A case study of A_∞-structure.

Ainhoa Berciano, Julio Rubio, Francis Sergeraert

December 2008

1 Introduction.

The notion of A_∞-object is now classical. Initiated by Jim Stasheff [29], it can be applied to most algebraic structures, when the required algebraic properties are only satisfied up to homotopy, which homotopy in turn must satisfy some properties, only up to homotopy and so on. The A_∞-structures have in particular been intensively studied and used by Tornike Kadeishvili [14, 15, 16] to whom this paper is dedicated.

Kadeishvili himself wrote a convenient presentation of the subject, as an essential step toward operads, for the recent Ictp-Map Summer School [17, Section 5]. We will follow here the definitions and notations of this text. The "retrospective" [30, Section 2] describes in particular how this subject is strongly related to the problem of algebraic deformations. See [27, 4] for other recent interesting points of view. As just mentioned above, the subject of A_∞-structures is strongly connected to the modern notion of *operad*, algebraic or topological, the A_∞-structures being in fact an important particular case of operads, see for example [20].

As in the paper [4], we use the so-called basic perturbation lemma to measure the complexity of the A_∞-structures and to study them in a convenient direct global process. In [4], this method was used to study the A_∞-structures underlying the natural elementary algebras produced by Cartan's solution for the homology of Eilenberg-MacLane spaces [8]. Here we do an analogous work around the homology of *iterated* loop spaces.

This allows us to exhibit two chain complexes A_* and B_* with the following properties:

- A_* is the chain complex *canonically* associated to a simplicial set, namely the Kan model of $\Omega^2(P^\infty\mathbb{R}/P^3\mathbb{R})$. Therefore A_* is a differential Hopf algebra, in particular a differential coalgebra.
- The chain complex B_* is of finite type, so that its homology groups are elementarily computable. A homology equivalence $A_* \ll\!\!\gg B_*$ proves they are in fact the homology groups of $\Omega^2(P^\infty\mathbb{R}/P^3\mathbb{R})$.

- The chain complex B_* therefore inherits of an A_∞-coalgebra structure:

$$\{\Delta_n : B_* \to B_*^{\otimes n}\}_{n\geq 1}$$

  Each Δ_n has degree $n-2$ and in particular Δ_1 is the differential of B_*.
- *Every* Δ_n is *non-trivial*.

To obtain these *experimental* results, we use the Kenzo program [10] based on the so-called *Effective Homology* theory [24]. The first sections of the paper present as simply as possible the main ideas of Effective Homology. The Kenzo machine illustrations then naturally lead to the announced result. We hope this paper shows how such a machine program can be used to experimentally study the complex structures of Algebraic Topology in various particular cases, when pen and paper do not allow the topologist to go very far.

2 Effective homology.

Definition 1 — A *reduction* $\rho = (f, g, h) : \widehat{C}_* \Rightarrow C_*$ is a diagram:

$$\rho = \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftleftarrows}} C_*}$$

where:

- The nodes $\widehat{C}_*$ and C_* are chain complexes, the first one $\widehat{C}_*$ being the *big* one, the second one C_* the *small* one;
- The arrows f and g are two chain complex morphisms;
- The self-arrow h is a homotopy operator (degree $+1$);
- The following relations are satisfied:

$$\begin{aligned} fg &= \mathrm{id}_{C_*} \\ gf + dh + hd &= \mathrm{id}_{\widehat{C}_*} \\ fh &= 0 \\ hg &= 0 \\ hh &= 0 \end{aligned}$$

This reduction describes the big chain complex $\widehat{C}_*$ as the direct sum of the small one $C_* \cong g(C_*)$ and an acyclic complement $\ker(f)$.

Theorem 2 (Basic Perturbation "Lemma" [28, 6]) — *Let* ρ *be a reduction*

$$\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rightarrow (C_*, d)$$

and $\widehat{\delta}$ *a* perturbation *of the differential* $\widehat{d}$ *of* $\widehat{C}_*$*:* $(\widehat{d} + \widehat{\delta})^2 = 0$*. We assume the* nilpotency condition *is satisfied:*

For every $x \in \widehat{C}_*$, *there exists* $n \in \mathbb{N}$ *with* $(\widehat{\delta}h)^n x = 0$.

Then perturbations, δ_f, δ_g, δ_h *and* δ_d, *respectively of* f, g, h *and* d, *can be computationnally constructed defining a new reduction:*

$$\rho' = (f + \delta_f, g + \delta_g, h + \delta_h) : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rightarrow\!\!\!\!\Rightarrow (C_*, d + \delta_d)$$

It is an *implicit function theorem*: the various maps $\widehat{d}$, d, f, g and h are tied together by a set of relations, those which describe the required reduction properties. Perturbing the differential $\widehat{d}$ leads to perturb the other morphisms to keep the same set of relations. Simple examples show the nilpotency condition is necessary.

A reduction $\widehat{C}_* \Rightarrow\!\!\!\!\Rightarrow C_*$ often allows the user to determine the homology of $\widehat{C}_*$, which maybe is a complex *not* of finite type, thanks to the other complex which for example could be on the contrary of finite type. This is a frequent situation. In such a case, if the basic perturbation lemma can be applied, the homology of the *perturbed* big chain complex $(\widehat{C}_*, \widehat{d} + \widehat{\delta})$ can be determined as well through the *perturbed* small chain complex $(C_*, d + \delta_d)$

Definition 3 — An *equivalence* $\varepsilon : C_* \Leftarrow\!\!\!\Rightarrow C'_*$ between two chain complexes is an extra chain complex $\widehat{C}_*$ and a pair of reductions $\rho = (f, g, h) : \widehat{C}_* \Rightarrow\!\!\!\!\Rightarrow C_*$ and $\rho' = (f', g', h') : \widehat{C}_* \Rightarrow\!\!\!\!\Rightarrow C'_*$.

