A combinatorial tool for computing the effective homotopy of iterated loop spaces

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Abstract

This paper is devoted to the *Cradle Theorem*. It is a combinatorial contraction discovered when studying a crucial point of the *effective* Bousfield-Kan spectral sequence, an unavoidable step to make *effective* the famous Adams spectral sequence. The homotopy equivalence $\mathrm{TOP}(S^p, \mathrm{TOP}(S^q, X)) \sim \mathrm{TOP}(S^{p+q}, X)$ is obvious in ordinary topology, not surprising in combinatorial topology, but it happens the proof in the last case is relatively difficult, it is essentially our Cradle Theorem. Based on a simple and natural discrete vector field, it produces also new tools to understand and efficiently implement the Eilenberg-Zilber theorem, the usual one and the twisted one as well. Once the Cradle Theorem is proved, we quickly explain its role in the Bousfield-Kan spectral sequence, via the notion of effective homotopy. An interested reader must know the present paper is elementary, in particular no knowledge of the Bousfield-Kan and Adams spectral sequences is required. The cradle theorem could also be a good opportunity to enter the subject of discrete vector fields and to understand its power in an unexpected field: the structure of simplicial products.

1 Introduction

Given a topological space X, its loop space ΩX is defined as the pointed function space $\Omega X := \mathrm{TOP}(S^1, X) := \mathrm{TOP}((S^1, \star), (X, \star))$, where S^1 is the sphere of dimension 1; that is, the elements of ΩX are continuous maps $\alpha : S^1 \to X$ such that $\alpha(\star) = \star$. If one applies recursively the construction, we obtain the iterated loop space $\Omega^p X$ given by $\Omega^p X = \Omega(\Omega^{p-1} X)$. It can be proved that $\Omega^p X \sim \mathrm{TOP}(S^p, X)$, where S^p is the sphere of dimension p.

A similar construction can be considered in the category of simplicial sets, denoted SS. Given a simplicial set K (with a base point $\star \in K_0$) satisfying the Kan extension property [5] and $p \geq 1$, one can construct the pointed function space $\mathrm{SS}(S^p, K)$; here S^p denotes the simplicial model for the p-sphere, with only two non-degenerate simplices: the base point $\star$ in dimension 0 and s_p in dimension p. It is well-known that $\mathrm{SS}(S^p, -)$ can be seen as a model for the topological iterated loop space Ω^p, and in particular, the homotopy groups of $\mathrm{SS}(S^p, K)$ satisfy $\pi_n(\mathrm{SS}(S^p, K)) \cong \pi_{n+p}(K)$ (see [4]). However, the isomorphism so indirectly obtained is not *combinatorially constructive* so that we do not have the explicit relation between the elements of $\pi_n(\mathrm{SS}(S^p, K))$ and those of $\pi_{n+p}(K)$.

The simplicial functional model for the iterated loop space of a simplicial set K, that is, the function spaces $\mathrm{SS}(S^p, K)$ for $p \geq 1$, appear in the construction of the Bousfield-Kan spectral sequence, an essential Algebraic Topology tool introduced in [2], designed for computing homotopy groups of simplicial sets[1]. More concretely, in order to produce an algorithm computing the different levels of the Bousfield-Kan spectral sequence of a (1-reduced) simplicial set X, one needs to *constructively* compute the homotopy groups of some $\mathrm{SS}(S^p, G_X)$, where G_X is a particular 1-reduced simplicial Abelian group obtained from the initial simplicial set X and $\Omega^p G_X := SS(S^p, G_X)$ is the simplicial functional model for the simplicial iterated loop space of a simplicial set. The definition of G_X is given in [1, Chapter X], it is not used here, our result may be applied to arbitrary reduced simplicial groups. The Cradle theorem is used to *constructively* obtain the necessary correspondence between the *combinatorial* homotopy groups of G_X and those of $\Omega^p G_X$. As already said, it is well-known that $\pi_n(\mathrm{SS}(S^p, G_X)) \cong \pi_{n+p}(G_X)$, but it is necessary to remark here that an explicit isomorphism between these groups is needed to produce an algorithm computing the Bousfield-Kan spectral sequence.

In order to construct the desired explicit isomorphism $\pi_n(\mathrm{SS}(S^p, G_X)) \cong \pi_{n+p}(G_X)$, we apply our new effective homotopy theory [8], an organization designed to *constructively* compute the homotopy groups of simplicial sets. Obtaining the desired isomorphism was expected to be a routine exercise, but its construction is in fact relatively difficult, leading us to use Forman's Discrete Vector Fields [3], revealing also the power of this tool in a totally new domain. The main point is the so-called *Cradle Theorem*, a sophisticated algorithm constructively describing a combinatorial contraction of a general prism over its corresponding *cradle*. The Cradle theorem, or more precisely its underlying technology, has also been used in [9] to give a striking new understanding of basic theorems of Algebraic Topology, namely the Eilenberg- Zilber theorem, the twisted Eilenberg-Zilber theorem, and the Eilenberg-MacLane correspondance between the Classifying Space and Bar constructions.

