Discrete Vector Fields and Fundamental Algebraic Topology.

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Chapter 1

Introduction.

We show in this text how the most important homology equivalences of fundamental Algebraic Topology can be obtained as *reductions* associated to *discrete vector fields*. Mainly the homology equivalences whose *existence* — most often non-constructive – is proved by the main spectral sequences, the Serre and Eilenberg-Moore spectral sequences. On the contrary, the *constructive* existence is here systematically looked for and obtained.

Algebraic topology consists in applying *algebraic* methods to study *topological* objects. Algebra is assumed to be more tractable than Topology, and the motivation of the method is clear.

Algebraic Topology sometimes reduces non-trivial topological problems to some algebraic problems which, in favourable cases, can be solved. For example, the Brouwer theorem is reduced to the impossibility of factorizing the identity $\mathbb{Z} \to \mathbb{Z}$ into a composition $\mathbb{Z} \to 0 \to \mathbb{Z}$. Magic!

Algebraic topology leads to more and more sophisticated algebraic translations, think for example of the spectral sequences, derived categories, E_∞-operads, ... Almost all the topologists are somewhat transformed into algebraists, specialists in *Homological Algebra*. This is so true that sometimes it happens some elementary topological methods are neglected. These "elementary" methods can however be very powerful, in particular when reexamining Algebraic Topology with a *constructive* view.

The *constructive* point of view, so new in this area, forces the algebraic topologist to carefully reexamine the very bases of his methods. Because of the deep connection "constructive" = "something which can be processed on a computer" and because a computer is unable to solve sophisticated problems by itself, a constructive version of some theory must be split into elementary steps, elementary enough to be in the scope of a computer programming language.

For example in the paper [19], the classical Serre and Eilenberg-Moore spectral sequences have been replaced by a more elementary tool, the *Homological Perturbation Theorem*[1]. Elementary enough to be easily installed on a computer and

[1]Usually called the *Basic Perturbation Lemma*, a strange terminology when we observe its

it was so possible to obtain homology and homotopy groups so far unreachable. Working with this elementary result allowed us to replace the – non-constructive – Serre and Eilenberg-Moore spectral sequences by a *constructive* process, more elementary but constructive, and finally more powerful.

This paper goes along the same line. The most elementary tool in homotopy, the Whitehead contraction, is systematically studied to obtain the main results of fundamental algebraic topology.

The basic tool is a direct adaptation of the so-called *discrete Morse theory*. More precisely the notion of discrete vector field is described here in an algebraic setting; it is nothing but a rewriting of the main part of Robin Forman's wonderful paper [9]. A rewriting taking account of the long experience learned when designing our methods of *effective homology* [24, 4, 19, 20, 21, 22]. The main result of this part is Theorem 19, the Vector-Field Reduction theorem; it is implicitly contained in [9], but we hope the presentation given here through the essential notion of (homological) reduction, see Section 2.4, should interest the reader. We use again here the Homological Perturbation Theorem to obtain a convenient, direct and efficient proof of this result.

Then the various homology equivalences which are the very bases of Algebraic Topology are restudied and proved to be in fact direct consequences of this elementary theorem. Mainly, the normalization theorems, the Eilenberg-Zilber theorem – the ordinary one and the *twisted* one as well –, the Bar and Cobar reductions which are essential in the Eilenberg-Moore spectral sequences to compute the *effective* homology of classifying spaces and loop spaces. Obvious applications to the homological analysis of digital images are also included.

This gives a very clear and simple understanding of all these results under a form of combinatorial game playing with the *degeneracy operators*. Since the remarkable works by Sam Eilenberg and Saunders MacLane in the fifties, by Daniel Kan in the sixties, these operators have somewhat been neglected. We hope the results obtained here show the work around the combinatorial nature of these operators is far from being finished.

This is of course interesting for our favourite theory, but other applications are expected: this simple way to understand our main homology equivalences gives also new methods of programming: the heart of the method is extremely simple and allows the programmer to carefully concentrate his work on the very kernel of the method. Compare for example the method which was used up to now to program the Eilenberg-Moore spectral sequences, for example [2, Section 4], with the *direct reductions* which are now very simply obtained in Chapters 7 and 8. The programming experiences already undertaken with respect to the Eilenberg-Zilber reduction, in particular in the twisted case, are very encouraging.

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importance now, fifty years after its discovery by Shih Weishu [26].

Chapter 2

Discrete vector fields.

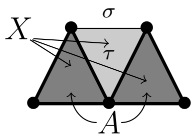
2.1 W-contractions.

Definition 1 — An *elementary W-contraction* is a pair (X, A) of simplicial sets, satisfying the following conditions:

1. The component A is a *simplicial subset* of the simplicial set X.
2. The difference $X - A$ is made of exactly *two* non-degenerate simplices $\tau \in X_n$ and $\sigma \in X_{n-1}$, the second one σ being a *face* of the first one τ.
3. The incidence relation $\sigma = \partial_i \tau$ holds for a *unique* index $i \in 0...n$.

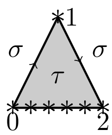
It is then said A is obtained from X by an elementary W-contraction, and X is obtained from A by an elementary W-extension. ♣

For example, X could be made of three triangles and A of two only as in the next figure.



(2.1)

The condition 3 is necessary – and sufficient – for the existence of a topological contraction of X on A. Think for example of the minimal triangulation of the real projective plane $P^2\mathbb{R}$ as a simplicial set X, see the next figure: one vertex $*$, one edge σ and one triangle τ; no choice for the faces of σ; the faces of τ must be $\partial_0 \tau = \partial_2 \tau = \sigma$ and $\partial_1 \tau = \eta_0 *$ is the degeneracy of the base point. The realization of X is homeomorphic to $P^2\mathbb{R}$. If you omit the condition 3 in the definition of W-contraction, then $(X, *)$ would be a W-contraction, but $P^2\mathbb{R}$ is not contractible.



(2.2)

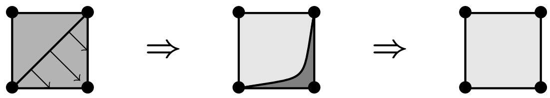
Definition 2 — A *W-contraction* is a pair (X, A) of simplicial sets satisfying the following conditions:

1. The component A is a *simplicial subset* of the simplicial set X.
2. There exists a sequence $(A_i)_{0 \leq i \leq m}$ with:
   (a) $A_0 = A$ and $A_m = X$.
   (b) For every $0 < i \leq m$, the pair (A_i, A_{i-1}) is an elementary W-contraction. ♣

In other words, a W-contraction is a finite sequence of elementary contractions. If (X, A) is a W-contraction, then a topological contraction $X \to A$ can be defined.

'W' stands for J.H.C. Whitehead, who undertook [28] a systematic study of the notion of *simple homotopy type*, defining two simplicial objects X and Y as having the same simple homotopy type if they are equivalent modulo the equivalence relation generated by the elementary W-contractions and W-extensions.

Another kind of modification when examining a topological object can be studied. Let us consider the usual triangulation of the square with two triangles, the square cut by a diagonal. Then it is tempting to modify this triangulation by pushing the diagonal onto two sides as roughly described in this figure.


(2.3)

Why not, but this needs other kinds of cells, here a square with *four* edges, while in a simplicial framework, the only objects of dimension 2 that are provided are the triangles Δ^2. Trying to overcome this essential obstacle leads to two major subjects:

1. The Eilenberg-Zilber theorem, an algebraic translation of this idea, which consists in *algebraically* allowing the use of simplex products.
2. The CW-complex theory, where the added cells are attached to the previously constructed object through *arbitrary* attaching maps.

This paper systematically reconsiders these essential ideas through the notion of *discrete vector field*.

2.2 Algebraic discrete vector fields.

The notion of discrete vector field (DVF) is due to Robin Forman [9]; it is an essential component of the so-called *discrete Morse theory*. It happens the notion of DVF will be here the major tool to treat the *fundamental* problems of algebraic topology, more precisely to treat the general problem of *constructive* algebraic topology.

This notion is usually described and used in combinatorial *topology*, but a purely algebraic version can also be given; we prefer this context.

Definition 3 — An *algebraic cellular complex* (ACC) is a family:

$$C = (C_p, d_p, \beta_p)_{p \in \mathbb{Z}}$$

of *free* $\mathbb{Z}$-modules and boundary maps. Every C_p is called a *chain group* and is provided with a *distinguished* $\mathbb{Z}$-basis β_p; every basis component $\sigma \in \beta_p$ is a *p-cell.* The boundary map $d_p : C_p \to C_{p-1}$ is a $\mathbb{Z}$-linear map connecting two consecutive chain groups. The usual boundary condition $d_{p-1}d_p = 0$ is satisfied for every $p \in \mathbb{Z}$. ♣

Most often we omit the index of the differential, so that the last condition can be denoted by $d^2 = 0$. The notation is redundant: necessarily, $C_p = \mathbb{Z}[\beta_p]$, but the standard notation C_p for the group of p-chains is convenient.

The chain complex associated to any sort of topological cellular complex is an ACC. We are specially interested in the chain complexes associated to simplicial sets.

Important: we *do not* assume *finite* the distinguished bases β_p, the chain groups are not necessarily of *finite type.* This is not an artificial extension to the traditional Morse theory: this point will be often essential, but this extension is obvious.

Definition 4 — Let C be an ACC. A $(p-1)$-cell σ is said to be a *face* of a p-cell τ if the coefficient of σ in $d\tau$ is non-null. It is a *regular face* if this coefficient is +1 or -1. ♣

If Δ^p is the standard simplex, every face of every subsimplex is a regular face of this subsimplex. We gave after Definition 1 an example of triangulation of the real projective plane as a simplicial set; the unique non-degenerate 1-simplex σ is *not* a regular face of the triangle τ, for $d\tau = 2\sigma$.

Note also the *regular* property is *relative*: σ can be a regular face of τ but also a non-regular face of another simplex τ'.

Definition 5 — A *discrete vector field* V on an algebraic cellular complex $C = (C_p, d_p, \beta_p)_{p \in \mathbb{Z}}$ is a collection of pairs $V = \{(\sigma_i, \tau_i)\}_{i \in \beta}$ satisfying the conditions:

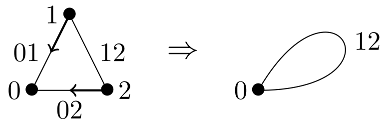
1. Every σ_i is some p-cell, in which case the other corresponding component τ_i is a $(p+1)$-cell. The degree p depends on i and in general is not constant.
2. Every component σ_i is a *regular* face of the corresponding component τ_i.
3. A cell of C appears *at most one time* in the vector field: if $i \in \beta$ is fixed, then $\sigma_i \neq \sigma_j$, $\sigma_i \neq \tau_j$, $\tau_i \neq \sigma_j$ and $\tau_i \neq \tau_j$ for every $i \neq j \in \beta$. ♣

It is not required all the cells of C appear in the vector field V. In particular the void vector field is allowed. In a sense the remaining cells are the most important.

Definition 6 — A cell χ which does not appear in a discrete vector field $V = \{(\sigma_i, \tau_i)\}_{i \in \beta}$ is called a *critical* cell. A component (σ_i, τ_i) of the vector field V is a *p-vector* if σ_i is a p-cell. ♣

We do not consider in this paper the traditional vector fields of differential geometry, which allows us to call simply a *vector field* which should be called a *discrete vector field*.

In case of an ACC coming from a topological cellular complex, a vector field is a recipe to cancel "useless" cells in the underlying space, useless with respect to the homotopy type. A component (σ_i, τ_i) of a vector field can vaguely be thought of as a "vector" starting from the center of σ_i, going to the center of τ_i. For example $\partial\Delta^2$ and the circle have the same homotopy type, which is described by the following scheme:

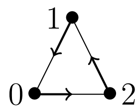


(2.4)

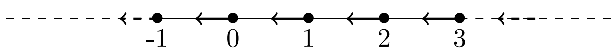
The initial simplicial complex is made of three 0-cells 0, 1 and 2, and three 1-cells 01, 02 and 12. The drawn vector field is $V = \{(1, 01), (2, 02)\}$, and this vector field defines a homotopy equivalence between $\partial\Delta^2$ and the minimal triangulation of the circle as a simplicial set. The last triangulation is made of the critical cells 0 and 12, attached according to a process which deserves to be seriously studied in the general case. This paper is devoted to a systematic use of this idea.

2.3 V-paths and admissible vector fields.

By the way, what about this vector field in $\partial\Delta^2$?



No critical cell and yet $\partial\Delta^2$ does not have the homotopy type of the void object. We must forbid possible *loops*. This is not enough. Do not forget the infinite case must be also covered; but look at this picture:



(2.5)

representing an infinite vector field on the real line triangulated as an infinite union of 1-cells connecting successive integers. No critical cell and yet the real line does not have the homotopy type of the void set. We must also forbid the possible *infinite* paths.

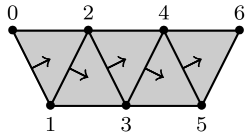
The notions of V-paths and admissible vector fields are the appropriate tools to define the necessary restrictions.

Definition 7 — If $V = \{(\sigma_i, \tau_i)\}_{i\in\beta}$ is a vector field on an algebraic cellular complex $C = (C_p, d_p, \beta_p)_p$, a V-path of degree p is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0\leq k<m}$ satisfying:

1. Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of the vector field V and the cell τ_{i_k} is a p-cell.

2. For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.

If $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ is a V-path, and if σ is a face of $\tau_{i_{m-1}}$ different from $\sigma_{i_{m-1}}$, then π *connects* σ_{i_0} and σ through the vector field V. ♣



(2.6)

A V-path connecting the edges 01 and 56.

In a V-path $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ of degree p, a $(p-1)$-cell σ_{i_k} is a regular face of τ_{i_k}, for the pair $(\sigma_{i_k}, \tau_{i_k})$ is a component of the vector field V, but the same σ_{i_k} is non-necessarily a regular face of $\tau_{i_{k-1}}$.

Definition 8 — The *length* of the path $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ is m. ♣

If (σ, τ) is a component of a vector field, in general the cell τ has *several* faces different from σ, so that the possible paths starting from a cell generate an oriented graph.

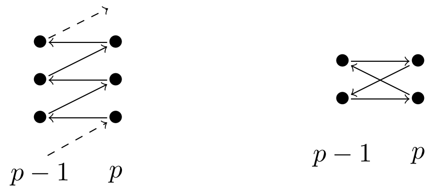
Definition 9 — A discrete vector field V on an algebraic cellular complex $C = (C_p, d_p, \beta_p)_{p \in \mathbb{Z}}$ is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$. ♣

Excluding infinite paths is almost equivalent. The difference between both possibilities is measured by Markov's principle; we prefer our more constructive statement.

A circular path would generate an infinite path and is therefore excluded.

The next diagram, an *oriented bipartite graph*, can help to understand this notion of admissibility for some vector field V. This notion makes sense degree by degree. Between the degrees p and $p-1$, organize the *source* $(p-1)$-cells (resp. *target* p-cells) as a lefthand (resp. righthand) column of cells. Then every vector $(\sigma, \tau) \in V$ produces an oriented edge $\sigma \rightarrowtail \tau$. In the reverse direction, if τ is a target p-cell, the boundary $d\tau$ is a finite linear combination $d\tau = \sum \alpha_i \sigma_i$, and some of these σ_i's are source cells, in particular certainly the corresponding V-source cell σ. For every such source component σ_i, be careful *except* for the corresponding source σ, you install an oriented edge $\sigma_i \leftarrowtail \tau$.

Then the vector field is admissible between the degrees $p-1$ and p if and only if, starting from every source cell σ, all the (oriented) paths have a length bounded by some integer $\lambda_p(\sigma)$. In particular, the loops are excluded. We draw the two simplest examples of vector fields non-admissible. The lefthand one has an infinite path, the righthand one has a loop, a particular case of infinite path.



(2.7)

Definition 10 — Let $V = \{(\sigma_i, \tau_i)_{i\in\beta}\}$ be a vector field on an ACC. A *Lyapunov function* for V is a function $L : \beta \to \mathbb{N}$ satisfying the following condition: if σ_j is a face of τ_i different from σ_i, then $L(j) < L(i)$. ♣

It is the natural translation in our discrete framework of the traditional notion of Lyapunov function in differential geometry. It is clear such a Lyapunov function proves the admissibility of the studied vector field. Obvious generalizations to ordered sets more general than $\mathbb{N}$ are possible.

2.4 Reductions.

2.4.1 Definition.

Definition 11 — A (homology) *reduction*[1] ρ is a diagram:

$$\rho = \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*} \tag{2.8}$$

where:

1. The nodes $\widehat{C}_*$ and C_* are chain complexes;
2. The arrows f and g are chain complex morphisms;
3. The self-arrow h is a homotopy operator, of degree $+1$;
4. The following relations are satisfied:

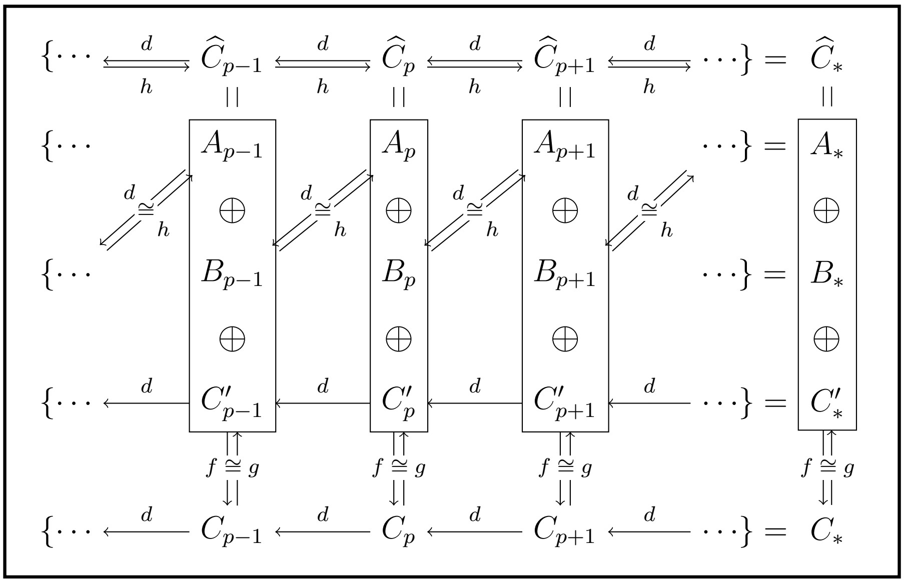
$$\begin{aligned} fg &= \mathrm{id}_{C_*} \\ gf + dh + hd &= \mathrm{id}_{\widehat{C}_*} \\ fh &= 0 \\ hg &= 0 \\ hh &= 0 \end{aligned} \tag{2.9}$$

♣

The relation $fg = \mathrm{id}$ implies that g identifies the *small* chain complex C_* with the subcomplex $C'_* := g(C_*)$ of the *big* chain complex $\widehat{C}_*$. Furthermore the last

[1] Often called *contraction*, but this terminology is not appropriate: it is important to understand such a reduction has an *algebraic* nature, like when you *reduce* $6/4 \mapsto 3/2$. When reducing a rational number, you cancel the opposite roles of a common factor in numerator and denominator; in our homology reductions we cancel the opposite roles of the A_* and B_* components in the big chain complex to obtain the small one, see Figure 2.10.

one gets a canonical decomposition $\widehat{C}_* = \mathrm{im}(g) \oplus \ker(f)$. The relations $hg = 0$ and $fh = 0$ imply the homotopy operator h is null on $\mathrm{im}(g) = C'_*$ and its image is entirely in $\ker(f)$: the h map is in fact defined on $\ker(f)$, extended by the zero map on C'_*. Finally $dh + hd$ is the identity map on $\ker(f)$. Also, because of the relation $h^2 = 0$, the homotopy h is a codifferential and the pair (d, h) defines a Hodge decomposition $\ker(f) = A_* \oplus B_*$ with $A_* = \mathrm{im}(h) = \ker(f) \cap \ker(h)$ and $B_* = \ker(f) \cap \ker(d) = \ker(f) \cap \mathrm{im}(d)$. The direct sum $\ker(f) = A_* \oplus B_*$ is a subcomplex of $\widehat{C}_*$, but both components A_* and B_* are only graded modules. These properties are illustrated in this diagram.



(2.10)

$A_* = \ker f \cap \ker h$ $\quad$ $C'_* = \mathrm{im}\, g$ $\quad$ $B_* = \ker f \cap \ker d$

We will simply denote such a reduction by $\rho = (f_\rho, g_\rho, h_\rho) : \widehat{C}_* \Rrightarrow C_*$ or simply by $\rho : \widehat{C}_* \Rrightarrow C_*$.

A reduction $\rho = (f_\rho, g_\rho, h_\rho) : \widehat{C}_* \Rrightarrow C_*$ establishes a strong connection between the chain complexes $\widehat{C}_*$ and C_*: it is a particular quasi-isomorphism between chain complexes describing the big one $\widehat{C}_*$ as the direct sum of the small one $C'_* = C_*$ and another chain-complex $\ker(f)$, the latter being provided with an *explicit* null-reduction h. The morphisms f and g are inverse homology equivalences[2].

The main role of a reduction is the following. It often happens the big complex $\widehat{C}_*$ is so enormous that the homology groups of this complex are out of scope of computation; we will see striking examples where the big complex is *not of finite type*, so that the homology groups $H_*\widehat{C}_*$ are not computable from $\widehat{C}_*$, even in theory. But if on the contrary the small complex C_* has a reasonable size, then its homology groups are computable and they are canonically isomorphic to those of the big complex.

More precisely, if the *homological problem* is solved in the small complex, then

[2] Often called *chain equivalences*, yet they do not define an equivalence between *chains* but between *homology* classes.

the reduction produces a solution of the same problem for the big complex. See [22, Section 4.4] for the definition of the notion of *homological problem*. In particular, if $z \in C_p$ is a cycle representing the homology class $\mathfrak{h} \in H_p(C_*)$, then gz is a cycle representing the corresponding class in $H_p(\widehat{C}_*)$. Conversely, if z is a cycle of $\widehat{C}_p$, then the homology class of this cycle corresponds to the homology class of fz in $H_p(C_*)$. If ever this homology class is null and if $c \in C_{p+1}$ is a boundary-preimage of fz in the small complex, then $gc + hz$ is a boundary-preimage of z in the big complex.

Theorem 12 — *If $\rho = (f, g, h) : \widehat{C}_* \Rrightarrow C_*$ is a reduction between the chain complexes $\widehat{C}_*$ and C_*, then any homological problem in the big complex $\widehat{C}_*$ can be solved through a solution of the same problem in the small complex C_*.* ♣

2.4.2 The Homological Perturbation Theorem.

This theorem is often called the Basic Perturbation "Lemma". It was introduced in a (very important) particular case by Shih Weishu [26], allowing him to obtain a *more effective* version of the Serre spectral sequence: Shih so obtained an *explicit* homology equivalence between the chain complex of the total space of a fibration and the corresponding Hirsch complex. The general scope of this lemma was signalled by Ronnie Brown [3]. Our organization of *Effective Homology*, see [24, 4, 19, 20, 21, 22], consists in combining this lemma with functional programming, more precisely with the notion of locally effective object.

Theorem 13 (Homological Perturbation Theorem) — *Let $\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)$ be a reduction and let $\widehat{\delta}$ be a perturbation of the differential $\widehat{d}$ of the big chain-complex. We assume the* nilpotency hypothesis *is satisfied: for every $c \in \widehat{C}_n$, there exists $\nu \in \mathbb{N}$ satisfying $(\widehat{\delta}h)^\nu(c) = 0$. Then a perturbation δ can be defined for the differential d and a new reduction $\rho' = (f', g, h') : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)$ can be constructed.*

The nilpotency hypothesis states the composition $\widehat{\delta}h$ is pointwise nilpotent. The process described by the Theorem perturbs the differential d of the small complex, becoming $d + \delta$, and the components (f, g, h) of the reduction which so becomes a new reduction $\rho' = (f', g', h') : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)$.

Which is magic in the BPL is the fact that a sometimes complicated perturbation of the "big" differential can be accordingly reproduced in the "small" differential; in general it is not possible, unless the nilpotency hypothesis is satisfied.

♣ See [26, §1 and 2] and [3]. A detailed proof in the present context is at [22, Theorem 50]. ♣

This theorem[3] is so important in *effective* Homological Algebra that Julio Rubio's team at Logroño decided to write a proof in the language of the Isabelle

[3] Usually called a "lemma"!

theorem prover, and succeeded [1]; it is the starting point to obtain *proved* programs using this crucial result, a fascinating challenge.

2.5 A vector field generates a reduction.

Let $C_* = (C_*, d_*, \beta_*)$ be an algebraic cellular complex provided with an *admissible* discrete vector field V. Then a reduction $\rho : C_* \Rightarrow\!\!\!\Rightarrow C_*^c$ can be constructed where the small chain complex C_*^c is the *critical* complex; it is also a cellular complex but generated only by the *critical* cells of β_*, those which do not appear in the vector field V, with a differential appropriately defined, combining the initial differential d of the initial complex and the vector field.