Definition 4 — An object *with effective homology* is a 4-tuple $(X, C_*X, EC_*, \varepsilon)$ where:

- X is some object studied from a homological point of view, thanks to the canonical chain complex C_*X associated to it in the current context: simplicial homology, homology of groups, Hochschild homology, cyclic homology...
- EC_* is a chain complex of finite type, the homology of which being therefore elementarily computable (E for effective);
- ε is an equivalence $\varepsilon : C_*X \Leftarrow\!\!\!\Rightarrow EC_*$

The equivalence ε defines in particular a homology equivalence between C_* and EC_*; a canonical isomorphism is defined $H_*X := H_*C_*X \cong H_*EC_*$: the homology groups of X are *computable*.

Much more important, this data type is *stable*, which is explained now.

Meta-Theorem 5 — *Let* χ *be a constructor:*

$$\chi : (X_1, \ldots, X_n) \mapsto X$$

producing an object X *from various objects* $X_1, \ldots, X_n$. *Then, under appropriate conditions, an* algorithm χ^{EH}*:*

$$\chi^{EH} : (X_1^{EH}, \ldots, X_n^{EH}) \mapsto X^{EH}$$

can be written down. This algorithm χ^{EH} *is called a* version with effective homology *of the constructor* χ.

Each input object X_i^{EH} is assumed to be an object with effective homology $X_i^{EH} = (X_i, C_*X_i, EC_{*,i}, \varepsilon_i)$ and the algorithm χ^{EH} produces an object $X^{EH} = (X, C_*X, EC_*, \varepsilon)$, also an object with effective homology.

So that, if interested by the homology groups of X, you can use the effective chain complex EC_* to elementarily compute them. More important, if you intend to use the output object X as input for another constructor χ', the same process can be applied in turn to χ'^{EH} and X^{EH}; in particular, iterations become easy, the key point to obtain a simple and powerful solution of the Adams' problem.

3 The fibration construction.

The fibrations were invented by Charles Ehresmann, intensively used by Jean-Pierre Serre when computing homotopy groups of spheres, formalized in the simplicial framework by Daniel Kan [18], a good reference for simplicial fibrations being [21].

Let B and F be simplicial sets and G a simplicial group acting on F. A (simplicial) fibration of base space B, fiber space F, structural group G is defined by a twisting map $\tau : B \to G$ expressing how the trivial product $F \times B$ is twisted to obtain $F \times_\tau B$ instead; the twisting map is equivalent to a classifying map into the classifying space for the group G, denoted by the same symbol $\tau : B \to BG$. The structure of the twisting map is described for example in [21, §18]; the key point is that the simplex sets $(F \times B)_n$ and $(F \times_\tau B)_n$ are the same, only the face operators are modified. The associated chain complexes $C_*(F \times B)$ and $C_*(F \times_\tau B)$ have the same underlying graded modules, only the differentials are... different. We are in a situation where the perturbation lemma could be applied.

Theorem 6 — *A general algorithm can be written down:*

- **Input**: $(F, C_*F, EC_*^F, \varepsilon^F)$, $(B, C_*B, EC_*^B, \varepsilon^B)$, G, τ.
- **Output**: $(E, C_*E, EC_*^E, \varepsilon^E)$

where:

- *F and B are simplicial sets with effective homology, B being 1-reduced;*
- *G is a simplicial group acting on F;*
- *$\tau : B \to G$ is a twisting map defining a simplicial fibration:*

$$F \hookrightarrow (E = F \times_\tau B) \to B$$

- *The output is a version with effective homology of the total space of the fibration $E = F \times_\tau B$.*

♣ [Sketch of proof, details in [25, Section 8]] The Eilenberg-Zilber theorem produces a reduction:

$$EZ : C_*(F \times B) \Rightarrow C_*(F) \otimes C_*(B)$$

and elementary tensor product calculations produce an equivalence:

$$\varepsilon^{F\times B} : C_*F \otimes C_*B \Longleftarrow \widehat{C}_* \Longrightarrow EC_*^F \otimes EC_*^B.$$

We will use two times the basic perturbation lemma along the path:

$$C_*(F \times B) \Longrightarrow C_*F \otimes C_*B \Longleftarrow \widehat{C}_* \Longrightarrow EC_*^F \otimes EC_*^B.$$

First, replacing the left-hand object $C_*(F \times B)$ by $C_*(F \times_\tau B)$ is nothing but introducing a perturbation in the differential of the big chain complex $C_*(F \times B)$. The nilpotency condition is satisfied, the perturbation lemma can be applied, producing a new reduction:

$$EZ_\tau : C_*(F \times_\tau B) \Longrightarrow C_*(F) \otimes_t C_*(B)$$

where the right-hand tensor product is also *twisted* according to a *twisting cochain* computed by the perturbation lemma. This is the so-called *twisted* Eilenberg-Zilber theorem [5, 28, 6], at the origin of the perturbation lemma.

The non-twisted tensor product is isomorphic to a subcomplex of $\widehat{C}_*$ in the equivalence $\varepsilon^{F\times B}$. The perturbation replacing the non-twisted tensor product $C_*(F) \otimes C_*(B)$ by the twisted one $C_*(F) \otimes_t C_*(B)$ can be trivially transfered to $\widehat{C}_*$, leaving unchanged the complementary chain complex, producing a new differential and a new chain complex $\widehat{C}_{t,*}$.

Again applying the perturbation lemma between $\widehat{C}_*$ and $\widehat{C}_{t,*}$ produces a new reduction $\widehat{C}_{t,*} \Longrightarrow (EC_*^F \otimes EC_*^B)_t$. For this application of the perturbation lemma, the nilpotency condition requires B is 1-reduced.

Combining these results, we obtain a path:

$$C_*(F \times_\tau B) \Longrightarrow C_*(F) \otimes_t C_*(B) \Longleftarrow \widehat{C}_{t,*} \Longrightarrow (EC_*^F \otimes EC_*^B)_t$$

and elementary computations can transform this path into an equivalence:

$$\varepsilon_\tau : C_*(F \times_\tau B) \Longleftrightarrows (EC_*^F \otimes EC_*^B)_t$$

where the right-hand chain complex is *effective*. ♣

This is the version *with effective homology* of the Serre spectral sequence. It is just an adaptation to a computational environment of Shih Weishu's crucial work [28].