2 A combinatorial tool: the Cradle Theorem

In this section we present a combinatorial result, named the *Cradle Theorem*, that we will use in Section 3 to develop an algorithm computing the effective homotopy of iterated loop spaces. This combinatorial tool is based on the notion of *Discrete Vector Field*, which is an essential component of Forman's Discrete Morse Theory [3], and has been adapted to the algebraic setting in [7].

In [9] we briefly introduced the Cradle Theorem to present a different understanding of some classical results in Algebraic Topology such as the Eilenberg- Zilber theorem, the twisted Eilenberg-Zilber theorem, and the Eilenberg-MacLane correspondance between the Classifying Space and Bar constructions. The proof of the Cradle Theorem was not given in

[1]Pedro Real's algorithm computing the homotopy groups [6] uses the Whitehead tower, requiring an iterative use of Serre and Eilenberg-Moore spectral sequences, while Bousfield-Kan produces a *unique* spectral sequence, with a rich algebraic structure, leading in particular, if localized at a prime, to the module structure with respect to the corresponding Steenrod algebra, that is, the Adams spectral sequence.

that paper; we use the opportunity of our homotopical problem to give here a detailed proof; it was in fact discovered when processing this essential step of the Bousfield-Kan spectral sequence explained in Section 3.

2.1 Collapses and Discrete Vector Fields

Let us begin by recalling some definitions which will be necessary for our combinatorial result.

Definition 1. *An* elementary collapse *is a pair* (X, A) *of simplicial sets, satisfying the following conditions:*

1. *The component* A *is a* simplicial subset *of the simplicial set* X.
2. *The difference* $X - A$ *is made of exactly* two *non-degenerate simplices* $\tau \in X_n$ *and* $\sigma \in X_{n-1}$, *the second one* σ *being a* face *of the first one* τ.
3. *The incidence relation* $\sigma = \partial_k \tau$ *holds for a* unique *index* $k \in 0...n$.

If the condition 3 is not satisfied, the homotopy types of A and X could be different.

Definition 2. *A* collapse *is a pair* (X, A) *of simplicial sets satisfying the following conditions:*

1. *The component* A *is a* simplicial subset *of the simplicial set* X.
2. *There exists a sequence* $(A_i)_{0 \leq i \leq m}$ *with:*
   (a) $A_0 = A$ *and* $A_m = X$.
   (b) *For every* $0 < i \leq m$, *the pair* (A_i, A_{i-1}) *is an elementary collapse.*

In other words, a collapse is a finite sequence of elementary collapses. If (X, A) is a collapse, then a topological contraction $X \to A$ can be defined.

Definition 3. *Let* (X, A) *be a collapse. A* sequence of elementary collapses *is an ordering* $(\sigma_1, \sigma_2, \ldots, \sigma_{2r-1}, \sigma_{2r})$ *of all non-degenerate simplices of the difference* $X - A$ *satisfying the following properties. Let* $A_0 = A$ *and* $A_i = A_{i-1} \cup \sigma_i$ *for* $1 \leq i \leq 2r$. *Then:*

1. *Every face of* σ_i *is in* A_{i-1}.
2. *The simplex* σ_{2i-1} *is a face of the simplex* σ_{2i}, *so that the pair* (A_{2i}, A_{2i-2}) *is an elementary collapse.*
3. $A_{2r} = X$.

Such a description is a particular case of Forman's *Discrete Vector Field*. In our case the vector field is $V = \{(\sigma_{2i-1}, \sigma_{2i})_{0<i\leq r}\}$.

2.2 The Cradle Theorem

Definition 4. *Let p, q be two natural numbers. The prism $\Delta^{p,q}$ is the simplicial set $\Delta^{p,q} := \Delta^p \times \Delta^q$.*

We have to define the cradle $C^{p,q}$, a simplicial subcomplex of the prism $\Delta^{p,q}$.

Definition 5. *The q-horn Λ^q is the subcomplex $\Lambda^q \subset \Delta^q$ made of all the faces of Δ^q except the ∂_0-face.*

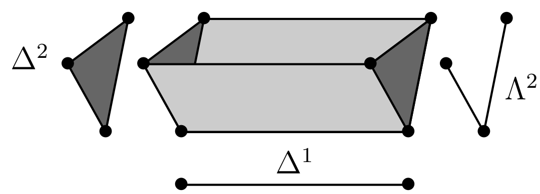
Let us observe that the horn Λ^q is the union of q simplices of dimension $q-1$; adding the missing face $\partial_0 \Delta^q$ would produce the boundary $\partial \Delta^q$ of the q-simplex.

Generalizing the previous definition, one can define $\Lambda^q_j \subset \Delta^q$ for $0 \leq j \leq q$ as the subcomplex made of all the faces of Δ^q except the ∂_j-face. In particular, $\Lambda^q_0 = \Lambda^q$.

Definition 6. *The (p,q)-cradle is the simplicial subcomplex $C^{p,q} \subset \Delta^{p,q}$ defined by:*

$$C^{p,q} := (\Delta^p \times \Lambda^q) \cup (\partial \Delta^p \times \Delta^q)$$

This designation *cradle*, due to Julio Rubio, is inspired by the particular case $p = 1$ and $q = 2$.