This result is due to Robin Forman [9, Section 8]. It is here extended to the complexes not necessarily of finite type, but the extension is obvious. Many (slightly) different proofs are possible. Forman's proof uses an intermediate chain complex, the *Morse* chain complex, which depends in fact only on the vector field, it is made of the chains that are invariant for the flow canonically associated to the vector field. We give here two other organizations, each one having its own interest.

The first one reduces the result to the standard Gauss elimination process for the linear systems. It has the advantage of identifying the heart of the process, the component $d_{2,1}^{-1}$ of the formulas (2.18). It is nothing but the homotopy component h of the looked-for reduction. For critical time applications, storing the values of this component when it is calculated could be a good strategy, the rest being a direct and quick consequence. Our $d_{2,1}^{-1}$ is the operator L of [9, Page 122].

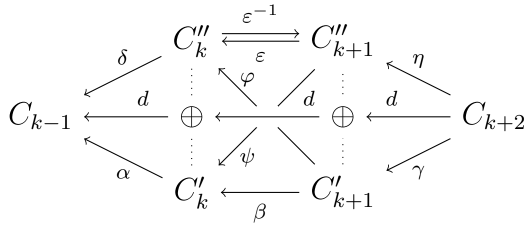
The difference between *ordinary* and *effective* homology is easy to see here. A "simple" user of Forman's paper could skip the construction of this operator, used only to prove the Morse complex gives the right homology groups. On the contrary, in effective homology, this operator is crucial, from a theoretical point of view and for a computational point of view as well: the profiler experiments show most computing time is *then* devoted to this operator, which is *not* the case in ordinary homology.

The second proof uses the Homological Perturbation Theorem 13. It is much faster and highlights the decidedly wide scope of this result. The general style of application of this theorem is again there: starting from a particular case where the result is obvious, the general case is viewed as a *perturbation* of the particular case. The nilpotency condition must be satisfied, which amounts to requiring the vector field is admissible. The critical complex of Forman is the bottom chain complex of the obtained reduction, and the Morse complex is its canonical image in the top complex. How to be simpler?

2.5.1 Using Gauss elimination.

Proposition 14 (Hexagonal lemma) — *Let $C = (C_p, d_p)_p$ be a chain complex.*

For some $k \in \mathbb{Z}$, the chain groups C_k and C_{k+1} are given with decompositions $C_k = C'_k \oplus C''_k$ and $C_{k+1} = C'_{k+1} \oplus C''_{k+1}$, so that between the degrees $k-1$ and $k+2$ this chain complex is described by the diagram:



(2.11)

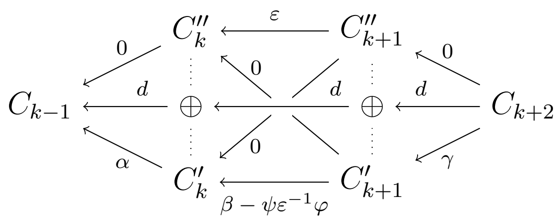
The partial differential $\varepsilon : C''_{k+1} \to C''_k$ is assumed to be an isomorphism. Then a canonical reduction can be defined $\rho : C \Rrightarrow C'$ where C' is the same chain complex as C except between the degrees $k-1$ and $k+2$:

$$\leftarrow\text{-}\text{-}\text{-}\; C_{k-2} \longleftarrow C_{k-1} \xleftarrow{\alpha} C'_k \xleftarrow{\beta - \psi\varepsilon^{-1}\varphi} C'_{k+1} \xleftarrow{\gamma} C_{k+2} \longleftarrow C_{k+3} \;\leftarrow\text{-}\text{-}\text{-} \tag{2.12}$$

♣ An integer matrix $\begin{bmatrix} \varepsilon & \varphi \\ \psi & \beta \end{bmatrix}$ is equivalent to the matrix $\begin{bmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{bmatrix}$ if $|\varepsilon| = 1$, it is the simplest case of Gauss' elimination. More generally, the following matrix relation is always satisfied, even if the matrix entries are in turn coherent linear maps:

$$\begin{bmatrix} \varepsilon & \varphi \\ \psi & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \psi\varepsilon^{-1} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{bmatrix} \begin{bmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{bmatrix} \tag{2.13}$$

The lateral matrices of the right-hand term can be considered as basis changes. These matrices define an isomorphism $\rho' : C \to \overline{C}$ between the initial chain complex C and the chain complex $\overline{C}$ made of the same chain groups but the differentials displayed on this diagram:



(2.14)

Throwing away the component $\varepsilon : C''_{k+1} \to C''_k$ from this chain complex $\overline{C}$ produces a reduction $\rho'' : \overline{C} \Rrightarrow C'$ to the announced chain complex C'. The desired reduction is $\rho = \rho''\rho' : C \Rrightarrow C'$ with an obvious interpretation of the composition $\rho''\rho'$. Finally $\rho = (f, g, h)$ with:

1. The morphism f is the identity except:

$$f_k = \begin{bmatrix} -\psi\varepsilon^{-1} & 1 \end{bmatrix} \qquad f_{k+1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{2.15}$$

2. The morphism g is the identity except:

$$g_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad g_{k+1} = \begin{bmatrix} -\varepsilon^{-1}\varphi \\ 1 \end{bmatrix} \tag{2.16}$$

3. The homotopy operator h is the null operator except:

$$h_k = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.17}$$

matrices to be interpreted via appropriate block decompositions. ♣

Note the boundary components α and γ are not modified by the reduction process. So that if independent "hexagonal" decompositions are given for every degree, the process can be applied to every degree simultaneously.

Theorem 15 — *Let $C = (C_p, d_p)_p$ be a chain complex. We assume every chain group is decomposed $C_p = D_p \oplus E_p \oplus F_p$. The boundary maps d_p are then decomposed in 3×3 block matrices $[d_{p,i,j}]_{1 \leq i,j \leq 3}$. If every component $d_{p,2,1} : D_p \to E_{p-1}$ is an isomorphism, then the chain complex can be canonically reduced to a chain complex (F_p, d'_p).*

♣ Simultaneously applying the formulas produced by the hexagonal lemma gives the desired reduction, the components of which are:

$$\begin{array}{ll} d'_p = d_{p,3,3} - d_{p,3,1} d_{p,2,1}^{-1} d_{p,2,3} & f_p = \begin{bmatrix} 0 & -d_{p,3,1} d_{p,2,1}^{-1} & 1 \end{bmatrix} \\ g_p = \begin{bmatrix} -d_{p,2,1}^{-1} d_{p,2,3} \\ 0 \\ 1 \end{bmatrix} & h_{p-1} = \begin{bmatrix} 0 & d_{p,2,1}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \tag{2.18}$$

♣

It is an amusing exercise to check the displayed formulas satisfy the required relations: $d'f = fd$, $dg = gd'$, $fg = 1$, $dh + hd + fg = 1$, $fh = 0$, $hg = 0$, $hh = 0$, stated in Definition 11. The components $d_{p,1,1}$, $d_{p,1,2}$, $d_{p,1,3}$, $d_{p,22}$ and $d_{p,3,2}$ do not play any role in the homological nature of C, but these components are not independent of the others, because of the relation $d_{p-1} d_p = 0$.

2.5.2 A vector field generates a reduction, first proof.

Let $C = (C_p, d_p, \beta_p)_p$ be an algebraic cellular complex and $V = \{(\sigma_i, \tau_i)\}_{i \in \beta}$ an *admissible* discrete vector field on C. These data are fixed in this section. We intend to apply the hexagonal lemma to obtain a canonical reduction of the initial chain complex C to a reduced chain complex C^c where the generators are the critical cells of C with respect to V.

Definition 16 — If σ (resp. τ) is a $(p-1)$-cell (resp. a p-cell) of C, then the *incidence number* $\varepsilon(\sigma, \tau)$ is the coefficient of σ in the differential $d\tau$. This incidence number is non-null if and only if σ is a face of τ; it is ± 1 if and only if σ is a *regular* face of τ. ♣

In particular, for the pairs (σ_i, τ_i) of our vector field, the relation $|\varepsilon(\sigma_i, \tau_i)| = 1$ is satisfied.

Definition 17 — If $v = (\sigma_i, \tau_i)$ is a component of our vector field V, we call σ_i the *source* of v, we call τ_i the *target* of v. We also write $\tau_i = V(\sigma_i)$ and $\sigma_i = V^{-1}(\tau_i)$. ♣

A cell basis β_p is canonically divided by the vector field V into three components $\beta_p = \beta_p^t + \beta_p^s + \beta_p^c$ where β_p^t (resp. β_p^s, β_p^c) is made of the target (resp. source, critical) cells. For the condition 3 of Definition 5 implies a cell cannot be simultaneously a source cell and a target cell.

The vector field V defines a bijection between β_{p-1}^s and β_p^t. The decompositions of the bases β_* induce a corresponding decomposition of the chain groups $C_p = C_p^t \oplus C_p^s \oplus C_p^c$, so that every differential d_p can be viewed as a 3×3 matrix[4].

Proposition 18 — *Let $d_{p,2,1} : C_p^t \to C_{p-1}^s$ be the component of the differential d_p starting from the target cells, going to the source cells. Then $d_{p,2,1}$ is an isomorphism.*

♣ If $\sigma \in \beta_{p-1}^s$, the length of the V-paths between σ and the critical cells is bounded. Let us call $\lambda(\sigma) \geq 1$ the maximal length of such a path. This length function λ is a grading: $\beta_{p-1}^{s,\ell} = \{\sigma \in \beta_{p-1}^s \;\underline{\textbf{st}}\; \lambda(\sigma) = \ell\}$. In the same way $C_{p-1}^{s,\ell} = \mathbb{Z}[\beta_{p-1}^{s,\ell}]$, which in turn defines a filtration $\widehat{C}_{p-1}^{s,\ell} = \oplus_{k\leq\ell} C_{p-1}^{s,k}$.

The V-bijection between β_{p-1}^s and β_p^t defines an isomorphic grading on β_p^t and an analogous filtration on $C_p^t = \mathbb{Z}[\beta_p^t]$; we denote in the same way $C_p^{t,\ell} = \mathbb{Z}[\beta_p^{t,\ell}]$ and $\widehat{C}_p^{t,\ell} = \oplus_{k\leq\ell} C_p^{t,k}$.

If $\sigma \in \beta_{p-1}^{s,\ell}$, that is, if $\lambda(\sigma) = \ell$, then the corresponding V-image $\tau \in \beta_p^{t,\ell}$ has certainly σ as a *regular* face and in general other *source* faces; the grading for these last faces is $< \ell$ and if $\ell > 1$, one of these source faces has the grading $\ell - 1$, if the longest V-path is followed.

A consequence of this description is the following. The partial differential $d_{p,2,1}$ can in particular be restricted to $d_{p,2,1} : \widehat{C}_p^{t,\ell} \to \widehat{C}_{p-1}^{s,\ell}$; dividing by the same map between degrees $(\ell - 1)$ then produces a quotient map $\overline{d}_{p,2,1} : C_p^{t,\ell} = \mathbb{Z}[\beta_p^{t,\ell}] \to \mathbb{Z}[\beta_{p-1}^{s,\ell}] = C_{p-1}^{s,\ell}$, which is a V-isomorphism of $\mathbb{Z}$-modules *with distinguished basis*.

The standard recursive argument then proves $d_{p,2,1}$ is an isomorphism. ♣

Theorem 19 (Vector-Field Reduction Theorem) — *Let $C = (C_p, d_p, \beta_p)_p$ be an algebraic cellular complex and $V = \{(\sigma_i, \beta_i)\}_{i\in\beta}$ be an* admissible *discrete vector field on C. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rrightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}[\beta_p^c]$ is the free $\mathbb{Z}$-module generated by the critical p-cells.*

♣ Theorem 15 + Proposition 18. ♣

[4]Our choice to put C_p^t before C_p^s is intended to produce a situation close to which is described in Figure (2.10). Think the homotopy operator starts from source cells and go to target cells, while the differential goes in the opposite direction.

The bottom chain complex (C^c_*, d'_*) and its image $g(C^c_*) \subset C_*$ are both versions of the Morse complex described in Forman's paper [9, Section 7].

The inverse $d^{-1}_{p,2,1}$ is crucial when concretely using this proposition to solve the homological problem for (C_p, d_p) thanks to a solution of the same problem for the *critical complex* (C^c_p, d'_p): this matrix can be – and will sometimes be in this text – not of finite type!

If $\tau \in \beta^t_p$, the value $d_{p,2,1}(\tau)$ is defined by:

$$d_{p,2,1}(\tau) = \sum_{\sigma \in \beta^s_{p-1}} \varepsilon(\sigma, \tau)\sigma \tag{2.19}$$

where $\varepsilon(V^{-1}(\tau), \tau) = \pm 1$, but the other coefficients are arbitrary. We so obtain the recursive formula:

$$d^{-1}_{p,2,1}(\sigma) = \varepsilon(\sigma, V(\sigma)) \left(V(\sigma) - \sum_{\sigma' \in \beta^s_{p-1} - \{\sigma\}} \varepsilon(\sigma', V(\sigma))\, d^{-1}_{p,2,1}(\sigma') \right) \tag{2.20}$$

with $\varepsilon(\sigma, V(\sigma)) = \pm 1$, the other incidence numbers being arbitrary.

This formula is easily recursively programmed.

2.5.3 Using the Homological Perturbation Theorem.

We give now a direct proof of Theorem 19 based on the Homological Perturbation Theorem 13.

♣ Instead of considering the right differential d of our chain complex (C_*, d), we start with a different chain complex (C_*, δ); the underlying graded module C_* is the same, but the differential is "simplified". The new differential $\delta_p : C_p \to C_{p-1}$ is roughly defined by the formula $\delta = \varepsilon V^{-1}$.

More precisely, we take account of the canonical decomposition defined by the vector field V:

$$C_p = C^t_p \oplus C^s_p \oplus C^c_p = \mathbb{Z}[\beta^t_p] \oplus \mathbb{Z}[\beta^s_p] \oplus \mathbb{Z}[\beta^c_p] \tag{2.21}$$

as follows. If $\tau \in \beta_p$ is an element of β^s_p or β^c_p, then we decide $\delta_p(\tau) = 0$. If $\tau \in \beta^t_p$, that is, if τ is a target cell, then there is a unique vector $(\sigma, \tau) \in V$ with τ as the second component and we define $\delta_p(\tau) = \varepsilon(\sigma, \tau)\sigma$ "=" $(\varepsilon V^{-1})(\tau)$.

The homotopy operator $h = \varepsilon V$ defined in the same way in the reverse direction obviously defines an initial *reduction* $(C_p, \delta_p) \Rrightarrow (C^c_p, 0)$ of this initial chain complex over the chain complex with a null differential generated by the critical cells. Look again at the diagram (2.10): the reverse diagonal arrows d and h of the diagram correspond to our simplified differential δ and the homotopy operator h, while the horizontal arrows d are null.

Restoring the right differential $(d_p)_p$ on $(C_p, d_p)_p$ can be considered as introducing a *perturbation* $(d_p - \delta_p)_p$ of the differential $(\delta_p)_p$. The basic perturbation lemma can be applied if the *nilpotency condition* is satisfied, that is, if $(d - \delta) \circ h$ is pointwise nilpotent. The perturbation is nothing but the right differential except for the target cells; if τ is such a target cell, then $(d-\delta)(\tau) = d(\tau) - \varepsilon(\sigma,\tau)\sigma$ if (σ,τ) is the corresponding vector.

Let us examine this composition $(d-\delta)h$. It is non-trivial only for a source cell σ and $h(\sigma) = V(\sigma)$. Because of the definition of δ, the value of $(d-\delta)h(\sigma)$ is $d(\tau) - \sigma$, it is made of the faces of τ *except the starting cell* σ. Considering next $h(d-\delta)h(\sigma)$ amounts to selecting the source cells of $d(\tau) - \sigma$ and applying again the vector field V. You understand we are just following all the V*-paths* starting from the initial source cell σ, see Definition 7. If the vector field is admissible, the length of these V-paths is bounded, the process terminates, $(d-\delta)h$ is pointwise nilpotent and the *nilpotency condition*, required to apply the Homological Perturbation Theorem, is satisfied.

It happens the explicit formulas for the new reduction produced by this theorem are exactly the formulas (2.18) combined with the recursive definition (2.20) of the homotopy operator h' of the perturbed reduction. ♣

This proof is not only more direct, but also much easier to program, at least if the Homological Perturbation Theorem is implemented in your programming environment.

Definition 20 — A *W-reduction* is the reduction produced by Theorem 19 when an algebraic cellular complex is provided with an admissible discrete vector field. ♣

2.6 Composition of vector fields.

2.6.1 Introduction.

Homology is more commutative than homotopy. Typically, a π_1 group can be non-abelian while the corresponding H_1 is the abelianization of this π_1.

So that the homological reductions defined by vector fields maybe also enjoy commutativity properties. Compare Definition 2 for a *W-contraction* and Definition 5 for a *discrete vector field*. They are somewhat the same with just a difference: the vectors of a W-contraction are *ordered* while those of a vector field are not: the index set β in $V = \{(\sigma_i,\tau_i)\}_{i\in\beta}$ is not ordered; the collection of vectors defining a W-contraction is a *sequence*, the collection of vectors defining a vector field is a non-ordered *set*. We already noticed this is an avatar of the difference between homotopy and homology.

Situations[5] could happen where the following game is productive. A vector field V on C_* can be divided in two sub-vector fields V' and V''. If V is admissible, then V' and V'' are admissible too, while the converse is false in general. Instead

[5]We hoped the commutativity property detailed in this section could give an elegant proof of the naturality of the Eilenberg-Zilber reduction, but finally we failed in designing such a proof, see Section 5.12. Anyway this commutativity property seems to have its own interest and finally we keep it in this text. The present section can be therefore skipped without any drawback later.

of considering the homological reduction $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\!\Rightarrow C''_*$ defined by V, we can at first consider the reduction $\rho' = (f', g', h') : C_* \Rightarrow\!\!\!\!\!\Rightarrow C'_*$ defined by V'. The vectors of V'' are critical with respect to C'_* and maybe V'' is also admissible with respect to V', which could define another reduction $\rho'' = (f'', g'', h'') : C'_* \Rightarrow\!\!\!\!\!\Rightarrow C''_*$. Question: if so, are both C''_* the same? More generally, can we define and prove the relation $\rho = \rho''\rho'$? The answer is positive.

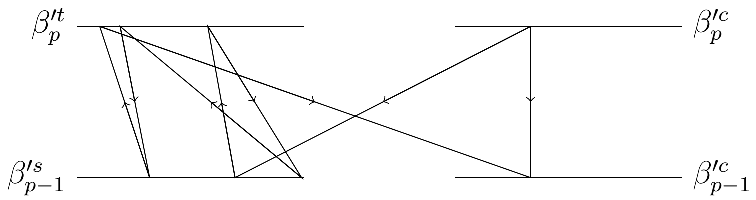
But we could as well *at first* use the vector field V'', producing another reduction $\overline{\rho}''$, and afterwards the vector field V', producing a reduction $\overline{\rho}'$. If the composition result sketched in the previous paragraph is correct, we obtain $\rho''\rho' = \rho = \overline{\rho}'\overline{\rho}''$. In other words, the vector fields V' and V'' *commute*, which could be roughly written $V'V'' = V''V'$.

2.6.2 Dividing a vector field.

Let C_* be a cellular chain complex provided with an *admissible* discrete vector field V. We divide this vector field in two parts V' and V''.

The vector field V being admissible defines a reduction $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\!\Rightarrow C''_*$. The vector field V' is certainly admissible and defines a reduction $\rho' = (f', g', h') : (C_*, d) \Rightarrow\!\!\!\!\!\Rightarrow (C'_*, d')$. The vectors of V'' are critical with respect to V' and can be also vectors in C'_*.

Does V'' *remain* an *admissible* vector field in C'_*? Yes. The key point here is the formula (2.18) for the differential of C'_*: $d'_p = d_{p,3,3} - d_{p,3,1}d^{-1}_{p,2,1}d_{p,2,3}$; the corrector term $d_{p,3,1}d^{-1}_{p,2,1}d_{p,2,3}$ consists in applying the following process. From $\tau \in \beta'^c_p$, try to go to some element $\sigma \in \beta'^s_{p-1}$ thanks to the differential d, then follow an arbitrary path to be considered when determining the admissibility of V', see Figure (2.7), and finally, when you are at some cell in β'^t_p, finally go to some cell of β'^c_{p-1}. You understand the β'^*_* correspond to the division of cells defined by the vector field V'.



(2.22)

Now we have to study the admissiblity of V'' with respect to this *new* differential d'. Again we must think of the corresponding oriented bipartite graph roughly sketched in Figure (2.7). But it is clear if an infinite path is present in this new diagram, it would imply an infinite path for the original diagram with respect to the total vector field V: in fact the vector field V'' is also admissible with respect to the new differential d'.

The vector field V'' therefore defines in turn a reduction $\rho'' = (f'', g'', h'') : (C'_*, d') \Rightarrow\!\!\!\!\!\Rightarrow (C''_*, d'')$. We do not know at this time whether d'' is also the differential produced in C''_* by the *total* reduction ρ.

Definition 21 — Let $\rho' = (f', g', h') : (C_*, d) \Rightarrow\!\!\!\!\!\Rightarrow (C'_*, d')$ and $\rho'' = (f'', g'', h'') : (C'_*, d') \Rightarrow\!\!\!\!\!\Rightarrow (C''_*, d'')$ be two homological reductions. Then the *composition* of these reductions is the reduction $\rho = (f, g, h) : (C_*, d) \Rightarrow\!\!\!\!\!\Rightarrow (C''_*, d'')$ defined by:

$$\begin{array}{rcl} f & = & f''f' \\ g & = & g'g'' \\ h & = & h' + g'h''f' \end{array} \tag{2.23}$$

♣

Verifying the coherence of the definition is elementary. Note that in the reductions ρ and ρ'', the chain complex C''_* has the same differential d''.

Theorem 22 — *If the admissible vector field V for the cellular complex C_* is divided in two parts V' and V'', then the reductions $\rho : C_* \Rightarrow\!\!\!\!\!\Rightarrow C''_*$ (resp. $\rho' : C_* \Rightarrow\!\!\!\!\!\Rightarrow C'_*$, $\rho'' : C'_* \Rightarrow\!\!\!\!\!\Rightarrow C''_*$) defined by V (resp. V', and* after *V' by V'') satisfy $\rho = \rho''\rho'$. In particular both definitions of C''_* so obtained are the same.*

The vector field V' *defines* the critical chain complex C'_*. The vector field V'' is defined on C'_* and defines in turn a new critical chain complex C''_*. Restarting from C_*, you can also use at once the vector field V to obtain an a priori *other* critical chain complex C''_*. As stated, both C''_* are in fact the same chain complexes, that is, both processes define the same differential.

♣ It is enough to consider which happens between two particular degrees p and $p-1$. The vector fields V' and V'' divide the bases $\beta_p = \beta'^t_p + \beta''^t_p + \beta^c_p$ and $\beta_{p-1} = \beta'^s_{p-1} + \beta''^s_{p-1} + \beta^c_{p-1}$. The differential $d_p : \mathbb{Z}[\beta_p] \to \mathbb{Z}[\beta_{p-1}]$ is a 3×3 block matrix:

$$d_p = \begin{bmatrix} \alpha & d_{12} & d_{13} \\ d_{21} & \beta & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \tag{2.24}$$

Proposition 18 proves that α, β and the 2×2 top lefthand block are invertible. For the last one, this produces an inverse matrix and a relation:

$$\begin{bmatrix} \alpha & d_{12} \\ d_{21} & \beta \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \alpha & d_{12} \\ d_{21} & \beta \end{bmatrix} = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} \tag{2.25}$$

from which eight relations are deduced, for example $\alpha e_{11} + d_{12}e_{21} = \mathrm{id}$. The reductions defined by V, V' and V'' are obtained by applying the formulas (2.15-2.17). The non-trivial components of the reduction ρ are:

$$f_{p-1} = \left[\ -d_{31}e_{11} - d_{32}e_{21} \mid -d_{31}e_{12} - d_{32}e_{22} \mid 1\ \right] \tag{2.26}$$

$$g_p = \begin{bmatrix} -e_{11}d_{13} - e_{12}d_{23} \\ -e_{21}d_{13} - e_{22}d_{23} \\ \mathrm{id} \end{bmatrix} \tag{2.27}$$

$$h_{p-1} = \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.28}$$

$$d''_p = d_{33} - d_{31}e_{11}d_{13} - d_{32}e_{21}d_{13} - d_{31}e_{12}d_{23} - d_{32}e_{22}d_{23} \tag{2.29}$$

The non-trivial components of the reduction ρ' are:

$$f'_{p-1} = \begin{bmatrix} -d_{21}\alpha^{-1} & \mathrm{id} & 0 \\ -d_{31}\alpha^{-1} & 0 & \mathrm{id} \end{bmatrix} \tag{2.30}$$

$$g'_p = \begin{bmatrix} -\alpha^{-1}d_{12} & -\alpha^{-1}d_{13} \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix} \tag{2.31}$$

$$h'_{p-1} = \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.32}$$

$$d'_p = \begin{bmatrix} \beta - d_{21}\alpha^{-1}d_{12} & d_{23} - d_{21}\alpha^{-1}d_{13} \\ d_{32} - d_{31}\alpha^{-1}d_{12} & d_{33} - d_{31}\alpha^{-1}d_{13} \end{bmatrix} \tag{2.33}$$

The restriction of the differential to the subspaces generated by β''^t_p and β''^s_{p-1} is no more β, but $\beta - d_{21}\alpha^{-1}d_{12}$, certainly invertible, for the vector field V'' remains admissible in the critical subcomplex produced by V'; and in fact it is easy to see this inverse is simply e_{22}. This allows us to construct in the same way the reduction ρ'' whose non-trivial components are:

$$f''_{p-1} = \begin{bmatrix} -d_{32}e_{22} + d_{31}\alpha^{-1}d_{12}e_{22} \mid \mathrm{id} \end{bmatrix} \tag{2.34}$$

$$g''_p = \begin{bmatrix} -e_{22}d_{23} + e_{22}d_{21}\alpha^{-1}d_{13} \\ \mathrm{id} \end{bmatrix} \tag{2.35}$$

$$h''_{p-1} = \begin{bmatrix} e_{22} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.36}$$

$$d''_p = d_{33} - d_{31}\alpha^{-1}d_{13} - (d_{32} - d_{31}\alpha^{-1}d_{12})e_{22}(d_{23} - d_{21}\alpha^{-1}d_{13}) \tag{2.37}$$

There remains to compare ρ and $\rho''\rho'$; obvious but lengthy computations must be done, mainly by taking account of the relations satisfied by the blocks e_{**} with α, β, d_{12} and d_{21}; no surprise: the relation $\rho = \rho''\rho'$ is satisfied and also both definitions of d'' are the same. ♣

A more conceptual proof would be welcome, but its nature is clear: it is visible it is just a matter of associativity and distributivity when decomposising trees in sub-trees. Defining the appropriate terminology probably would require still more time.

