Computing all maps into a sphere*

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Abstract

We present an algorithm for computing $[X, Y]$, i.e., all homotopy classes of continuous maps $X \to Y$, where X, Y are topological spaces given as finite simplicial complexes, Y is $(d-1)$-connected for some $d \geq 2$ (for example, Y can be the d-dimensional sphere S^d), and $\dim X \leq 2d-2$. These conditions on X, Y guarantee that $[X, Y]$ has a natural structure of a finitely generated Abelian group, and the algorithm finds generators and relations for it. We combine several tools and ideas from homotopy theory (such as *Postnikov systems, simplicial sets, and obstruction theory*) with algorithmic tools from effective algebraic topology (*objects with effective homology*).

We hope that a further extension of the methods developed here will yield an algorithm for computing, in some cases of interest, the $\mathbb{Z}_2$-*index*, which is a quantity playing a prominent role in Borsuk–Ulam style applications of topology in combinatorics and geometry, e.g., in topological lower bounds for the chromatic number of a graph. In a certain range of dimensions, deciding the embeddability of a simplicial complex into $\mathbb{R}^d$ also amounts to a $\mathbb{Z}_2$-index computation. This is the main motivation of our work.

We believe that investigating the computational complexity of questions in homotopy theory and similar areas presents a fascinating research area, and we hope that our work may help bridge the cultural gap between algebraic topology and theoretical computer science.

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1 Introduction

The problem. One of the central themes in algebraic topology is understanding the structure of all *continuous* maps $X \to Y$, for given topological spaces X and Y (all maps between topological spaces in this paper are assumed to be continuous). For topological purposes, two maps $f, g\colon X \to Y$ are usually considered equivalent if they are *homotopic*, i.e., if one can be continuously deformed into the other[1]; thus, the object of interest is $[X, Y]$, the set of all homotopy classes of maps $X \to Y$.

Many of the celebrated results throughout the history of topology can be cast as information about $[X, Y]$ for particular spaces X and Y. An early example is a famous theorem of Hopf from the 1930s, asserting that the homotopy class of a map $f\colon S^n \to S^n$, between two spheres of the same dimension, is in one-to-one correspondence with an integer parameter, the *degree* of f. Another great discovery of Hopf, with ramifications in modern physics and elsewhere, was a map $S^3 \to S^2$, now called by his name, that is not homotopic to a constant map.

These are early results in the theory of *higher homotopy groups*. For our purposes, the *kth homotopy group* $\pi_k(Y)$, $k \geq 2$, of a space Y can be thought of as the set $[S^k, Y]$ (which is, moreover, equipped with a suitable group operation).[2] In particular, the *homotopy groups of spheres* $\pi_k(S^n)$ are among the most puzzling objects of mathematics, and many respected papers have been devoted to computing them in special cases (see, e.g., the book [14]).

Related to the problem of determining $[X, Y]$ is the *extension problem*: given $A \subset X$ and a map $f\colon A \to Y$, can it be extended to a map $X \to Y$? For example, the famous *Brouwer fixed-point theorem* can be re-stated as non-extendability of the identity map $S^n \to S^n$ to the ball D^{n+1}. A number of topological concepts, which may look quite advanced and esoteric to a newcomer in algebraic topology, e.g. *Steenrod squares*, have a natural motivation in an attempt at a stepwise solution of the extension problem.

Earlier developments around the extension problems are described in Steenrod's paper [27], which we can recommend, for readers with a moderate topological background, as an exceptionally clear and accessible, albeit somewhat outdated, introduction to this area. In that paper, Steenrod asks for an effective procedure for (some aspects of) the extension problem.

There has been an enormous amount of work in homotopy theory since the 1960s, with a wealth of new concepts and results, some of them opening completely new areas or reaching to distant branches of mathematics. However, as far as we could find out, the *algorithmic part* of the program discussed in [27] has not been explicitly completed up until now.

The only algorithmic paper concerning the computation of $[X, Y]$ we are aware of is that by Brown [2] from 1957(!). Brown showed that $[X, Y]$ is computable under the assumption that Y is 1-connected[3] and all the higher homotopy groups $\pi_k(Y)$, $2 \leq k \leq \dim X$, are *finite* (this is a rather strong assumption, *not* satisfied by spheres, for example). Then he went on to show the *computability of the higher homotopy groups* $\pi_k(Y)$, $k \geq 2$, for every 1-connected Y. To do this, he overcame the problem of infinite homotopy groups (which we will discuss below) by a somewhat ad-hoc method, which does not seem to generalize to the $[X, Y]$ setting.

On the negative side, it is well known that the problem of computing $[X, Y]$, in full generality, is *algorithmically unsolvable*. Indeed, for Y connected, $[S^1, Y]$ is nontrivial exactly if $\pi_1(Y) \neq 0$, where $\pi_1(Y)$ is the *fundamental group* of Y, and the undecidability of $\pi_1(Y) \neq 0$ is a celebrated result of Adjan and of Rabin (see, e.g., the survey by Soare [24]). Actually, this is the only

[1] More precisely, f and g are defined to be homotopic, in symbols $f \sim g$, if there is a continuous $F\colon X \times [0,1] \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. With this notation, $[X, Y] = \{[f] : f\colon X \to Y\}$, where $[f] = \{g : g \sim f\}$ is the *homotopy class* of f.

[2] Strictly speaking, the isomorphism $\pi_k(Y) \cong [S^k, Y]$ needs mild assumptions on Y; e.g., it holds if Y is a path-connected CW-complex.

[3] A *k-connected* space Y is one whose first k homotopy groups vanish; in other words, every map $S^i \to Y$ can be extended to D^{i+1}, the ball bounded by the S^i, $0 \leq i \leq k$.

hardness result known to us.[4] For undecidability results concerning numerous more loosely related problems we refer to [24], [13], [12] and references therein.

Effective algebraic topology. In the 1990s, three independent collections of works appeared with the goal of making various more advanced methods of algebraic topology *effective* (algorithmic): by Schön [20], by Smith [23], and by Rubio, Sergeraert, Dousson, and Romero (e.g., [21, 16, 15, 17]; also see [19] for an exposition). These obtain general *computability* results, and in the case of Rubio et al., a *practical implementation* as well, but none of them provides any running time bounds.