Except in simple situations, the Serre spectral is *not* an algorithm: the higher differentials are mathematically defined, but not computationnally reachable with the data usually available. The above theorem is on the contrary a "closed" algorithm entirely computing a version with effective homology of the total space when versions with effective homology of the base space and the fiber space are given. We must also signal Ana Romero proved [23] our algorithm can be completed to obtain as a *by-product* the *entire* description of the corresponding Serre spectral sequence, in particular any higher differential, and also the structure of the filtration of the homology groups at the abutment.

We use this opportunity to give a simple small Kenzo demo, to prepare the reader to more sophisticated demos. Let $B = P^\infty\mathbb{R}/P^1\mathbb{R}$ be the infinite real projective space stunted at dimension 2. The $\mathbb{Z}$-homology of B is obvious: $H_0 = H_2 = \mathbb{Z}$, $H_{2n+1} = \mathbb{Z}/2$ for $n \geq 1$, and $H_n = 0$ otherwise. The fundamental cohomology class of B in dimension 2, a simplicial map $\tau : B \to K(\mathbb{Z}, 2)$, defines a fibration:

$$K(\mathbb{Z}, 1) \hookrightarrow (F \times_\tau B = E) \to B$$

and produces a new space E, the 3-stage of the Whitehead tower of our stunted projective space B. What about this total space ? Let us ask this question to the Kenzo program.

```
> (setf B (r-proj-space :infinity 2)) ✠
[K1 Simplicial-Set]
> (setf ch2 (chml-clss B 2)) ✠
[K12 Cohomology-Class on K1 of degree 2]
> (setf fibration (z-whitehead B ch2)) ✠
[K25 Fibration K1 -> K13]
> (setf E (fibration-total fibration)) ✠
[K31 Simplicial-Set]
> (efhm E) ✠
[K112 Equivalence K31 <= K102 => K98]
> (homology E 0 7) ✠
Homology in dimension 0 :
Component Z
---done---
Homology in dimension 1 :
---done---
Homology in dimension 2 :
---done---
Homology in dimension 3 :
Component Z
---done---
Homology in dimension 4 :
---done---
Homology in dimension 5 :
---done---
Homology in dimension 6 :
---done---
```

The Lisp prompt is the greater character '`>`' and the user then enters a Lisp *statement* to be executed. On this display, the end of the Lisp statement is marked by the maltese character '✠', in fact not visible on the user's screen. Then the Kenzo program works and *returns* some result.

For example the Lisp statement `(setf B (r-proj-space :infinity 2))` is in fact double; on the one hand `(r-proj-space :infinity 2)` constructs and *returns* the desired stunted projective space $B = P^\infty\mathbb{R}/P^1\mathbb{R}$; on the other hand a statement `(setf B xxx)` *assigns* the object returned by `xxx` to the machine symbol `B`, probably for later use. The external display of the projective space is `[K1 Simplicial-Set]` to be read *the Kenzo object #1, a simplicial set*; the internal object, a *complete* description of a simplicial model for B is of course a little more complicated.

Then the fundamental cohomology class of B in degree 2 is computed and assigned to the symbol `ch2`; the corresponding fibration is constructed (`fibration`) and the associated total space E, the simplicial set `K31` is produced. The simplicial model of B has only one simplex for every dimension, so that the effective homology is direct. In this context, the Kan "minimal" model of $K(\mathbb{Z},1)$ is required, which model is not of finite type; but a classical result gives the effective homology of the fiber space $K(\mathbb{Z},1)$, more precisely a reduction $C_*K(\mathbb{Z},1) \Rightarrow C_*S^1$ with the ordinary model of the circle, one simplex in dimensions 0 and 1. So that Kenzo may apply the effective version of the Serre spectral sequence to compute the effective homology (`efhm`) of the total space. Which allows us to ask for the homology groups $H_n(B)$ for example up to $n = 6$.

Only one non-trivial homology group, $H_3E = \mathbb{Z}$. This suggests maybe our total space could have the homotopy type of S^3. Correct? Yes, but not so easy to prove, see [9] for a solution of this problem and many others.

4 Eilenberg-Moore spectral sequence.

The Serre spectral sequence is not an algorithm, the Eilenberg-Moore spectral sequence is not an algorithm either. Given a fibration:

$$F \hookrightarrow E \to B$$

the Eilenberg-Moore spectral sequence describes relations between the homology groups of F, E and B; in *some* favourable cases, the knowledge of H_*E and H_*B allows the user to compute H_*F. But with the data usually available, the higher differentials are unreachable, and also the extension problems at abutment can be dreadful.

The methods of effective homology on the contrary give the desired algorithm.

Theorem 7 — *A general algorithm can be written down:*

- **Input:** F, $(B, C_*B, EC^B_*, \varepsilon^B)$, G, τ, $(E, C_*E, EC^E_*, \varepsilon^E)$.
- **Output:** $(F, C_*F, EC^F_*, \varepsilon^F)$

where:

- *F and B are simplicial sets, B being 1-reduced.*
- *G is a simplicial group acting on F.*
- *τ is a twisting map $\tau : B \to G$ defining a fibration:*

$$F \hookrightarrow (E = F \times_\tau B) \to B$$

- *The total space E and the base space B are provided with their effective homology.*
- *The output is a version with effective homology of the fiber space F.*

♣ [Sketch of proof, details in [25, Section 9]] As in the standard Eilenberg-Moore spectral sequence, we must use the Cobar construction. The *Bar* construction is due to Eilenberg and MacLane [11, Chapter II]; the dual *Cobar* construction was firstly used by Frank Adams [1]; a detailed definition corresponding to the present context can be found in particular at [25, p.126] and [17, Section 4]. If C_* is a differential coalgebra and A_* (resp. B_*) is a differential left (resp. right) C_*-comodule, then the Cobar construction produces a chain complex $\mathrm{Cobar}^{C_*}(A_*, B_*)$.