2.7 Computing vector fields.

2.7.1 Introduction.

A vector field is most often an *initial* tool to construct an interesting *reduction*, deduced from the vector field by the Vector-Field Reduction Theorem. For example we will see the so fundamental Eilenberg-Zilber reduction is a direct consequence of a simple vector field. The plan is then:

$$\text{Vector field} \Longrightarrow \text{Reduction} \tag{2.38}$$

But it happens most interesting results of this text in fact have been obtained with the reverse plan:

$$\text{Reduction} \Longrightarrow \text{Vector field} \tag{2.39}$$

The story is the following. The *discrete vector field* point of view is relatively recent. In many situations, the *reductions* we are interested in are known for a long time. The problem is then the following: is it possible in fact to obtain such a reduction thanks to an appropriate discrete vector field? We explain in this section there exists a systematic method to *deduce* such a vector field from the reduction, of course *if it exists*, which is not necessarily the case.

The reader can then wonder which can motivate such a research, when the corresponding reduction is already known! The point is the following: The *already known* reduction has therefore been defined by a different process, often relatively sophisticated. These reductions are important in *constructive* algebraic homology, and the algorithms implementing them can be complex, therefore time and space consuming.

Several experiments then show that if a vector field giving directly such a reduction is finally found, then the algorithm computing this reduction can be much more efficient, in time *and* space complexity, in particular for the terrible homotopy operators, so important from a constructive point of view, almost always neglected by "classical" topologists. The algorithm explained here to identify a vector field possibly defining a *known* reduction can therefore be considered as a program systematically *improving* other programs *previously written.*

There are also some cases where some reduction is *conjectured*, but not yet proved. We obtain in such a context a relatively spectacular result: Eilenberg and MacLane sixty years ago conjectured[6] the existence of a direct reduction $C_*(BG) \Rrightarrow \text{Bar}(C_*(G))$ for a simplicial group G, see the comments after the statement of [5, Theorem 20.1]. Numerous calculations using repeatedly the Homological Perturbation Theorem led us to suspect we had obtained such a direct reduction, but we were unable to prove it. Using the below algorithm, we easily obtained a vector field producing this *possible* reduction. It happens the structure of this vector field is very easy to understand, which this time produces a *proof* of the Eilenberg-MacLane conjecture. Furthermore, the calculation of the *effective* homology of the Eilenberg-MacLane spaces is so much more efficient, a key point for the quick calculation of homotopy groups of *arbitrary* simply connected spaces.

2.7.2 The unique possible vector field.

We consider in this section a fixed *given* reduction:

$$\rho = (f, g, h) : \widehat{C}_* \Rrightarrow C_* \tag{2.40}$$

[6]Eilenberg and MacLane's terminology is a little confusing: what *we* call a *reduction* is a *contraction* in *their* paper, while *their reduction* is a chain complex morphism inducing a weak homology equivalence.

between two *cellular* chain complexes. And we want to study whether some admissible discrete vector field could "explain" this reduction.

The problem is the following: we have to divide the cells elements of β_* in $\widehat{C}_*$ in *source*, *target* and *critical* cells, and we have also to define a coherent *pairing* between source and target cells. In such a way the reduction then obtained by the Vector-Field Reduction Theorem 19 is just the given reduction ρ.

The key point is the collection of formulas (2.18).

Observe firstly the homotopy operator h is non-null only for source cells. More precisely, a cell $\sigma \in \beta_p$ is a source cell if and only if $h(\sigma) \neq 0$. No circularity in a V-path, for the vector field must be *admissible*. So that if (σ, τ) is the vector starting from σ, the occurence of $\tau = V(\sigma)$ in the formula (2.20) cannot be cancelled by other terms. So the reduction ρ gives a decision algorithm for the source property of a cell.

Let us now consider a cell σ which has been proved being a source cell. What is the corresponding target cell? The value $h(\sigma)$ is a linear combination of cells, and at least one of them has σ in its boundary. In fact only one, for again, otherwise there will be a circular V-path. So no choice for the corresponding target cell τ. If the incidence number $\varepsilon(\sigma, \tau)$ is not ± 1, possible, no vector field is possible. If such a vector field on the contrary exists, we have an algorithm defining the pairing source cell $\mapsto$ target cell.

In the finite case, this is enough. The first algorithm gives the list of the source cells, the second one gives the list of the target cells, and the remaining cells are the critical ones. There remains to verify the formulas (2.20).

In the infinite case, in general there is no algorithm allowing one to decide whether a given reduction comes from a vector field. Because of the usual obstacle forbidding for example to solve the halting problem.

Nevertheless you can also decide whether *some* cell τ is a target cell. First, you examine if it is a source cell, and if it is, it is not a target cell. If τ is not a source cell, then you consider the boundary $d\tau = \sum \alpha_i \sigma_i$, it is a finite linear combination, and you just have to examine if some σ_i in the boundary is a source cell, and if the corresponding target cell is this cell τ.

Finally if a cell is neither a source cell nor a target cell, it is a critical cell. Of course if the rank of your chain groups are infinite, you cannot verify the formulas (2.20) for all the cells.

Another point is useful. In most interesting reductions, the small chain complex C_* is of finite type, even if the big one $\widehat{C}_*$ is not. It is then important to identify and to understand the "nature" of the critical cells. These cells can be easily obtained thanks to the formula (2.18) for g_p. Take some generator a of the small complex C_*, apply the lifting operator $g : C_* \to \widehat{C}_*$ to obtain $g(a)$; the formula for g proves this image is made of exactly *one* critical cell and an arbitrary combination of target cells; so that running the generators present in $g(a)$ allows you to identify the critical cell of $\widehat{C}_*$ corresponding to the generator a of C_*.

Once these simple tricks are applied, the game becomes the following, when it makes sense. Examining the "general style" of as many source, target and critical cells as reasonably possible, you have to *guess* the unknown vector field in fact your reduction ρ has "followed" to be defined. Experience shows it can be quite amusing. We will see this rule is quite simple for the Eilenberg-Zilber reduction, at least if the good point of view is understood, a little less simple for the direct reduction $C_*(BG) \Rrightarrow \mathrm{Bar}(C_*(G))$, very sophisticated for the direct reduction $C_*(\Omega X) \Rrightarrow \mathrm{Cobar}(C_*(X))$.

Before defining the Eilenberg-Zilber vector field in Section 5.11.3, we will play this game with the Eilenberg-Zilber reduction obtained sixty years ago.

Chapter 3

Two simple examples.

This section is devoted to two classical results of algebraic topology proved here thanks to simple discrete vector fields. The first one is the *normalization theorem* for simplicial homology. The second one is devoted to the "simplest" Eilenberg-MacLane space, $K(\mathbb{Z}, 1)$, simple but unfortunately not of finite type! So that a homological reduction $K(\mathbb{Z}, 1) \Rrightarrow S^1$ on the *very small* standard circle, one vertex and one edge, is quite useful.

3.1 Simplicial sets and their chain complexes.

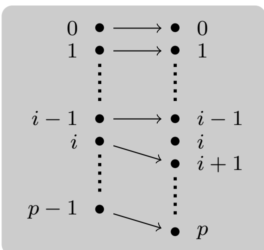
The reader is assumed to be familiar with the elementary definitions, properties and results about the *simplicial sets*. Not to be confused with simplicial *complexes*. The main references are maybe [14, 10]; the notes [22, Section 7] or [25] can also be useful.

The basic category in this context is the category $\underline{\Delta}$. An object $\underline{p}$ of $\underline{\Delta}$ is the set of the integers $0 \leq i \leq p$, also denoted by $[0 \ldots p]$. A $\underline{\Delta}$-morphism $\alpha : \underline{p} \to \underline{q}$ is an increasing function: $i \leq j \Rightarrow \alpha(i) \leq \alpha(j)$.

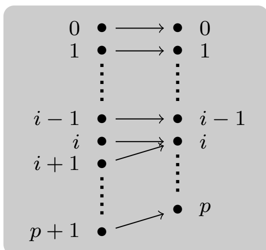
Let X be a simplicial set. For every natural number $p \in \mathbb{N}$ the set of p-simplices X_p is defined. For every $\underline{\Delta}$-morphism $\alpha : \underline{p} \to \underline{q}$, a corresponding map $\alpha^* : X_q \to X_p$ is defined. The simplicial set X can be viewed as a *contravariant* functor $X : \underline{\Delta} \to \underline{\mathrm{Set}}$.

The face $\underline{\Delta}$-morphisms $\partial_i^p : [0 \ldots (p-1)] \to [0 \ldots p]$ are defined for $p \geq 1$ and $0 \leq i \leq p$. The (elementary) degeneracy $\underline{\Delta}$-morphisms $\eta_i^p : [0 \ldots (p+1)] \to [0 \ldots p]$ are defined for $0 \leq i \leq p$.

$\partial_i^p =$



$\eta_i^p =$



(3.1)

An arbitrary composition of (elementary) degeneracies $\eta = \eta^q_{i_{p-q}} \eta^{q+1}_{i_{p-q-1}} \cdots \eta^{p-1}_{i_1}$ is a *degeneracy*; this expression is unique if the inequalities $i_{p-q} < \cdots < i_1$ are required. The degeneracies are nothing but the *surjective* $\underline{\Delta}$-morphisms.

Most often the sup-index of the face and degeneracy operators are omitted, we write simply ∂_i (resp. η_i) instead of ∂_i^p (resp. η_i^p).

For every simplicial set X, every $p > 0$ and every $0 \le i \le p$, a face operator $\partial_i : X_p \to X_{p-1}$ is defined, this is the map which applies a simplex to its i-th face, thought of as opposite to the i-th vertex; an elementary degeneracy operator $\eta_i : X_p \to X_{p+1}$ is defined for $0 \le i \le p$, it is the map which applies a simplex to the same but the i-th vertex is "repeated", replaced by a degenerate edge, increasing the dimension by 1. Every $\underline{\Delta}$-morphism is a composition of face and (elementary) degeneracy operators, so that the face and degeneracy operators between the simplex sets X_p are enough to define the simplicial structure, at least if they satisfy appropriate compatibility conditions.

Definition 23 — A p-simplex $\sigma \in X_p$ is *degenerate* if there exists an integer $q < p$, a $\underline{\Delta}$-morphism $\alpha : \underline{p} \to \underline{q}$ and a q-simplex $\tau \in X_q$ satisfying $\sigma = \alpha^* \tau$. We denote by X_p^{D} (resp. X_p^{ND}) the set of degenerate (resp. non-degenerate) p-simplices. ♣

The Eilenberg-Zilber lemma gives for every simplex a canonical expression from a unique non-degenerate simplex.

Theorem 24 (Eilenberg-Zilber lemma) — *Let σ be a p-simplex of a simplicial set X. Then there exists a unique triple (q, η, τ), the* Eilenberg triple *of σ, satisfying:*

1. $0 \le q \le p$.
2. $\eta : \underline{p} \to \underline{q}$ *is a surjective $\underline{\Delta}$-morphism.*
3. $\tau \in X_q$ *is non-degenerate and $\alpha^* \tau = \sigma$.*

♣ [7, (8.3)] ♣

In short, every simplex comes from a unique non-degenerate simplex, by a unique degeneracy, and every non-degenerate simplex $\sigma \in X_p$ generates a collection of degenerate simplices in any dimension $> p$: $\{\eta^* \sigma\}_{\eta \in \underline{\Delta}^{\mathrm{surj}}(-,p) - \{\mathrm{id}_{[0\ldots p]}\}}$.

In a sense the degenerate simplices are somewhat *redundant* and the *normalization theorem* explains you can neglect them when defining – and *computing* – the homology of the underlying simplicial set.

Definition 25 — Let X be a simplicial set. The (non-normalized) chain complex $C_* X$ associated to X is the algebraic cellular complex $(\mathbb{Z}[X_p], d_p, X_p)_p$ where the differential d_p is defined by $d_p \sigma = \sum_{i=0}^p (-1)^i \partial_i \sigma$.

The normalized chain complex $C_*^{\mathrm{ND}} X$ is the algebraic cellular complex $(\mathbb{Z}[X_p^{\mathrm{ND}}], d_p, X_p^{\mathrm{ND}})_p$, using only the non-degenerate cells; the differential is defined by the same formula, except every degenerate face is cancelled. ♣

In fact it is better to observe the degenerate simplices generate a chain *sub*complex $C_*^{\mathrm{D}}X$ and the normalized chain complex is nothing but the quotient $C_*^{\mathrm{ND}}X = C_*X/C_*^{\mathrm{D}}X$. The following result is classical and fundamental.

Theorem 26 (Normalization Theorem) — *The projection $C_*X \to C_*^{ND}X$ is a homology equivalence.* ♣

The proof is not difficult, see for example [12, Section VIII.6].

3.2 A vector field proof of the Normalization Theorem.

The standard proof of the Normalization Theorem is not difficult but however requires some lucidity. We intend to give a slightly different proof based on a discrete vector field. Consider this as an opportunity to illustrate our technique in a very simple case. Maybe you will find this proof simpler and, why not, funnier. The announced homology equivalence is a direct consequence of the next proposition.

Proposition 27 — *The degenerate cellular complex C_*^D is acyclic.*

♣ Let us recall a degenerate simplex $\sigma \in X_p^{\mathrm{D}}$ of a simplicial set X has a unique expression: $\sigma = \eta\tau$ with $\tau \in X_q^{\mathrm{ND}}$ a non-degenerate q-simplex, with $q < p$, and $\eta : \underline{p} \to \underline{q}$ a $\underline{\Delta}$-surjection. We represent such a surjection as an increasing sequence of $(p+1)$ integers of $\underline{q}$. For example 0122234 represents the unique surjection $\eta : \underline{6} \to \underline{4}$ satisfying $\eta(2) = \eta(3) = \eta(4) = 2$.

We will construct an *admissible* discrete vector field on C_*^{D} where *every* cell is source or target of the vector field: no remaining cell, that is, no critical cell, which at once proves our complex is acyclic: homology equivalent to the null complex.

We must divide the set of cells in two "equal" parts in bijection by the vector field to construct. The process is simple. Necessarily, because a degeneracy operator η is a non-injective surjection, some images are repeated. We take the first occurence of a repetition, which has a *multiplicity*: for example the multiplicity of the first repetition of 012222345556 is 4. If this multiplicity is *even*, we decide the corresponding cell is a source of the vector field to be defined, and the corresponding target is the cell obtained by adding an extra repetition to this first one. For example if $\sigma = \eta\tau$ with $\eta = 012222345556$, we decide this σ is a source and the corresponding target is $\sigma' = \eta'\tau$ with $\eta' = 0122222345556$, first multiplicity 5; this new simplex is also degenerate.

Symmetrically, if the multiplicity of the first repetition is odd, therefore at least 3, we decide the cell is a target, and removing one repetition gives back the corresponding source. The division of the degenerate cells in two "equal" parts is clear, how to be simpler?

If a cell is a source, then applying the boundary formula shows this source actually is a *regular* face of the corresponding target. Observe you cannot exchange the choices of odd and even when processing the number of repetitions, otherwise the regularity condition would not hold.

The most important remains to do: we must prove the *admissibility* of this vector field. We construct a Lyapunov function, see Definition 10. Instead of a general definition for the desired Lyapunov function, we prefer a unique example which illustrates all the cases to be considered. Let τ be some non-degenerate 4-simplex and $\sigma = \eta\tau$ with $\eta = 01123334 = \eta_1\eta_4\eta_5$. We define the value $L_p(\sigma) = 4$, the "genuine" dimension of σ, that is the dimension of the non-degenerate simplex τ associated to σ by the Eilenberg-Zilber lemma.

We defined $V(\sigma) = \sigma' = (011123334)\tau = \eta_6(\eta_5(\eta_2(\eta_1(\tau)))) = (\eta_1\eta_2\eta_5\eta_6)\tau$, do not forget a simplicial set is a *contravariant* functor. We must consider the faces $\partial_i\sigma'$ for $0 \leq i \leq 8$. In fact the faces 1, 2 and 3 are not to be considered, for such a face is the initial cell σ, the incidence number $\varepsilon(\sigma, \sigma')$ being in this case -1: from the *algebraic* point of view, it is a unique face, a regular one. The faces 5, 6, and 7 are $(01112334)\tau$, not to be considered either, for this is not a source cell: the multiplicity of 1 is 3, odd. There remains the faces 0, 4 and 8.

For example $\partial_4\sigma' = \partial_4\eta_6\eta_5\eta_2\eta_1\tau = \eta_5\eta_4\eta_2\eta_1\partial_2\tau$ and the genuine dimension of this simplex is ≤ 3. It is possible this face is not a source cell, but anyway, if it is, the value of the Lyapunov function has decreased. The same for any face operator which is not swallowed by a degeneracy, in this case, for the faces 0, 4 and 8. We have proved that for any face σ'' of $V(\sigma) = \sigma'$, except the initial face σ, if σ'' is a source cell, then the inequality $L_p(\sigma'') < L_p(\sigma)$ holds.

The Lyapunov function L_p can be used for the bounding function λ_p required by Definition 9. Our vector field is admissible and the proposition is proved. ♣

3.3 The Eilenberg-MacLane space $K(\mathbb{Z}, 1)$.

It is an abelian simplicial group defined as follows. The set of n-simplices is simply the abelian group $\mathbb{Z}^n$, and a simplex is traditionally denoted as a *bar*-object $\sigma = [a_1|\cdots|a_n]$. In particular only one vertex (0-cell), the void bar object $[\,]$, while for every positive n, the simplex set $\mathbb{Z}^n$ is infinite. The face operators $\partial_i : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ are defined as follows:

$$\begin{array}{rcll} \partial_0([a_1|\cdots|a_n]) & = & [a_2|\cdots|a_n] & \\ \partial_i([a_1|\cdots|a_n]) & = & [a_1|\cdots|a_{i-1}|(a_i + a_{i+1})|a_{i+2}|\cdots|a_n] & \text{if } 1 < i < n \\ \partial_n([a_1|\cdots|a_n]) & = & [a_2|\cdots|a_{n-1}] & \end{array} \quad (3.2)$$

The degeneracy operator $\eta_i : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ consists in putting an extra null component in position i:

$$\eta_i([a_1|\cdots|a_n]) = [a_1|\cdots|a_i|\,0\,|a_{i+1}|\cdots|a_n] \quad (3.3)$$

so that the collection of non-degenerate simplices K_n^{ND} is $\mathbb{Z}_*^n$: no component of the bar expression must be null.

This definition can be generalized to any group G, non-necessarily abelian, then producing the simplicial set $K(G, 1)$. The topological *realization* $X = |K(G, 1)|$ of this simplicial object has a characteristic property up to homomotopy equivalence: all its homotopy groups are null except $\pi_1 X = G$, but this fact is not used here. In general, $K(G, 1)$ *is not* a simplicial *group*; but if G is abelian, then $K(G, 1)$ is an abelian simplicial group.

It happens $K(\mathbb{Z}, 1)$ is the minimal model of the circle S^1, minimal in the sense of Kan [14, §9]. Not really concretely minimal in the ordinary sense: as many non-degenerate n-cells as elements in $\mathbb{Z}_*^n$. On one hand such a big model is unavoidable if you want to *compute* the homotopy groups of an *arbitrary* simplicial set; on the other hand a *reduction* is easily obtained $C_* K(\mathbb{Z}, 1) \Rrightarrow C_* S^1$ where this time S^1 is the usual model of the circle with one vertex $*$ and one (non-degenerate) edge s^1, so that the (normalized) chain complex of $C_* S^1$ is simply: $0 \leftarrow \mathbb{Z}[*] \stackrel{0}{\leftarrow} \mathbb{Z}[s^1] \leftarrow 0$.

The *ordinary* homology of $K(\mathbb{Z}, 1)$ is obvious: $\mathbb{Z}$ in degrees 0 and 1, nothing else. But in *effective* homology, it is *mandatory* to keep in one's environment the reduction $C_* K(\mathbb{Z}, 1) \Rrightarrow C_* S^1$. It happens this reduction can be deduced from a discrete vector field.

Proposition 28 — *An* admissible discrete vector field V *can be defined on* $C_*^{ND}(K(\mathbb{Z}, 1))$ *as follows:*

1. *The critical cells are the unique* 0*-simplex* $[\,]$ *and the* 1*-simplex* $[1]$.
2. *The source cells are the non-critical cells* $[a_1|\cdots]$ *satisfying* $a_1 \neq 1$.
3. *The target cells are the cells* $[a_1|a_2|\cdots]$ *of dimension* ≥ 2 *satisfying* $a_1 = 1$.
4. *The pairing* [source cell $\leftrightarrow$ target cell] *associates to the source cell* $[a_1|a_2|\cdots]$ *the target cell* $[1|a_1 - 1|a_2|\cdots]$ *if* $a_1 > 1$ *and* $[1|a_1|a_2|\cdots]$ *if* $a_1 < 0$.

♣ The admissibility property comes from the following observation: any V-path decreases the absolute value of the first component a_1. Two examples:

$$\begin{array}{l} [3|6] \mapsto [1|2|6] \mapsto [2|6] \mapsto [1|1|6] \mapsto \text{halt!} \\ [-3|6] \mapsto [1|-3|6] \mapsto [-2|6] \mapsto [1|-2|6] \mapsto [-1|6] \mapsto [1|-1|6] \mapsto \text{halt!} \end{array} \tag{3.4}$$

For example the four faces of $[1|2|6]$ are $[2|6]$, a source cell which is the continuation of the path, $[3|6]$, the source cell matching $[1|2|6]$, not to be considered, $[1|8]$ not a source cell, and $[1|2]$ not a source cell. For the last target cell of this path, only one face $[2|6]$ is a source cell but it just matches $[1|1|6]$, so that the path cannot be continued. We let the reader play the same game with the second example.

The critical chain complex is therefore $0 \leftarrow \mathbb{Z}[\,] \stackrel{d'}{\leftarrow} \mathbb{Z}[1] \leftarrow 0$, but what is the differential d'? We must use the first formula (2.18). The differential $d[1] = \partial_0[1] - \partial_1[1] = [\,] - [\,] = 0$ is null, so that $d_{p,3,3}$ and $d_{p,2,3}$ are null as well, and finally $d' = 0$. ♣

Our two examples are in a sense symmetric. In the first case, for the normalization theorem, the studied complex is an acyclic subcomplex, so that the reduction

$C_*X \twoheadrightarrow C_*^{ND}X$ has a quotient complex as the small complex. In the second case $K(\mathbb{Z}, 1)$, the critical cells [ ] and [1] generate a subcomplex which then is always the critical complex, for the components $d_{p,2,3}$ of the boundary matrices are null; it is the quotient by this subcomplex which is acyclic.

It is interesting to compare the previous proposition with the formula (14.4) of [6]: which shows Eilenberg and MacLane already had a perfect knowledge of our vector field, sixty years ago, even if they did not find useful to introduce the corresponding terminology.

Chapter 4

W-reductions of digital images.