Roughly speaking, Rubio et al. provide algorithms that can construct basic topological spaces, such as finite simplicial complexes or *Eilenberg–MacLane spaces* (discussed below), and then obtain new spaces from them by various operations, e.g., the Cartesian product, the *loop space* and the *bar construction*, the *total space of a fibration*, etc. These objects and constructions are often of infinitary nature, which means that the resulting spaces have to be represented in a certain implicit manner. Yet one can compute homology and cohomology groups (of given dimensions) of the resulting objects; one speaks of *objects with effective homology*.

The problem of computing $[X, Y]$ and the extension problem were not addressed in those papers, but we build on them to some extent, relying on objects with effective homology for implementing certain operations in our algorithm.

Our work. We are generally interested in the *computational complexity* of the problem of computing $[X, Y]$. We assume that X and Y are given as finite simplicial complexes (or, more generally, *simplicial sets* with finitely many nondegenerate simplices, as discussed below).

We would like to find, on the one hand, sufficient conditions on X and Y, as weak as possible, making the problem decidable, or even polynomial-time solvable, and on the other hand, interesting settings where the problem can be proved algorithmically intractable (undecidable or NP-hard, say). We also believe that similar methods may bring results for the extension problem and for other related questions.

Here we prove the following positive result:

Theorem 1.1. *Let $d \geq 2$. Assuming that $Y = S^d$ or, more generally, that Y is $(d-1)$-connected, and that $\dim X \leq 2d - 2$, the set $[X, Y]$ is computable, in the following sense: It is known that, under the above conditions on X and Y, $[X, Y]$ can be naturally endowed with a structure of a (finitely generated) Abelian group, in an essentially unique way. The algorithm computes the structure of this group (i.e., expresses it as a direct product of cyclic groups). Moreover, given two simplicial maps $f, g\colon X \to Y$, it can be decided whether they are homotopic.*

We establish Theorem 1.1 mainly by combining ideas and tools that have been essentially known. We see our main contribution as that of synthesis: identifying suitable methods, putting them all together, and organizing the result in a hopefully accessible way, so that it can be built on in the future. Some technical steps are apparently new; in this direction, our main technical contribution is probably a suitable implementation of the group operation on $[X, Y]$ and recursive testing of nullhomotopy.

Applications, motivation. We consider the fundamental nature of the algorithmic problem of computing $[X, Y]$ a sufficient motivation of our research (e.g., because $[X, Y]$ is indeed one of the most basic objects of study in algebraic topology). However, we also believe that work in this area will bring various connections and applications, also in other fields, possibly including practically usable software, e.g., for aiding research in topology.

[4] There is a also result of Anick [1] on #P-hardness of computing the higher homotopy groups. However, the way he presents it, it is not immediately relevant for spaces given as simplicial complexes, since his reduction uses a very compact representation of the input space—roughly speaking, he needs to encode degrees of attaching maps as binary integers. Perhaps with some more work one could also use his method to show hardness of computing $\pi_k(Y)$ for Y given as a simplicial complex, say.

A nice concrete application comes from the paper by Franek et al. [5]. They provide an algorithm testing if a given system of equations involving analytic functions has a "robust zero", and in order to extend their result to more general situations, they ask for an algorithm testing *nullhomotopy* (i.e., homotopy to a constant map) of a map into S^d. Our Theorem 1.1 provides such an algorithm in a certain range of dimensions.

Our motivation for starting this project was the computation of the $\mathbb{Z}_2$-*index* (or *genus*) $\mathrm{ind}(X)$ of a $\mathbb{Z}_2$-space[5] X, i.e., the smallest d such that X can be equivariantly mapped into S^d. We hope that by extending the methods of the present paper, one can obtain an algorithm for deciding whether $\mathrm{ind}(X) \leq d$, provided that $\dim(X) \leq 2d-2$.

The problem of computing $\mathrm{ind}(X)$ arises, among others, in the problem of *embeddability* of topological spaces, which is a classical and much studied area (see, e.g., the survey by Skopenkov [22]). One of the basic questions here is, given a k-dimensional finite simplicial complex K, can it be (topologically) embedded in $\mathbb{R}^d$? The celebrated *Haefliger–Weber theorem* from the 1960s asserts that, in the *metastable range of dimensions*, i.e., for $k \leq \frac{2}{3}d - 1$, embeddability is equivalent to $\mathrm{ind}(K^2_\Delta) \leq d-1$, where K^2_Δ is a certain $\mathbb{Z}_2$-space constructed from K (the *deleted product*). Thus, in this range, the embedding problem is, computationally, a special case of $\mathbb{Z}_2$-index computation; see [10] for a study of algorithmic aspects of the embedding problem, where the metastable range was left as one of the main open problems.

The $\mathbb{Z}_2$-index also appears as a fundamental quantity in combinatorial applications of topology. For example, the celebrated result of Lovász on Kneser's conjecture can nowadays be restated as $\chi(G) \geq \mathrm{ind}(B(G))+2$, where $\chi(G)$ is the chromatic number of a graph G, and $B(G)$ is a certain simplicial complex constructed from G (see, e.g., [9]). We find it striking that *nothing* seems to be known about the computability of such an interesting quantity as $\mathrm{ind}(B(G))$.

Further work. Besides the problem of adapting the machinery behind Theorem 1.1 to the equivariant setting, or to the setting of the extension problem, there are number of other open questions related to our work.