We must again work with the trivial product $F \times B$ and the twisted one $F \times_\tau B$; the twisted Eilenber-Zilber theorem gives a reduction:

$$C_*(F \times_\tau B) \twoheadrightarrow C_*(F) \otimes_t C_*B$$

for an approriate twisting cochain $t : C_*B \to C_*G$ defining the twisted tensor product.

The objects $C_*F \otimes C_*B$, $C_*F \otimes_t C_*B$ and $C_*(F \times_\tau B)$ have natural right C_*B-comodule structures, induced by the natural projections over C_*B.

The Cobar object $\mathrm{Cobar}^{C_*B}(C_*B, \mathbb{Z})$ is acyclic, for this is nothing but the standard cobar resolution of $\mathbb{Z}$ with respect to the C_*B-comodule structure induced by the inclusion $* \hookrightarrow B$ of the base point into the base space. See for example [19, Section 5.8] for the dual case with the bar resolution. More precisely there is a canonical reduction:

$$\mathrm{Cobar}^{C_*B}(C_*B, \mathbb{Z}) \twoheadrightarrow \mathbb{Z}.$$

The tensor product functor $C_*F \otimes <?>$ can be applied, producing a reduction:

$$\mathrm{Cobar}^{C_*B}(C_*F \otimes C_*B, \mathbb{Z}) \twoheadrightarrow C_*F.$$

But we prefer the twisted product, which leads again to apply the perturbation lemma. It happens in this case, the differential of the small chain complex is not perturbed, which gives a reduction:

$$\mathrm{Cobar}^{C_*B}(C_*F \otimes_t C_*B, \mathbb{Z}) \twoheadrightarrow C_*F. \tag{1}$$

The twisted Eilenberg-Zilber reduction $C_*(F \times_\tau B) \twoheadrightarrow C_*(F) \otimes_t C_*B$ is compatible with the C_*B-comodule structures, which allows us to obtain a reduction:

$$\mathrm{Cobar}^{C_*B}(C_*F \times_\tau C_*B, \mathbb{Z}) \twoheadrightarrow \mathrm{Cobar}^{C_*B}(C_*F \otimes_t C_*B, \mathbb{Z}). \tag{2}$$

The natural composition of the two last reductions is a reduction:

$$\mathrm{Cobar}^{C_*B}(C_*F \times_\tau C_*B, \mathbb{Z}) \twoheadrightarrow C_*F. \tag{3}$$

The left-hand argument of the Cobar, $C_*F \times_\tau C_*B$, is C_*E and it would be tempting to continue the same process to obtain a new equivalence:

$$\mathrm{Cobar}^{C_*B}(C_*E, \mathbb{Z}) \twoheadleftarrow\!\!\!\!\twoheadrightarrow \mathrm{Cobar}^{EC_*^B}(EC_*^E, \mathbb{Z})$$

but this does not make sense: no coalgebra structure on EC_*^B! We must use another technique.

When a chain complex C_* is augmented, that is, when a 0-generator $* \in C_0$ is distinguished, a *trivial* coalgebra structure can be defined by $\Delta(c) = c \otimes * + * \otimes c$. If C'_* is another chain complex, a trivial right C_*-comodule structure can be defined on C'_* by $\Delta(c) = c \otimes *$; the same for a trivial left structure.

Instead of the canonical coalgebra and comodule structures on C_*B and C_*E, we can for a moment prefer the trivial structures, obtaining:

$$\mathrm{PreCobar}^{C_*B}(C_*E, \mathbb{Z}) := \mathrm{Cobar}^{(C_*B)_0}((C_*E)_0, \mathbb{Z})$$

the 0-indices applied to C_*B and C_*E meaning the trivial structures are applied. The so-called PreCobar is nothing but the Cobar where the simplicial differential associated to the coproducts is cancelled. The PreCobar is a direct sum of various ordinary tensor products, which gives, using the effective homologies of B and E an equivalence:

$$\mathrm{PreCobar}^{(C_*B)_0}((C_*E)_0, \mathbb{Z}) \Longleftarrow\!\!\!\Longrightarrow \mathrm{PreCobar}^{(EC_*^B)_0}((EC_*)_0, \mathbb{Z})$$

Now restoring the actual Cobar at the left-hand term of this equivalence is nothing but perturbing the differential of the Precobar. Again the perturbation lemma can be applied, producing an equivalence:

$$\mathrm{Cobar}^{C_*B}(C_*E, \mathbb{Z}) \Longleftarrow\!\!\!\Longrightarrow \widetilde{\mathrm{Cobar}}^{EC_*^B}(EC_*^E, \mathbb{Z}). \tag{4}$$

The notation $\widetilde{\mathrm{Cobar}}$ means it is not an actual Cobar which is defined here: a sophisticated multi-differential, automatically constructed by the perturbation lemma, is installed on the initial PreCobar to obtain this object. This is nothing but defining the A_∞-coalgebra structure on EC_*^B, deduced from the equivalence $C_*B \Longleftarrow\!\!\!\Longrightarrow EC_*^B$, and the A_∞-comodule structure on EC_*^E, deduced from the equivalence $C_*E \Longleftarrow\!\!\!\Longrightarrow EC_*^E$.

Combining the reduction (3) and the equivalence (4) produces the desired equivalence:

$$C_*F \Longleftarrow\!\!\!\Longrightarrow \widetilde{\mathrm{Cobar}}^{EC_*^B}(EC_*^E, \mathbb{Z}).$$

where the right-hand term is effective, due to the 1-reduced property of the base space B. ♣

Victor Gugenheim and others, see for example [12], worked also in the same direction. Precisely they used the perturbation lemma to obtain which is called now the cobar-tilde $\widetilde{\mathrm{Cobar}}^{EC_*^B}(EC_*^E, \mathbb{Z})$, a chain complex computing the homology of the fiber space. The key complement given here is an *explicit* homology equivalence with the chain complex of the fiber space C_*F; this is essential to use this fiber space as input in a version with effective homology of some other constructor. We obtain in particular in this way a solution to Adams' problem.