4.1 Introduction.

Definition 29 — A *digital image*, an *image* in short, is a *finite* algebraic cellullar complex $(C_p, d_p, \beta_p)_{p\in\mathbb{Z}}$: every β_p is finite and furthermore every β_p is empty outside an interval $[0 \ldots P]$, the smallest possible P being the *dimension* of the image. ♣

For example the various techniques of scientific imaging produce *images*, some finite objects, typically finite sets of pixels. If you are interested in some *homological* analysis of such an image, you associate to it a *geometrical* cellular complex, again various techniques can be used, and finally this defines an algebraic cellular complex. There remains to compute the homology groups of this complex; in fact computing the *effective* homology [19] of this complex is much better: think for example of this interesting notion of *persistent* homology where the homology groups are not enough, you must *exhibit* cycles representing the homology classes, for example to study if such a cycle "remains alive" when the image is modified; it so happens effective homology is exactly designed to construct such cycles.

The Vector-Field Reduction Theorem (Theorem 19) is then particularly welcome. The bases β_p of the initial complex can be enormous, but appropriately choosing a vector field can produce, applying this Reduction Theorem, a new complex which is homology equivalent, with small *critical* bases β_p^c; so that the homology computations are then fast. More important: because this result produces a *reduction* between both complexes, the fast solution of the homological problem for the critical complex gives also at once a solution for the same problem in the big initial complex. Let us recall these methods of *effective* homology were designed for initial complexes *not of finite type*; which is successful for infinite objects should also work for big ones, big but finite! He who can do more can do less.

The simplest case is the case of a chain complex with only two consecutive chain groups, and the general case is easily reduced to this one. It is the problem known as the reduction of an integer matrix to the Smith form about which much work has already been done, often impressive and fascinating. The matter here is just of taking account of the frequent presence of terms ± 1 in the matrices produced by imaging, and to examine if this method of vector fields can be used. It is nothing but the following game: let M be a given integer matrix; please extract a square triangular submatrix, as large as possible, where all the diagonal terms are ± 1; triangular with respect to some order of rows and columns to be appropriately chosen. This has certainly already been extensively studied by the specialists in imaging; we are not such specialists and we do not pretend invent anything. But this point of view of *effective* homology could after all be of some usefulness.

4.2 Vector fields and integer matrices.

Let M be an initial matrix $M \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, with m rows and n columns. Think of M as the unique non-null boundary matrix of the chain complex:

$$\cdots \leftarrow 0 \leftarrow \mathbb{Z}^m \stackrel{M}{\longleftarrow} \mathbb{Z}^n \leftarrow 0 \leftarrow \cdots \tag{4.1}$$

A *vector field* V for this matrix is nothing but a set of integer pairs $\{(a_i, b_i)\}_i$ satisfying these conditions:

1. $1 \leq a_i \leq m$ and $1 \leq b_i \leq n$.
2. The entry $M[a_i, b_i]$ of the matrix is ± 1.
3. The indices a_i (resp. b_i) are pairwise different.

This clearly corresponds to a vector field, and constructing such a vector field is very easy. But there remains as usual the main problem: is this vector field *admissible*? An interesting but serious problem!

Because the context is finite, it is just a matter of avoiding *loops*. If the vector field is admissible, it defines a *partial* order between source cells: the relation $a > a'$ is satisfied between source cells if and only if a V-path goes from a to a'. The non-existence of loops guarantees this is actually a partial order.

Conversely, let V be a vector field for our matrix M. If $1 \leq a, a' \leq m$, with $a \neq a'$, we can decide $a > a'$ if there is an *elementary* V-path from a to a', that is, if a vector (a, b) is present in V and the entry $M[a', b]$ is non-null; for this corresponds to a cell b with in particular a as regular face and a' as an arbitrary face. We so obtain a binary relation. Then the vector field V is admissible if and only if this binary relation actually transitively generates a partial order, that is, if again there is no loop $a_1 > a_2 > \cdots > a_k = a_1$.

We can therefore summarize the reduction problem by a vector field as follows: given the matrix M, what process could produce a vector field as large as possible, but admissible, that is, without any loops? Finding such a vector field of maximal size seems much too difficult in real applications. Finding a maximal admissible vector field, not the same problem, is more reasonable but still serious. We start with some simple heuristic strategies to obtain significant admissible vector fields, large but most often not maximal.

4.2.1 Using a predefined order.

A direct way to quickly construct an admissible vector field consists in *pre*defining an order between row indices, and to collect all the indices for which some column is "above this index". Let us play with this toy-matrix given by our random generator:

$$M = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \end{bmatrix} \tag{4.2}$$

If we take simply the index order between row indices, we see the columns 1, 4 and 5 can be selected, giving the vector field $\{(5,1),(3,4),(4,5)\}$. This leads to reorder rows and colums in the respective orders $(5,4,3,1,2)$ and $(1,5,4,2,3)$, rewriting the matrix M as:

$$M = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \tag{4.3}$$

where the 3×3 top left-hand block is triangular. The block decomposition which follows then corresponds to the one used in the Hexagonal Lemma 14:

$$\varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \ \varphi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \ \psi = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{4.4}$$

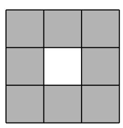
The Hexagonal Lemma produces a reduction of $M : \mathbb{Z}^5 \leftarrow \mathbb{Z}^5$ to $M' : \mathbb{Z}^2 \leftarrow \mathbb{Z}^2$ with $M' = \beta - \psi\varepsilon^{-1}\varphi$, where the computation of ε^{-1} is easy, for ε is triangular unimodular, a computation directly given by the formula 2.20. This gives:

$$M' = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \tag{4.5}$$

which is at once triangular unimodular, so that in fact our random matrix M is an automorphism of $\mathbb{Z}^5$.

4.2.2 Geometric orders.

For chain complexes coming from actual digital images, partial orders coming from geometrical properties can be very convenient. We take again a toy example, a screen with a 3×3 "resolution" and this image, eight pixels black and one white.

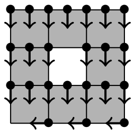


(4.6)

The bases of the corresponding cellular complex are made of 16 vertices, 24 edges and 8 squares.

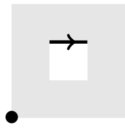
$$0 \leftarrow \mathbb{Z}^{16} \leftarrow \mathbb{Z}^{24} \leftarrow \mathbb{Z}^8 \leftarrow 0 \tag{4.7}$$

The boundary matrices are a little complicated. To design an *admissible* vector field, we can decide the only allowed vectors are oriented leftward or downward, this is enough to avoid *loops*. Various systematic methods are possible. Such a method could for example produce this vector field:



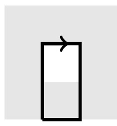
(4.8)

There remains only two critical cells, one vertex and one edge:



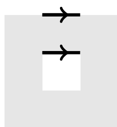
(4.9)

The reduced chain complex is $0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$ with the null map between both copies of $\mathbb{Z}$. The reduction described by Theorem 19 gives in particular a chain map $g : C^c_* \to C_*$ from the critical chain complex to the initial one; the generator of $H_1(C^c_*)$ is the only critical edge and its image in the initial chain complex is this cycle:



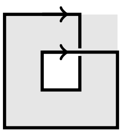
(4.10)

This reduction informs that $H_1(C_*) = \mathbb{Z}$ and produces a representant for the generating homology class. The f_1-component of the reduction $f_1 : C_1 \to C^c_1$ is null except for these two edges where the image is the unique critical edge:



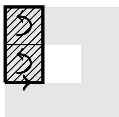
(4.11)

So that if some cycle is given in the initial chain complex:



(4.12)

the f-image gives its homology class, here twice the generator of $H_1(C^c_*)$. If ever the homology class so calculated is null, again the reduction produces a boundary preimage. For example the homology class of this rectangle 1-cycle is null and the image shows the boundary preimage, the sum of two squares, *computed* by the vector field.



(4.13)

All these comments can here be directly read from the vector field, for this "image" is very small. But this process can easily be made automatic for the actual images produced by computers, describing the *effective* homology of this image, with the same amount of information: the *homological problem* for this image is *solved.*

4.2.3 Constructing an appropriate order.

A more sophisticated strategy consists, given an *admissible* vector field already constructed, in trying to add a new vector to obtain a better reduction. The already available vector field defines a partial order between the source cells with respect to this vector field and the game now is to search a new vector to be added, but keeping the admissibility property. This process is applied by starting from the void vector field.

Let us try to apply this process to the same matrix as before:

$$M = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \end{bmatrix} \tag{4.14}$$

We start with the void vector field $V_0 = \{\}$. Running the successive rows in the usual reading order, we find $M[1,3] = -1$, and we add the vector (1,3), obtaining $V_1 = \{(1,3)\}$. Only one source cell 1, but we must note that it is from now on forbidden to add a vector which would produce the relation $4 > 1$ or $5 > 1$: this will generate a loop $1 > 4 > 1 > \cdots$ and the same for 5. In other words, the partial order to be recorded is $1 > 4$ and $1 > 5$, even if 4 and 5 are not yet source cells. Also the row 1 and the column 3 are now used and cannot be used anymore.

We read the row 2 and find $M[2,2] = -1$, which suggests to add the vector $(2,2)$, possible, with the same restrictions as before. Now $V_2 = \{(1,3),(2,2)\}$.

Reading the row 3 suggests to add the vector $(3,4)$ where 4 has 1 as a face, because $M[1,4] = -1$. This does not create any cycle, and we define $V_3 = \{(1,3),(2,2),(3,4)\}$. We note also that $3 > 1$.

Reading the row 4, the only possibility would be the new vector $(4,5)$, but 2 is a face of 5 and this would generate the loop $4 > 2 > 4 > \cdots$, forbidden. It is impossible to add a vector $(4,-)$.

Finally we can add the vector $(5,1)$, convenient, for 1 has no other face than 5; adding this vector certainly keeps the admissibility property.

This leads to the vector field $V_4 = \{(1,3),(2,2),(3,4),(5,1)\}$, which generates the partial order on $1,2,3,5$ where the only non-trivial relations are $3 > 1 > 5$ and $2 > 5$. In particular $\lambda(3) = 3$, $\lambda(1) = \lambda(2) = 2$ and $\lambda(5) = 1$, see the comments preparing Proposition 18. Reordering the rows and columns in the respective orders $(3,1,2,5,4)$ and $(4,3,2,1,5)$ gives the new form for our matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \tag{4.15}$$

The vector field has 4 components, and the 4×4 top left-hand submatrix is triangular unimodular. The reduction is better. The corresponding blocks are:

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}, \ \varphi = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \psi = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} -1 \end{bmatrix}. \tag{4.16}$$

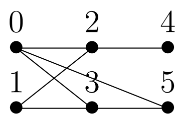
The same formula as before gives a reduction producing the matrix $[-1]$ which of course can be reduced to the void matrix.

Both examples show a first step of reduction produces a smaller matrix which in turn can sometimes be also reduced, even if the used vector field is maximal.

4.2.4 In more realistic situations.

The toy example of the previous section in fact is misleading. For such a small matrix, it is easy to maintain the state of the situation on one's draft sheet, but when the matrix is for example $100{,}000 \times 100{,}000$ with 10 non-null entries on every column, the work becomes harder.

Let V_k be a vector field already obtained, which we intend to enrich by a new vector v. This vector field generates an order graph, such as:



(4.17)

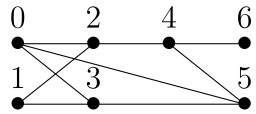
to be read as follows: with respect to V_k, the cells 0, 1, 2 and 3 are source cells, 4 and 5 are not source cells, and $a > b$ if a is connected to b by a path going rightward. For example $0 > 4$ and 5 are true, but $2 > 5$ is false, which of course in this context does not imply $2 \leq 5$!

The cells of such a graph are divided in two parts: the *source* cells, 0, 1, 2 and 3 in our example; the *minimal* cells, here 4 and 5, certainly not source cells with respect to V_k.

A new vector to be added is something like $a_0 > \{a_1, a_2, a_3\}$. We decide adding this vector is allowed if one of these conditions is satisfied:

1. a_0 is a minimal cell and a_1, a_2 and a_3 are not source cells of the previous graph, otherwise a loop *could* be generated.
2. a_0 is not present in our graph order.

The new vector to be added cannot have a source already used, but it can be a minimal cell of the previous graph. With respect to our example, adding the vector $4 > \{5, 6\}$ is correct, generating the new order graph:

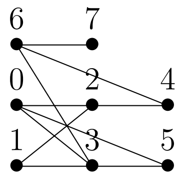


(4.18)

The status of 4 is changed from minimal to source, and the new cell 6 gets the minimal status.

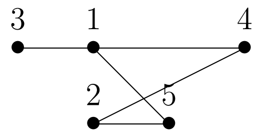
It would be illegal to add the vector $4 > 1$, because this would generate a loop. It would be legal to add the vector $4 > 3$, this would not generate any loop, but such a possibility is missed by our simplified method, for this would need a complete analysis of the order relation, too time consuming for real examples.

Also, the vector $6 > \{3, 4, 7\}$ can certainly be added, for its source 6 is not in the previous graph.



(4.19)

If the reader would like a toy example illustrating this example, it is enough to reexamine the example of Section 4.2.3. The vector field which was obtained by a careful examination of the situation is in fact also produced by the above automatic heuristic method! The corresponding order graph for this example is this one:



(4.20)

where 4 is the unique remaining minimal cell.

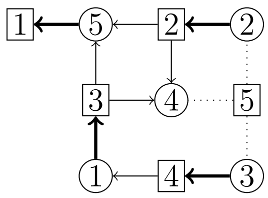
4.2.5 The corresponding graph problem.

To finish this short study of the chain complexes connected to images, let M be an integer $r \times c$ matrix. First we consider the particular case where all the entries are elements of $\{-1, 0, +1\}$. Such a matrix generates a graph G with nodes of two colors: the row nodes, indexed by $[1 \dots r]$ and the column nodes indexed by $[1 \dots c]$. Every matrix entry ± 1 generates an edge connecting the corresponding row and column nodes. The coloring of the graph is compatible with the incidence relations. The game is then the following: construct a collection of edges $V = \{(r_i, c_i)\}$ satisfying the following requirements. Every first component r_i is a row index, every second component c_i is a column index. The r_i's (resp. the c_i's) are pairwise different. Consider the subgraph made of all these c_i's and their *neighbouring* row nodes. Orient the edge $(r_i, c_i) \in V$ from the row node to the column node for every selected edge, *and* for all the *other* edges starting from c_i, orient these edges from c_i to the row node. Then you must not have any loop in this graph.

Let us take again our toy matrix:

$$M = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \end{bmatrix} \tag{4.21}$$

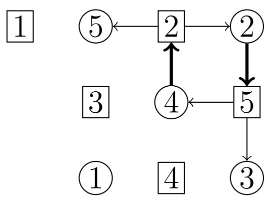
The graph corresponding to the vector field constructed in Section 4.2.3 is:



(4.22)

where the row nodes are circles and the column nodes are squares. The thick arrows are the elements of our vector field; the thin arrows go from one column to the neighbouring row nodes, when the column has been selected, except the component of the vector field itself, oriented in the reverse direction. Finally the column node 5 is not used.

It is then not difficult to prove that in our example there *does not exist* any admissible vector field of size 5. It would be necessary to "use" every node. You cannot select the vector $(4, 2)$, for the last possibility for the row node 2 would be $(2, 5)$ generating a loop:



(4.23)

In the same way, you cannot use the vector $(4, 5)$. Necessarily, the vector $(4, 3)$ must be used. But it would be necessary to use the vectors $(1, 4)$ and $(3, 5)$, again generating a loop of period 3. In other words, no reorder of rows and columns can make our matrix triangular. Which does not prevent our random matrix from being an isomorphism.

The vector field of size 4 is in fact in this example the vector field of maximal size. Solving such a problem for giant graphs of this sort is probably intractable.

For matrices with arbitrary integer entries, not necessarily in $\{-1, 0, +1\}$, the only difference is the following. If an entry $a_{r,c}$ of the matrix has an absolute value > 1, then the vector (r, c) is forbidden. If the column c is used in a vector field, the orientation of the edge (r, c) is necessarily $[c \to r]$.

Chapter 5

The Eilenberg-Zilber W-reduction.

5.1 Introduction.

The Eilenberg-Zilber theorem is essential in combinatorial topology, usually presented as a result of algebraic topology. The systematic use of *discrete vector fields* gives a more precise analysis: this result is in fact a particular case of deformations à la Whitehead, which gives as a *by-product* the usual homological result.

The subject is the following. Let Δ^p and Δ^q be two standard simplices of respective dimensions p and q. What about the product $\Delta^{p,q} = \Delta^p \times \Delta^q$? The lazy solution consists in enriching your collection of *elementary* models by all the possible products of simplices, so that $\Delta^{p,q}$ is then viewed as "elementary". Why not, but the penalty is not far: the numerous results patiently obtained in *simplicial* topology are nomore valid for the spaces made of these less elementary models.

Another solution consists in *triangulating* this product $\Delta^{p,q}$, feasible but not so easy. With a drawback, it is then difficult to read the product structure in this triangulation.

The Eilenberg-Zilber theorem settles the right connection between both solutions, certainly one of the most important results in Algebraic Topology: it is in fact the heart of the fundamental Serre and Eilenberg-Moore spectral sequences, and our analysis based on discrete vector fields gives a precise description of this interpretation.

5.2 Triangulations.

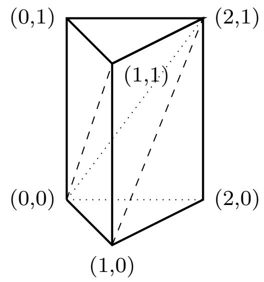
We have to work in the simplicial complex $\Delta^{p,q} = \Delta^p \times \Delta^q$. A vertex of Δ^p is an integer in $\underline{p} = [0 \ldots p]$, a (non-degenerate) d-simplex of Δ^p is a strictly increasing sequence of integers $0 \leq v_0 < \ldots < v_d \leq p$. The same for our second factor Δ^q.

The canonical triangulation of $\Delta^p \times \Delta^q$ is made of (non-degenerate) simplices $((v_0, v'_0), \ldots, (v_d, v'_d))$ satisfying the relations:

- $0 \leq v_0 \leq v_1 \leq \cdots \leq v_d \leq p$.
- $0 \leq v'_0 \leq v'_1 \leq \cdots \leq v'_d \leq q$.
- $(v_i, v'_i) \neq (v_{i-1}, v'_{i-1})$ for $1 \leq i \leq d$.

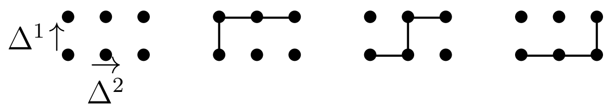
In other words, the canonical triangulation of $\Delta^{p,q} = \Delta^p \times \Delta^q$ is associated to the poset $\underline{p} \times \underline{q}$ endowed with the *product order* of the factors. For example the three maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ are:

- $((0,0),(0,1),(1,1),(2,1))$.
- $((0,0),(1,0),(1,1),(2,1))$.
- $((0,0),(1,0),(2,0),(2,1))$.



5.3 Simplex = s-path.

We can see the poset $\underline{p} \times \underline{q}$ as a lattice where we arrange the first factor $\underline{p}$ in the horizontal direction and the second factor $\underline{q}$ in the vertical direction. The first figure below is the lattice $\underline{2} \times \underline{1}$ while the other figures are representations of the maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ as *increasing* paths in the lattice.



(5.1)

Definition 30 — An *s-path* π of the lattice $\underline{p} \times \underline{q}$ is a finite sequence $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ of elements of $\underline{p} \times \underline{q}$ satisfying $(a_{i-1}, b_{i-1}) < (a_i, b_i)$ for every $1 \leq i \leq d$ with respect to the product order. The d-simplex σ_π represented by the path π is the convex hull of the points (a_i, b_i) in the prism $\Delta^{p,q}$. ♣

The simplices Δ^p and Δ^q have affine structures which define a product affine structure on $\Delta^{p,q}$, and the notion of convex hull is well defined on $\Delta^{p,q}$.

"S-path" stands for "path representing a simplex", more precisely a non-degenerate simplex. Replacing the strict inequality between two successive vertices by a non-strict inequality would lead to analogous representations for degenerate simplices, but such simplices are not to be considered in this section.

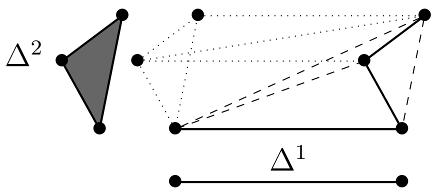
This representation of a simplex as an s-path running in a lattice is the key point to master the relatively complex structure of the canonical prism triangulations.

Definition 31 — The *last simplex* $\lambda_{p,q}$ of the prism $\Delta^{p,q}$ is the $(p+q)$-simplex defined by the path:

$$\lambda_{p,q} = ((0,0),(1,0),\ldots,(p,0),(p,1),\ldots,(p,q)). \tag{5.2}$$

♣

The path runs some edges of $\Delta^p \times 0$, visiting all the corresponding vertices in the right order; next it runs some edges of $p \times \Delta^q$, visiting all the corresponding vertices also in the right order. Geometrically, the last simplex is the convex hull of the visited vertices. The last simplex of the prism $P_{1,2} = \Delta^1 \times \Delta^2$ is shown in the figure below. The path generating the last simplex is drawn in full lines, the other edges of this last simplex are dashed lines, and the other edges of the prism are in dotted lines.



(5.3)

5.4 Subcomplexes.

Definition 32 — The *hollowed prism* $H\Delta^{p,q} \subset \Delta^{p,q}$ is the difference:

$$H\Delta^{p,q} := \Delta^{p,q} - \text{int(last simplex)}. \tag{5.4}$$

♣

The faces of the last simplex are retained, but the interior of this simplex is removed.

Definition 33 — The *boundary* $\partial\Delta^{p,q}$ of the prism $\Delta^{p,q}$ is defined by:

$$\partial\Delta^{p,q} := (\partial\Delta^p \times \Delta^q) \cup (\Delta^p \times \partial\Delta^q) \tag{5.5}$$

♣

It is the geometrical Leibniz formula.

We will give a detailed description of the pair $(H\Delta^{p,q}, \partial\Delta^{p,q})$ as a *W-contraction*, cf. Definition 1; it is a combinatorial version of the well-known topological contractibility of $\Delta^{p,q} - \{*\}$ on $\partial\Delta^{p,q}$ for every point $*$ of the interior of the prism. A very simple admissible vector field will be given to homologically annihilate the difference $H\Delta^{p,q} - \partial\Delta^{p,q}$. In fact, carefully ordering the components of this vector field will give the desired W-contraction.

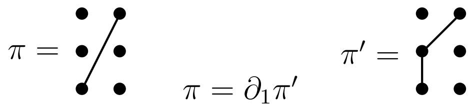
5.5 Interior and exterior simplices of a prism.

Definition 34 — A simplex σ of the prism $\Delta^{p,q}$ is said *exterior* if it is included in the boundary of the prism: $\sigma \subset \partial\Delta^{p,q}$. Otherwise the simplex is said *interior*. We use the same terminology for the s-paths, implicitly referring to the simplices coded by these paths. ♣

The faces of an exterior simplex are also exterior, but an interior simplex can have faces of both sorts.

Proposition 35 — *An s-path π in $\underline{p} \times \underline{q}$ is interior if and only if the projection-paths π_1 on $\underline{p}$ and π_2 on $\underline{q}$ run all the respective vertices of $\underline{p}$ and $\underline{q}$.*

The first s-path π in the figure below represents a 1-simplex *in* $\partial P_{1,2}$, for the point 1 is missing in the projection π_2 on the second factor $\underline{2}$: π is an *exterior* simplex. The second s-path π' represents an *interior* 2-simplex of $P_{1,2}$, for both projections are surjective.

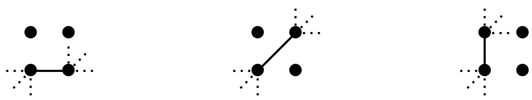


(5.6)

In particular, if $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ is an interior simplex of $\Delta^{p,q}$, then necessarily $(a_0, b_0) = (0, 0)$ and $(a_d, b_d) = (p, q)$: an s-path representing an interior simplex of $\Delta^{p,q}$ starts from $(0, 0)$ and arrives at (p, q).

♣ If for example the first projection of π is not surjective, this means the first projection of the generating path does not run all the vertices of Δ^p, and therefore is included in one of the faces $\partial_k \Delta^p$ of Δ^p. This implies the simplex σ_π is included in $\partial_k \Delta^p \times \Delta^q \subset \partial \Delta^{p,q}$. ♣

We so obtain a simple descrition of an interior simplex $((a_i, b_i))_{0 \leq i \leq d}$: it starts from $(a_0, b_0) = (0, 0)$ and arrives at $(a_d, b_d) = (p, q)$; furthermore, for every $1 \leq i \leq d$, the difference $(a_i, b_i) - (a_{i-1}, b_{i-1})$ is $(0, 1)$ or $(1, 0)$ or $(1, 1)$: both components of this difference are non-negative, and if one of these components is ≥ 2, then the surjectivity property is not satisfied. In a geometrical way, the only possible *elementary steps* for an s-path π describing an *interior* simplex of $\Delta^{p,q}$ are:



(5.7)

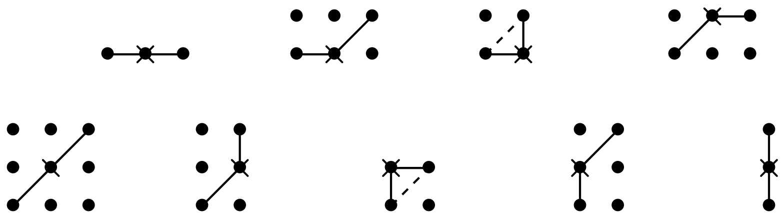
5.6 Faces of s-paths.