Polynomiality. At present we do not state any bounds on the running time of the algorithm. A critical part for the running time are subroutines for building a *Postnikov system* of Y and evaluating *Postnikov classes*. For this, one can use algorithms sketched in [18, 17] (which we intend to present in more detail in a companion paper), but these appear to be at least exponential. More precisely, we believe that all of the steps in these algorithms are polynomial for *fixed* dimension, with the single exception of an *effective homology* reduction for the simplicial Eilenberg–MacLane space $K(\mathbb{Z}, 1)$ (see, e.g., [19] for these notions; some of them are also discussed later in the present paper). This is a very concrete algorithmic problem, although too technical to be stated here explicitly. Of course, all of the remaining steps in the algorithm have to be analyzed carefully as well, but the hoped-for outcome should be, in the setting of Theorem 1.1, a running time polynomial in the size of X and Y for every *fixed* d (on the other hand, we consider polynomial dependence on d highly unlikely—for example, because Theorem 1.1 includes the computation of the *stable* homotopy groups $\pi_{d+k}(S^d)$, $k \leq d-2$; these are unlikely to be easily computable, in view of their notorious mathematical difficulty).

Hardness? We suspect that once the assumptions in Theorem 1.1 are weakened, the problem of deciding, say, nontriviality of $[X, Y]$ may become intractable. This is, in our opinion, one of the most interesting open problems related to our work.

General remarks. *Algorithmic* or *computational topology* has been a blooming discipline in recent years (see, e.g., [3, 28]). Our work addresses issues different from those investigated in the current mainstream of this field. We study *homotopic* questions, generally regarded as much less tractable than, e.g., homology computations.

[5] A $\mathbb{Z}_2$-*space* is a topological space X with an action of the group $\mathbb{Z}_2$; the action is described by a homeomorphism $\nu\colon X \to X$ with $\nu \circ \nu = \mathrm{id}_X$. A primary example is a sphere S^d with the antipodal action $x \mapsto -x$. An *equivariant map* between $\mathbb{Z}_2$-spaces is a continuous map that commutes with the $\mathbb{Z}_2$ actions.

Although such questions have been thoroughly studied from a topological perspective already in the 1950s and 1960s, we are not aware of any work in this direction in theoretical computer science, with the perspective of computational complexity. We believe that questions similar to those studied here offer an exciting field for complexity-theoretic study.

Given the number of topological concepts and tools employed in our algorithm, we cannot present much of the technical contents in a short talk or in a ten-page abstract, at least without assuming a substantial topological background. We hope to convey the message that there is an interesting area on the borderline of computer science and topology, and motivate readers to reach for the full version of this paper (where we aim at general accessibility) and additional sources.

An outline of the methods. In the rest of this section, we sketch the main ideas and tools in the algorithm. Some topological notions are left undefined here; we will introduce them later.

Conceptually, the basis of the algorithm is classical *obstruction theory* [4]. For a first encounter, it is probably easier to consider a version of obstruction theory which proceeds by constructing maps $X \to Y$ inductively on the i-dimensional *skeleta*[6] of X, extending them one dimension at a time. (For the actual development, we use a different version of obstruction theory, where we lift maps from X through stages of a Postnikov system of Y.)

In a nutshell, at each stage, the extendability of a map from the i-skeleton to the $(i+1)$-skeleton is characterized by vanishing of a certain *obstruction*, which can, more or less by known techniques, be evaluated algorithmically.

Textbook expositions may give the impression that obstruction theory is a general algorithmic tool for testing the extendability of maps. However, the extension at each step is generally not unique, and extendability at higher stages may depend, in a nontrivial way, on the choices made earlier. Thus, in principle, one needs to search an infinitely branching tree of extensions.[7] In our setting, we make essential use of the group structure on the sets $[X, Y]$ (mentioned in Theorem 1.1), as well as on some related ones, for a finite encoding of the set of all possible extensions at a given stage.

The description of our algorithm has several levels. On the top level, we talk about operations on Abelian groups, whose elements are homotopy classes of maps (and we need to be careful in distinguishing "how explicitly" the relevant groups are available to us). On a lower level, the group operation and other primitives are implemented by computations with *concrete representatives* of the homotopy classes; interestingly, on the level of the representatives, the operations are generally non-associative.

The space Y enters the computation in the form of a *Postnikov system.* This is a topological concept from the 1950s (usually considered unsuitable for concrete computations by topologists; see, e.g., [8]); roughly speaking, it provides a way of building Y from "canonical pieces", called *Eilenberg–MacLane spaces*, whose homotopy structure is the simplest possible, although they are not that simple combinatorially.

Our main data objects are *simplicial sets*, an ingenious generalization of simplicial complexes. They are suitable for algorithmic representation of Eilenberg–Mac Lane spaces and other infinite objects in the algorithm. The stages P_i of the Postnikov system are built as simplicial sets in such a way that every *continuous* map $X \to P_i$ is homotopic to a *simplicial* map. The proof of Theorem 1.1 will rely on two facts: that for $\dim X \leq 2d-2$, there is an isomorphism $[X, Y] \cong [X, P_{2d-2}]$, and that we can compute $[X, P_i]$ inductively for $i \leq 2d-2$.

Then, due to the properties of the Eilenberg–MacLane spaces, simplicial maps into P_i can be compactly represented by certain sequences of *cochains* on X. Concretely, a map appears in the algorithm as a labeling of the simplices of X by elements of various Abelian groups.

An important component of the algorithm are subroutines, not treated in detail in this

[6]The k-skeleton of a simplicial complex X consists of all simplices of X of dimension at most k.

[7]Brown's result mentioned earlier, on computing $[X, Y]$ with the $\pi_k(Y)$'s finite, is based on a complete search of this tree, where the assumptions on Y guarantee the branching to be finite.

paper, for evaluating k_i's, the *ith Postnikov classes* of Y, $d \leq i \leq 2d-2$. The input to k_i is represented as a simplex with faces labeled by elements of appropriate Abelian groups, and the output lies in yet another Abelian group.

For Y fixed, these subroutines can be hard-wired once and for all. In some particular cases, they are given by known explicit formulas. In particular, for $Y = S^d$, k_d corresponds to the famous *Steenrod square* [26, 27], and k_{d+1} to *Adem's secondary cohomology operation*.[8] However, in the general case, the only way of evaluating the k_i we are aware of is using *objects with effective homology* mentioned earlier.

2 Operations with Abelian groups

On the top level, our algorithm works with finitely generated Abelian groups. In our setting, an Abelian group A is represented by a set $\mathcal{A}$, whose elements are called *representatives*; we also assume that the representatives can be stored in a computer. For $\alpha \in \mathcal{A}$, let $[\alpha]$ denote the element of A represented by α. The representation is generally non-unique; we may have $[\alpha] = [\beta]$ for $\alpha \neq \beta$.