Theorem 7. *(Cradle Theorem) An algorithm can be written down:*

- ***Input:***
  - *Integers $p, q \in \mathbb{N}$.*
- ***Output:*** *A sequence of elementary collapses $(\sigma_1, \sigma_2, \ldots, \sigma_{2r-1}, \sigma_{2r})$ for the pair $(X, A) = (\Delta^{p,q}, C^{p,q})$.*

The proof of the Cradle Theorem is "elementary", but relatively difficult, needing a small package of auxiliary notions having their own interest. These notions correspond to the standard terminology for products of simplicial sets (for details, see `http://ncatlab.org/nlab/show/product+of+simplices` or [11, Section 8]).

2.2.1 Triangulations

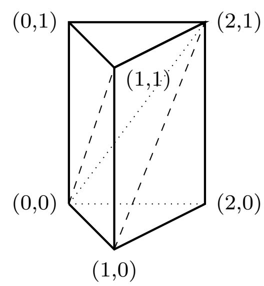
We have to work in the simplicial complex $\Delta^{p,q} = \Delta^p \times \Delta^q$. A vertex of Δ^p is an integer in $\underline{p} = [0 \dots p]$ and a (non-degenerate) d-simplex of Δ^p is a strictly increasing sequence of integers $0 \leq v_0 < \cdots < v_d \leq p$. The same for our second factor Δ^q.

The canonical triangulation of $\Delta^{p,q}$ is made of (non-degenerate) simplices $((v_0, v'_0), \dots, (v_d, v'_d))$ satisfying the relations:

- $0 \leq v_0 \leq v_1 \leq \cdots \leq v_d \leq p$.
- $0 \leq v'_0 \leq v'_1 \leq \cdots \leq v'_d \leq q$.
- $(v_i, v'_i) \neq (v_{i-1}, v'_{i-1})$ for $1 \leq i \leq d$.

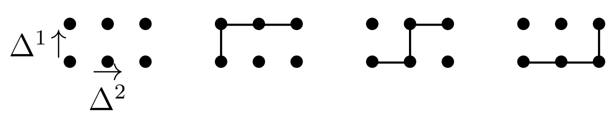
In other words, the canonical triangulation of $\Delta^{p,q} = \Delta^p \times \Delta^q$ is associated to the poset $\underline{p} \times \underline{q}$ endowed with the *product order* of the factors. For example the three maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ are:

- $((0,0),(0,1),(1,1),(2,1))$.
- $((0,0),(1,0),(1,1),(2,1))$.
- $((0,0),(1,0),(2,0),(2,1))$.



2.2.2 Simplex = s-path

We can see the poset $\underline{p} \times \underline{q}$ as a lattice where we arrange the first factor $\underline{p}$ in the horizontal direction and the second factor $\underline{q}$ in the vertical direction. The first figure below is the lattice $\underline{2} \times \underline{1}$ while the other figures are representations of the maximal simplices of $\Delta^{2,1} = \Delta^2 \times \Delta^1$ as *increasing* paths in the lattice.



Definition 8. An *s-path* π of the lattice $\underline{p} \times \underline{q}$ is a finite sequence $\pi = ((a_i, b_i))_{0 \leq i \leq d}$ of elements of $\underline{p} \times \underline{q}$ satisfying $(a_{i-1}, b_{i-1}) < (a_i, b_i)$ for every $1 \leq i \leq d$ with respect to the product order. The d-simplex σ_π represented by the path π is the convex hull of the points (a_i, b_i) in the prism $\Delta^{p,q}$. □

The simplices Δ^p and Δ^q have affine structures which define a product affine structure on $\Delta^{p,q}$, and the notion of convex hull is well defined on $\Delta^{p,q}$.

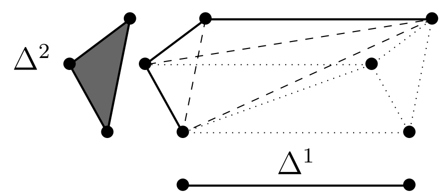
"S-path" stands for "path representing a simplex", more precisely a non-degenerate simplex. Replacing the strict inequality between two successive vertices by a non-strict inequality would lead to analogous representations for degenerate simplices, but such simplices are not to be considered in this section.

This representation of a simplex as an s-path running in a lattice is the key point to master the relatively complex structure of the canonical prism triangulations.

Definition 9. The *last simplex* $\lambda^{p,q}$ of the prism $\Delta^{p,q}$ is the $(p+q)$-simplex defined by the path:

$$\lambda^{p,q} = ((0,0),(0,1),\ldots,(0,q),(1,q),\ldots,(p,q))$$

The path runs some edges of $0 \times \Delta^q$, visiting all the corresponding vertices in the right order; next it runs some edges of $\Delta^p \times q$, visiting all the corresponding vertices also in the right order. Geometrically, the last simplex is the convex hull of the visited vertices. The last simplex of the prism $\Delta^{1,2} = \Delta^1 \times \Delta^2$ is shown in the figure below. The path generating the last simplex is drawn in full lines, the other edges of this last simplex are dashed lines, and the other edges of the prism are in dotted lines.