If $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ represents a d-simplex σ_π of $\Delta^{p,q}$, the face $\partial_k \sigma_\pi$ is represented by the same s-path except the k-th component (a_k, b_k) which is removed: we could say this point of $\underline{p} \times \underline{q}$ is *skipped.* For example in Figure (5.6) above, $\partial_1 \pi' = \pi$. In particular a face of an interior simplex is not necessarily interior.

Proposition 36 — *Let $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ be an s-path representing an interior d-simplex of $\Delta^{p,q}$. The faces $\partial_0 \pi$ and $\partial_d \pi$ are certainly* not *interior. For $1 \leq k \leq d-1$, the face $\partial_k \pi$ is interior if and only if the point (a_k, b_k) is a right-angle*  *or*  *of the s-path π in the lattice $\underline{p} \times \underline{q}$.*

♣ Removing the vertex $(a_0, b_0) = (0, 0)$ certainly makes non-surjective a projection π_1 or π_2 (or both if $(a_1, b_1) = (1, 1)$). The same if the last point (a_d, b_d) is removed.

If we examine now the case of $\partial_k\pi$ for $1 \le k \le d-1$, nine possible configurations for two consecutive elementary steps before and after the vertex (a_k, b_k) to be removed:



(5.8)

In these figures, the intermediate point ✱ of the displayed part of the considered s-path is assumed to be the point (a_k, b_k) of the lattice, to be removed to obtain the face $\partial_k\pi$. In the cases 1, 2, 4 and 5, skipping this point makes non-surjective the first projection π_1 on $\underline{p}$ '=' Δ^p. In the cases 5, 6, 8 and 9, the second projection π_2 on $\underline{q}$ becomes non-surjective. There remain the cases 3 and 7 where the announced right-angle bend is observed. ♣

5.7 From vector fields to W-contractions.

Let (X, A) be an elementary W-contraction, cf Definition 1. The difference $X - A$ is made of two non-degenerate simplices σ and τ, the first one being a face of the second one with a *unique* face index. The pair (σ, τ) is nothing but the unique vector of a vector field V, a vector field which, via the Vector-Field Reduction Theorem 19, defines also the W-reduction $C_*^{ND}X \Rrightarrow C_*^{ND}A$

Definition 37 — A simplicial pair (X,A) is an elementary *filling* if the difference $X - A$ is made of a *unique* non-degenerate simplex σ, all the faces of which are therefore simplices of A. ♣

You might think $\partial\sigma$ is the initial state of a decayed tooth in the body A, to be restored by adding $\mathrm{int}(\sigma)$, obtaining X.

Definition 38 — Let (X, A) be a simplicial pair. A *description by a filling sequence* of this pair, more precisely of the difference $X - A$, is an ordering $(\sigma_i)_{0<i\le r}$ of the non-degenerate simplices of $X - A$ satisfying the following condition: if $A_i = A \cup (\cup_{j=1}^{i}\sigma_j)$, then every pair (A_i, A_{i-1}) is an elementary filling. ♣

Every pair (X, A) with a finite number of non-degenerate simplices in $X - A$ can be described by a filling sequence: order the missing simplices according to their dimension. In particular, adding an extra vertex is a particular filling.

It is convenient to describe the general W-contractions, see Definition 2, by *special* filling sequences.

Proposition 39 — *Let (X,A) be a simplicial pair. This pair is a* W-contraction *if and only if it admits a description by a filling sequence $F = (\sigma_i)_{0<i\leq 2r}$ satisfying the extra condition: for every* even *index $2i$, the simplex σ_{2i-1} is a face of σ_{2i} with a unique face index.*

♣ Such a description is nothing but the vector field $V = \{(\sigma_{2i-1}, \sigma_{2i})_{0<i\leq r}\}$ with an *extra information*: the vectors are *ordered* in such a way they justify also the W-contraction property. Such a vector field is necessarily *admissible*: all the V-paths go to A and cannot loop. ♣

This extra information given by the order on the elements of the vector field is an avatar of the traditional difference between homotopy and homology.

5.8 The theorem of the hollowed prism.

Theorem 40 — *The pair: $(H\Delta^{p,q}, \partial\Delta^{p,q})$ is a W-contraction.*

The hollowed prism can be W-contracted on the boundary of the same prism.

♣ The proof is recursive with respect to the pair (p, q). If $p = 0$, the boundary of $\Delta^0 = *$ is void, so that the boundary of $\Delta^{0,q}$ is simply $\partial\Delta^q$; the last simplex is the unique q-simplex, the hollowed prism $H\Delta^{0,q}$ is also $\partial\Delta^q$: the desired W-contraction is trivial, more precisely the corresponding vector field is empty. The same if $q = 0$ for the pair $(H\Delta^{p,0}, \partial\Delta^{p,0})$.

Now we prove the general case (p, q) with $p, q > 0$, assuming the proofs of the cases $(p-1, q-1)$, $(p, q-1)$ and $(p-1, q)$ are available. Three justifying filling sequences are available; it is more convenient to see the sequences of simplices as sequences of *s-paths*:

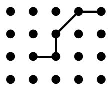
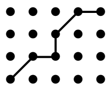
- $F_1 = (\pi_i^1)_{0<i\leq 2r_1}$ for $\Delta^{p-1,q-1} = \partial_0\Delta^p \times \partial_0\Delta^q$.
- $F_2 = (\pi_i^2)_{0<i\leq 2r_2}$ for $\Delta^{p,q-1} = \Delta^p \times \partial_0\Delta^q$.
- $F_3 = (\pi_i^3)_{0<i\leq 2r_3}$ for $\Delta^{p-1,q} = \partial_0\Delta^p \times \Delta^q$.

All the components of these filling sequences can be viewed as s-paths starting from $(1, 1)$ (resp. $(0, 1)$, $(1, 0)$), going to (p, q).

These filling sequences are made of all the non-degenerate s-paths (simplices) of the difference $H\Delta^{*,*} - \partial\Delta^{*,*}$, ordered in such a way every face of an s-path is either interior *and* present *beforehand* in the list, or exterior; furthermore, for the s-paths of even index, the previous one is one of its faces. Using these sequences, we must construct an analogous sequence for the bidimension (p, q).

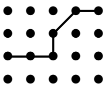
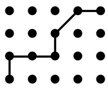
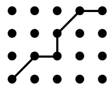
Every s-path π_i^j of dimension d can be completed into an interior s-path $\overline{\pi}_i^j$ of dimension $d+1$ in $\underline{p} \times \underline{q}$ in a unique way, adding a first diagonal step $((0, 0), (1, 1))$ if $j = 1$, or a first vertical step $((0, 0), (0, 1))$ if $j = 2$, or a first horizontal step $((0, 0), (1, 0))$ if $j = 3$. Conversely, every interior s-path of $\Delta^{p,q}$ can be obtained from an interior s-path of $\Delta^{p-1,q-1}$, $\Delta^{p,q-1}$ or $\Delta^{p-1,q}$ in a unique way by this completion process.

For example, in the next figure, we illustrate how an s-path π_i^1 of $\underline{3} \times \underline{2}$ can be completed into an s-path $\overline{\pi}_i^1$ of $\underline{4} \times \underline{3}$:

$\pi_i^1 =$  $\overline{\pi}_i^1 =$  (5.9)

Adding such a last diagonal step *does not add* any right-angle bend in the s-path, so that the assumed incidence properties of the initial sequence $\Sigma^1 = (\pi_1^1, \ldots, \pi_{2r_1}^1)$ are essentially preserved in the completed sequence $\overline{\Sigma}^1 = (\overline{\pi}_1^1, \ldots, \overline{\pi}_{2r_1}^1)$: the faces of each s-path are already present in the sequence or are exterior; in the even case $\pi_{2i}^1 \in \Sigma^1$, the previous s-path π_{2i-1}^1 is a face of π_{2i}^1 and hence $\overline{\pi}_{2i-1}^1$ is a face of $\overline{\pi}_{2i}^1$. For example in the illustration above, if i is even, certainly $\partial_1 \pi_i^1 = \pi_{i-1}^1$ (for this face is the only interior face) and this implies also $\partial_2 \overline{\pi}_i^1 = \overline{\pi}_{i-1}^1$.

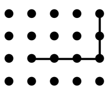
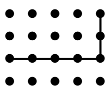
On the contrary, in the case $j = 2$, the completion process *can* add one right-angle bend, nomore. For example, in this illustration:

$\pi_i^2 =$  $\overline{\pi}_i^2 =$  $\partial_1 \overline{\pi}_i^2 =$  (5.10)

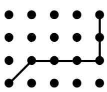
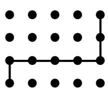
If the index i is even, then $\partial_2 \pi_i^2 = \pi_{i-1}^2$ and the relation $\partial_3 \overline{\pi}_i^2 = \overline{\pi}_{i-1}^2$ is satisfied as well. But another face of $\overline{\pi}_i^2$ is interior, namely $\partial_1 \overline{\pi}_i^2$, generated by the new right-angle bend; because of the diagonal nature of the first step of this face, this face is present in the list $\overline{\Sigma}^1$, see the previous illustration.

Which is explained about $\overline{\Sigma}^2$ with respect to $\overline{\Sigma}^1$ is valid as well for $\overline{\Sigma}^3$ with respect to $\overline{\Sigma}^1$.

The so-called last simplices, see Definition 31, must not be forgotten! The last simplex $\lambda_{p-1,q-1}$ (resp. $\lambda_{p,q-1}$) *is not* in the list Σ_1 (resp. Σ_2): these lists describe the contractions of the *hollowed* prisms over the corresponding boundaries: all the interior simplices are in these lists except the last ones. The figure below gives these simplices in the case $(p, q) = (4, 3)$:

$\lambda_{3,2} =$  $\lambda_{4,2} =$  (5.11)

Examining now the respective completed paths:

$\overline{\lambda}_{3,2} =$  $\overline{\lambda}_{4,2} =$  (5.12)

shows that $\partial_1 \overline{\lambda}_{p,q-1} = \overline{\lambda}_{p-1,q-1}$; also the faces $\partial_p \overline{\lambda}_{p-1,q-1}$ and $\partial_{p+1} \overline{\lambda}_{p,q-1}$ are respectively in $\overline{\Sigma}^1$ and $\overline{\Sigma}^2$.

Putting together all these facts leads to the conclusion: If Σ^1, Σ^2 and Σ^3 are respective filling sequences for $(H\Delta^{(*,*)} - \partial\Delta^{(*,*)})$, with $(*, *) = (p-1, q-1)$,

$(p, q-1)$ and $(p-1, q)$ then the following list *is a filling sequence proving the desired W-contraction property* for the indices (p, q):

$$\overline{\Sigma}^1 \;||\; \overline{\Sigma}^2 \;||\; (\overline{\lambda}_{p-1,q-1}, \overline{\lambda}_{p-1,q}) \;||\; \overline{\Sigma}^3 \tag{5.13}$$

where '||' is the list concatenation. Fortunately, the last simplex $\lambda_{p,q} = \overline{\lambda}_{p-1,q}$ is the only interior simplex of $\Delta^{p,q}$ missing in this list. ♣

5.9 Examples.

The reader can apply himself the above algorithm for the small dimensions. The table below gives all the results for $(p, q) \leq (2, 2)$.

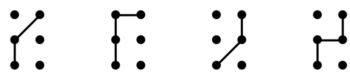
$(p, q) = (1, 1)$

(5.14)

$(p, q) = (2, 1)$

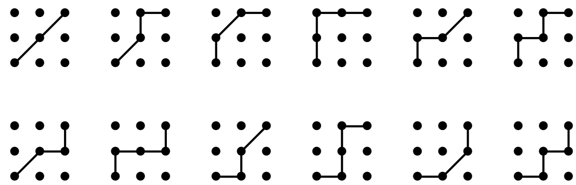
(5.15)

$(p, q) = (1, 2)$



(5.16)

$(p, q) = (2, 2)$



(5.17)

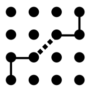
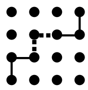
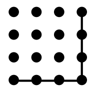
For significantly bigger values of (p, q), only a program can produce the corresponding filling sequences. A short Lisp program (45 lines) can produce the justifying list for reasonably small values of p and q. For example, if $p = q = 8$, the filling sequence is made of 265,728 paths, a list produced in 4 seconds on a modest laptop. But if $p = q = 10$, the number of paths is 8,097,452; and the same laptop is then out of memory. A typical behaviour in front of exponential complexity: the necessary number of paths is $> 3^p$ if $p = q > 1$.

5.10 If homology is enough.

The proposed proof of Theorem 40 is elementary but a little technical. If you are only interested by the *homological* Eilenberg-Zilber theorem, a simple proof analogous to Proposition 27's is sufficient.

Definition 41 — Let σ be an *interior* simplex of $\Delta^{p,q}$, represented by an s-path denoted by π. Then the *status* of σ (or π), *source* or *target* or *critical*, is defined as follows. You run the examined path π *from* (p, q) *backward* to $(0, 0)$ in the lattice $\underline{p} \times \underline{q}$ and you are interested by the *first* "event":

1. Either you run a diagonal elementary step $\swarrow$, in which case the path π is a *source s-path*;
2. Or you pass a bend  (not a bend ) in which case the path π is a *target s-path.*
3. Otherwise it is a *critical s-path* and only one *interior* s-path has this status, it is the s-path corresponding to the *last simplex* of $\Delta^{p,q}$. ♣

$\pi_1 =$  $\in S$ $\quad$ $\pi_2 =$  $\in T$ $\quad$ $\pi_3 =$  $\in C$ (5.18)

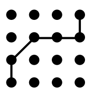
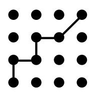
The figure above displays one example in either case when $p = q = 3$; the set S (resp. T, C) is the set of the *source* (resp. *target*, *critical*) cells. The deciding "event" is signalled by dotted lines. Observe $\partial_3\pi_2 = \pi_1$. More generally it is clear the operator assigning to every path of T the face in S corresponding to the bend  is a bijection organizing all these paths by pairs defining a discrete vector field, a good candidate to construct an interesting W-reduction: the unique path without any event fortunately is the last simplex!

Proposition 42 — *The relative chain complex* $C_*(H\Delta^{p,q}, \partial\Delta^{p,q})$ *admits a W-reduction to the null complex.*

♣ This relative chain complex is generated by all the interior simplices σ of $\Delta^{p,q}$ except the last one. Representing such a simplex σ by the corresponding s-path π allows us to divide all these simplices into two disjoint sets S (source) and T (target).

There remains to prove this vector field is admissible. It is a consequence of the organization of this vector field as a filling sequence given in Section 5.8, but it is possible to prove it directly and simply.

The example of the vector (π_1, π_2) above is enough to understand. We have to consider the faces of π_2 which are *sources*, in other words which are in S, therefore in particular *interior*, and *different* from π_1. In general at most two faces satisfy these requirements, here these faces $\pi_4 = \partial_2\pi_2$ and $\pi_5 = \partial_5\pi_2$:

$\pi_4 = \partial_2\pi_2 =$  $\in S$ $\quad$ $\pi_5 = \partial_5\pi_2 =$  $\in S$ (5.19)

This gives a Lyapunov function, see Definition 10. If $\pi \in S$, decide $L(\pi)$ is the number of points of the (p, q)-lattice strictly *above* the path. Observe $L(\pi_1) = 5$ while $L(\pi_3) = L(\pi_4) = 4$. A reader having reached this point of the text will probably prefer to play in ending the proof by himself. ♣

5.11 The Eilenberg-Zilber W-reduction.

5.11.1 The (p, q)-Eilenberg-Zilber reduction.

Proposition 42 can be arranged to produce a sort of top-dimensional Eilenberg-Zilber W-reduction for the prism $\Delta^{p,q}$.

Proposition 43 — *The discrete vector field used in Proposition 42 induces a W-reduction $C_*(\Delta^{p,q}) \Rrightarrow C_*^c(\Delta^{p,q})$ where in particular the chain group $C_{p+q}(\Delta^{p,q})$ of rank $\binom{p+q}{p,\, q}$ is replaced by a critical chain group with a* unique *generator, the so-called last simplex $\lambda_{p,q}$.*

♣ This vector field makes sense as well in this context as in Proposition 42 and the admissibility property remains valid. A W-reduction is therefore generated, where in dimension $(p+q)$ the only critical simplex is the last simplex $\lambda_{p,q}$. ♣

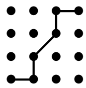
5.11.2 Products of simplicial sets.

Let us recall the product $X \times Y$ of two simplicial sets X and Y is very simply defined. These simplicial sets X and Y are nothing but contravariant functors $\underline{\Delta} \to \underline{\text{Sets}}$, and the simplicial set $X \times Y$ is the *product functor*. In particular $(X \times Y)_p = X_p \times Y_p$ and if $\alpha : \underline{p} \to \underline{q}$ is a $\underline{\Delta}$-morphism, then $\alpha^*_{X \times Y} = \alpha^*_X \times \alpha^*_Y : (X \times Y)_q \to (X \times Y)_p$. It is not obvious when seeing this definition for the first time this actually corresponds to the standard notion of topological product but, except in esoteric cases when simplex sets are not countable, the topological realization of the product is homeomorphic to the product of realizations. In these exceptional cases, the result remains true under the condition of working in the category of compactly generated spaces [27].

A p-simplex of the product $X \times Y$ is therefore a *pair* $\rho = (\sigma, \tau)$ of p-simplices of X and Y, and it is important to understand when this simplex is degenerate or not. Taking account of the Eilenberg-Zilber Lemma 24, we prefer to express both components of this pair as the degeneracy of some non-degenerate simplex, which produces the expression $(\eta_{i_{s-1}} \cdots \eta_{i_0} \sigma', \eta_{j_{t-1}} \cdots \eta_{j_0} \tau')$ for our p-simplex ρ, where σ' (resp. τ') is a non-degenerate $(p-s)$-simplex of X (resp. $(p-t)$-simplex of Y); also the sequences i_* and j_* must be *strictly increasing* with respect to their indices. Then the algebra of the elementary degeneracies η_i shows the simplex $\rho = (\sigma, \tau)$ is non-degenerate if and only if the intersection $\{i_{s-1}, \dots, i_0\} \cap \{j_{t-1}, \dots, j_0\}$ is empty.

The next definition is a division of all the non-degenerate simplices of the product $Z = X \times Y$ into three parts: the target simplices Z^t_*, the source simplices Z^s_* and the critical simplices Z^c_*. This division corresponds to a discrete vector field V, the natural extension to the whole product $Z = X \times Y$ of the vector field constructed in Sections 5.8 and 5.10 for the top bidimension (p, q) of the prism $\Delta^{p,q}$.

We must translate the definition of the vector field V in Section 5.10 into the language of non-degenerate product simplices expressed as pairs of possible degeneracies. To prepare the reader at this translation, let us explain the recipe which translates an s-path into such a pair. Let us consider this s-path:



(5.20)

This s-path represents an interior 5-simplex of $\Delta^{3,3}$ to be expressed in terms of the maximal simplices $\sigma, \tau \in \Delta^3_3$. Run this s-path from $(0,0)$ to $(3,3)$; every vertical elementary step, for example from $(1,0)$ to $(1,1)$ produces a degeneracy in the first factor, the index being the time when this vertical step is started, here 1. Another vertical step starts from $(2,2)$ at time 3, so that the first factor will be $\eta_3\eta_1\sigma$. In the same way, examining the horizontal steps produces the second factor $\eta_4\eta_0\tau$. Finally our s-path codes the simplex $(\eta_3\eta_1\sigma, \eta_4\eta_0\tau)$. The index 2 is missing in the degeneracies, meaning that at time 2 the corresponding step is diagonal: the lists of degeneracy indices are directly connected to the structure of the corresponding s-path. We will say the *degeneracy configuration* of this simplex is $((3,1),(4,0))$; a degeneracy configuration is a pair of disjoint decreasing integer lists.

Conversely, reading the indices of the degeneracy operators in the canonical writing $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ of a simplex ρ of $X \times Y$ unambiguously describes the corresponding s-path.

This process settles a canonical bijection between $S_{p,q}$ and $D_{p,q}$ if:

1. The set $S_{p,q}$ is the collection of all the *interior* s-paths running from $(0,0)$ to (p,q) in the (p,q)-lattice.
2. The set $D_{p,q}$ is the collection of all the configurations of degeneracy operators which can be used when writing a non-degenerate simplex in its canonical form $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$, when $\sigma \in X^{ND}_p$ and $\tau \in Y^{ND}_q$. A configuration is a pair of integer lists $((i_{s-1}, \ldots, i_0), (j_{t-1}, \ldots, j_0))$ satisfying the various coherence conditions explained before: $p + s = q + t$, every component i_- and j_- is in $[0 \ldots (p+q-1)]$, both lists are disjoint, and their elements are increasing with respect to their respective indices.

5.11.3 The Eilenberg-Zilber vector field.

Definition 44 — Let $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ be a non-degenerate p-simplex of $X \times Y$ written in the canonical form. The degeneracy configuration $((i_{s-1}, \ldots, i_0), (j_{t-1}, \ldots, j_0))$ is a well-defined element of $D_{p-s,p-t}$ which in turn defines an s-path $\pi(\rho) \in S_{p-s,p-t}$. Then ρ is a *target* (resp. *source*, *critical*) simplex if and only if the s-path $\pi(\rho)$ has the corresponding property. ♣

Definition 45 — If $\rho = (\eta_{i_{s-1}} \cdots \eta_{i_0}\sigma, \eta_{j_{t-1}} \cdots \eta_{j_0}\tau)$ is a p-simplex of $X \times Y$ as in the previous definition, the pair $(p-s, p-t)$ is called the *bidimension of* ρ. It is the bidimension of the smallest prism of $X \times Y$ containing this simplex. ♣

For example the diagonal of the square $\Delta^{1,1}$ has the bidimension $(1, 1)$. The sum of the components of the bidimension can be bigger than the dimension.

Definition 46 — Let $X \times Y$ be the product of two simplicial sets. The division of the non-degenerate simplices of $X \times Y$ according to Definition 44 into *target* simplices, *source* simplices and *critical* simplices, combined with the pairing described in the proof of Proposition 42, defines the *Eilenberg-Zilber vector field* $V_{X\times Y}$ of $X \times Y$. ♣

Theorem 47 (Eilenberg-Zilber Theorem) — *Let $X \times Y$ be the product of two simplicial sets. The Eilenberg-Zilber vector field $V_{X\times Y}$ induces the Eilenberg-Zilber W-reduction:*

$$EZ : C_*^{ND}(X \times Y) \Rrightarrow C_*^{ND}(X) \otimes C_*^{ND}(Y). \tag{5.21}$$

The reader may wonder why all these technicalities to reprove a well-known sixty years old theorem. Two totally different reasons.

On the one hand, the Eilenberg-Zilber reduction is time consuming when concretely programmed. In particular, profiler examinations of the effective homology programs show the terrible homotopy component $h : C_*(X \times Y) \to C_*(X \times Y)$ of the Eilenberg-Zilber reduction, rarely seriously considered[1], is the kernel program unit the most used in concrete computations. Our description of the Eilenberg-Zilber reduction makes the corresponding program unit simpler and more efficient.

On the other hand, maybe more important, the *same* (!) vector field will be soon used to process in the same way the *twisted* products, leading to totally elementary *effective* versions of the Serre and Eilenberg-Moore spectral sequences. Maybe the same for the Bousfield-Kan spectral sequence.

♣ The vector field $V_{X\times Y}$ has a layer for every bidimension (p, q). The admissibility proof given in Proposision 42 shows that every V-path starting from a source simplex of bidimension (p, q) goes after a finite number of steps to sub-layers. The Eilenberg-Zilber vector field is admissible.

If σ (resp. τ) is a non-degenerate p-simplex of X (resp. q-simplex of Y), we can denote by $\sigma \times \tau$ the corresponding (generalized) prism in $X \times Y$, made of all the simplices of bidimension (p, q) with respect to σ and τ. The collection of the *interior* simplices of this prism $\sigma \times \tau$ is nothing but an *exact* copy of the collection of the *interior* simplices of the standard prism $\Delta^{p,q}$. In particular only one *interior* critical cell in every prism. You are attending the birth of the tensor product $C_*^{ND}(X) \otimes C_*^{ND}(Y)$: exactly one generator $\sigma \otimes \tau$ for every prism $\sigma \times \tau$, namely the last simplex of this prism: $\sigma \otimes \tau$ "=" $\lambda_{p,q}(\sigma \times \tau)$.