We call A represented in this way *semi-effective* if algorithms for the following three tasks are available: provide an element $o \in \mathcal{A}$ with $[o] = 0$ (the neutral element); given $\alpha, \beta \in \mathcal{A}$, compute $\gamma \in \mathcal{A}$ with $[\gamma] = [\alpha] + [\beta]$; given $\alpha \in \mathcal{A}$, compute $\beta \in \mathcal{A}$ with $[\beta] = -[\alpha]$. We call a semi-effective Abelian group A *fully effective* if the following are explicitly available: a finite list of generators $a_1, \ldots, a_k$ of A (given by representatives) and their orders $q_1, \ldots, q_k \in \{2, 3, \ldots\} \cup \{\infty\}$ (so that each a_i generates a cyclic subgroup of A of order q_i, $i = 1, 2, \ldots, k$, and A is the direct sum of these subgroups); and an algorithm that, given $\alpha \in \mathcal{A}$, computes integers $z_1, \ldots, z_k$ so that $[\alpha] = \sum_{i=1}^{k} z_i a_i$.

Let X, Y be sets. We call a mapping $\varphi\colon X \to Y$ *locally effective* if there is an algorithm that, given an arbitrary $x \in X$, computes $\varphi(x)$. For semi-effective Abelian groups A, B, with sets $\mathcal{A}, \mathcal{B}$ of representatives, respectively, we call a mapping $f\colon A \to B$ *locally effective* if there is a locally effective mapping $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $[\varphi(\alpha)] = f([\alpha])$ for all $\alpha \in \mathcal{A}$. In particular, we speak of a *locally effective homomorphism* if f is a group homomorphism.

The proofs of the following three lemmas are not difficult (given an algorithm for computing the Smith normal form of an integer matrix) and are omitted from this extended abstract.

Lemma 2.1 (Kernel)**.** *Let $f\colon A \to B$ be a locally effective homomorphism of fully effective Abelian groups. Then* $\ker(f) = \{a \in A : f(a) = 0\}$ *can be represented as fully effective.*

Lemma 2.2 (Cokernel)**.** *Let A, B be fully effective Abelian groups with sets of representatives $\mathcal{A}, \mathcal{B}$, respectively, and let $f\colon A \to B$ be a locally effective homomorphism. Then we can obtain a fully effective representation of the factor group $C := \operatorname{coker}(f) = B/\operatorname{im}(f)$, again with the set $\mathcal{B}$ of representatives. Moreover, there is an algorithm that, given a representative $\beta \in \mathcal{B}$, tests whether β represents 0 in C, and if yes, returns a representative $\alpha \in \mathcal{A}$ such that $[f(\alpha)] = [\beta]$ in B.*

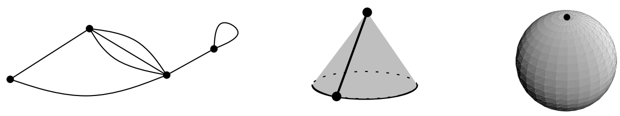
Lemma 2.3 (Short exact sequence)**.** *Let A, B, C be Abelian groups, with A, C fully effective and B semi-effective, and let $f\colon A \to B$ and $g\colon B \to C$ be locally effective homomorphisms such that f is injective, g is surjective, and $\operatorname{im} f = \ker g$ (in other words, $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence). Assume, moreover, that the following locally effective maps are given: $r\colon \operatorname{im} f = \ker g \to A$ such that $f(r(b)) = b$ for every $b \in B$ with $g(b) = 0$; and $\xi\colon \mathcal{C} \to \mathcal{B}$ (where $\mathcal{B}, \mathcal{C}$ are the sets of representatives for B, C, respectively) that behaves like a* section *for g, i.e., such that $g([\xi(\gamma)]) = [\gamma]$ for all $\gamma \in \mathcal{C}$. Then we can obtain a fully effective representation of B.*

[8] It is worth remarking that the k_i's represent a "nonlinear part" of the algorithm, which otherwise, on the bottom level, deals mostly with solving systems of linear Diophantine equations. For example, a Steenrod square can be thought of as a quadratic form.

3 Topological preliminaries

Here we briefly summarize the main topological notions and tools from the literature.

Simplicial sets. A *simplicial set* can be thought of as a generalization of simplicial complexes. Similar to a simplicial complex, a simplicial set is a space built of vertices, edges, triangles, and higher-dimensional simplices, but simplices are allowed to be glued to each other and to themselves in more general ways. For example, one may have several 1-dimensional simplices connecting the same pair of vertices, a 1-simplex forming a loop, two edges of a 2-simplex identified to create a cone, or the boundary of a 2-simplex all contracted to a single vertex, forming an S^2.



However, unlike for the still more general *CW-complexes*, a simplicial set can be described purely combinatorially. Another new feature of a simplicial set, in comparison with a simplicial complex, is the presence of *degenerate simplices*. For example, the edges of the triangle with a contracted boundary (in the last example above) do not disappear—formally, each of them keeps a phantom-like existence of a degenerate 1-simplex.

A simplicial set X is represented as a sequence $(X_0, X_1, X_2, \ldots)$ of mutually disjoint sets, where the elements of X_m are called the *m-simplices of* X. Moreover, for every m, there are maps $\partial_0, \ldots, \partial_m \colon X_m \to X_{m-1}$ (the *face operators*) and $s_0, \ldots, s_m \colon X_m \to X_{m+1}$ (the *degeneracy operators*); they have to satisfy natural axioms, which we won't need explicitly.

We let $|X|$ denote the *geometric realization* of a simplicial set X (this is the topological space described by X, generalizing the polyhedron of a simplicial complex). The *cone* CX is obtained from X by adding a new vertex $*$ and erecting a cone with apex $*$ over every simplex of X, and the *suspension* SX is constructed from CX by contracting all of X into a single vertex. The *product* $X \times Y$ of simplicial sets is a simplicial set with $|X \times Y| \cong |X| \times |Y|$; we have $(X \times Y)_m = X_m \times Y_m$, $m = 0, 1, 2, \ldots$.