5 Adam's problem.

Frank Adams, at the beginning of the paper [1], states the problem of computing the homology groups of an *iterated* loop space, and solves it. Starting from a simplicial set K_0, Adams obtains a cubical set K_1' having the homotopy type of the loop space. Subdividing this complex K_1' into a simplicial set K_1 in principle allows to iterate; if the theoretical feasability is clear, the practical difficulties are enormous and fifty years later, to our knowledge, this method has never been used for concrete computations.

On the contrary, Theorem 7 gives a simple solution to Adams' problem, with a *strictly* wider scope.

Theorem 8 — *A general algorithm can be written down:*

- **Input:** $(X, C_*X, EC_*^X, \varepsilon^X)$.
- **Output:** $(GX, C_*GX, EC_*^{GX}, \varepsilon^{GX})$.

where:

- *The simplicial set X is 1-reduced (one vertex, no non-degenerate edge) and provided with effective homology.*
- *The simplicial set GX is the Kan model of the loop space $\Omega|X|$, returned with its effective homology.*

♣ The Kan model GX of the loop space is a simplicial *group*, the fiber space of a simplicial principal fibration:

$$GX \hookrightarrow (GX \times_\tau X = E) \to X$$

where the total space E, playing the role of the *path space* in the standard Serre fibration $\Omega|X| \hookrightarrow P|X| \to |X|$, is *contractible*. Its ordinary homology is trivial, and its *effective* homology is an explicit reduction $C_*E \Rightarrow C_*(*) = \mathbb{Z}$. Such a reduction is explicitely described at [21, Theorem 26.6]. Combined with the provided effective homology of the base space B, Theorem 7 computes the effective homology of the fiber space GX. ♣

Corollary 9 (Solution of Adam's problem) — *A general algorithm can be written down:*

- **Input:** k, $(X, C_*X, EC_*^X, \varepsilon^X)$.
- **Output:** $(G^kX, C_*G^kX, EC_*^{G^kX}, \varepsilon^{G^kX})$.

where:

- $k \in \mathbb{N}$

- *The simplicial set X is k-reduced (one vertex, no non-degenerate simplex in dimensions $\leq k$) and provided with effective homology.*
- *The simplicial set G^kX is the Kan model of the* iterated *loop space $\Omega^k|X|$, returned with its effective homology.*

♣

With respect to the previous results of Adams [1], Milgram [22] and Baues [3] (see also [7]), the progress of this result is multiple:

- The proof is natural and remarkably simple.
- The integer k is arbitrary, of course modulo the usual connectivity condition for the original space.
- The original space is not necessarily of finite type, it can be itself the result of a long sequence of constructions, with a unique condition: it is given *with effective homology*. For example you can consider the standard model of $K(\mathbb{Z},4)$, "highly" infinite, you attach to it a finite 3-reduced simplicial set A by an arbitrary map $f : A \supset B \to K(\mathbb{Z},4)$; the result $C = K(\mathbb{Z},4) \cup_f A$ is a 3-reduced simplicial set with effective homology. You can therefore apply the previous result to compute the effective homology of its third loop space $\Omega^3 C$, and also its ordinary homology.
- In particular no topological condition such as the suspension condition in [22].

As an illustration, to be used also in the last section, let us consider the simplest case where other classical results, mainly [22, 3, 7], fail: what about $H_*\Omega^3(P^\infty R/P^3\mathbb{R})$? The stunted projective space $P^\infty R/P^3\mathbb{R}$ is *not* a suspension, compare with [22], and we want to consider the *third* loop space, compare with [3] and the beginning of [7, Section 6].

The stunted projective space is constructed and assigned to the symbol `stunted-4`:

```
> (setf stunted-4 (r-proj-space :infinity 4)) ✠
[K1 Simplicial-Set]
```

The standard model of this stunted projective space has only one vertex and exactly one simplex in dimensions ≥ 4, the corresponding chain complex has one generator in these dimensions and is effective, so that the effective homology of this space is trivial.

```
> (efhm stunted-4)
[K9 Equivalence K1 <= K1 => K1]
```

Let us construct now the first loop space and its effective homology.

```
> (setf loop-space-1 (loop-space stunted-4)) ✠
[K10 Simplicial-Group]
> (efhm loop-space-1) ✠
[K118 Equivalence K10 <= K108 => K25]
```

The right-hand chain complex K25 is of finite type, it is simply $\texttt{K25} = \mathrm{Cobar}^{\texttt{K1}}(\mathbb{Z}, \mathbb{Z})$. The effective homology this time is not trivial, for the Kan model K10 is made of non-commutative free groups, not at all finite. But this does not matter, Theorem 8 can be applied as well to obtain the effective homology of the second loop space.

```
> (setf loop-space-2 (loop-space loop-space-1)) ✠
[K119 Simplicial-Group]
> (efhm loop-space-2) ✠
[K261 Equivalence K119 <= K251 => K247]
```

Note the simplex sets of K119 are made of non-commutative free groups, the *generators* of which are the *elements* of the free groups defining K10. If G is some set of *generators*, and if $\mathbb{Z}^{*G}$ denotes the free *non-commutative* group *generated* by G, think of something like $\mathbb{Z}^{*(\mathbb{Z}^{*G})}$. No interpretation of the right-hand chain complex K247 as a Cobar.