[1]With two notable exceptions. In the landmark papers by Eilenberg and... MacLane [5, 6], more useful than the standard reference [8], a nice recursive description of this homotopy operator is given. Forty years later (!), when a computer program was at last available to make experiments, Julio Rubio found a closed formula for this operator, proved by Frédéric Morace a little later [17]. We reprove this formula and others in the next section, by a totally elementary process depending only on our vector field, independent of Eilenberg-MacLane's and Shih's recursive formulas, reproved as well.

There remains to prove the small chain complex so obtained is not only the right graded module $C_*^{ND}(X) \otimes C_*^{ND}(Y)$, but is endowed by the reduction process with the right differential. This is a corollary of the next section, devoted to a detailed study of the Eilenberg-Zilber vector field. ♣

5.12 Naturality of the Eilenberg-Zilber reduction.

We consider here four simplicial sets X, X', Y and Y' and two simplicial morphisms $\varphi : X \to Y$ and $\varphi' : X' \to Y'$. These morphisms induce a simplicial morphism $\varphi \times \varphi' : X \times X' \to Y \times Y'$. Also the products $X \times X'$ and $Y \times Y'$ carry their respective Eilenberg-Zilber vector fields V and W.

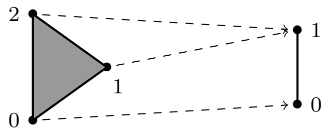
Theorem 48 — *With these data, the morphisms φ and φ' induce a natural morphism between both Eilenberg-Zilber reductions:*

$$\begin{aligned}\varphi \times \varphi' : [\rho = (f, g, h) : C_*(X \times X') \Rrightarrow C_*(X) \otimes C_*(X')] \longrightarrow \\ [\rho' = (f', g', h') : C_*(Y \times Y') \Rrightarrow C_*(Y) \otimes C_*(Y')]\end{aligned} \tag{5.22}$$

The chain complexes are normalized: C_* stands for C_*^{ND}. At this time of the process, we do not have much information for the small chain complexes: we know the underlying graded modules are (isomorphic to) those of $C_*(X) \otimes C_*(Y)$ and $C_*(X') \otimes C_*(Y')$, but we do not yet know their differentials.

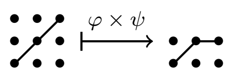
It is... natural to ask for such a result, but another goal is looked for. As it is common in a simplicial environment, once such a naturality result is available, it is often enough to prove some desired result in the particular case of an appropriate *model*, maybe a prism $\Delta^{p,q}$, and then to use an obvious simplicial morphism to transfer this result to an arbitrary product and obtain the general result. In fact, this method will also be an essential ingredient of the proof.

The statement of Theorem 48 is... natural, but the proof is more difficult than we could expect. The morphism $\varphi \times \psi$ can be *not at all compatible* with the respective Eilenberg-Zilber vector fields of $X \times Y$ and $X' \times Y'$, which generates essential obstacles. Consider for example the morphisms $\varphi = \mathrm{id} : \Delta^2 \to \Delta^2$ and $\psi : \Delta^2 \to \Delta^1$ defined by $\psi(012) = (011)$; the map ψ is nothing but the map $\psi = \eta_1 : \Delta^2 \to \Delta^1$ canonically associated to the $\underline{\Delta}$-map $\eta_1 : [0 \dots 2] \to [0 \dots 1]$, see Figure (3.1).



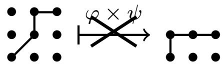
(5.23)

Then the simplicial morphism $\varphi \times \psi$ sends the "diagonal" 2-simplex σ of $\Delta^{2,2} = \Delta^2 \times \Delta^2$ to some 2-simplex of $\Delta^{2,1}$ as displayed in the next figure, where a simplex is represented by an s-path.



(5.24)

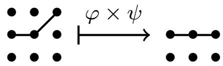
Both simplices are *source* cells, but for "reasons" which do no match. The lefthand cell is source because of the diagonal component $(1,1)-(2,2)$ of the s-path, which component is sent by $\varphi \times \psi$ to the horizontal component $(1,1)-(2,1)$. While the righthand cell is source because of the diagonal component $(0,0)-(1,1)$. The corresponding target cells are below:



(5.25)

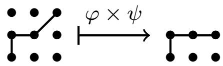
and they do not match by $\varphi \times \psi$: the image by $\varphi \times \psi$ of the lefthand target cell is in fact degenerate.

It can also happen the image of a source cell is critical:



(5.26)

or target:



(5.27)

In other words the tempting relation $(\varphi \times \psi)V(\sigma) = V(\varphi \times \psi)(\sigma)$ in general *is false*, or even does not make sense. It is then clear the "natural" Theorem 48 requires a "real" proof. It depends on a very general result, the statement of which is rather amazing.

Definition 49 — A *cellular morphism* $\varphi : A_* \to B_*$ between two cellular (chain) complexes A_* and B_* is a chain complex morphism which maps every cell of A_* to *zero* or to a *cell* of B_*.

Definition 50 — A *vectorious* cellular complex is a pair (A_*, V) where A_* is a cellular complex A_* provided with an *admissible* discrete vector field V.

Definition 51 — Let $\varphi : (A_*, V) \to (B_*, W)$ be a cellular morphism between the vectorious cellular complexes (A_*, V) and (B_*, W). The morphism φ is an *admissible morphism* if the following conditions are satisfied:

- The morphism φ maps every critical cell of V to 0 or to a critical cell of W.
- The morphism φ maps every target cell of V to 0 or to a target cell of W.

In particular, no condition is required for a source cell of V!

Theorem 52 — *Let $\varphi : (A_*, V) \to (B_*, W)$ be an* admissible *morphism between the vectorious cellular complexes $(A_*, V) \to (B_*, W)$. Then the morphism φ induces a natural morphism between the reductions $\rho : A_* \Rrightarrow A_*^c$ and $\rho' : B_* \Rrightarrow B_*^c$.*

♣[52] Let $h : A_* \to A_*$ and $h' : B_* \to B_*$ be the respective homotopies of ρ and ρ'. We must in particular prove $\varphi h = h' \varphi$. We use the recursive construction of h given by the formula (2.20), taking account also of the formulas given at (2.18).

The vector field V is there viewed as a set theoretic map [source cell $\mapsto$ target cell]; this map V induces a codifferential $v : A_* \to A_*$ of degree $+1$ defined by $v(a) = 0$ if a is a critical or target cell, and $v(a) = \varepsilon(a, V(a))V(a)$ if a is a source cell. Then the homotopy h defined by the vector field V is defined by $h(a) = 0$ if a is a critical or target cell, and:

$$h(a) = v(a) - h(dv(a) - a) \tag{5.28}$$

if a is a source cell. A source cell a has a *level* $\lambda(a)$, the length of the longest vector path issued from this cell, and any cell in $dv(a) - a$ has necessarily a level $< \lambda(a)$. If $\lambda(a) = 1$, then all the cells of $dv(a) - a$ are critical or target cells, which coherently starts the recursive process.

The relation $\varphi h(a) = h'\varphi(a)$ is obvious if a is a critical or target cell: the morphism φ is admissible and maps such a cell to a cell of the same sort, or to 0; in any case, both members are null.

Lemma 53 — *Let b be a target cell of V; then $b = hd(b)$.*

♣[53] Let a be the corresponding source cell. Then $h(a) = v(a) - h(dv(a) - a) = v(a) - hdv(a) + h(a)$, so that $v(a) - hdv(a) = 0$ where $v(a)$ is b up to sign. ♣[53]

Let us consider now a cell a source of the vector $(a, V(a))$ in (A_*, V). We inductively prove the relation $\varphi h(a) = h'\varphi(a)$, assuming the result is already known for the source cells of level $< \lambda(a)$. The difference $dv(a) - a$ is made of cells of level $< \lambda(a)$, which justifies the calculation: $\varphi h(a) = \varphi v(a) - \varphi h(dv(a) - a) = \varphi v(a) - h'\varphi(dv(a) - a) = (1 - h'd')(\varphi v(a)) + h'\varphi(a)$. The admissibility of φ implies the cell $\varphi v(a)$ is 0, or is a target cell of W, therefore cancelled by $(1 - h'd')$, because of the lemma above. In any case $\varphi h(a) = h'\varphi(a)$ and the compatibility between φ and the homotopy operators of ρ and ρ' is obtained.

The formula (2.18, g_p) implies $g = 1 - hd$ for the injection of the critical complex A^c_* in the whole complex A_*. Therefore: $\varphi g = \varphi(1 - hd) = (1 - h'd')\varphi = g'\varphi$ and the morphism φ is also compatible with the injections g and g' of ρ and ρ'.

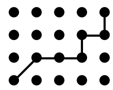
There remains to obtain $\varphi f = f'\varphi$ for the respective projections of A_* and B_* over the critical complexes A^c_* and B^c_*. The map g' is injective and the desired relation is equivalent to $g'\varphi f (= \varphi g f) \stackrel{?}{=} g' f' \varphi$. But $gf = 1 - dh - hd$, the same for $g'f'$ and the relation is a consequence of $\varphi d = d'\varphi$ and $\varphi h = h'\varphi$. ♣[52]

♣[48] We may work simplex by simplex, more precisely prism by prism; if $\sigma \in X$ (resp. $\sigma' \in X'$) is a non-degenerate simplex, then the prism $\sigma \times \sigma' \in X \times X'$ is mapped inside a prism $\tau \times \tau' \in Y \times Y'$ depending on the image simplices $\varphi(\sigma)$ and $\varphi'(\sigma')$.

A map $\varphi : \sigma \to \tau$ between simplices is a composition of face and degeneracy operators. So that it is enough to consider the case with only a face operator, or only a degeneracy operator. A face operator is injective and the result is then obvious.

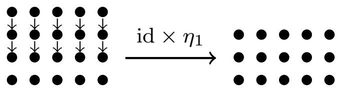
There remains to consider for example the case $X = Y = \Delta^p$, $X' = \Delta^q$, $Y' = \Delta^{q-1}$, $\varphi = \mathrm{id}_{\Delta^p}$ and φ' is the degeneracy $\eta_i : \Delta^q \to \Delta^{q-1}$ for some $0 \leq i < q$ which maps the vertex $\#j$ to itself if $j \leq i$ and to the vertex $\#(j-1)$ if $j > i$. The hoped-for result is then *not at all obvious.*

The technology of s-paths explained at Section 5.3 leads to understand a simplex of $\Delta^p \times \Delta^q$ as an oriented path in the lattice $[0 \ldots p] \times [0 \ldots q]$. For example, the s-path:



(5.29)

represents the 6-simplex spanned by the vertices $(0,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $(3,2)$, $(4,2)$ and $(4,3)$ in the prism $\Delta^4 \times \Delta^3$. We refer the reader to Definition 41 for the process dividing these simplices in source, target and critical simplices. A map $\mathrm{id} \times \eta_i$ is the identity for the vertices (j,k) satisfying $k \leq i$ and maps (j,k) to $(j,k-1)$ for $k > i$. For example the map $\mathrm{id} \times \eta_1 : \Delta^4 \times \Delta^3 \to \Delta^4 \times \Delta^2$ could be represented as follows:



(5.30)

We let the reader check himself the image of a critical cell is critical or degenerate, therefore in this case null in the normalized chain complex. The same for a target cell: essentially a morphism $\mathrm{id} \times \eta_i$ cannot destroy the bend characterizing a target cell without mapping it to a degenerate cell, cancelled in the normalized cell complex. ♣[48]

5.13 Two Eilenberg-Zilber reductions.

Let X and Y be two simplicial sets. We have now in our toolbox two reductions $EZ_1 = (f_1, g_1, h_1) : C_*(X \times Y) \Rightarrow\!\!\!\!\!\Rightarrow C_*(X) \otimes C_*(Y)$ and $EZ_2 = (f_2, g_2, h_2) : C_*(X \times Y) \Rightarrow\!\!\!\!\!\Rightarrow C_*(X) \otimes C_*(Y)$. The first one was determined by Eilenberg and MacLane in [5, 6], the second one is obtained from our "Eilenberg-Zilber" vector field, see Definition 46 and Theorem 47. In fact both reductions are the same, it is the goal of this section.

Both reductions have quite different definitions and a real task is in front of us. The structure of the proof is as follows:

1. We recall the standard formulas for the Eilenberg-MacLane reduction EZ_1, that is, the AW-formula for f_1 (Alexander-Whitney), the EML-formula for g_1 (Eilenberg-MacLane) and the RM-formula for h_1 (Rubio-Morace).
2. The homotopy operator h_2 defined by our vector field satisfies also the RM-formula, it is the key point.

3. We *deduce* from this fact that $g_2 = g_1$ and $f_2 = f_1$: our vector field reduction also satisfies the traditional Alexander-Whitney and Eilenberg-MacLane formulas.
4. We give the appropriate formula for the composition $g_2 f_2 = g_1 f_1$.
5. We prove the RM-formula satisfies the recursive Eilenberg-MacLane definition of h_1.

The last point is redundant with respect to which is available elsewhere, but the vector field understanding of the Eilenberg-MacLane recursive formula becomes too simple to be omitted.

5.13.1 Alexander-Whitney, Eilenberg-MacLane and Rubio-Morace.

The three components of the reduction EZ_1 are:

$$f_1 = AW : C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \quad (AW = \text{Alexander-Whitney})$$
$$g_1 = EML : C_*(X) \otimes C_*(Y) \to C_*(X \times Y) \quad (EML = \text{Eilenberg-MacLane})$$
$$h_1 = RM : C_*(X \times Y) \to C_*(X \times Y) \quad (RM = \text{Rubio-Morace})$$

The explicit formulas for these components are:

$$AW(x_p \times y_p) = \sum_{i=0}^{p} \partial_{p-i+1} \cdots \partial_p x_p \otimes \partial_0 \cdots \partial_{p-i-1} y_p \tag{5.31}$$

$$EML(x_p \otimes y_q) = \sum_{(\eta,\eta') \in Sh(p,q)} \varepsilon(\eta, \eta')\, (\eta' x_p \times \eta y_q) \tag{5.32}$$

$$RM(x_p \times y_p) = \sum_{0 \leq r \leq p-1, 0 \leq s \leq p-r-1, (\eta,\eta') \in Sh(s+1,r)} (-1)^{p-r-s}\, \varepsilon(\eta, \eta') \tag{5.33}$$
$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

where x_p and y_p are respective p-simplices of X and Y, which can be degenerate, but their cartesian product $(x_p \times y_p)$ is not. We denote $(x_p \times y_p)$ the *simplex* of $X \times Y$ defined by its *projections* x_p and y_p, with the separator $\times$, more readable in our relatively complex formulas than the traditional comma in (x_p, y_p). So that $(x_p \times y_p)$ denotes here a *simplex*, not a prism.

The diagonal map $C_*(X) \to C_*(X \times X)$ composed with the AW formula defines in particular the standard coproduct $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ of a simplicial chain complex.

Two symmetric formulas are possible for AW, and any affine combination of both is also possible. The paper [16] gives a further condition with respect to some order compatibility to obtain a unique choice for AW. The EML formula is unique [15].

It is clear four symmetric possible choices are possible for our Eilenberg-Zilber vector field. Examine the process described in Definition 41 to decide whether an s-path is source, target or critical. Instead of running the s-path backward from (p, q)

to $(0,0)$ you could run this path forward from $(0,0)$ to (p,q). Instead of replacing the diagonal $\swarrow$ by a bend $\ulcorner$ to define the target associated to some source s-path, you could once for all prefer to replace the diagonal by a bend $\lrcorner$, modifying accordingly the criterion for a target s-path. Finally four possible natural vector fields. Giving four different symmetric RM-formulas, two different symmetric AW-formulas and only one EML-formula. Our choice gives the most standard AW-, EML- and RM-formulas given above.

In the EML formula above, The set $Sh(p,q)$ is made of all the (p,q)-*shuffles* of $(0\cdots(p+q-1))$, that is, all the partitions of these $p+q$ integers in two increasing sequences of length p and q. Every shuffle produces in turn a pair of multi-degeneracy operators denoted in the same way; for example the shuffle $((0\,3),(1\,2\,4))$ produces the pair $(\eta,\eta') = (\eta_3\eta_0, \eta_4\eta_2\eta_1)$: the factors are to be in the right degeneracy order. Such a pair (η,η') is so associated to a permutation, producing a signature $\varepsilon(\eta,\eta')$. For example, the shuffle $((0\,1\,5),(2\,3\,4))$ in the case $p=q=3$ produces the term $-(\eta_4\eta_3\eta_2x_3 \times \eta_5\eta_1\eta_0y_3)$, for the permutation $(0\,1\,5\,2\;3\,4)$ is negative.

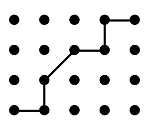
Julio Rubio, using numerical results computed by the EAT program [23], produced without proof in his thesis [18] the lovely formula RM (there called SHI) for the Eilenberg-Zilber homotopy operator. This formula was also called SHI in [17], this time with a proof (due to Frédéric Morace) based on the recursive formula given for Φ_n at [26, Page 25]. In fact this recursive formula is already at [6, Formula (2.13)].

The $\uparrow$-operator in the RM-formula shifts the indices of the multi-degeneracy operator, for example $\uparrow(\eta_3\eta_1) = \eta_4\eta_2$, $\uparrow^3(\eta_4\eta_2) = \eta_7\eta_5$.

This section is devoted to a careful analysis of the Eilenberg-Zilber vector field, leading to new proofs of all these formulas. Nothing more than a combinatorial game with the s-paths of prisms, that is, with the degeneracy operators. The key point is to obtain first the RM-formula, the others being in fact consequences.

5.13.2 Collisions between degeneracies.

A simplex in a prism is represented by an s-path such as this one:



(5.34)

This is the s-path representation of the 6-simplex spanning the vertices $(0\times 0) - (1\times 0) - (1\times 1) - (2\times 2) - (3\times 2) - (3\times 3) - (4\times 3)$ of the (4×3)-prism $\Delta^4 \times \Delta^3$.

The last vertical or horizontal segments of an s-path will play an essential role. For example, for the above s-path, the fact the last two segments are a vertical one followed by a horizontal one implies the RM-value for this simplex is null. A sequence of lemmas are to be devoted to various situations.

Let us carefully examine the generic term of the RM-formula, where we forget the sign:

$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s}\cdots\partial_{p-r-1}y_p) \tag{5.35}$$

Its dimension is $p+1$ and the configuration of the degeneracy operators will be essential. The pair (η, η') is a $(s+1, r)$-shuffle, invoking the indices $0\cdots(r+s)$, so that $\uparrow^{p-r-s}(\eta')$ *and* $\uparrow^{p-r-s}(\eta)$ *collectively* invoke the indices $(p-r-s)\cdots p$. Taking account also of the isolated operator $\eta_{p-r-s-1}$, finally all the indices of $(p-r-s-1)\cdots p$ are *explicitly* invoked in the formula.

But in general the initial factors x_p and y_p can also contain degeneracy operators, often generating "collisions" with the just considered explicit operators, then cancelling the corresponding term. For example if you consider a term $(\eta_6\eta_4 x', \eta_5\eta_3 y')$ of dimension 7, then if ever x' or y' contains an η_4 in his canonical expression, then this term is degenerate. If $x' = \eta_4 x''$, then $(\eta_6\eta_4 x', \eta_5\eta_3 y') = (\eta_6\eta_4\eta_4 x'', \eta_5\eta_3 y') = (\eta_6\eta_5\eta_4 x'', \eta_5\eta_3 y') = \eta_5(\eta_5\eta_4 x'', \eta_3 y')$ is degenerate; something analogous if $y' = \eta_4 y''$. This is due to the permutation rule $\eta_i\eta_j = \eta_{j+1}\eta_i$ if $j \geq i$ which tends to *increase* the indices when you sort the degeneracy operators. These examples allow us to state without any proof the next lemma.

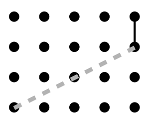
Lemma 54 (Collision lemma) — *In an expression* $(\eta' x' \times \eta y')$ *of dimension* $p+1$ *where the multidegeneracies* η' *and* η *contain all the indices of* $(p-r-s-1)\cdots p$, *then, if* x' *or* y' *contains a degeneracy with an index in the same range, the term* $(\eta' x', \eta y')$ *is in fact degenerate.* ♣

5.13.3 Null terms in the *RM*-formula.

A sequence of elementary lemmas playing with degeneracies will give *shorter RM*-formulas according to the nature of the simplex $(x_p \times y_p)$ considered. The general nature of these lemmas is roughly the following: if x_p and/or y_p contains degeneracy operators with high indices, then the corresponding terms of the RM-formula, because of the collision lemma, will be null *unless* the face operators previously annihilate these degeneracies.

Lemma 55 — *If* $x_p = \eta_{p-1} x'$, *then the terms* $r = 0$ *of the RM-formula are null.*

♣ This corresponds to this situation:

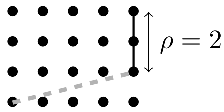


(5.36)

where the pale dashed line between $(0,0)$ and $(4,2)$ means we do not have any specific information about the s-path between the points $(0,0)$ and $(4,2)$.

Anyway, the relation $p-r-s-1 \leq p-1$ is satisfied in the generic term of the RM-formula. Because of the collision lemma, if the η_{p-1} of $x_p = \eta_{p-1}x'$ is not swallowed by a face operator, the term of the RM-formula will be null, which required $r \geq 1$. ♣

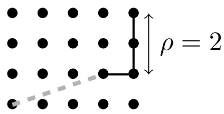
Lemma 56 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$, then the terms $r < \rho$ of the RM-formula are null.*



(5.37)

♣ Induction with respect to ρ, the previous lemma being the case $\rho = 1$. If the lemma is known for $\rho - 1$, this implies a non-null term satisfies $r \geq \rho - 1$; also the degeneracies $\eta_{p-1}\cdots\eta_{p-\rho+1}$ are annihilated by the face operators $\partial_{p-\rho+2}\cdots\partial_p$. Then the relation $p - r - s - 1 \leq p - \rho$ is satisfied. To avoid a collision, it is necessary to annihilate also the $\eta_{p-\rho}$, which requires $r \geq \rho$. ♣

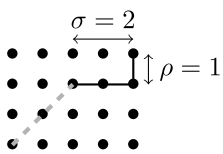
Lemma 57 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ and $y_p = \eta_{p-\rho-1}y'$ then the terms $r \neq \rho$ or $s = 0$ of the RM-formula are null. This is valid even if $\rho = 0$.*



(5.38)

♣ We already know a non-null term satisfies $r \geq \rho$. So that $p-r-s-1 \leq p-\rho-1$ and the $\eta_{p-\rho-1}$ in the expression of y_p is to be annihilated to avoid a collision, which requires $r = \rho$ and $s > 0$. Note the "missing" face operator ∂_{p-r} in the RM-formula is crucial here. ♣

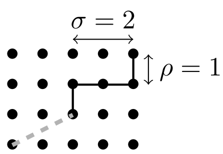
Lemma 58 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ and $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y'$ with $\sigma > 0$ then the terms $r \neq \rho$ or $s < \sigma$ of the RM-formula are null. This is valid even if $\rho = 0$.*



(5.39)

♣ Induction with respect to σ. ♣

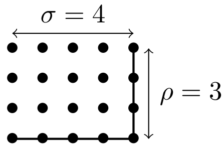
Lemma 59 — *If $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}\eta_{p-\rho-\sigma-1}x'$ and $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y'$ with $\sigma > 0$ then $RM(x_p, y_p) = 0$. This is valid even if $\rho = 0$.*



(5.40)

♣ A non-null term would satisfy $r = \rho$ and $s \geq \sigma$, so that the relation $p - r - s - 1 \leq p - \rho - \sigma - 1$ is satisfied. The $\eta_{p-\rho-\sigma-1}$ must therefore be annihilated to produce a non-null term, but nomore face operator is available in the corresponding component. Note $\sigma > 0$ is crucial while on the contrary $\rho = 0$ is possible. ♣

Lemma 60 — *If* $x_p = \eta_{p-1}\eta_{p-2}\cdots\eta_{p-\rho}x'$ *and* $y_p = \eta_{p-\rho-1}\cdots\eta_0 y'$, *then* $RM(x_p \times y_p) = 0$. *This is valid even if* $\rho = 0$ *or* $\sigma = 0$.



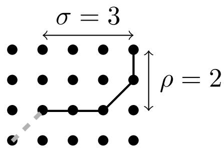
(5.41)

♣ A non-null term would satisfy $r \geq \rho$ and $s \geq \sigma$ but the required inequality $r + s \leq p - 1$ cannot be satisfied. ♣

Corollary 61 — *If* $(x_p \times y_p)$ *is a target or critical cell in the product* $X \times Y$, *then* $RM(x_p \times y_p) = 0$.

♣ Restatement of the last two lemmas. ♣

Lemma 62 — *If* $x_p = \eta_{p-1}\cdots\eta_{p-\rho+1}x'$ *and* $y_p = \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma+1}y'$ *for* $\sigma \geq 2$, *then a non-null term in the* RM*-formula must satisfy* $r = \rho$ *and* $\sigma - 1 \leq s \leq p - \rho - 1$, *or* $r = \rho - 1$ *and* $0 \leq s \leq p - \rho$.