A *simplicial map* $s \colon X \to Y$ of simplicial sets sends every m-simplex of X to an m-simplex of Y, $m = 0, 1, \ldots$, and respects the face and degeneracy operators. We let $\mathrm{SMap}(X, Y)$ stand for the set of all simplicial maps $X \to Y$. A very important feature in our algorithm is that the targets of maps are *Kan simplicial sets*, i.e., simplicial sets Y such that every continuous map $f \colon |X| \to |Y|$ is homotopic to a *simplicial* map $X \to Y$. Thus, maps into such a Y have a discrete representation; the price to pay is that Y has to have infinitely many simplices in some dimensions of interest, so its computer representation is not straightforward.

In the above, we have omitted many details; for precise definitions we refer to the very friendly introduction by Friedman [6], or to May [11] as a standard reference.

Eilenberg–MacLane spaces. For an Abelian group π and an integer $n \geq 1$, an Eilenberg–MacLane space $K(\pi, n)$ is a space with $\pi_n(K(\pi, n)) \cong \pi$ and $\pi_i(K(\pi, n)) = 0$ for all $i \neq n$. The circle S^1 is a $K(\mathbb{Z}, 1)$, but otherwise, the $K(\pi, n)$ are typically infinite-dimensional. For us, it is important that maps *into* Eilenberg–MacLane spaces are simple to describe and to represent: namely, for every CW-complex X, the set $[X, K(\pi, n)]$ is in a bijective correspondence with the cohomology group[9] $H^n(X; \pi)$.

[9] Here are the basic definitions concerning cohomology, which we will need in the sequel: For a simplicial complex (or set) X and an Abelian group π, $C^n(X; \pi)$ denotes the group of all *n-cochains* on X, i.e., labelings of n-simplices of X by elements of π. The *coboundary operator* $\delta \colon C^n(X; \pi) \to C^{n+1}(X; \pi)$ is given by $\delta c(\tau) = \sum_{i=0}^{n+1} (-1)^i c(\partial_i \tau)$, where τ is an $(n+1)$-simplex of X. We also need the group $Z^n(X; \pi) := \ker \delta$ of *cocycles* and $B^{n+1}(X; \pi) := \operatorname{im} \delta$ of *coboundaries*. The *nth cohomology group* is $H^n(X; \pi) := Z^n(X; \pi)/B^n(X; \pi)$.

We will use $K(\pi,n)$ represented by a Kan simplicial set. The set of m-simplices is given by the amazing formula $K(\pi,n)_m := Z^n(\Delta^m;\pi)$, where Δ^m is the standard m-simplex. Thus, an m-simplex σ can be regarded as a labeling of the n-dimensional faces of Δ^m by elements of π; moreover, the labels (with appropriate signs) must add up to 0 on the boundary of every $(n+1)$-face. We also need a related simplicial set $E(\pi,n)$ with $E(\pi,n)_m := C^n(\Delta^m;\pi)$.

For every simplicial set X, $\mathrm{SMap}(X, K(\pi,n))$ is in a bijective correspondence with $Z^n(X;\pi)$, and $\mathrm{SMap}(X, E(\pi,n)) \cong C^n(X;\pi)$. Two maps $s_1, s_2 \in \mathrm{SMap}(X, K(\pi,n))$ represented by $c_1, c_2 \in Z^n(X;\pi)$ are homotopic iff $c_1 - c_2 \in B^n(X;\pi)$. All of this can be found in [11].

Simplicial Postnikov systems. A Postnikov system for a space Y consists of spaces $P_0, P_1, \dots$ (the *stages*), maps $p_i\colon P_i \to P_{i-1}$, and maps $\varphi_i\colon Y \to P_i$.[10] The P_i can be thought of as successive stages in a process of building Y (or rather, a space homotopy equivalent to Y) "layer by layer" from the Eilenberg–MacLane spaces $K(\pi_i, i)$, where $\pi_i := \pi_i(Y)$.

A key fact for our use of Postnikov systems is that if X is a CW-complex with $\dim X \le i$, then there is a bijection between $[X,Y]$ and $[X,P_i]$ (induced by composition with φ_i). Thus, for computing $[X,Y]$ in Theorem 1.1, it suffices to compute $[X, P_{2d-2}]$.

In a simplicial Postnikov system, P_i is a simplicial subset of the product $P_{i-1} \times E_i \subseteq E_0 \times E_1 \times \cdots \times E_i$, where $E_j := E(\pi_j, j)$. We remark that for a $(d-1)$-connected Y the stages $P_0, \dots, P_{d-1}$ are trivial, since $\pi_0, \dots, \pi_{d-1}$ are trivial. We will usually write an m-simplex of P_i as $(\boldsymbol{\sigma}, \sigma^i)$, where $\boldsymbol{\sigma} \in P_{i-1}$ and $\sigma^i \in C^i(\Delta^m, \pi_i)$ is a simplex of E_i. The projection map $p_i\colon P_i \to P_{i-1}$ is given by $p_i(\boldsymbol{\sigma}, \sigma^i) = \boldsymbol{\sigma}$.

We will also need the *Postnikov classes* $k_{i-1} \in \mathrm{SMap}(P_{i-1}, K_{i+1})$ (with $K_{i+1} := K(\pi_i, i{+}1)$), which can also be represented by a cocycle in $Z^{i+1}(P_{i-1}, \pi_i)$. They are used to "cut out" P_i from the product $P_{i-1} \times E_i$, as follows: $P_i := \{(\boldsymbol{\sigma}, \sigma^i) \in P_{i-1} \times E_i : k_{i-1}(\boldsymbol{\sigma}) = \delta\sigma^i\}$, where $\delta\colon E_i \to K_{i+1}$ is induced by the coboundary operator.

