**

✓ **Effective chain complexes.**

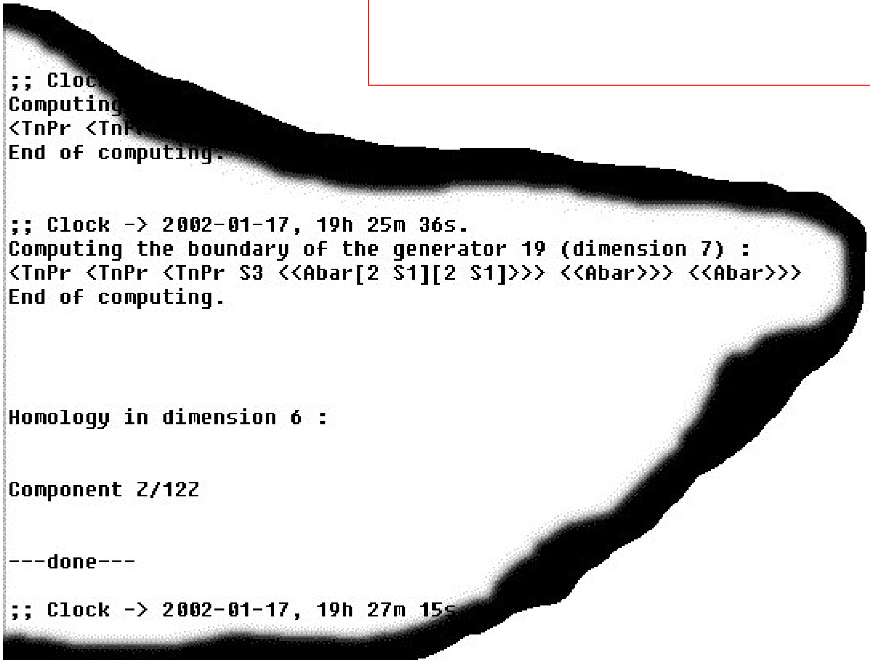
✓ **Locally effective chain complexes.**

Functional Programming.

Homological reductions.

Homological perturbation theory.

The END



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