(5.42)

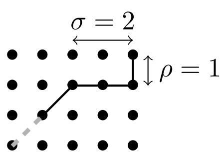
♣ If $r > \rho$, the first degeneracy $\eta_{p-\rho-1}$ of the second factor remains alive and becomes $\eta_{p-\rho-s-1}$; the condition $p - \rho - s - 1 < p - r - s - 1$ is required to avoid a collision, that is, $r < \rho$, contradiction. Now if $r = \rho$ and $s < \sigma - 1$, then a degeneracy remains in the second factor, the first one being again $\eta_{p-\rho-s-1}$, requiring also $r < \rho$ to obtain a non-null term. ♣

5.13.4 Examining a source cell.

The important part of a source cell is the last diagonal, certainly followed *first* by horizontal segments, *then* by vertical segments; but these vertical and/or horizontal parts can be missing. A generic expression for such a source cell is therefore:

$$(\eta_{p-1}\ldots\eta_{p-\rho}x' \times \eta_{p-\rho-1}\cdots\eta_{p-\rho-\sigma}y') \tag{5.43}$$

with the degeneracy operator $\eta_{p-\rho-\sigma-1}$ absent in x' and y', so that the segment of the s-path between times $p - \rho - \sigma - 1$ and $p - \rho - \sigma$ is *diagonal*:



(5.44)

We noticed above the non-totally symmetric role of ρ and σ. This is the reason why two different cases are to be considered in our final expression for the RM-formula when evaluated on a source cell.

Proposition 63 — *Let* $(x_p \times y_p)$ *be a* source *p-simplex of a simplicial product* $X \times Y$*, detailed as above. Then there are two different situations. If* $\sigma = 0$*, then an RM-term is non null only if* $r \geq \rho$*. The RM-formula therefore is then:*

$$RM(x_p \times y_p) = \sum_{\rho \leq r \leq p-1, 0 \leq s \leq p-r-1, (\eta,\eta') \in Sh(s+1,r)} (-1)^{p-r-s} \varepsilon(\eta, \eta') \tag{5.45}$$
$$(\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_p x_p \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

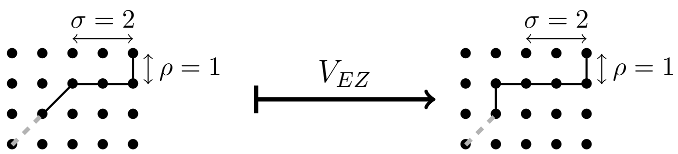
If $\sigma > 0$*, then to obtain a non-null term, the conditions* $r = \rho$ *and* $s \geq \sigma$ *are required The RM-formula therefore is then:*

$$RM(x_p \times y_p) = \sum_{\sigma \leq s \leq p-\rho-1, (\eta,\eta') \in Sh(s+1,\rho)} (-1)^{p-\rho-s} \varepsilon(\eta, \eta') \tag{5.46}$$
$$(\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1}\partial_{p-\rho+1} \cdots \partial_p x_p, \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s} \cdots \partial_{p-\rho-1} y_p)$$

Some of the preserved terms can be also null because we do not have any information for the faces of x_p and y_p, and also because of other collisions as it will be observed later. But at least one term is certainly non-null, we call it the *principal term*, the one corresponding to the parameters $r = \rho$, $s = \sigma$ and (η, η') the trivial shuffle $((0 \cdots \sigma), ((\sigma + 1) \cdots (\rho + \sigma)))$; it is:

$$(-1)^{p-\rho-\sigma}(\eta_p \cdots \eta_{p-\rho+1}\eta_{p-\rho-\sigma-1}x_{p-\rho} \times \eta_{p-\rho} \cdots \eta_{p-\rho-\sigma} y_{p-\sigma}) \tag{5.47}$$

which is nothing but the *target cell* associated to our source cell in our Eilenberg-Zilber vector field V_{EZ}.



(5.48)

In other words, the "first non-null term" of the RM-formula is the corresponding target cell. Someone who knows only the RM-formula can *guess* our Eilenberg-Zilber vector field, in fact unique to be compatible with a homotopy operator, see Section 2.7.2.

5.13.5 *EZ*-Homotopy = *RM*-formula

Theorem 64 — *Let* $X \times Y$ *be the product of two simplicial sets* X *and* Y*. The Eilenberg-Zilber vector field* V_{EZ} *defines a reduction* (f_2, g_2, h_2) *and in particular a homotopy operator* h_2*. Then* $h_2 = RM = h_1$*.*

Cf. Definition 44 for the Eilenberg-Zilber vector field V_{EZ}.

♣ The naturality property obtained in Section 5.12 allows us to consider only $X = Y = \Delta^p$ and to prove the RM-formula for the operator h for the diagonal simplex $(\delta_p \times \delta_p)$. But the necessary recursive process leads to consider more generally any non-degenerate simplex $(x_p \times y_p) \in \Delta^p \times \Delta^p$ where x_p and/or y_p can be degenerate.

If $(x_p \times y_p)$ is a target or critical cell, we say its *level* is 0. If it is a source cell, its *level* is the length of the longest V_{EZ}-path starting from this simplex. The proof is recursive with respect to the level of $(x_p \times y_p)$. Corollary 61 proves the result if the level is null.

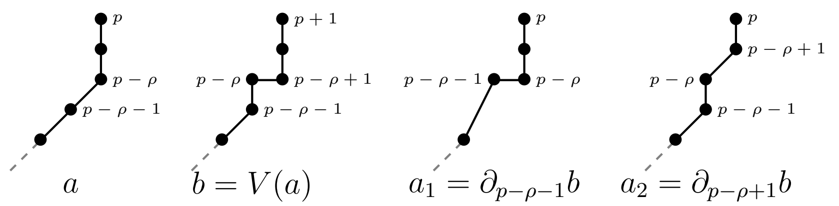
We assume now the result is known for a source cell of level $\ell - 1$ and we assume $(x_p \times y_p)$ is a source cell of level ℓ. The homotopy operator is defined by the recursive formula (5.28): $h(a) = v(a) - h(dv(a) - a)$ if a is a source cell, if v is the function associating to a source cell the corresponding target cell with the right sign. The term $dv(a) - a$ is made of cells of levels $<$ level(a), which justifies the recursive process.

In our Eilenberg-Zilber situation, in fact a maximum of two components of $dv(a) - a$ are source cells, certainly with a smaller level. The game now is the following: we (recursively) assume $h(dv(a) - a)$ can be computed by the RM-formula and we will prove $h(a)$ can be computed by the same formula. Which can seem a priori a little strange, but it will be a consequence of the results detailed in Section 5.13.3.

The case $\sigma = 0$.

A source cell admits a canonical presentation as explained before Proposition 63, producing in particular two important indices ρ and σ. The situation is very different in the cases $\sigma = 0$ or $\sigma \neq 0$. We begin with the first one $\sigma = 0$.

The essential part of the corresponding s-paths are drawn below.



(5.49)

The s-path a represents a simplex denoted also by a finishing by a vertical segment of length ρ, preceded by a diagonal segment between times $p-\rho-1$ and $p-\rho$. Its associated target cell b is obtained by replacing this diagonal segment by a vertical one followed by a horizontal one. All the faces of b are also target cells except the faces of index $p-\rho-1$ and $p-\rho+1$. So that the recursive formula for the homotopy operator is simply in this case: $h(a) = b + h(a_1) + h(a_2)$ if we forget the signs. The values of $h(a_1)$ and $h(a_2)$ are recursively given by the RM-formula: $h(a_1) = RM(a_1)$, $h(a_2) = RM(a_2)$ and we must prove $h(a) = RM(a)$.

We must appropriately express b, a_1, a_2, $h(a_1)$ and $h(a_2)$ and observe which is finally obtained is exactly which is expected for $h(a)$. We collect first the simplices:

$$a = (\eta_{p-1} \cdots \eta_{p-\rho} x' \times y_p) \tag{5.50}$$
$$b = (\eta_p \cdots \eta_{p-\rho+1} \eta_{p-\rho-1} x' \times \eta_{p-\rho} y) \tag{5.51}$$

$$a_1 = (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \partial_{p-\rho-1} y_p) \quad (5.52)$$
$$a_2 = (\eta_{p-1} \cdots \eta_{p-\rho+1} \eta_{p-\rho-1} x' \times y_p) \quad (5.53)$$

The components x' and y_p can contain other degeneracies, but they do not concern the rest of the computation.

We systematically forget the signs, always easily checked correct. The hoped-for formula for $h(a)$ is:

$$h(a) = \sum (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_{p-\rho} x' \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

the parameters satisfying $\rho \leq r \leq p-1$, $0 \leq s \leq p-r-1$, $(\eta, \eta') \in Sh(s+1, r)$.

First the simplex b is the term in this formula corresponding to $r = \rho$, $s = 0$ and $(\eta, \eta') = ((0), (1 \cdots \rho))$.

The simplex a_1 has $\sigma > 0$, which implies the *known* formula for $h(a_1)$ is:

$$h(a_1) = \sum (\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s} \cdots \partial_{p-\rho-1} y_p)$$

because of a happy relation $\partial_{p-\rho-1}\eta_{p-\rho-1} = \mathrm{id}$ in the second factor. The valid parameters are $1 \leq s \leq p-\rho-1$ and $(\eta, \eta') \in Sh(s+1, \rho)$. This implies $h(a_1)$ produces all the desired terms for the $h(a)$-formula satisfying $r = \rho$ and $s \geq 1$.

Processing $h(a_2)$ is more complex. The initial formula for $h(a_2)$ is:

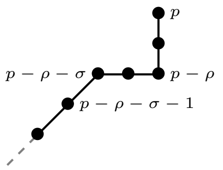
$$h(a_2) = \sum (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_{p-\rho+1}\eta_{p-\rho-1} x' \times$$
$$\times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1} y_p)$$

for $\rho - 1 \leq r \leq p-1$, $0 \leq s \leq p-r-1$ and $(\eta, \eta') \in Sh(s+1, r)$. For $r = \rho - 1$, the degeneracy $\eta_{p-\rho-1}$ survives in the first factor, so that the condition $p - \rho - 1 < p - r - s - 1$ is required to avoid a collision, that is, $s = 0$. This produces all the terms of the desired formula for $h(a)$ satisfying $\rho = r$, $s = 0$, except the term b. Observe in particular how the shuffles in $Sh(1, \rho - 1)$ in $h(a_2)$ produce the shuffles in $Sh(1, \rho)$ in $h(a)$.

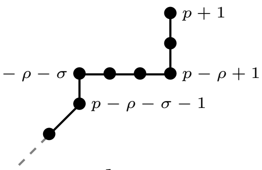
If $r = \rho$, the degeneracy $\eta_{p-\rho-1}$ in the first factor again remains alive, but this time, the collision cannot be avoided and all the corresponding terms in fact are null. Finally, if $r > \rho$, the degeneracy $\eta_{p-\rho+1}$ is annihilated by a face operator and an elementary computation shows all the obtained terms of this sort are exactly the same as those with the same indices in $h(a)$.

The case $\sigma > 0$.

Consider the figures below to understand the involved degeneracies. Again the simplex a produces a target cell b, and only two faces of b are source cells, so that we again have to deduce $h(a) = RM(a)$ from $h(a_1) = RM(a_1)$ and $h(a_2) = RM(a_2)$ and from the recursive formula $h(a) = b + h(a_1) + h(a_2)$, signs omitted.

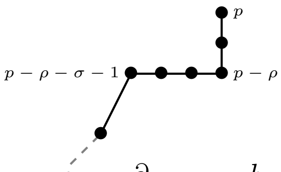


a

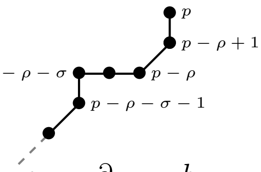


b

(5.54)



$a_1 = \partial_{p-\rho-\sigma-1} b$



$a_2 = \partial_{p-\rho+1} b$

(5.55)

The simplices are:

$$a = (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y') \tag{5.56}$$
$$b = (\eta_p \cdots \eta_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' \times \eta_{p-\rho} \cdots \eta_{p-\rho-\sigma} y') \tag{5.57}$$
$$a_1 = (\eta_{p-1} \cdots \eta_{p-\rho} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma-1} \partial_{p-\rho-\sigma-1} y') \tag{5.58}$$
$$a_2 = (\eta_{p-1} \cdots \eta_{p-\rho+1} \eta_{p-\rho-\sigma-1} x' \times \eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y') \tag{5.59}$$

Again the subcomponents x' and y' can have other degeneracies. The simplex a has $\sigma > 0$, so that the formula to be proved is:

$$h(a) = \sum (\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s} \cdots \partial_{p-\rho-\sigma-1} y')$$

for $\sigma \leq s \leq p-\rho-1$ and $(\eta, \eta') \in Sh(s+1, \rho)$. The corresponding target cell b is the term of this sum with $s = \sigma$ and $(\eta, \eta') = ((0 \cdots \sigma), ((\sigma+1) \cdots (\rho+\sigma))$.

The simplex a_1 has also $\sigma > 0$, which gives us the expression:

$$h(a_1) = \sum (\uparrow^{p-\rho-s}(\eta')\eta_{p-\rho-s-1} x' \times \uparrow^{p-\rho-s}(\eta)\partial_{p-\rho-s} \cdots \partial_{p-\rho-\sigma-1} y')$$

for $\sigma+1 \leq s \leq p-\rho-1$ and $(\eta, \eta') \in Sh(s+1, \rho)$; always at least one face operator $\partial_{p-\rho-\sigma-1}$ in the second factor. We see this $h(a_1)$ gives all the terms of the desired expression for $h(a)$ satisfying $s \geq \sigma + 1$.

Again processing $h(a_2)$ is more complex, for a_2 has $\sigma = 0$. The initial expression for $h(a_2)$ is:

$$h(a_2) = \sum (\uparrow^{p-r-s}(\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_{p-\rho+1}\eta_{p-\rho-\sigma-1} x' \times \uparrow^{p-r-s}(\eta)\partial_{p-r-s} \cdots \partial_{p-r-1}\eta_{p-\rho-1} \cdots \eta_{p-\rho-\sigma} y')$$

for $r = \rho - 1$ and $0 \leq s \leq p - \rho$, or $r = \rho$ and $\sigma \leq s \leq p - \rho - 1$, see Lemma 62.

For $r = \rho$, the tail of the first factor is $\partial_{p-\rho+1}\eta_{p-\rho-\sigma-1} x' = \eta_{p-\rho-\sigma-1}\partial_{p-\rho} x'$, and to avoid a collision, the relation $p-\rho-\sigma-1 < p-\rho-s-1$ is necessary, that is, $s < \sigma$, contradiction.

For $r = \rho - 1$, all the face operators of the first factor are absent and there remains for the first component: $\uparrow^{p-\rho-s+1}(\eta')\eta_{p-\rho-s}\eta_{p-\rho-\sigma-1} x'$. Again, to avoid a collision, the relation $p - \rho - \sigma - 1 < p - \rho - s$ is necessary, that is, $s \leq \sigma$. So that all the face operators are also absent of the second factor which becomes: $\uparrow^{p-\rho-s+1}(\eta)\eta_{p-\rho-s-1} \cdots \eta_{p-\rho-\sigma} y'$. A careful examination then shows we have so found all the remaining terms of the desired formula for $h(a)$ satisfying $s = \sigma$, except the term corresponding to b. In particular a term $s = \sigma$, $(\eta, \eta') \in Sh(\sigma+1, \rho)$ of $h(a)$ corresponds to a term of $h(a_2)$ with $s = \sigma - \ell$ for ℓ the length of the longest "trivial" initial sequence $(0 \cdots (\ell - 1)) \subset \eta$, the maximal length $\sigma + 1$ being excluded, coming on the contrary of the target cell b. Also the position of this removed trivial initial sequence does not modify the signature of the underlying permutation. ♣

5.13.6 Eilenberg-MacLane formula.

Theorem 65 — *Let (f_2, g_2, h_2) be the reduction associated to the Eilenberg-Zilber vector field on the product $X \times Y$ of two simplicial sets. Then the chain complex morphism g_2 is given by the Eilenberg-MacLane formula:*

$$EML(x_\sigma \otimes y_\rho) = \sum_{(\eta,\eta') \in Sh(\sigma,\rho)} \varepsilon(\eta,\eta')\,(\eta' x_\sigma \times \eta y_\rho) \qquad (5.60)$$

We prefer our favorite indices σ and ρ instead of the traditional p and q: they had essentially the same interpretations as in the previous sections.

Taking account of [15], Theorem 65 was already proved thirty years ago. But it is good training for further tasks to obtain the result directly using this game of vector fields and s-paths.

♣ The g-component of our vector field reduction (f, g, h) is a chain complex morphism $g : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$. More precisely, a generator $x_\sigma \otimes y_\rho \in C_\sigma(X) \otimes C_\rho(Y)$ of the source chain complex must first be translated into the so-called "last simplex" $\lambda(x_\sigma, y_\rho) = (\eta_{p-1} \cdots \eta_\sigma x_\sigma \times \eta_{\sigma-1} \cdots \eta_0 y_\rho) \in C_*(X \times Y)$; see the explanations after Theorem 47.

(5.61)

The formula (2.18, g_p), taking account of the other formula (2.18, h_{p-1}), can be read $g = \mathrm{id} - hd$. In particular $\mathrm{id}(\lambda(x_\sigma, y_\rho))$ is the term with $(\eta, \eta') = ((0 \cdots (\sigma - 1)), (\sigma \cdots (\rho + \sigma - 1)))$ of the EML-formula.

The $hd(\lambda(x_\sigma, y_\rho))$ must produce the other terms with the right signs. The homotopy operator is null except for the source cells, and it happens we have only one source cell in $d(\lambda(x_\sigma, y_\rho))$, drawn above, with the expression:

$$d(\lambda(x_\sigma, y_\rho)) = (\eta_{\rho+\sigma-2} \cdots \eta_\sigma x_\sigma, \eta_{\sigma-2} \cdots \eta_0 y_\rho) \qquad (5.62)$$

We apply the RM-formula to this term to obtain:

$$\sum(\uparrow^{\rho+\sigma-r-s-1}(\eta')\eta_{\rho+\sigma-r-s-2}\partial_{\rho+\sigma-r}\cdots\partial_{\rho+\sigma-1}\eta_{\rho+\sigma-2}\cdots\eta_\sigma x_\sigma\times \uparrow^{\rho+\sigma-r-s-1}(\eta)\partial_{\rho+\sigma-r-s-1}\cdots\partial_{\rho+\sigma-r-2}\eta_{\sigma-2}\cdots\eta_0 y_\rho) \tag{5.63}$$

for $r=\rho$ and $\sigma-1\leq s\leq \rho+\sigma-r-2=\sigma-2$, impossible, and there remain only the parameters $r=\rho-1$, $0\leq s\leq\sigma-1$. The face operators of the first factor are exactly cancelled by the following degeneracies, and the face operators of the second factor are totally cancelled by the following degeneracies, but some of these degeneracies in general remain alive. We obtain finally:

$$\sum(\uparrow^{\sigma-s}(\eta')\eta_{\sigma-s-1}x_\sigma\times\uparrow^{\sigma-s}(\eta)\eta_{\sigma-s-2}\cdots\eta_0 y_\rho) \tag{5.64}$$

for $0\leq s\leq\sigma-1$ and $(\eta,\eta')\in Sh(s+1,\rho-1)$. The generic term of this sum corresponds to the term $(\overline{\eta}'x_\sigma\times\overline{\eta}y_\rho)$ of the standard Eilenberg-MacLane formula for $\sigma-s-1$ the length of the maximal "trivial" initial segment $(0\cdots(\sigma-s-2))$ in $\overline{\eta}$. The last simplex $(\eta_{\rho+\sigma-1}\cdots\eta_\sigma x_\sigma\times\eta_{\sigma-1}\cdots\eta_0 y_\rho)$ would correspond to $s=-1$, not possible, but this last simplex had been previously produced by the id term of the formula $g=\mathrm{id}-hd$. For further reference, we give the formula obtained:

$$hd(\lambda(x_\sigma,y_\rho)=\sum(\eta' x_\sigma\times\eta x_\rho) \tag{5.65}$$

where $(\eta,\eta')\in Sh(\rho,\sigma)-\{\mathrm{id}\}$, that is, all the shuffles are to be used except the trivial one.

We let the reader check the signs are also correct. ♣

5.13.7 Alexander-Whitney formula.

Theorem 66 — *Let (f_2,g_2,h_2) be the reduction associated to the Eilenberg-Zilber vector field on the product $X\times Y$ of two simplicial sets. Then the chain complex morphism f_2 is given by the Alexander-Whitney formula:*

$$AW(x_p\times y_p)=\sum_{i=0}^{p}\partial_{p-i+1}\cdots\partial_p x_p\otimes\partial_0\cdots\partial_{p-i-1}y_p \tag{5.66}$$

Same remark with respect to [16] as with respect to [15] in the previous section for the EML-formula.

♣ This morphism f_2 is *natural* and it is sufficient to consider the particular case $X=Y=\Delta^p$ and $(x_p\times y_p)=(\delta_p\times\delta_p)$ if δ_p is the maximal simplex of Δ_p and $(\delta_p\times\delta_p)$ the diagonal p-simplex of $\Delta_p\times\Delta_p$.

The formula (2.18, f_p), taking account of (2.18, h_{p-1}), can be read:

$$f=\mathrm{pr}_3(\mathrm{id}-dh)$$

with pr_3 being the canonical projection on the critical subcomplex. For example $\mathrm{pr}_3\mathrm{id}(\delta_p\times\delta_p)=0$, for this diagonal simplex is a source cell, giving a null projection on the critical complex. Except if $p = 0$ where the AW-formula is then obvious, for $h(\delta_0 \times \delta_0) = 0$.

Now we have to compute $dh(\delta_p \times \delta_p)$ and to extract from a lot of terms those that are critical. Figure 5.1 should help to understand a generic term of $h(\delta_p \times \delta_p)$, that is:

$$(\uparrow^{p-r-s} (\eta')\eta_{p-r-s-1}\partial_{p-r+1} \cdots \partial_p \delta_p \times \uparrow^{p-r-s} (\eta)\partial_{p-r-s} \cdots \partial_{p-r-1}\delta_p)$$

The figure represents the particular case $p = 7$, $r = 2$ and $s = 2$. In general the face operators of this generic term cancel the vertices $(p - r + 1) \cdots (p)$ of the first factor and $(p - r - s) \cdots (p - r - 1)$ for the second factor, so that the s-path is allowed to run among a subset of $(0 \cdots p) \times (0 \cdots p)$ represented by •'s in the figure, whereas the forbidden points are represented by ×'s.

The "generic" simplex is represented by an s-path with essentially three parts:

1. A diagonal part starting at (0×0) up to the point $(p-r-s-1 \times p-r-s-1)$; this diagonal part can be a point, if $r + s = p - 1$.
2. A *vertical* segment, always present, devoted to this very particular degeneracy of the first factor $\eta_{p-r-s-1}$, starting at $(p - r - s - 1 \times p - r - s - 1)$ up to $(p - r - s - 1 \times p - r)$ in a *unique step*, for the expected intermediary vertices are in fact cancelled by the face operators of the second factor. We call this segment the *pole* of the s-path, think of the pole of a flag, a flag to be described soon.
3. Finally an "Eilenberg-MacLane" step starting from $(p - r - s - 1 \times p - r)$ up to $(p - r \times p)$ by an arbitrary combination of vertical (toward north)) and horizontal segments (toward east), combination depending only on the shuffle (η, η'). For example, in our figure, it is the shuffle $((1 \cdot 2 \cdot 4)(0 \cdot 3))$. We call "$EML$-flag" the rectangle between $(p-r-s-1 \times p-r)$ and $(p-r \times p)$, for this part of the path, parametrized by a shuffle, mimics an s-path produced by the EML-formula. And this flag is "carried" by the "pole" $\eta_{p-r-s-1}$.

Remember we have to select all the critical components of the expression $dh(\delta_p \times \delta_p)$. The figure represents one term of $h(\delta_p \times \delta_p)$. A component of $dh(\delta_p \times \delta_p)$ is obtained by cancelling one of the vertices of the s-path, and which remains must be a *critical* cell, that is, an s-path made first only of horizontal segments and then only of vertical segments.

For example, for the simplex represented by Figure 5.1, no face of this simplex is critical, for a simple reason: there certaily will remain at least one diagonal component if we apply a face operator. In other words we just have to consider the cases $p - r - s - 1 = 0$ or 1.

Another obstacle for a face to be critical is the vertical segment $\eta_{p-r-s-1}$ followed by an Eilenberg-MacLane path made of horizontals and verticals joining the extreme points of the rectangle $(((p - r - s - 1) \cdots (p - r)) \times ((p - r) \cdots p))$: it is difficult for such a path not to have a bend qualifying this path as a target cell, and the same for its faces.