2.2.3 Subcomplexes

Definition 10. The *hollowed prism* $H\Delta^{p,q} \subset \Delta^{p,q}$ is the difference:

$$H\Delta^{p,q} := \Delta^{p,q} - \mathrm{int}(\text{last simplex}).$$

The faces of the last simplex are retained, but the interior of this simplex is removed.

Definition 11. The *boundary* $\partial\Delta^{p,q}$ of the prism $\Delta^{p,q}$ is defined by:

$$\partial\Delta^{p,q} := (\partial\Delta^p \times \Delta^q) \cup (\Delta^p \times \partial\Delta^q)$$

It is the geometrical Leibniz formula.

We will give a detailed description of the pair $(H\Delta^{p,q}, \partial\Delta^{p,q})$ as a *collapse*, cf. Definition 1; it is a combinatorial version of the well-known topological contractibility of $\Delta^{p,q} - \{*\}$ on $\partial\Delta^{p,q}$ for every point $*$ of the interior of the prism. A very simple admissible vector field will be given to homologically annihilate the difference $H\Delta^{p,q} - \partial\Delta^{p,q}$. In fact, carefully ordering the components of this vector field will give the desired collapse.

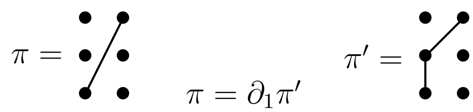
2.2.4 Interior and exterior simplices of a prism

Definition 12. A simplex σ of the prism $\Delta^{p,q}$ is said to be *exterior* if it is included in the boundary of the prism: $\sigma \subset \partial\Delta^{p,q}$. Otherwise the simplex is said to be *interior*. We use the same terminology for the s-paths, implicitly referring to the simplices coded by these paths.

The faces of an exterior simplex are also exterior, but an interior simplex can have faces of both sorts.

Proposition 13. *An s-path π in $\underline{p} \times \underline{q}$ is interior if and only if the projection-paths π_1 on $\underline{p}$ and π_2 on $\underline{q}$ run all the respective vertices of $\underline{p}$ and $\underline{q}$.*

The first s-path π in the figure below represents a 1-simplex *in* $\partial\Delta^{1,2}$, for the point 1 is missing in the projection π_2 on the second factor $\underline{2}$: π is an *exterior* simplex. The second s-path π' represents an *interior* 2-simplex of $\Delta^{1,2}$, for both projections are surjective.

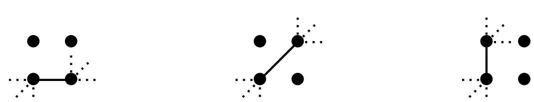


(1)

In particular, if $\pi = ((a_i, b_i))_{0 \le i \le d}$ is an interior simplex of $\Delta^{p,q}$, then necessarily $(a_0, b_0) = (0, 0)$ and $(a_d, b_d) = (p, q)$: an s-path representing an interior simplex of $\Delta^{p,q}$ starts from $(0, 0)$ and arrives at (p, q).

Proof. If for example the first projection of π is not surjective, this means the first projection of the generating path does not run all the vertices of Δ^p, and therefore is included in one of the faces $\partial_k\Delta^p$ of Δ^p. This implies the simplex σ_π is included in $\partial_k\Delta_p \times \Delta_q \subset \partial\Delta^{p,q}$. □

We so obtain a simple description of an interior simplex $((a_i, b_i))_{0 \le i \le d}$: it starts from $(a_0, b_0) = (0, 0)$ and arrives at $(a_d, b_d) = (p, q)$; furthermore, for every $1 \le i \le d$, the difference $(a_i, b_i) - (a_{i-1}, b_{i-1})$ is $(0, 1)$ or $(1, 0)$ or $(1, 1)$: both components of this difference are non-negative, and if one of these components is ≥ 2, then the surjectivity property is not satisfied. In a geometrical way, the only possible *elementary steps* for an s-path π describing an *interior* simplex of $\Delta^{p,q}$ are:



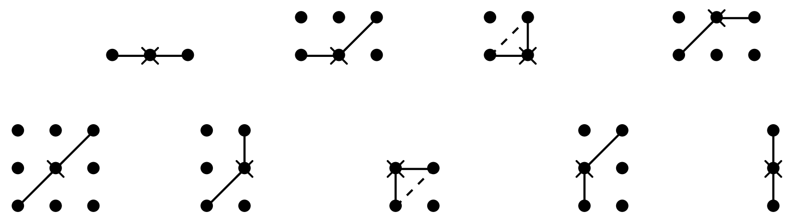
2.2.5 Faces of s-paths

If $\pi = ((a_i, b_i))_{0 \le i \le d}$ represents a d-simplex σ_π of $\Delta^{p,q}$, the face $\partial_k\sigma_\pi$ is represented by the same s-path except the k-th component (a_k, b_k) which is removed: we could say this point of $\underline{p} \times \underline{q}$ is *skipped*. For example in Figure (1) above, $\partial_1\pi' = \pi$. In particular a face of an interior simplex is not necessarily interior.