The next step of the iteration is the same.

```
> (setf loop-space-3 (loop-space loop-space-2)) ✠
[K262 Simplicial-Group]
> (efhm loop-space-3) ✠
[K404 Equivalence K262 <= K394 => K390]
```

Think this time to simplex sets looking like $\mathbb{Z}^{*(\mathbb{Z}^{*(\mathbb{Z}^{*G})})}$, generating the chain complex K262. But the effective homology K404 is an equivalence between this giant chain complex K262 and the *effective* chain complex K390. Let us examine the rank of the chain groups of K390 in the dimensions < 8.

```
> (dotimes (i 8)
    (princ (length (basis (k 390) i)))
    (princ " - ")) ✠
1 - 1 - 2 - 5 - 13 - 33 - 84 - 214 -
NIL
```

For example the boundary matrix between the degrees 4 and 3 can be computed and displayed.

```
> (chcm-mat (k 390) 4) ✠
========== MATRIX 5 lines + 13 columns =====
L1=
L2=
L3=[C1=4][C3=1][C4=-1][C5=2]
L4=[C1=4][C3=1][C4=-1][C8=2]
L5=[C5=4][C6=2][C8=-4][C9=-2]
========== END-MATRIX
```

which would be displayed in traditional mathematical notation:

$$d_4^{\text{K390}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 & 0 & -4 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The same boundary matrices, for example between degrees 6 and 5 and between degrees 7 and 6, are a little larger, 33×84 and 84×214, but this is not a problem for a machine, and elementary methods then give the $H_6(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R}); \mathbb{Z})$.

```
> (homology loop-space-3 6) ✠
Component Z/2Z
Component Z/2Z
[... 7 lines deleted ...]
Component Z/2Z
```

In other words, $H_6(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R}); \mathbb{Z}) = (\mathbb{Z}/2)^{10}$.

6 Examining A_∞-structures.

Most authors consider only A_∞-structures only for *vector spaces* with respect to some ground *field*. The simple and direct point of view used here allows us to consider the most general situation: A_∞-structures over $\mathbb{Z}$-modules with respect to the ground *ring* $\mathbb{Z}$.

Let A_* and B_* be two chain complexes (made of $\mathbb{Z}$-modules), the first one being provided with a structure of differential coalgebra. We suppose these chain complexes are homology equivalent. Then the chain complex B_* inherits of an A_∞-coalgebra structure. This structure on B_* allows one to construct the so-called cobar-tilde construction:

$$[B_* + A_\infty\text{-coalgebra structure}] \mapsto \widetilde{\text{Cobar}}^{B_*}(\mathbb{Z}, \mathbb{Z}).$$

with the main property that this cobar-tilde is homology equivalent to $\text{Cobar}^{A_*}(\mathbb{Z}, \mathbb{Z})$.

But the process can be reversed. Let $\varepsilon : A_* \twoheadleftarrow\!\!\!\!\!\rightarrow B_*$ be a (strong) equivalence as defined in Definition 3. Then the process explained to obtain the equivalence (4)

in Section 4 can be applied, this constructs an equivalence:

$$\mathrm{Cobar}^{A_*}(\mathbb{Z},\mathbb{Z}) \Longleftarrow\!\!\Longrightarrow \widetilde{\mathrm{Cobar}}^{B_*}(\mathbb{Z},\mathbb{Z})$$

and we can *read* inside the differential of $\widetilde{\mathrm{Cobar}}^{B_*}(\mathbb{Z},\mathbb{Z})$ so calculated the A_∞-coalgebra structure of B_*. The right-hand term of this equivalence has not been constructed with the help of the components of the A_∞-coalgebra as it is usually understood; here the right-hand term of the equivalence has been entirely constructed by the perturbation lemma and this "lemma" has in particular *produced* all the components of the A_∞-coalgebra structure of B_* which are usually used to construct the cobar-tilde.

Our Kenzo program so becomes, thanks to the perturbation lemma, a tool to construct A_∞-structures and possibly to easily detect "exotic" components with a high simplicial degree in the A_∞-coalgebra structure, the usual coproduct being considered as having simplicial degree 2.

It is natural to suspect the loop spaces the homology of which is "difficult" to be calculated according to simple methods are good candidates for such exotic components. This is a good reason to consider again the iterated loop spaces of the previous section. The Kenzo example of this section, centered around the iterated loop space $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$, can be summarized in the next diagram, where the key object Ω_{EH} is the version *with effective homology* of the loop space functor, in other words, the Theorem 8.

$$\begin{array}{rcl}
C_*(P^\infty\mathbb{R}/P^3\mathbb{R}=X)=\texttt{K1} & \Longleftarrow\!\!\Longrightarrow & \texttt{K1} \\
\Big\downarrow{\scriptstyle \Omega_{EH}} & & \\
C_*(\Omega X)=\texttt{K10} & \Longleftarrow\!\!\Longrightarrow & \texttt{K25}=\mathrm{Cobar}^{\texttt{K1}}(\mathbb{Z},\mathbb{Z}) \\
\Big\downarrow{\scriptstyle \Omega_{EH}} & & \\
A_*=C_*(\Omega^2X)=\texttt{K119} & \Longleftarrow\!\!\Longrightarrow & \texttt{K247}=\widetilde{\mathrm{Cobar}}^{\texttt{K25}}(\mathbb{Z},\mathbb{Z})=B_* \\
\Big\downarrow{\scriptstyle \Omega_{EH}} & & \\
C_*(\Omega^3X)=\texttt{K262} & \Longleftarrow\!\!\Longrightarrow & \texttt{K390}=\widetilde{\mathrm{Cobar}}^{\texttt{K247}}(\mathbb{Z},\mathbb{Z})
\end{array}$$

Let A_* be the chain complex canonically associated to the loop space $\Omega^2(P^\infty\mathbb{R}/P^3\mathbb{R})$, in other words the chain-complex `K119` of our Kenzo environment. Its effective homology is an equivalence with the effective chain complex $B_*=$ `K247` The chain complex A_* is a genuine coalgebra, it is even a Hopf algebra, while the chain complex B_* is only an A_∞-coalgebra. To examine the last structure, it is enough to remark the effective homology of the *third* loop space $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$ is an equivalence between the chain complex of this loop space `K262` and the effective chain complex:

$$\texttt{K390} =: \widetilde{\mathrm{Cobar}}^{\texttt{K247}}(\mathbb{Z},\mathbb{Z}).$$

The symbol '=:' meaning we intend to *read* the A_∞-coalgebra structure of `K247` from the differential in `K390` obtained by a simple application of the perturbation lemma.