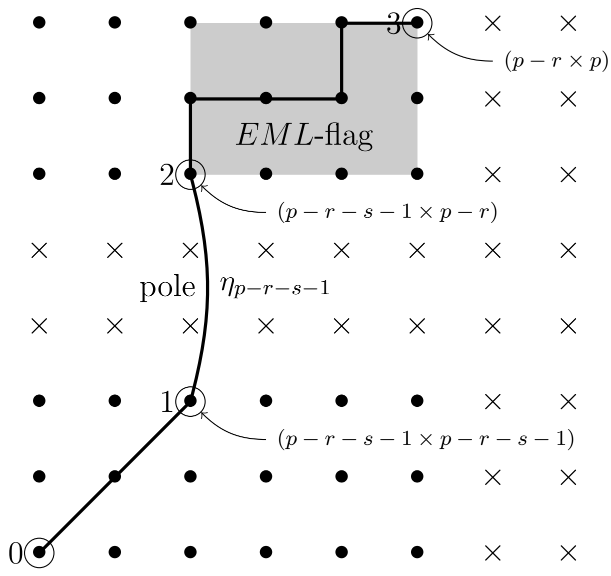
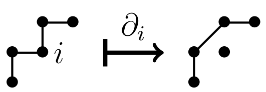


Figure 5.1: Understanding $RM(\delta_p \times \delta_p)$

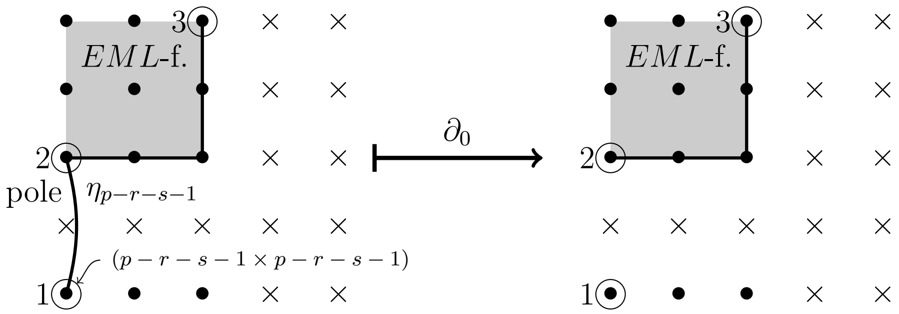
The width of the EML-area is $(p-r)-(p-r-s-1) = s+1 \geq 1$, which implies the certain presence of at least one horizontal segment for our path in this area. More precisely, after the pole, there certainly will be a bend $\ulcorner$. This bend, or another possible one, later in the flag, qualifies our s-path as a target cell, which is not amazing: a value of the RM-formula is a combination of target cells. This already implies the terms for which $p-r-s-1=1$ can be given up, because for these terms we will have to destroy by *one* face operator a diagonal *and* a bend $\ulcorner$, impossible. For $p-r-s-1=0$, that is, for $r+s=p-1$, no initial diagonal part for our s-path.

But we have to apply now to this *target* cell a face operator to get a *critical* cell, not easy! First we must have in our target cell a *unique* bend $\ulcorner$, for it is impossible to destroy *two* such bends with *one* face operator without creating a diagonal:



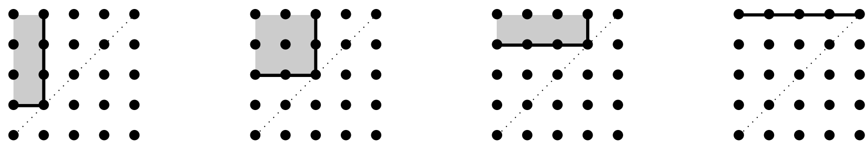
(5.67)

Remember also a critical cell has a unique bend $\lrcorner$ or no bend at all. If you remove the precise point of the unique bend $\ulcorner$ of our target cell, you create a diagonal and the result is a source cell, forbidden. If you remove another point, then you first think this cannot suppress the bend $\ulcorner$? Correct, except a special case, when the bend has before (resp. after) the bend the *initial* vertical (resp. *final* horizontal) segment of the path. This is the key point of our subject which, once understood, immediately products the Alexander-Whitney formula. An example of this sort is below $(p = 4, r = 2, s = 1, (\eta, \eta') = (0 \cdot 1, 2 \cdot 3))$:



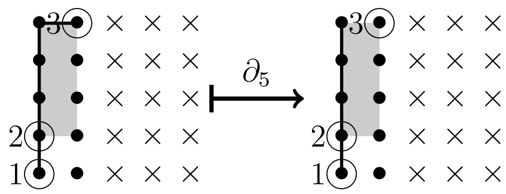
(5.68)

For an arbitrary $p \geq 1$, we so obtain p terms of the Alexander-Whitney formula, those corresponding in the RM-formula to the parameters $0 \leq r \leq p-1$, $s = p-r-1$, $(\eta, \eta') = ((0 \cdots s), ((s+1) \cdots (p-1)))$, to which we apply the ∂_0-operator. For example we obtain for $p = 4$:



(5.69)

One term of the Alexander-Whitney formula is still missing. It is obtained by another *unique* particular case, when $r = p-1$, $s = 0$, $(\eta, \eta') = ((p-1), (0 \cdots (p-2)))$. The figure when $p = 4$:



(5.70)

There remains to apply the canonical correspondance between critical cells of the product $\Delta^p \times \Delta^p$ and the generators of $C_*\Delta^p \otimes C_*\Delta^p$. That is:

$$(\eta_{p-1} \cdots \eta_{p-r}\partial_{p-r+1} \cdots \partial_p\delta_p \times \eta_{p-r-1} \cdots \eta_0\partial_0 \cdots \partial_{p-r-1}\delta_p) \longmapsto \partial_{p-r+1} \cdots \partial_p\delta_p \otimes \partial_0 \cdots \partial_p\delta_p.$$

and also to check the final sign is always positive... ♣

5.13.8 The critical differential.

Several times in the previous sections we used the expression "critical subcomplex" to designate the graded *submodule* generated by the critical cells. Which submodule in general *is not* a subcomplex. Consider again the formulas 2.18, in particular the formula for d'_p. With the notations of Theorem 15, if $d_{p,1,3}$ or $d_{p,2,3}$ is non-null, then our submodule *is not* a subcomplex.

The situation is a little different. We have to install on F_* a *specific* differential, defined as d', and the chain complex (F_*, d') is isomorphic to a *subcomplex* of C_*, namely $g(F_*, d')$; this time the word subcomplex is properly used.

In the Eilenberg-Zilber particular case, we must now study the differential to be installed on $C^c_*(X \times Y)$ and check, taking account of the isomorphism $C^c(X \times Y) \cong C_*(X) \otimes C_*(Y)$, that we find the usual differential of the tensor product. It is as funny as in the previous sections. Because of the naturality of our process, it is sufficient to consider the case of $C_*(\Delta^p \times \Delta^p)$.

A generic critical cell to be considered is:

$$s^c = (\eta_{p-1} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0 \delta_\rho) \cong \delta_\sigma \otimes \delta_\rho \tag{5.71}$$

where as usual δ_σ and δ_ρ are simplices of respective dimensions σ and ρ with $\sigma + \rho = p$.

This time, Theorem 15 provides the formula $d' = \mathrm{pr}_3(d - dhd)$to be applied to our critical generic simplex s^c. The game is similar to the one of the previous section for Alexander-Whitney. Anyway we must compute the initial differential ds^c (signs omitted):

$$ds^c = \sum_{i=0}^{p} \partial_i(\eta_{p-1} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0 \delta_\rho) \tag{5.72}$$

and study which happens according to the situation of i with respect to σ. If $i < \sigma$, the face operator on the one hand is to be applied to the δ_σ of the first factor, and on the other hand annihilates a degeneracy of the second factor, to produce:

$$\partial_i s^c = (\eta_{p-2} \cdots \eta_{\sigma-1} \partial_i \delta_\sigma \times \eta_{\sigma-2} \cdots \eta_0 \delta_\rho) \cong \partial_i \delta_\sigma \otimes \delta_\rho \tag{5.73}$$

where we recognize the expected $\partial_i \delta_\sigma \otimes \delta_\rho$; this is valid for $i < \sigma$, just one face of this sort $\partial_\sigma \delta_\sigma \otimes \delta_\rho$ is missing.

Symmetrically, if $i > \sigma$, in the same way, we obtain:

$$\partial_i s^c = (\eta_{p-2} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-1} \cdots \eta_0 \partial_{i-\sigma} \delta_\rho) \cong \delta_\sigma \otimes \partial_{i-\sigma} \delta_\rho \tag{5.74}$$

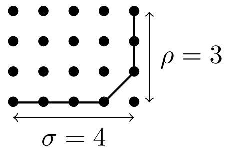
We recognize the expected $\delta_\sigma \otimes \partial_j \delta_\rho$ for $j = i - \sigma > 0$; one face of this sort is missing: $\delta_\sigma \otimes \partial_0 \delta_\rho$.

All these simplices are critical cells and they contribute to the studied differential d' only via the component $\mathrm{pr}_3 d$ in the expression $d' = \mathrm{pr}_3(d - dhd)$, for the homotopy operator h is null on the critical cells.

Processing the face of index σ is very particular. First the face operator cancels the last degeneracy of the first factor and the first one of the second factor, without forgetting how the degeneracies of the first factor are renumbered:

$$\partial_\sigma s^c =: s' := (\eta_{p-2} \cdots \eta_\sigma \delta_\sigma \times \eta_{\sigma-2} \cdots \eta_0 \delta_\rho) \tag{5.75}$$

We are again in front of this "almost critical" simplex which generates the EML-formula, in fact a source simplex. Example for $\sigma = 4$ and $\rho = 3$:

$s' =$ 

(5.76)

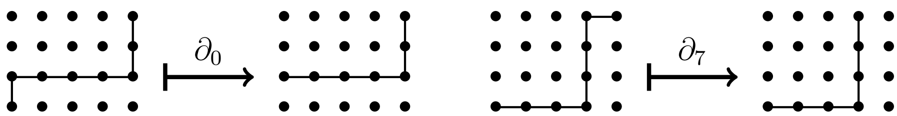
It is a source cell, so that the contribution of this face in $d's^c = \mathrm{pr}_3(d - dhd)(s^c)$ is only $-\mathrm{pr}_3 dh\partial_\sigma s^c = dhs'$, sign omitted. We rewrite as follows the formula (5.65):

$$hs' = \sum (\eta' x_\sigma \times \eta x_\rho) \tag{5.77}$$

for$(\eta, \eta') \in Sh(\rho, \sigma) - \{\mathrm{id}\}$. The final step consists in computing the differential of this sum and keeping in the obtained terms the critical cells.

Every term of this sum is a target cell; again it is a little funny, for in the Eilenberg-MacLane formula, all the terms are target cells except the trivial one which is critical. In the case 4×3 drawn above at (5.76), an Eilenberg-MacLane term must go from (0×0) to (4×3) following only horizontals toward east and verticals toward north. A unique path of this sort is without a bend, the critical cell $(0 \times 0) \to (4 \times 0) \to (4 \times 3)$. All the other paths necessarily have somewhere a bend and this is why they are target cells.

Now we have to differentiate such a cell and keep the critical cells. Remember the discussion page 71 for the Alexander-Whitney formula where we already meet such a situation. A target cell containing a critical cell in his differential must have a unique bend and this bend must be either preceded by the initial vertical segment of the path or followed by its final horizontal segment. Only two possibilities drawn below in the case 4×3:



(5.78)

which fortunately exactly produces the two terms missing in the differential of our critical complex. ♣

5.13.9 Eilenberg-MacLane's recursive formula.

The recursive formula [6, (2.13)]:

$$\Phi c_q = -\Phi' c_q + h' D_0 c_q \tag{5.79}$$

is essential when Eilenberg and MacLane process the Eilenberg-Zilber equivalence. The pseudo-derivative operator to be applied to Φ and h in the second member, these operators Φ and h being mainly made of face and degeneracy operators, consists in fact in taking the same operator for one dimension less and to shift all the face and degeneracy indices by +1. For example, if $k(s) := \eta_2\partial_3 s + \eta_1\partial_4 s$ for a 5-simplex s, then $k'(s) = \eta_3\partial_4 s + \eta_2\partial_5 s$ for a 6-simplex s. The Φ of Eilenberg-MacLane is our homotopy operator $h = RM$ and *their* h is our composition $gf = EML \circ AW$. Finally D_0 is our degeneracy η_0.

So that the translation of Eilenberg-MacLane's recursive formula into our notations is:

$$RM(x_p \times y_p) = -RM'(x_p \times y_p) + EML'AW'\eta_0(x_p \times y_p) \tag{5.80}$$

Also, possibly degenerate terms in the final result are to be cancelled.

We deduce from the AW formula (5.31) and the EML-formula (5.32) the following expression for the composition $EML \circ AW$ (signs omitted):

$$\sum(\eta'\partial_{p-r+1}\cdots\partial_p x_p \times \eta\partial_0\cdots\partial_{p-r-1}y_p) \tag{5.81}$$

for $0 \le r \le p$ and $(\eta, \eta') \in Sh(p-r, r)$. Replacing x_p and y_p by $\eta_0 x_p$ and $\eta_0 y_p$, and shifting the indices to take account of the pseudo-derivations gives:

$$\sum(\uparrow(\eta')\partial_{p-r+2}\cdots\partial_{p+1}\eta_0 x_p \times \uparrow(\eta)\partial_1\cdots\partial_{p-r}\eta_0 y_p) \tag{5.82}$$

We must now install the η_0's at the right place. The right one is always killed by a face operator, except for $r = p$; the left one always remains alive, so that the unique term corresponding to $r = p$ disappears, it is 0-degenerate, and there remains:

$$\sum(\uparrow(\eta')\eta_0\partial_{p-r+1}\cdots\partial_p x_p \times \uparrow(\eta)\partial_1\cdots\partial_{p-r-1}y_p) \tag{5.83}$$

for $0 \le r \le p-1$ and $(\eta, \eta') \in Sh(p-r, r)$. We recognize here all the terms of the RM-formula (5.33) for which $s = p-r-1$ or, simpler, all the terms for which $p-r-s-1 = 0$, that is, exactly these terms where this very specific degeneracy operator $\eta_{p-r-s-1}$ is in fact η_0.

Which orients the study of the Eilenberg-MacLane recursive formula to Figure 5.1. All the terms of the RM-formula can be divided in two simple classes. The first one contains all the terms starting by a vertical, that is, those terms where $p-r-s-1 = 0$. An example of this sort is the lefthand figure of (5.68). These terms are produced by $EML'AW'\eta_0(x_p \times y_p)$ in (5.80). The second class of terms are those satisfying $p-r-s-1 > 0$, that is, those starting by a diagonal segment in Figure 5.1; they are produced by $-RM'(x_p \times y_p)$ in (5.80), for a possible pattern for the s-path after the first diagonal segment is a RM-pattern for one dimention less. ♣

Chapter 6

The twisted Eilenberg-Zilber W-reduction.

6.1 Twisted products.

If X and Y are two simplicial sets, the usual definition of the product $X \times Y$ was recalled in Section 5.11.2. A *fibration* is a sort of "twisted" product, a notion having a major role in algebraic topology. We follow here the terminology and the notations of [14, §18].

Definition 67 — A *simplicial morphism* $f : X \to Y$ between two simplicial sets X and Y is a collection $(f_p : X_p \to Y_p)_{p \in \mathbb{N}}$ of maps between simplex sets, compatible with the $\underline{\Delta}$-operators: for every $\underline{\Delta}$-morphism $\alpha : \underline{p} \to \underline{q}$, the relation $\alpha^* f_q = f_p \alpha^*$ is satisfied. ♣

Think this diagram must be commutative:

$$\begin{array}{ccc} X_p & \xrightarrow{f_p} & Y_p \\ \alpha^* \uparrow & & \uparrow \alpha^* \\ X_q & \xrightarrow{f_q} & Y_q \end{array} \tag{6.1}$$

See Section 3.1. The face and degeneracy operators generate all the $\underline{\Delta}$-morphisms, so that it is enough this compatibility condition is satisfied for face and degeneracy operators.

Definition 68 — A *simplicial group* G is a simplicial set provided with two simplicial morphisms, a group law $\mu_G : G \times G \to G$ and an inversion map $\iota_G : G \to G$; every homogeneous p-dimensional pair $(\mu_{G,p}, \iota_{G,p})$ must satisfy the usual group axioms. ♣

Every homogeneous simplex set G_p is endowed with a group structure, and the collection of groups $(G_p)_p$ is compatible with face and degeneracy operators. We most often simply write in a multiplicative way $\mu_G(g, g') = g.g'$ or gg' and $\iota_G(g) = g^{-1}$, or sometimes in an additive form $\mu_G(g, g') = g + g'$ and $\iota_G(g) = -g$ if the group law is abelian.

Definition 69 — A simplicial action γ of the simplicial group G on a simplicial set X is a simplicial morphism $\gamma : G \times X \to X$ satisfying the usual axioms of a group action. ♣

For every $p \in \mathbb{N}$, a group action $\gamma_p : G_p \times X_p \to X_p$ is defined and all these actions are compatible with face and degeneracy operators. Most often we will not denote the action γ by a letter, using simply the product notation: $\gamma(g, x)$ will be simply denoted by $g.x$ or gx.

Definition 70 — Let F and B be two simplicial sets, the *fiber space* and the *base space* of the fibration to be defined. Let G be a simplicial group, the *structural group* and let $G \times F \to F$ be some action of the structural group on the fiber space. A *twisting function* is a collection of maps $(\tau_p : B_p \to G_{p-1})_{p>0}$ satisfying the conditions:

$$\begin{array}{rcll} \partial_0(\tau b) & = & \tau(\partial_0 b)^{-1}.\tau(\partial_1 b) & \\ \partial_i \tau(b) & = & \tau(\partial_{i+1} b) & i > 0 \\ \eta_i \tau(b) & = & \tau(\eta_{i+1} b) & i \geq 0 \\ e_p & = & \tau(\eta_0 b) & b \in B_p \end{array} \tag{6.2}$$

if e_p is the neutral element of the group G_p. ♣

Definition 71 — If G, F, B and τ are as in the previous definition, the *twisted product* $F \times_\tau B$ is the simplicial set defined as follows. Every homogeneous simplex set $(F \times_\tau B)_p$ is *the same as* for the non-twisted product: $(F \times_\tau B)_p = (F \times_\tau B)_p = F_p \times B_p$. Only the 0-face operator is different:

$$\partial_0(f, b) = (\tau(b).\partial_0 f, \partial_0 b). \tag{6.3}$$

The other face and degeneracy operators therefore are simply $\partial_i(f, b) = (\partial_i f, \partial_i b)$ for $i > 0$ and $\eta_i(f, b) = (\eta_i f, \eta_i b)$ for arbitrary relevant i. ♣

In other words, the twisting function τ is used only to *perturb* the 0-face operator in the *vertical* direction if, as usual, we think of the base space as the horizontal component of the (twisted) product and the fiber space as the vertical component.

Unfortunately, our reference book [14] does not give any easily understandable example. The reader could profitably examine the notes [25], in particular Sections 7 and 12.

6.2 The twisted Eilenberg-Zilber vector field.

The notion of twisted product, due to Daniel Kan [11, Section 6], recalled in the previous section, is really a *miracle*. The point is that only the 0-face operator is modified by the twisting process, while the Eilenberg-Zilber vector field described in Section 5.11.3 invokes "vectors" (σ, σ') where the face-index i in the regular face relation $\partial_i \sigma' = \sigma$ always satisfies $i > 0$: the 0-face is never concerned in this vector field.

Theorem 72 — *Let $F \times_\tau B$ be a twisted product as defined in the previous section. Then the Eilenberg-Zilber vector field defined in Theorem 47 for the non-twisted product $F \times B$ can be used as well for the twisted product $F \times_\tau B$. It is admissible and therefore defines a homological reduction:*

$$TEZ : C_*(F \times_\tau B) \Rrightarrow C_*(F) \otimes_t C_*(B). \tag{6.4}$$

The small chain complex of this reduction has the same underlying graded module as $C_*(F) \otimes C_*(B)$; only the differential is modified, which is indicated by the index t of the twisted product symbol '$\otimes$': it is a twisted *tensor product.*

♣ We reuse the terminology and the notations of Chapter 5. Every r-simplex $\rho = (\varphi, \beta) = (\eta_{i_{s-1}} \cdots \eta_{i_0} \varphi', \eta_{j_{t-1}} \cdots \eta_{j_0} \beta')$ of the non-twisted cartesian product $F \times B$ is as well an r-simplex of the twisted product $F \times_\tau B$. We decide the degeneracy configuration $((i_{s-1}, \ldots, i_0), (j_{t-1}, \ldots, j_0))$ determines the nature of the simplex ρ in the vector field $V_{F \times_\tau B}$ to be defined on $F \times_\tau B$, source, target or critical, exactly like in Definition 46: the respective status of ρ with respect to the vector fields $V_{F \times B}$ (resp. $V_{F \times_\tau B}$) of $F \times B$ (resp. $F \times_\tau B$) are *the same*.

In particular, if a pair $(\rho_1, \rho_2) = ((\varphi_1, \beta_1), (\varphi_2, \beta_2))$ is a *vector* of the Eilenberg-Zilber vector field $V_{F \times B}$ of $F \times B$, we decide it is as well a vector ot the twisted Eilenberg-Zilber vector field $V_{F \times_\tau B}$ we are defining. The incidence relation $\rho_1 = \partial_i \rho_2$ is satisfied in $F \times B$ for a unique face index i satisfying $0 < i < r$ if r is the dimension of ρ_2; so that the *same* incidence relation is also satisfied in the twisted cartesian product $F \times_\tau B$, for only the 0-face is modified in the twisted product.

However the 0-face operator plays a role when studying whether a vector field is *admissible*. The admissibility proof of the vector field $V_{X \times Y}$ is based on the Lyapunov function L defined and used in Proposition 42 and on the partition of all the simplices of $X \times Y$ in layers indexed by the bidimension (p, q), see Definition 42. This proof amounts to defining a new Lyapunov function $L(\rho_1) = (p, q, L(\chi))$ if the bidimension of the simplex ρ_1 is (p, q), certainly the same as the bidimension of ρ_2, and if χ is the degeneracy configuration of ρ_1; we use the lexicographic order to compare the images of this function.

We claim the same Lyapunov function can be used for the twisted product $F \times_\tau B$. Let us assume $(\rho_1, \rho_2) \in V_{F \times_\tau B}$ with $\rho_1 = \partial_i \rho_2$ and $\rho_3 = \partial_j \tau$ with $j \neq i$. If $j > 0$, then the j-face $\partial_j \tau$ is the same in both products, twisted or non-twisted, so that $L'(\rho_3)$ has the same value whatever the product you consider, twisted or non-twisted.

Let $\rho_2 = (\varphi_2, \beta_2) = (\eta_{i_{s-1}} \cdots \eta_{i_0}\varphi'_2, \eta_{j_{t-1}} \cdots \eta_{j_0}\beta'_2)$ be the canonical expression of $\rho_2 = (\varphi_2, \beta_2)$ using the non-degenerate simplices $\varphi'_2 \in F_p$ and $\beta'_2 \in B_q$ if the bidimension of ρ_2 is (p, q). The first formula below gives the 0-face in $F \times B$ while the second one gives the 0-face in $F \times_\tau B$:

$$\begin{aligned} \partial_0 \rho_2 &= (\partial_0 \eta_{i_{s-1}} \cdots \eta_{i_0}\varphi'_2, \partial_0 \eta_{j_{t-1}} \cdots \eta_{j_0}\beta'_2) \\ \partial_{0,\tau} \rho_2 &= (\tau(\beta_2)\partial_0 \eta_{i_{s-1}} \cdots \eta_{i_0}\varphi'_2, \partial_0 \eta_{j_{t-1}} \cdots \eta_{j_0}\beta'_2) \end{aligned} \tag{6.5}$$

It has been observed in Proposition 36 the 0-face of an *interior* simplex of $\Delta^{p,q}$ is always *exterior*. In other words, the 0-face of a non-degenerate simplex always has a stictly smaller bidimension. The group operator $\tau(\beta_2)$ is a simplicial *isomorphism*, so that the *geometrical* dimensions of the *first* components of $\partial_0\rho_2$ and $\partial_{0,\tau}\rho_2$, that is, the dimensions of the corresponding non-degenerate simplices given by the Eilenberg-Zilber lemma 24, are the same. This implies the bidimension of $\partial_0\rho_2$ and $\partial_{0,\tau}\rho_2$ are the same, therefore stictly smaller than the bidimension (p, q) of ρ_2.

This implies the Lyapunov function L' can be used for both products and their respective vector fields $V_{F\times B}$ and $V_{F\times_\tau B}$. This vector field $V_{F\times_\tau B}$ therefore is *admissible*. ♣

Chapter 7

The Classifying Space W-reduction.

Chapter 8

The Loop Space W-reduction.

Chapter 9

The Adams model of a loop space.

Chapter 10

The Bousfield-Kan W-reduction (??).

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