Proposition 14. *Let $\pi = ((a_i, b_i))_{0 \le i \le d}$ be an s-path representing an interior d-simplex of $\Delta^{p,q}$. The faces $\partial_0\pi$ and $\partial_d\pi$ are certainly* not *interior. For $1 \le k \le d-1$, the face $\partial_k\pi$ is interior if and only if the point (a_k, b_k) is a right-angle bend of the s-path π in the lattice $\underline{p} \times \underline{q}$.*

Proof. Removing the vertex $(a_0, b_0) = (0, 0)$ certainly makes non-surjective a projection π_1 or π_2 (or both if $(a_1, b_1) = (1, 1)$). The same if the last point (a_d, b_d) is removed.

If we examine now the case of $\partial_k \pi$ for $1 \leq k \leq d-1$, nine possible configurations for two consecutive elementary steps before and after the vertex (a_k, b_k) to be removed:



(2)

In these figures, the intermediate point of the displayed part of the considered s-path is assumed to be the point (a_k, b_k) of the lattice, to be removed to obtain the face $\partial_k \pi$. In the cases 1, 2, 4 and 5, skipping this point makes non-surjective the first projection π_1 on $\underline{p}$. In the cases 5, 6, 8 and 9, the second projection π_2 on $\underline{q}$ becomes non-surjective. There remain the cases 3 and 7 where the announced right-angle bend is observed. □

2.2.6 The Hollowed Prism Theorem

Theorem 15. *(Hollowed Prism Theorem) The pair $(H\Delta^{p,q}, \partial\Delta^{p,q})$ is a collapse.*

The hollowed prism can be collapsed on the boundary of the same prism.

Proof. The proof is recursive with respect to the pair (p, q). If $p = 0$, the boundary of $\Delta^0 = *$ is void, so that the boundary of $\Delta^{0,q}$ is simply $\partial\Delta^q$; the last simplex is the unique q-simplex, the hollowed prism $H\Delta^{0,q}$ is also $\partial\Delta^q$: the desired collapse is trivial, more precisely the corresponding vector field is empty. The same if $q = 0$ for the pair $(H\Delta^{p,0}, \partial\Delta^{p,0})$.

The simplices of $H\Delta^{p,q} - \partial\Delta^{p,q}$ are all the interior simplices of $\Delta^{p,q}$, except the last simplex. In particular, the corresponding s-paths satisfy the condition explained in Proposition 13.

Now we prove the general case (p, q) with $p, q > 0$, assuming the proofs of the cases $(p-1, q-1)$, $(p, q-1)$ and $(p-1, q)$ are available. Three justifying filling sequences are available; it is more convenient to see the sequences of simplices as sequences of *s-paths*:

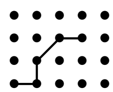
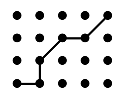
- $\Sigma^1 = (\pi_i^1)_{0<i\leq 2r_1}$ for $\Delta^{p-1,q-1} = \partial_p\Delta^p \times \partial_q\Delta^q$.
- $\Sigma^2 = (\pi_i^2)_{0<i\leq 2r_2}$ for $\Delta^{p,q-1} = \Delta^p \times \partial_q\Delta^q$.
- $\Sigma^3 = (\pi_i^3)_{0<i\leq 2r_3}$ for $\Delta^{p-1,q} = \partial_p\Delta^p \times \Delta^q$.

All the components of these filling sequences can be viewed as s-paths starting from $(0, 0)$ and going to $(p-1, q-1)$ (resp. $(p, q-1)$, $(p-1, q)$).

These filling sequences are made of all the non-degenerate s-paths (simplices) of the difference $H\Delta^{*,*} - \partial\Delta^{*,*}$, ordered in such a way every face of an s-path is either interior *and* present *beforehand* in the list, or exterior; furthermore, for the s-paths of even index, the previous one is one of its faces. Using these sequences, we must construct an analogous sequence for the bidimension (p, q).

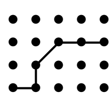
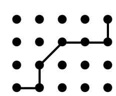
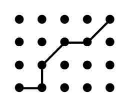
Every s-path π_i^j of dimension d can be completed into an interior s-path $\overline{\pi}_i^j$ of dimension $d+1$ in $\underline{p} \times \underline{q}$ in a unique way, adding a last diagonal step $((p-1, q-1), (p, q))$ if $j = 1$, or a last vertical step $((p, q-1), (p, q))$ if $j = 2$, or a last horizontal step $((p-1, q), (p, q))$ if $j = 3$. Conversely, every interior s-path of $\Delta^{p,q}$ can be obtained from an interior s-path of $\Delta^{p-1,q-1}$, $\Delta^{p,q-1}$ or $\Delta^{p-1,q}$ in a unique way by this completion process.