As explained in the previous section, the chain complex `K390` is of finite type and its generators look like cobar terms. It is useful to explain here the initial stunted projective space has exactly one non-degenerate simplex in dimensions ≥ 4 and such a simplex in our program is coded as an integer, its dimension. For example the "basis" of this simplicial set in dimension 5 is the list (parentheses) made of the unique element 5.

```
> (basis stunted-4 5) ✠
(5)
```

For the first loop space, the cobar-tilde `K25` is a simple cobar, and if we ask for the basis in degree 7, we obtain:

```
> (basis (k 25) 7) ✠
(<<AlLp[7 8]>> <<AlLp[3 4][4 5]>> <<AlLp[4 5][3 4]>>)
```

which would be usually denoted by $([8], [4|5], [5|4])$: three elements, the first one of simplicial degree 1, the others of simplicial degree 2. In the Lisp result, `AlLp` must be read "algebraic loop", for a generator of the cobar is a sort of "algebraic" equivalent of a loop. Then for example `<<AlLp[3 4][4 5]>>` denotes $[4|5]$ where the brackets [3 4] and [4 5] of the Lisp notation mean the generator 4 (resp. 5) of the initial simplicial set contributes for a factor of degree 3 (resp. 4) in the cobar generator. This can seem a little cumbersome notation, but in general a generator of the initial simplicial set does not contain its dimension, so that it can be very useful for the user to see the cobar generator which would be usually denoted by $[a|b]$ as the Lisp object `<<AlLp[3 a][4 b]>>`, therefore with factors a and b of respective *initial* dimensions 4 and 5.

We can ask also for the *algebra* basis:

```
> (allp-basis (k 25) 7) ✠
(<<AlLp[7 8]>>)
```

that is, in this case, the unique generator of degree 7 having the simplicial degree 1, namely [8]. We have now the ingredients to verify the role of the usual coproduct in the cobar construction producing `K25`.

```
> (? (k 25) 9 (allp 9 10)) ✠
-------------------------------------------------------------------------{CMBN 8}
<-2 * <<AlLp[8 9]>>>
<-1 * <<AlLp[3 4][5 6]>>>
<-1 * <<AlLp[4 5][4 5]>>>
<-1 * <<AlLp[5 6][3 4]>>>
-------------------------------------------------------------------------------
```

For the faces of the unique n-simplex 'n' of our stunted projective space are $\partial_0 n = \partial_n n = (n-1)$ and $\partial_i n = \eta_{i-1}(n-2)$, except in dimensions 4 and 5 where the

non-existing simplices are replaced by the only remaining possibility, a degeneracy of the base point.

For the second and third loop spaces, the situation becomes more complicated. For example the five generators of the last cobar-tilde `K390` in dimension 3 are:

```
> (map nil #'print (basis (k 390) 3)) ✠
<<AlLp[3 <<AlLp[4 <<AlLp[5 6]>>]>>]>>
<<AlLp[3 <<AlLp[2 <<AlLp[3 4]>>][2 <<AlLp[3 4]>>]>>]>>
<<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>][2 <<AlLp[3 <<AlLp[4 5]>>]>>]>>
<<AlLp[2 <<AlLp[3 <<AlLp[4 5]>>]>>][1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>
<<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>][1 <<AlLp[2 <<AlLp[3 4]>>]>>]
      [1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>
NIL
```

Not so easy to read such a listing. In traditional mathematical notation, this basis would be:

$$([[[6]]], [[[4] \mid [4]]], [[[4]] \mid [[5]]], [[[5]] \mid [[4]]], [[[4]] \mid [[4]] \mid [[4]]])$$

with the respective simplicial degrees 1, 1, 2, 2 and 3; more readable?

Now the game to search for exotic components in the differentials of this cobar-tilde is the following. First consider only the generators of this *algebra*, that is, simplicial degree 1; then, for such a generator, compute its differential and on the contrary look in this differential for terms of high simplicial degrees. It is a little technical but not really difficult to design a little program automatically doing this work. And the *experimental* result is the following: for every odd integer $2n+1$, the boundary of the simplest generator $[[[2n+4]]]$ of the cobar-tilde in degree $2n+1$ contains an element of simplicial degree 2n, namely:

$$\underbrace{[[[4]] \mid \cdots \mid [[4]]]}_{2n \text{ times}}$$

The first dimension where this happens with an exotic diagonal is in degree 5 where the generator $[[[8]]]$ of the cobar-tilde has the element $[[[4]] \mid [[4]] \mid [[4]] \mid [[4]]]$ in its differential, of simplicial degree 4. This means the previous A_∞-coalgebra, namely the chain complex `K247`, has a non-trivial Δ_4 in degree 5. It is here the last term of the differential computed by the Kenzo program.

```
> (? (k 390) 5 (first (basis (k 390) 5))) ✠
------------------------------------------------------------------------{CMBN 4}
<-2 * <<AlLp[4 <<AlLp[5 <<AlLp[6 7]>>]>>]>>>
<-1 * <<AlLp[4 <<AlLp[5 <<AlLp[3 4][3 4]>>]>>]>>>
<-4 * <<AlLp[4 <<AlLp[2 <<AlLp[3 4]>>][3 <<AlLp[4 5]>>]>>]>>>
<6 * <<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>][3 <<AlLp[4 <<AlLp[5 6]>>]>>]>>>
<4 * <<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>]
           [3 <<AlLp[2 <<AlLp[3 4]>>][2 <<AlLp[3 4]>>]>>]>>>
<4 * <<AlLp[2 <<AlLp[3 <<AlLp[4 5]>>]>>][2 <<AlLp[3 <<AlLp[4 5]>>]>>]>>>
<6 * <<AlLp[3 <<AlLp[4 <<AlLp[5 6]>>]>>][1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>>
<4 * <<AlLp[3 <<AlLp[2 <<AlLp[3 4]>>][2 <<AlLp[3 4]>>]>>]
           [1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>>
<4 * <<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>][2 <<AlLp[3 <<AlLp[4 5]>>]>>]
           [1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>>
<-1 * <<AlLp[1 <<AlLp[2 <<AlLp[3 4]>>]>>][1 <<AlLp[2 <<AlLp[3 4]>>]>>]
            [1 <<AlLp[2 <<AlLp[3 4]>>]>>][1 <<AlLp[2 <<AlLp[3 4]>>]>>]>>>
------------------------------------------------------------------------------
```

More generally `K247` contains in degree $2n+1$ non-trivial higher "diagonals" Δ_i for $1 \leq i \leq 2n$.