We also introduce the notation $L_i := K(\pi_i, i)$, and $\lambda_i\colon L_i \to P_i$ is the insertion to the last component, $\lambda_i(\sigma^i) := (\mathbf{0}, \sigma^i) \in P_i$. Here $\mathbf{0} = (0, \dots, 0)$ denotes the zero m-simplex in P^{i-1}, made of a zero cochain in every component, where $m = \dim(\sigma^i)$.

A second key fact we need is that the stages P_i of the simplicial Postnikov system of a 1-connected Y are Kan simplicial sets (see, e.g., [2]). Thus, for every simplicial set X, there is a bijection between the set of simplicial maps $X \to P_i$ modulo simplicial homotopy and the set of homotopy classes of continuous maps between the geometric realizations. Slightly abusing notation, we will denote both sets by $[X, P_i]$ from now on.

A simplicial Postnikov system as above is *locally effective* if the homotopy groups $\pi_i(Y)$ are fully effective and algorithms are available for evaluating the simplicial maps $\varphi_i\colon Y \to P_i$ and the cocycles $k_{i-1} \in Z^{n+1}(P_{i-1}, \pi_i)$. The methods of *effective homology*, as explained, e.g., in [19], combined with the construction of a Postnikov system as given, e.g., in Spanier [25, Section 8.3] (in particular, Corollary 7 there), lead to the following result.

Theorem 3.1. *Let Y be a 1-connected simplicial set that has finitely many nondegenerate simplices (e.g., as obtained from a finite simplicial complex). Then, for every n, a locally effective Postnikov system for Y with n stages can be constructed.*

Induced maps and cochain representations. A simplicial map $s\colon P \to Q$ of arbitrary simplicial sets induces a map $s_*\colon \mathrm{SMap}(X,P) \to \mathrm{SMap}(X,Q)$ by composition, i.e., by $s_*(f)(\sigma) = (s \circ f)(\sigma)$ for each simplex $\sigma \in P$. If P and Q are Kan, we also get a well-defined map $[s_*]\colon [X,P] \to [X,Q]$. Moreover, if X has finitely many nondegenerate simplices and s is locally effective, then so is s_*.

A simplicial map $s\colon X \to P_i$ is, in particular, a simplicial map into the product $E_0 \times \cdots \times E_i$. So we have $s = (s_0, s_1, \dots, s_i)$, where $s_j\colon X \to E_j$, and each s_j can be represented by a cochain

[10] These should satisfy certain technical conditions; specifically, for every i, $p_i \circ \varphi_i = \varphi_{i-1}$, $\pi_j(P_i) = 0$ for $j > i$, and φ_i induces isomorphisms $\pi_j(Y) \cong \pi_j(P_i)$ for $i \le j$. However, these will not be explicitly used in this extended abstract.

$c^j \in C^j := C^j(X;\pi_j)$. Thus, s is represented by an $(i+1)$-tuple $\mathbf{c} = (c^0,\dots,c^i)$ of cochains, subject to the conditions $k_{(j-1)*}(c^0,\dots,c^{j-1}) = \delta c^j$, $j \le i$. Our algorithm uses this *cochain representation* of simplicial maps $s \in \mathrm{SMap}(X, P_i)$.

4 Defining and implementing the group operation on $[X, P_i]$

We recall that the device that allows us to handle the generally infinite set $[X, Y]$ of homotopy classes of maps, under the dimension/connectedness assumption of Theorem 1.1, is an Abelian group structure. We will actually use the group structure on the sets $[X, P_i]$, $d \le i \le 2d-2$. The *existence* of this group structure follows by standard topological considerations. However, for algorithmic use, we need the group operations represented by explicit, and locally effective, binary operations on the representatives, i.e., simplicial maps $X \to P_i$.

In the proof of the next proposition (omitted in this extended abstract) we construct such operations (which, interestingly, are non-associative in general) by induction on i. A key idea, which allows us to get the local effectivity, is to employ the Eilenberg–Zilber reduction as presented in [7].

Proposition 4.1. *Let Y be a $(d-1)$-connected simplicial set, $d \ge 2$, and let $P_d, P_{d+1}, \dots, P_{2d-2}$ be stages of a locally effective Postnikov system with $2d-2$ stages for Y. Then each P_i has an Abelian H-group structure,*[11] *given by locally effective simplicial maps $\boxplus_i\colon P_i \times P_i \to P_i$ and $\boxminus_i\colon P_i \to P_i$.*

In our subsequent use of this proposition, we also need several additional properties of $\boxplus_i$, $\boxminus_i$; most notably, that the induced maps $[k_{i*}]\colon [X, P_i] \to [X, K_{i+2}]$ and $[p_{i*}]\colon [X, P_i] \to [X, P_{i-1}]$ are homormorphisms. We also omit a precise statement of these additional properties from this extended abstract.

Once we have the operations $\boxplus_i$, $\boxminus_i$ on P_i, by the discussion at the end of Section 3, we obtain the desired locally effective Abelian group structure on $[X, P_i]$ immediately. Specifically, the group operations on $[X, P_i]$ are represented by locally effective maps $\boxplus_{i*}\colon \mathrm{SMap}(X, P_i) \times \mathrm{SMap}(X, P_i) \to \mathrm{SMap}(X, P_i)$ and $\boxminus_{i*}\colon \mathrm{SMap}(X, P_i) \to \mathrm{SMap}(X, P_i)$.

5 The main algorithm

Our main result, Theorem 1.1, is an immediate consequence of the following statement.

Theorem 5.1. *Let X be a simplicial set with finitely many nondegenerate simplices, and let Y be a $(d-1)$-connected simplicial set, $d \ge 2$, for which a locally effective Postnikov system with $2d-2$ stages $P_0, \dots, P_{2d-2}$ is available. Then, for every $i = d, d+1, \dots, 2d-2$, a fully effective representation of $[X, P_i]$ can be computed, with the cochain representations of simplicial maps $X \to P_i$ as representatives.*

Unlike in Theorem 1.1, there is no restriction on $\dim X$ (the assumption $\dim X \le 2d-2$ in Theorem 1.1 is needed only for the isomorphism $[X, Y] \cong [X, P_{2d-2}]$). Also, even if we want Theorem 1.1 only for a simplicial *complex* X, we need Theorem 5.1 with simplicial *sets* X, because of recursion.