For example, in the next figure, we illustrate how an s-path π_i^1 of $\underline{3} \times \underline{2}$ can be completed into an s-path $\overline{\pi}_i^1$ of $\underline{4} \times \underline{3}$:

$\pi_i^1 =$  $\overline{\pi}_i^1 =$ 

Adding such a last diagonal step *does not add* any right-angle bend in the s-path, so that the assumed incidence properties of the initial sequence $\Sigma^1 = (\pi_1^1, \ldots, \pi_{2r_1}^1)$ are essentially preserved in the completed sequence $\overline{\Sigma}^1 = (\overline{\pi}_1^1, \ldots, \overline{\pi}_{2r_1}^1)$: the faces of each s-path are already present in the sequence or are exterior; in the even case $\pi_{2i}^1 \in \Sigma^1$, the previous s-path π_{2i-1}^1 is a face of π_{2i}^1 and hence $\overline{\pi}_{2i-1}^1$ is a face of $\overline{\pi}_{2i}^1$. For example in the illustration above, if i is even, certainly $\partial_1 \pi_i^1 = \pi_{i-1}^1$ (for this face is the only interior face) and this implies also $\partial_1 \overline{\pi}_i^1 = \overline{\pi}_{i-1}^1$.

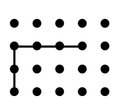
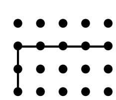
On the contrary, in the case $j = 2$, the completion process *can* add one right-angle bend, no more. For example, in this illustration:

$\pi_i^2 =$  $\overline{\pi}_i^2 =$  $\partial_5 \overline{\pi}_i^2 =$ 

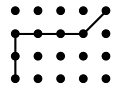
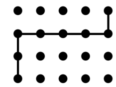
If the index i is even, then $\partial_1 \pi_i^2 = \pi_{i-1}^2$ and the relation $\partial_1 \overline{\pi}_i^2 = \overline{\pi}_{i-1}^2$ is satisfied as well. But another face of $\overline{\pi}_i^2$ is interior, namely $\partial_5 \overline{\pi}_i^2$, generated by the new right-angle bend; because of the diagonal nature of the first step of this face, this face is present in the list $\overline{\Sigma}^1$, see the previous illustration.

Which is explained about $\overline{\Sigma}^2$ with respect to $\overline{\Sigma}^1$ is valid as well for the list $\overline{\Sigma}^3$ with respect to $\overline{\Sigma}^1$.

The so-called last simplices, see Definition 9, must not be forgotten! The last simplex $\lambda^{p-1,q-1}$ (resp. $\lambda^{p,q-1}$) *is not* in the list Σ_1 (resp. Σ_2): these lists describe the collapses of the *hollowed* prisms over the corresponding boundaries: all the interior simplices are in these lists except the last ones. The figure below gives these simplices in the case $(p, q) = (4, 3)$:

$\lambda^{3,2} =$  $\lambda^{4,2} =$ 

Examining now the respective completed paths:

$\overline{\lambda}^{3,2} =$  $\overline{\lambda}^{4,2} =$ 

shows that $\partial_{p+q-1}\overline{\lambda}^{p,q-1} = \overline{\lambda}^{p-1,q-1}$; also the faces $\partial_{q-1}\overline{\lambda}^{p-1,q-1}$ and $\partial_{q-1}\overline{\lambda}^{p,q-1}$ are respectively in $\overline{\Sigma}^1$ and $\overline{\Sigma}^2$.

Putting together all these facts leads to the conclusion: if Σ^1, Σ^2 and Σ^3 are respective filling sequences for $(H\Delta^{(*,*)} - \partial\Delta^{(*,*)})$, with $(*,*) = (p-1,q-1)$, $(p,q-1)$ and $(p-1,q)$ then the following list *is a filling sequence proving the desired collapse property* for the indices (p,q):

$$\overline{\Sigma}^1 \ || \ \overline{\Sigma}^2 \ || \ (\overline{\lambda}^{p-1,q-1}, \overline{\lambda}^{p,q-1}) \ || \ \overline{\Sigma}^3$$

where '||' is the list concatenation. Fortunately, the last simplex $\lambda^{p,q} = \overline{\lambda}^{p-1,q}$ is the only interior simplex of $\Delta^{p,q}$ missing in this list. □

Let us remark that in [7] a *different* (but totally analogous) Hollowed Prism Theorem was proved; there the last simplex $\lambda^{p,q}$ is the $(p+q)$-simplex of $\Delta^{p,q}$

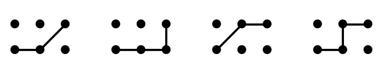
$$\lambda^{p,q} = ((0,0),(1,0),\ldots,(p,0),(p,1),\ldots,(p,q))$$

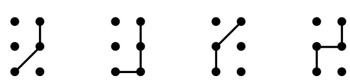
and the proof of the theorem is also done by induction, assuming the proofs of the cases $(p-1,q-1)$, $(p,q-1)$ and $(p-1,q)$ are available.

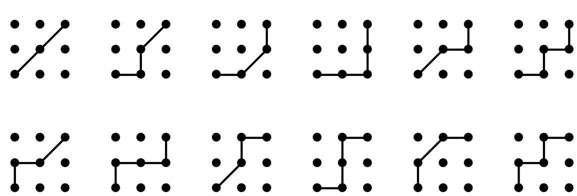
2.2.7 Examples

The reader can apply himself the above algorithm for the small dimensions. The table below gives all the results for $(p,q) \leq (2,2)$.