Of course such an *experimental* result is not very satisfactory, we would like to *prove* it, which seems relatively complicated. But at least the program produces the *text* of the *statement* to be proved! The statement of the result is relatively simple, and tracking the work of the perturbation lemma, it is probably possible to find out the exact phenomenon leading to this simple result. A subject for future work, where again a tool such as the Kenzo program could help.

References

[1] J. Frank Adams. *On the Cobar construction*. Proceedings of the National Academy of Science of the U.S.A., 1956, vol. 42, pp. 409-412.

[2] *Cohomological Methods in Homotopy Theory*. Jaume Aguadé, Carles Broto, Carles Casacuberta, editors. Birkhauser, 2001.

[3] Hans J. Baues. *Geometry of loop spaces and the cobar construction*. Memoirs of the American Mathematical Society, 1980, vol. 230.

[4] Ainhoa Berciano and Ronald Umble. *Some naturally occurring examples of A_∞-bialgebras*. `arXiv:0706.0703v4 [math.AT]`.

[5] Edgar H. Brown Jr.. *Twisted tensor products, I*. Annals of Mathematics, 1959, vol. 69, pp. 223-246.

[6] Ronnie Brown. *The twisted Eilenberg-Zilber theorem*. Celebrazioni Arch. Secolo XX, Simp. Top., 1967, pp. 34-37.

[7] Gunnar Carlsson and R. James Milgram. *Stable homotopy and iterated loop spaces*. in [13], pp. 505-583.

[8] Henri Cartan. *Algèbres d'Eilenberg-MacLane et homotopie*. Séminaire Henri Cartan, 1954-5. Or *Henri Cartan, Œuvres*, Springer, 1979.

[9] Frederick R. Cohen and Ran Levi. *On the homotopy type of infinite stunted projective spaces*. In [2], pp.79-90.

[10] Xavier Dousson, Julio Rubio, Francis Sergeraert and Yvon Siret. *The Kenzo program*. `http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`

[11] Samuel Eilenberg, Saunders MacLane. *On the groups $H(\pi, n)$, I*. Annals of Mathematics, 1953, vol. 58, pp. 55-106.

[12] V.K.A.M. Gugenheim. *On a perturbation theory for the homology of the loop space*. Journal of Pure and Applied Algebra, 1982, vol. 25, pp. 197-205.

[13] *Handbook of Algebraic Topology* (Edited by I.M. James). North-Holland (1995).

[14] T. Kadeishvili. *The A1-algebra Structure and Cohomology of Hochschild and Harrison*. Proc. of Tbil. Math. Inst., 1988, vol.91, pp.19-27.

[15] Tornike Kadeishvili. *A(∞)-algebra Structure in Cohomology and Rational Homotopy Type*. Forschungsswerpunkt Geometrie, Universitaet Heidelberg, Math. Inst., 1988, Heft 37, pp.1-64.

[16] T. Kadeishvili. *A_∞-algebra Structure in Cohomology and Rational Homotopy Type*, (in Russian). Proc. of Tbil. Mat. Inst., 1993, vol.107, pp. 1-94.

[17] Tornike Kadeishvili. *Ictp-Map 2008 Summer School*. `http://map.disi.unige.it/ictp/lectures_files/Kadeishvili_L.pdf`

[18] Daniel M. Kan. *A combinatorial definition of homotopy groups*. Annals of Mathematics. 1958, vol. 67, pp. 282-312.

[19] Saunders MacLane. *Homology*. Springer-Verlag, 1975.

[20] Martin Markl, Steve Shnider and Jim Stasheff. *Operads in algebra, topology and physics*. American Mathematical Society, 2002.

[21] J. Peter May. *Simplicial objects in algebraic topology*. Van Nostrand, 1967.

[22] R. James Milgram. *Iterated loop spaces*. Annals of Mathematics, 1966, vol. 84, pp. 386-403.

[23] A. Romero, J. Rubio, F. Sergeraert. *Computing spectral sequences*. Journal of Symbolic Computation, 2006, vol.41, pp.1059-1079.

[24] Julio Rubio, Francis Sergeraert. *Constructive Algebraic Topology*. Bulletin des Sciences Mathématiques, 2002, vol. 126, pp. 389-412.

[25] Julio Rubio, Francis Sergeraert. *Genova Lecture Notes*. `www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`

[26] Julio Rubio, Francis Sergeraert, Yvon Siret. *Overview of EAT, a System for Effective Homology Computation*. The SAC Newsletter, 1998, vol.3, pp. 68-79.

[27] Samson Saneblidze and Ronald Umble. *Diagonals on the Permutahedra, Multiplihedra and Associahedra*. Homology, Homotopy and Applications, 2004, vol.6, pp.363-411.

[28] Weishu Shih. *Homologie des espaces fibrés*. Publications Mathématiques de l'I.H.E.S., 1962, vol. 13.

[29] James D. Stasheff. *Homotopy associativity of H-spaces, I, II*. Transactions of the American Mathematical Society, 1963, vol.108, pp.275-292 and 293-312.

[30] Ronald N. Umble. *Higher Homotopy Hopf Algebras Found: A Ten Year Retrospective*. arXiv:0709.3436v3 [Math.AT].