Theorem 5.1 is proved by induction on i. The base case is $i = d$ (since $P_0, \dots, P_{d-1}$ are trivial for a $(d-1)$-connected Y), which presents no problem: we have $P_d = L_d = K(\pi_d, d)$, and so $[X, P_d] \cong H^d(X; \pi_d)$. This group is fully effective, since it is the cohomology group of

[11] An Abelian H-group structure on a CW-complex P with basepoint o is given by continuous maps $\mu\colon P \times P \to P$ and $\nu\colon P \to P$ (representing the binary group operation and the group inverse, respectively) such that $\mu(p, o) = \mu(o, p) = p$, μ is *homotopy associative* (meaning that the maps $(p, q, r) \mapsto \mu(p, \mu(q, r))$ and $(p, q, r) \mapsto \mu(\mu(p, q), r)$ are homotopic), *homotopy commutative* (μ is homotopic to $(p, q) \mapsto \mu(q, p)$), and ν is a *homotopy inverse* ($p \mapsto \mu(p, \nu(p))$ is nullhomotopic). We have $\boxplus_i$ in the role of μ and $\boxminus_i$ in the role of ν.

a simplicial set with finitely many nondegenerate simplices, with coefficients in a fully effective group.

So now we consider $i > d$, and we assume that a fully effective representation of $[X, P_{i-1}]$ is available, where the representatives of the homotopy classes $[f] \in [X, P_{i-1}]$ are (cochain representations of) simplicial maps $f : X \to P_{i-1}$. We want to obtain a similar representation for $[X, P_i]$. We describe this on an intuitive level, and then we formulate the algorithm, leaving several pages of a correctness proof for the full version.

Every map $g \in \mathrm{SMap}(X, P_i)$ yields a map $f = p_{i*}(g) = p_i \circ g \in \mathrm{SMap}(X, P_{i-1})$ by projection (forgetting the last coordinate in P_i). We first ask the question of *which* maps $f \in \mathrm{SMap}(X, P_{i-1})$ are obtained as such projections; this is traditionally called the *lifting problem* (and g is called a *lift* of f). Here the answer follows easily from the properties of the Postnikov system: the liftable maps in $[X, P_{i-1}]$ are obtained as the kernel of the homomorphism $[k_{(i-1)*}]$ induced by the Postnikov class. This is very similar to the one-step extension in the setting of obstruction theory, as was mentioned in the introduction.

Next, a single map $f \in \mathrm{SMap}(X, P_{i-1})$ may in general have many lifts g, and we need to describe their structure. This is reasonably straightforward to do on the level of *simplicial maps*. Namely, if $\mathbf{c} = (c^0, \ldots, c^{i-1})$ is the cochain representation of f and g_0 is a fixed lift of f, with cochain representation $(\mathbf{c}, c_0^i)$, then it turns out that all possible lifts g of f are of the form (again in the cochain representation) $(\mathbf{c}, c_0^i + z^i)$, $z^i \in Z^i(X, \pi_i) \cong \mathrm{SMap}(X, L_i)$. Thus, all of these lifts have a simple "coset structure".

This allows us to compute a list of generators of $[X, P_i]$. We also need to find all *relations* of these generators, and for this, we need to be able to test whether two maps $g_1, g_2 \in \mathrm{SMap}(X, P_i)$ are homotopic. Using the group structure, it suffices to test whether a given $g \in \mathrm{SMap}(X, P_i)$ is nullhomotopic. An obvious necessary condition for this is nullhomotopy of the projection $f = p_i \circ g$, which we test recursively. Then, if $f \sim 0$, we $\boxplus_{i*}$-add a suitable nullhomotopic map to g, and this reduces the nullhomotopy test to the case where g has a cochain representation of the form $(\mathbf{0}, z^i)$, $z_i \in Z^i(X, \pi_i) \cong \mathrm{SMap}(X, L_i)$.

Now $(\mathbf{0}, z^i)$ can be nullhomotopic, as a map $X \to P_i$, by an "obvious" nullhomotopy, namely, one "moving" only the last coordinate, or in other words, induced by a nullhomotopy in $\mathrm{SMap}(X, L_i)$. But there may also be "less obvious" nullhomotopies, and it turns out that these correspond to maps $SX \to P_{i-1}$. Thus, in order to be able to test homotopy of maps $X \to P_i$, we also need to compute $[SX, P_{i-1}]$ recursively.

The exact sequence. We will organize the computation of $[X, P_i]$ using an *exact sequence*, a basic tool in algebraic topology and many other branches of mathematics. It goes as follows:

$$[SX, P_{i-1}] \xrightarrow{[\mu_i]} [X, L_i] \xrightarrow{[\lambda_{i*}]} [X, P_i] \xrightarrow{[p_{i*}]} [X, P_{i-1}] \xrightarrow{[k_{(i-1)*}]} [X, K_{i+1}]. \tag{1}$$

This is a sequence of Abelian groups and homomorphisms of these groups, and exactness means that the image of each of the homomorphisms equals the kernel of the successive one.[12]

Here, for example, $k_{(i-1)*}$ is the mapping $\mathrm{SMap}(X, P_{i-1}) \to \mathrm{SMap}(X, K_{i+1})$ induced by the Postnikov class k_{i-1}, and $[k_{(i-1)*}]$ is its (simplicial) homotopy class. The only symbol we haven't yet encountered is the map $\mu_i \colon \mathrm{SMap}(SX, P_{i-1}) \to \mathrm{SMap}(X, L_i)$, which works as follows: given an $F \in \mathrm{SMap}(SX, P_{i-1})$, we compose it with k_{i-1}, which yields $k_{i-1} \circ F \in \mathrm{SMap}(SX, K_{i+1})$ represented by a cocycle in $Z^{i+1}(SX; \pi_i)$. We then re-interpret this cocycle[13] as a cocycle in $Z^i(X; \pi_i)$ representing a map in $\mathrm{SMap}(X, L_i)$, which we declare to be $\mu_i(F)$.

The algorithm for computing $[X, P_i]$ goes as follows.