$(p,q) = (1,1)$ 

$(p,q) = (2,1)$ 

$(p,q) = (1,2)$ 

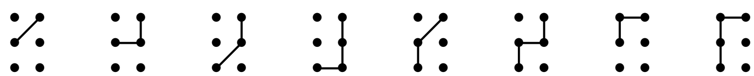
$(p,q) = (2,2)$ 

For significantly bigger values of (p,q), only a program can produce the corresponding filling sequences. A short Lisp program (45 lines) can produce the justifying list for reasonably small values of p and q. For example, if $p = q = 8$, the filling sequence is made of 265,728 paths, a list produced in 4 seconds on a modest laptop. But if $p = q = 10$, the number of paths is 8,097,452; and the same laptop is then out of memory. A typical behaviour in front of exponential complexity: the necessary number of paths is $> 3^p$ if $p = q > 1$.

2.2.8 Proof of the Cradle Theorem

Proof. We must again construct a justifying list. The difference $\Delta^{p,q} - C^{p,q}$ is made of the interior $\Delta^{p,q} - \partial\Delta^{p,q}$ of the prism $\Delta^{p,q}$ and the interior of the highest face $\Delta^p \times \partial_0\Delta^q$. The justifying list of $(H\Delta^{p,q-1}, \partial\Delta^{p,q-1})$ fills in this highest face except its last simplex $\lambda^{p,q-1}$; then the justifying list of $(H\Delta^{p,q}, \partial\Delta^{p,q})$ fills in the rest of the prism $\Delta^{p,q}$ except its last simplex $\lambda^{p,q}$; there remains to complete by both last simplices. □

For example, the s-path list for the pair $(\Delta^{1,2}, C^{1,2})$ with respect to the genuine cradle $C^{1,2}$ is the following.



Remark 16. *Similar collapses* $(\Delta^{p,q}, C_j^{p,q})$ *can be obtained for* $C_j^{p,q} := (\Delta^p \times \Lambda_j^q) \cup (\partial\Delta^p \times \Delta^q)$, *for* $0 \leq j \leq q$.

3 Effective homotopy of iterated loop spaces

In this section we briefly explain the role of the Cradle theorem for the computation of effective homotopy of iterated loop spaces, a necessary ingredient for the construction of an algorithm computing the Bousfield-Kan spectral sequence. See [10] for a more complete description of the results presented in this section.

The Bousfield-Kan spectral sequence is an Algebraic Topology tool designed for computing homotopy groups of simplicial sets. One of the ingredients for the development of an algorithm computing the different levels of the Bousfield-Kan spectral sequence of a (1-reduced) simplicial set X consists in *constructively* computing the homotopy groups of some $\Omega^p G_X$, where G_X is a particular 1-reduced simplicial Abelian group obtained from the initial simplicial set X (see in [1] the definition of these simplicial groups G_X and more information on the construction of the Bousfield-Kan spectral sequence) and $\Omega^p G_X := \mathrm{SS}(S^p, G_X)$ is the simplicial functional model for the iterated loop space of a simplicial set. The *Cradle theorem* is used to this aim.

The *effective homotopy* method was introduced in [8] trying to compute homotopy groups of Kan simplicial sets, a challenging problem in the field of Algebraic Topology. It is necessary that the Kan property [5] is satisfied in a constructive way, that is, the desired x is given explicitly by an algorithm (and then the simplicial set K is said to be *constructive*). In [8] the notion of *object with effective homotopy* was defined, consisting in four algorithms describing in a *constructive* way the homotopy groups of a constructive Kan simplicial set, which is said to have *effective homotopy*. The main point is the following: if a "reasonable" combinatorial topological construction is made using objects with effective homotopy, then the resulting object is also with effective homotopy. As a first work in this research, we presented in [8] a result allowing one to compute the effective homotopy of the total space of a *constructive Kan fibration* [8] if the base and fiber spaces are objects with effective homotopy. Similar algorithms have been developed computing the effective homotopy of the fiber (resp. base) space of a fibration when the effective homotopies of the base (resp. fiber) and the total space are known.

Given a simplicial set G (with a base point $\star$) and an integer $p \geq 1$, the pointed function space $\Omega^p G := \mathrm{SS}(S^p, G)$ is a simplicial set whose q-simplices are maps $\alpha : S^p \times \Delta^q = \Delta^p/\partial\Delta^p \times \Delta^q \to G$ such that $\alpha(\star, \sigma) = \star$ for every $\sigma \in \Delta^q$ (see [5] for the definition of face and degeneracy operators). For $p = 1$, $\Omega G := \mathrm{SS}(S^1, G)$ is known to be a model for the loop space of G, that is, it is the fiber of a fibration $\Omega G \hookrightarrow PG \xrightarrow{f} G$ where PG is a contractible space. In our case PG is the pointed function space $PG := \mathrm{SS}(\Delta^1, G)$, which is known to be contractible, the base space G is seen as $G \cong \mathrm{SS}(\Delta^0, G)$, and $f = (\partial^0)^* : PG = \mathrm{SS}(\Delta^1, G) \to \mathrm{SS}(\Delta^0, G) \cong G$ is induced by $\partial^0 : \Delta^0 \to \Delta^1$. From the long homotopy exact sequence of the fibration one can easily deduce that $\pi_*(\Omega G) \cong \pi_{*+1}(G)$. However, it is necessary to remark that the isomorphism obtained from the long exact sequence of homotopy is not constructive, and therefore we do not have the explicit correspondence between both groups. For $p > 1$, one has also fibrations $\Omega^p G \hookrightarrow P^p G \xrightarrow{(\partial^0)^*} \Omega^{p-1} G$ where $P^p G := \mathrm{SS}(\Delta^p/\Lambda^p, G)$ is a again a contractible simplicial Abelian group. As before, the long exact sequence of homotopy implies $\pi_*(\Omega^p G) \cong \pi_{*+1}(\Omega^{p-1} G)$ and then one has $\pi_*(\Omega^p G) \cong \pi_{*+p}(G)$, but again the isomorphism is not explicit.

In order to compute in a constructive way the homotopy groups of the spaces $\Omega^p G$, we determine in an iterative way the effective homotopy of these spaces. We start with the fibration $\Omega G \hookrightarrow PG \xrightarrow{f} G$. The three spaces in the fibration are simplicial Abelian groups and therefore their Kan property is constructive (see [5]). Then, on the one hand, the base space G is supposed to have effective homotopy. On the other hand, the total space PG is known to be contractible. If we make PG an object with effective homotopy (defining the four necessary algorithms, see [8]) and we prove that the fibration $f = (\partial^0)^*$ is a constructive Kan fibration, then our algorithms provide the effective homotopy of the fiber ΩG. For $p > 1$, one has the fibrations $\Omega^p G \hookrightarrow P^p G \xrightarrow{(\partial^0)^*} \Omega^{p-1} G$. As before, the three spaces in the fibration are simplicial Abelian groups and therefore constructive Kan simplicial sets. We suppose by induction that $\Omega^{p-1} G$ has effective homotopy. If results are given proving that the fibration $f = (\partial^0)^*$ is a constructive Kan fibration and that the total space $P^p G := \mathrm{SS}(\Delta^p/\Lambda^p, G)$ has effective homotopy, we could also obtain the desired effective homotopy of $\Omega^p G$. All these necessary intermediate results are obtained by means of the Cradle Theorem, which plays an essential role in the proofs of the two following lemmas (see [10] for details).

Lemma 17. *The fibrations $f = (\partial^0)^* : P^p G \to \Omega^{p-1} G$ are constructive Kan fibrations.*

Lemma 18. *An algorithm can be written down computing the effective homotopy of* $P^pG := SS(\Delta^p/\Lambda^p, G)$*.*

This leads to the following theorem, computing the desired effective homotopy of iterated loop spaces.

Theorem 19. *An algorithm can be written down:*

- ***Input:***
  - *A* 1*-reduced simplicial Abelian group* G*.*
  - *A solution for the homotopical problem of* G*,* $SHmtP_G$*.*
  - *An index* $p \geq 1$*.*
- ***Output:*** *A* $SHmtP_{\Omega^p G}$ *for the functional model of the iterated loop space* $\Omega^p G := SS(S^p, G)$*.*

This allows us to compute the effective homotopy of the spaces $\Omega^p G$ and then we obtain the following theorem, which will be used for the computation of the Bousfield-Kan spectral sequence.

Corollary 20. *Let* G *be a* 1*-reduced simplicial Abelian group* G *with effective homotopy and* $p, n \geq 1$*. Then one can construct an explicit isomorphism:*

$$\pi_n(\Omega^p G) \cong \pi_{n+p}(G)$$

In this way, the combination of the effective homotopy method and our combinatorial Cradle Theorem makes it possible to *constructively* determine the homotopy groups of iterated loop spaces. Once we have the space $\Omega^p G$ with effective homotopy, it could be used inside other fibrations or some constructions in Algebraic Topology to determine the effective homotopy of complicated spaces. Moreover, the effective homotopy of the iterated loop spaces $\Omega^p G$ will be used for the construction of an algorithm computing the different levels of the Bousfield-Kan spectral sequence in a forthcoming paper.

4 Conclusions

Given a simplicial group G and $p \geq 1$, the functional space $\Omega^p G := \mathrm{SS}(S^p, G)$ can be seen as a model for the iterated loop space of a topological space. In particular, it is well-known that $\pi_n(\Omega^p G) \cong \pi_{n+p}(G)$ for every $n \geq 1$, but this isomorphism is not constructive.

This paper has presented the *Cradle Theorem*, a combinatorial result based on the notion of Discrete Vector Field [3] introduced by Robin Forman, which was used in [9] to explain a totally new understanding of some basic results in Algebraic Topology. Now we combine this combinatorial result with our new *effective homotopy* theory [8] to compute in a constructive way the homotopy groups of iterated loop spaces. The result is an algorithm producing the effective homotopy of $\Omega^p G$ for every simplicial Abelian group G with effective homotopy. Once we have the effective homotopy of $\Omega^p G$, this space could be considered inside other fibrations or some constructions in Algebraic Topology to determine the effective homotopy of other spaces. Moreover, the effective homotopy of $\Omega^p G$ is one of the necessary ingredients for the development of an algorithm computing the different levels of the Bousfield-Kan spectral sequence [2] associated with a simplicial set.

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