[12] We remark that the exact sequence (1) can be obtained from the so-called *fibration sequence* in topology. However, since we need all the maps locally effective and also "effective inverses" for some of them, we actually provide (in the full version) a direct, elementary proof of the exactness.

[13] Here we use the fact that there is an obvious bijective correspondence between $C^{i+1}(SX; \pi_i)$ and $C^i(X; \pi)$, which is compatible with the coboundary operators (up to signs).

1. Compute $[X, P_{i-1}]$ fully effective, recursively.
2. Compute $N_{i-1} := \ker[k_{(i-1)*}] \subseteq [X, P_{i-1}]$ (so N_{i-1} consists of all homotopy classes of liftable maps), fully effective, using Lemma 2.1 and Theorem 3.1.
3. Compute $[SX, P_{i-1}]$ fully effective, recursively.
4. Compute the factor group $M_i := \operatorname{coker}[\mu_i] = [X, L_i]/\operatorname{im}[\mu_i]$ using Lemma 2.2, fully effective and including the possibility of computing "witnesses for 0" as in the lemma.
5. The exact sequence (1) can now be transformed to $0 \to M_i \xrightarrow{\ell_i} [X, P_i] \xrightarrow{[p_{i*}]} N_{i-1} \to 0$ (where ℓ_i is induced by exactly the same mapping λ_{i*} of representatives as $[\lambda_{i*}]$ in the original exact sequence (1)). Let $\mathcal{N}_{i-1} := \{f \in \mathrm{SMap}(X, P_{i-1}) : [k_{(i-1)*}(f)] = 0\}$ be the set of representatives of elements in N_{i-1}. Implement a locally effective "section" $\xi_i \colon \mathcal{N}_{i-1} \to \mathrm{SMap}(X, P_i)$ with $[p_{i*} \circ \xi_i] = \mathrm{id}$ and a locally effective "inverse" $r_i \colon \operatorname{im}[\lambda_{i*}] \to M_i$ with $\ell_i \circ r_i = \mathrm{id}$, as in Lemma 2.3, and compute $[X, P_i]$ fully effective using that lemma.

The maps ξ_i and r_i and recursive nullhomotopy testing. We now outline the implementation of the maps ξ_i and r_i, omitting the details and proofs.

The map ξ_i is easy to define. A map $\mathbf{c} \in \mathrm{SMap}(X, P_{i-1})$ (from now on, we don't distinguish between simplicial maps and their cochain representatives) has a lift iff $[k_{(i-1)*}(\mathbf{c})] = 0$, or in other words, iff there is a "witness" cochain c^i with $k_{(i-1)*}(\mathbf{c}) = \delta c^i$. If this holds, we can compute[14] such a c^i and set $\xi_i(\mathbf{c}) := (\mathbf{c}, c^i)$ This involves some arbitrary choice, but if we fix some (arbitrary) rule for choosing c^i, we obtain a locally effective ξ_i as needed.

As for r_i, we need an algorithm that evaluates a map ρ_i representing r_i on the level of simplicial maps. The input is a map $(\mathbf{c}, c^i) \in \mathrm{SMap}(X, P_i)$ with a guarantee that $(\mathbf{c}, c^i) \sim (\mathbf{0}, z^i)$ for some $z^i \in \mathrm{SMap}(X, L_i)$, and the goal is to *compute* some such z^i.

We use the (easy) fact that each nullhomotopy of a map $f \colon X \to Y$, for an arbitrary space Y, can equivalently be regarded as a map $CX \to Y$ extending f (and, if Y is a Kan simplicial set, this also works on the simplicial level). In our case, the assumption $(\mathbf{c}, c^i) \sim (\mathbf{0}, z^i)$ implies $\mathbf{c} \sim \mathbf{0}$, and the main step in evaluating ρ_i is the computation of a simplicial nullhomotopy $\mathbf{b} \in \mathrm{SMap}(CX, P_{i-1})$ for $\mathbf{c}$. Having such a $\mathbf{b}$, we compute an arbitrary lifting $(\mathbf{b}, b_i) \in \mathrm{SMap}(CX, P_i)$ of $\mathbf{b}$ (since CX is contractible, all maps in $\mathrm{SMap}(CX, P_{i-1})$ are liftable), and return $z^i := c^i - (b^i|_X)$ as the desired value of $\rho_i(\mathbf{c}, c^i)$.

It remains to provide an algorithm `NullHom`, which takes as input a $\mathbf{c} \in \mathrm{SMap}(X, P_i)$ with $\mathbf{c} \sim \mathbf{0}$ and returns a nullhomotopy $\mathbf{b} \in \mathrm{SMap}(CX, P_i)$ for it. In the above computation of ρ_i, `NullHom` is invoked with $i-1$ instead of i.

`NullHom` works as follows: It recursively computes a nullhomotopy $\mathbf{b}_0 \in \mathrm{SMap}(CX, P_{i-1})$ for $p_{i*}(\mathbf{c}) \in \mathrm{SMap}(X, P_{i-1})$, obtains an arbitrary lifting $(\mathbf{b}_0, b_0^i) \in \mathrm{SMap}(CX, P_i)$ for it, and sets $z^i := c^i - (b_0^i|_X)$. Then it uses the representation of M_i (coming from Lemma 2.2) to find a "witness for $[z^i] = 0$ in M_i". Concretely, it obtains a map $F \in \mathrm{SMap}(SX, P_{i-1})$ with $[z^i] = [\tilde{z}^i]$, where $\tilde{z}^i = \mu_i(F)$. Finally, `NullHom` computes a nullhomotopy for $z^i - \tilde{z}^i$ in $\mathrm{SMap}(CX, L_i)$, and combines it with $(\mathbf{b}_0, b_0^i)$ and with the map $CX \to P_{i-1}$ corresponding to F. This yields the desired nullhomotopy for $\mathbf{c}$. We refer to the full version for the details.

Having made ρ_i locally effective, we can implement Step 5 of the main algorithm. This completes the outline of the proof of Theorem 5.1.

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[14]Here we use that the cochain groups and coboundary operators of X with coefficients in π_i are fully effective.

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